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The Boundedness of Solutions to Nonlinear Third Order Differential Equations

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Abstract: In this paper, we establish some new sufficient conditions under which all solutions of nonlinear third order differential equations of the form

$$x''' + \psi(x, x')x'' + f(x, x') = p(t, x, x', x'')$$

are bounded. For illustrations, an example is also given on the bounded solutions.

Keywords: *nonlinear differential equations; third order; boundedness of solutions; Lyapunov's second method.*

Mathematics Subject Classification (2000): 34D20.

1 Introduction

In a recent paper, Omeike [5] considered the following nonlinear third order differential equation:

$$x''' + \psi(x, x')x'' + f(x, x') = 0. \tag{1.1}$$

He introduced a Lyapunov function and then discussed the global asymptotic stability of zero solution $x(t) = 0$ of this equation. By this work, the author proved under less restrictive conditions the stability result obtained by Qian [6] for equation (1.1). The Lyapunov function introduced in that paper, [5], raised this case. It should be noted that, first in 1970, Barbashin [2] proved some results related to the qualitative behaviors of solution of some systems of third order differential equation. Later, based on the results of Barbashin [2], some results have been improved concerning the qualitative behaviors of

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solutions of (1.1) and various nonlinear third order differential equations in the literature (see Omeike [5], Qian [6], Tunç [7, 8] and the references thereof). At the same time, for some papers published on the qualitative behaviors of solutions of various nonlinear third order differential equations and the stability and boundedness of nonlinear systems, we refer the reader to the papers of Aleksandrov and Platonov [1], Barbashin and Tabueva [3, 4], Tunç [9, 10, 11], Tunç and Ateş [12] and the references thereof. Now, we consider the following nonlinear third order differential equation

$$x''' + \psi(x, x')x'' + f(x, x') = p(t, x, x', x''). \tag{1.2}$$

This equation can be stated as the following equivalent system:

$$x' = y, \quad y' = z,$$

$$z' = -\psi(x, y)z - f(x, y) + p(t, x, y, z), \tag{1.3}$$

where $\psi \in C\left(\Re \times \Re,\ \Re\right)$, $f \in C\left(\Re \times \Re,\ \Re\right)$ and $p \in C\left([0,\infty) \times \Re \times \Re \times \Re,\ \Re\right)$. We also assume that the functions ψ, f and p depend only on the arguments displayed explicitly, and the primes in equation (1.2) denote differentiation with respect to t; the derivatives

$$\frac{\partial \psi(x, x')}{\partial x} \equiv \psi_x(x, x'), \quad \frac{\partial f(x, x')}{\partial x} \equiv f_x(x, x')$$

exist and are also continuous. The motivation for the present paper has been inspired basically by the papers of Barbashin [2], Omeike [5], Qian [6] and Tunç [7, 8] and the papers mentioned above. The principal aim of this paper is to improve the result achieved in Omeike [5] on the boundedness of solutions of nonlinear differential equation (1.2). It should also be noted that we prove our main result here by using the Lyapunov's second method.

2 Boundedness of Solutions

Our main result is the following theorem.

Theorem 2.1 *In addition to the basic assumptions imposed on the functions ψ, f and p appearing in equation* (1.2)*, we assume that there exist positive constants a, b and c such that the following conditions hold:*

(i) $\frac{f(x,0)}{x} \geq c$, *$(x \neq 0)$, $f_y(x, \theta y) \geq b$, $0 \leq \theta \leq 1$, $\psi(x, y) \geq a$ and*

$$a[f(x, y) - f(x, 0) - \int_0^y \psi_x(x, v)v dv]y \geq y \int_0^y f_x(x, v) dv;$$

(ii) $|p(t, x, y, z)| \leq q(t)$*, where $q \in L^1(0, \infty)$, $L^1(0, \infty)$ is a space of integrable Lebesgue functions.*

Then, there exists a finite positive constant K such that every solution $(x(t), y(t), z(t))$ of system (1.3) *satisfies*

$$|x(t)| \leq \sqrt{K}, \quad |y(t)| \leq \sqrt{K}, \quad |z(t)| \leq \sqrt{K}.$$

Proof The proof of this theorem depends on a scalar differentiable Lyapunov's function $V = V(x, y, z)$. This function and its time derivative satisfy some fundamental inequalities. We use here the Lyapunov's function V introduced in [5]:

$$V = \int_0^x f(u,0)du + \int_0^y \psi(x,v)vdv + a^{-1}\int_0^y f(x,v)dv + \frac{1}{2a}z^2 + yz. \tag{2.1}$$

