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On the PLF Construction for the Absolute Stability Study of Dynamical Systems with Non-Constant Gain

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Abstract: This paper deals with the absolute stability analysis of uncertain systems formulated in linear differential inclusion. It presents an approach based on the representation of a polyhedral positively invariant set by its vertices, allowing to construct the associated Lyapunov function. Efficiency of the method is discussed through a numerical example, where the absolute stability of a third order system has been analyzed via the construction of a Polyhedral Lyapunov Function (PLF). The flexibility of the proposed mesh and the check procedure of Molchanov–Pyatintskii conditions give a larger parameterized absolute stability domain than the one obtained by others existing in the literature.

Keywords: *absolute stability; uncertain systems; polyhedral Lyapunov function; sphere triangulation; linear programming.*

Mathematics Subject Classification (2000): 93D20, 93C10.

1 Introduction

Complex systems have always been difficult in their modeling and stability analysis since they may present nonlinearities and/or uncertainties. The problem has to do with non-linear systems formulated in differential inclusions, where it is worth to decide about their largest parameterized domain of variation of the non-constant gain without loss of their stability [19]. Several criteria have been developed as a solution of this problem such as the circle criterion [1], Popov criterion [2] and Borne and Gentina criterion [3]. However all these criteria give sufficient but not necessary conditions of stability.
The Second Lyapunov method is a powerful tool of the stability analysis for nonlinear or uncertain system. However, its implementation is dependent on the choice and the way

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of construction of the Lyapunov function. The well-known class of quadratic functions is the most common one [4]. However this kind of functions doesn't lead usually to the best solution, since the existence of a quadratic Lyapunov function is not a necessary condition of stability. Recently, a generalization of quadratic functions have been introduced in the context of constrained control and they are called composite quadratic Lyapunov functions [5]. Some classes of non-quadratic Lyapunov functions are introduced such as polynomial homogenous functions [6]. The class of piecewise linear functions [7] which is a universal class, since their construction represents necessary and sufficient condition of stability, was introduced for stability analysis and control [8]. A sub-class is the one of polyhedral Lyapunov functions, a set-induced functions, have positively invariant polyhedral sub-level sets [20]. Therefore, their construction is based on an operation of scaling of the set boundary.

Several approaches have been established for the construction of polyhedral Lyapunov functions, the plane representation of the sub-level set have been considered to determine the absolute stability boundary of a second order system [9, 18]. The symmetric representation of the set by its vertices is used to construct a polyhedral Lyapunov function for third order uncertain system [10]. The technique of Ray-gridding is another issue for scaling [11, 12] based on uniform partitions of the state space in terms of ray directions allowing stability analysis of linear switched systems.

This paper is devoted to the stability analysis of third order uncertain systems by constructing a polyhedral Lyapunov function. We propose to represent the positively invariant set by its vertices obtained by a surface sphere triangulation [16]. This kind of representation with an associated algorithm enables to enhance the set of parameters variations, the obtained boundaries of the uncertainty are larger than those obtained by existent approaches in the literature. The paper is organized as follows: First, we remind some properties of the polyhedral sets and of their associated Lyapunov functions. Then the procedure used for the computation of the Polyhedral Lyapunov Function is presented, the efficiency of the approach is illustrated by an example. Conclusions are summarized in the end.

2 Polyhedral Lyapunov Function

Our interest in this study is the construction of polyhedral Lyapunov functions, which are induced by polyhedral positively invariant sets. These sets present several theoretical and practical advantages over the ellipsoids, but they suffer from the problem of complexity of their representation.

We remind here that a polyhedral set can be represented by:

$$\mathcal{P}(F) = \{x : Fx \leq \bar{1}\} \tag{2.1}$$

or by its dual form:

$$\mathcal{V}(X) = \{\ x = Xz, \bar{1}^T z \leq 1, z \geq 0\}, \tag{2.2}$$

where $\bar{1} = [1, 1, ..., 1]^T$, F and X are $N \times n$-matrices.

The polyhedral set can be also represented by its rays, we denote by $R_N(\lambda), 0 < \lambda \leq 1$ the ray-polytope which is a scaled version of

$$R_N(\bar{1}) = \mathrm{conv}\big\{ \cos(\frac{2\pi k}{N}), \sin(\frac{2\pi k}{N}), 0 \leq k \leq N \big\},$$

where $\mathrm{conv}\{V\}$ denotes the convex hull of a set of vertices V.

Given a C-set $S \subset \mathbb{R}^n$ (a convex and compact subset of $\mathbb{R}^n$ including the origin as interior point), it is always possible to define a function, named Minkowski function, which is essentially the function whose sub-level sets are achieved by linearly scaling the set S.

Definition 2.1 [8] Given a C-set S, its Minkowski function is defined by :

$$\psi_S(x) = \inf\{\lambda \geq 0 : x \in \lambda S\}. \tag{2.3}$$

The Minkowski function ψ_S satisfies the following properties [13]:

- It is positive definite : $0 \leq \psi_S(x) \leq \infty$ and $\psi_S(x) > 0$ for all $x \neq 0$.
- It is positively homogeneous of order 1: $\psi_S(\lambda x) = \lambda \psi_S(x)$ for $\lambda \geq 0$.
- It is sub-additive: $\psi_S(x_1 + x_2) \leq \psi_S(x_1) + \psi_S(x_2)$.
- It is continuous.
- Its unit ball is $S = \{x : \psi_S(x) \leq 1\}$.
- It is convex.

If a polyhedral C-set is considered, the Minkowski functions deriving from the representations (2.1) and (2.2) are:

$$\psi_{\mathcal{P}(F)}(x) = \max\{Fx\} = \max_i\{F_i x\} \tag{2.4}$$

and

$$\psi_{\mathcal{V}(X)}(x) = \min\{\bar{1}\mu, x = X\mu, \mu \geq 0\}. \tag{2.5}$$

Consider a system (possibly resulting from a feedback connection) of the form:

$$\dot{x}(t) = f(x(t)). \tag{2.6}$$

For a convex (possibly non-differentiable) Lyapunov function $\psi(x)$, its Lyapunov derivative is defined by [14]:

$$D^+\psi(x) = \max_{i \in \mathcal{I}(x)} F_i f(x), \tag{2.7}$$

where $\mathcal{I} = \{i : F_i(x) = \psi_{\mathcal{P}(F)}(x)\}$ and D^+ denotes the upper-right Dini derivative defined by:

$$D^+\psi(x) = \limsup_{h \to 0^+} \frac{\psi(x + hf(x)) - \psi(x)}{h}.$$

3 Absolute Stability Theorem

We consider the following Linear Differential Inclusion (LDI) given by:

$$\dot{x} \in \Big\{Ax, A = \sum_{i=1}^{K} \alpha_i A_i, \alpha_i \geq 0, \sum_{i=1}^{K} \alpha_i = 1\Big\}. \tag{3.1}$$

The matrices $A_1, A_2, ...A_K \in \mathbb{R}^{n \times n}$ are vertices of the matrix polytope.

Theorem 3.1 [15] *The function $\psi_S(x)$ (2.3) induced by S represented as in (2.2) is a Lyapunov function for the system (3.1) which guarantees its absolute stability (respectively the polytope S is a positively invariant set) if and only if there exists K matrices $H_i \in \mathbb{R}^{N\times N}$, $i = 1, 2, ..., K$, each of them verifies:*

$$h_{kk}^{(i)} + \Sigma_{j=1_{j\neq k}}^{N} h_{kj}^{(i)} < 0 \tag{3.2}$$

for all $1 \le k \le N$, $h_{kj}^{(i)}$ denotes the elements of the matrix H_i,

$$A_i X = X H_i \tag{3.3}$$

where $X = [x_1, x_2, ..., x_N] \in \mathbb{R}^{n\times N}$ is the matrix containing the vertices of S.

4 Polyhedral Lyapunov Function Construction for Third Order System

First, the computation of Polyhedral Lyapunov Function needs the definition of an arbitrary set. The scaling of its vertices allows to get a positively invariant set which defines a sub-level set of the Lyapunov function.

4.1 Representation of the polyhedral set

The plane representation of the set for a third order system needs a tedious computation complexity. We propose to represent the set by its vertices, which are obtained by a surface triangulation of the unit sphere [16]. This triangulation is obtained by a Matlab function which uses recursive subdivision. The first approximation is a platonic solid, an octahedron (Figure 4.1).

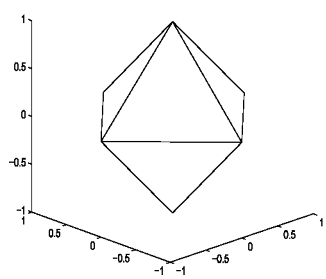

Figure 4.1: Octahedron.

This shape is defined by the vertices $[1, 0, 0]$, $[-1, 0, 0]$, $[0, 1, 0]$, $[0, -1, 0]$, $[0, 0, 1]$ and $[0, 0, -1]$. Each level of refinement subdivides each triangle face by a factor of 4 (Figure 4.2).

At each level of refinement, the vertices are projected to the sphere surface. Thus we define the arbitrary set S^A (Figure 4.3).

4.2 Generation of the Lyapunov function

After the definition of the arbitrary polytope S^A, the determination of the positively invariant set (level set of the associated Lyapunov function) is based on checking the

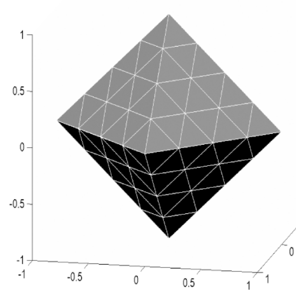

Figure 4.2: The octahedron obtained after two levels of refinement.

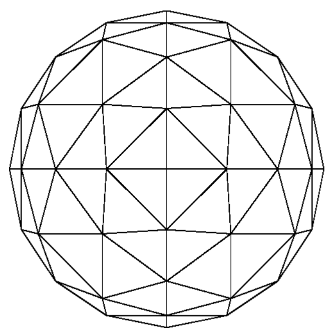

Figure 4.3: The polytope S^A obtained by a Surface Triangulation of the unit sphere.

two conditions (3.2) and (3.3) of theorem 3.1. Thus, the following linear program is formulated:

- For each vertex x_k, for all $k = 1, 2, ..., N$ we denote by $V(k)$ the matrix obtained by the neighbored vertices

$$V(k) = [-x_k, x_k, x_{1(k)}, x_{2(k)}, ..., x_{L(k)}] \tag{4.1}$$

  for all $k = 1, 2..., N$, where $x_{l(k)}$, for all $l(k) = 1, 2, ..., L(k)$ are the neighbored vertices of x_k.

- We resolve the following linear program:

$$\begin{array}{c} \max \quad FA_i x_k \\ FV(k) \leq \mathcal{J}^T \end{array} \tag{4.2}$$

  for all $k = 1, 2, ..., N$, where $\mathcal{J} = [-1, 1, 1...., 1]^T$ is a $\mathbb{R}^{L(k)+2}$ vector. The dual of the linear program (4.2) can be written:

$$\begin{array}{c} \min \quad \mathcal{J}^T \lambda(k), \\ V(k)\lambda(k) = A_i x_k \end{array} \tag{4.3}$$

where $\lambda(k) \in \mathbb{R}^{L(k)+2}$ is a vector containing the Lagrange multipliers relative to the linear program (4.3). We construct each column of the matrix H_i, for all $i = 1, 2, ..., K$ from the elements of the vectors $\lambda(k)$, $k = 1, 2, ..., N$:

$$h_{kk}^{(i)} = -\lambda_1(k), \quad h_{l(k)k}^{(i)} = \lambda_{l(k)+2}(k). \tag{4.4}$$

All the other components of H_i are equal to zero. With such a construction of H_i, $i = 1, 2, ..., K$, the condition (3.3) is well satisfied.

The computation of the matrices H_i, for all $i = 1, 2, ..., K$ followed by an operation of scaling the vertices of S^A leads to the construction of the modified polytope S^D with vertices contained in X^D. This operation consists in replacing the matrix X by $X^D = XD^{-1}$ where $D = \text{diag}(d_1, d_2, ..., d_N)$ is a diagonal matrix. The vector $d = [d_1, d_2, d_3, ..., d_N]^T$ is obtained by solving the following linear program:

$$\begin{gathered} \min \quad z, \\ \begin{bmatrix} |H_1|^T \\ |H_2|^T \\ \vdots \\ |H_K|^T \end{bmatrix} d - \mathbf{1}z \leq 0, \; d \geq 0, \; z \geq -100 \end{gathered} \tag{4.5}$$

where $|H_i|$ is the matrix obtained from H_i by replacing only the off-diagonal elements by their absolute values. $\mathbf{1}$ denotes the vector of appropriate dimension, of which all entries are equal to one.

5 Numerical Example

We consider the following system with nonlinear feedback gain defined by Figure 5.1. If we consider an output linear gain, we can prove that the stability condition is a positive unlimited gain. But where the gain is non-constant, we have to determine the largest domain $[k_{\min}, k_{\max}]$ in which the nonlinear gain $\frac{\sigma(y,t)}{y}$ may vary without loss of the system stability:

$$k_{\min} \leq \frac{\sigma(y,t)}{y} \leq k_{\max}, \quad y \neq 0. \tag{5.1}$$

The absolute stability of the considered system is equivalent to that of the Linear Differential Inclusion defined by the two vertices of the matrix polytopes:

$$A_1 = \begin{bmatrix} -10 & -10k_{\min} & -10k_{\min} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -10 & -10k_{\max} & -10k_{\max} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{5.2}$$

Let us set $k_{\min} = 0.2$. The problem is to determine $k_{\max}$ such that the system is absolutely stable. As long as the linear program (4.5) is feasible, we get an optimal solution $z_{opt} = -100$, which gives the associate scaling vector $d > 0$. Then the associated Lyapunov function is $\psi_{S^D}(x) = \inf\{\mu \geq 0 : x \in \mu S^D\}$.

With $N = 66$ vertices, the obtained upper boundary is $k_{\max} = 2.24$ which is upper than the values obtained by other developed criteria and approaches. Indeed the Circle

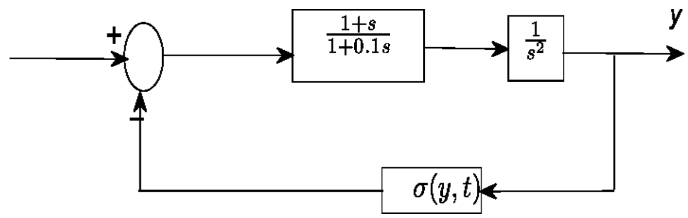

Figure 5.1: The studied system.

criterion leads to $k_{\max} = 0.5467$. The representation of the polytope with 6402 vertices [10], gives $k_{\max} = 1$ and the application of the ray-gridding technique [12] provides $k_{\max} = 1.5$. This comparison study shows the importance of the proposed procedure of PLF construction from the point of view of the width of the absolute stability domain and the reduction of the number of vertices which simplifies the computation complexity.

6 Conclusion

In this paper, we have dealt with the problem of the construction of a polyhedral lyapunov function for the absolute stability analysis of uncertain systems formulated on linear differential inclusion. It has been proved that the choice of a flexible representation of the polytope and the application of a suitable technique of scaling adjust its shape to some demands. The representation of the polytope by its vertices obtained by the proposed surface triangulation of the unit sphere associated with a suitable technique of scaling allows a convenient application of the Molchanov–Pyatintskii theorem. The comparison of the proposed procedure with other criteria and approaches has shown its availability and its efficiency.

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