This function, (2.1), can be rearranged as follows:

$$V = (1+a^{-1})\int_0^x f(u,0)du + \int_0^y \psi(x,v)vdv + a^{-1}\int_0^y f_v(x,\theta v)vdv + \frac{1}{2a}z^2 + yz$$

since

$$f_v(x,\theta v) = \frac{f(x,v) - f(x,0)}{v}, \quad (v \neq 0,\ 0 \leq \theta \leq 1),$$

that is

$$f(x,v) = f_v(x,\theta v)v + f(x,0), (v \neq 0,\ 0 \leq \theta \leq 1).$$

This arrangement implies

$$V = (1+a^{-1})\int_0^x \left[u^{-1}f(u,0) - c\right]\ udu + a^{-1}\int_0^y [f_v(x,\theta v) - b]\ vdv$$

$$+\int_0^y [\psi(x,v) - a]\ vdv + \frac{1}{2a}(z+ay)^2 + \frac{b}{2a}y^2 + \frac{(1+a^{-1})c}{2}x^2. \tag{2.2}$$

Obviously, it follows from (2.2) that there exist some positive constants D_i, $(i = 1,\ 2,\ 3)$, such that

$$\begin{aligned} V &\geq \frac{1}{2a}(z+ay)^2 + \frac{b}{2a}y^2 + \frac{(1+a^{-1})c}{2}x^2 \\ &\geq D_1x^2 + D_2y^2 + D_3z^2 \\ &\geq D_4(x^2+y^2+z^2), \end{aligned}$$

where $D_4 = \min\{D_1, D_2, D_3\}$. Now, let $(x, y, z) = (x(t), y(t), z(t))$ be any solution of system (1.3). Differentiating the function V, (2.1), along system (1.3) with respect to the independent variable t, we have

$$\begin{aligned} \frac{d}{dt}V(x,y,z) &= f(x,0)y + y\int_0^y \psi_x(x,v)vdv + a^{-1}y\int_0^y f_x(x,v)dv + z^2 \\ &\quad -a^{-1}\psi(x,y)z^2 - f(x,y)y + (y+a^{-1}z)p(t,x,y,z) \\ &= -[f(x,y) - f(x,0) - y\int_0^y \psi_x(x,v)vdv] + a^{-1}y\int_0^y f_x(x,v)dv \\ &\quad -[a^{-1}\psi(x,y) - 1]z^2 + (y+a^{-1}z)p(t,x,y,z). \end{aligned} \tag{2.3}$$

Making use of assumption (i) of the theorem, we obtain

$$\frac{d}{dt}V(x,y,z) \le (y + a^{-1}z)p(t,x,y,z).$$

By using assumption (ii) of the theorem, the inequality $2|uv| \le u^2 + v^2$ and the fact

$$y^2 + z^2 \le x^2 + y^2 + z^2 \le D_4^{-1}V(x,y,z), \tag{2.4}$$

one can easily obtain that

$$\begin{aligned}\frac{d}{dt}V(x,y,z) &\le \left(|y| + a^{-1}|z|\right)q(t)\\ &\le D_5\left(|y| + |z|\right)q(t)\\ &\le D_5\left(2 + y^2 + z^2\right)\ q(t).\\ &\le D_5\left(2 + D_4^{-1}V(x,y,z)\right)q(t)\\ &= 2D_5q(t) + D_5D_4^{-1}V(x,y,z)q(t),\end{aligned} \tag{2.5}$$

where $D_5 = \min\{1, a^{-1}\}$. Integrating (2.5) from 0 to t, using the assumption $q \in L^1(0,\infty)$ and the Gronwall–Reid–Bellman inequality, we have

$$\begin{aligned}V(x,y,z) &\le V(0,0,0) + 2D_5A + D_5D_4^{-1}\int_0^t (V(x(s),y(s),z(s)))q(s)ds\\ &\le (V(0,0,0) + 2D_5A)\exp\left(D_5D_4^{-1}\int_0^t q(s)ds\right)\\ &= (V(0,0,0) + 2D_5A)\exp\left(D_5D_4^{-1}A\right) = K_1 < \infty,\end{aligned} \tag{2.6}$$

where $K_1 > 0$ is a constant, $K_1 = (V(0,0,0) + 2D_5A)\exp\left(D_5D_4^{-1}A\right)$ and $A = \int_0^\infty q(s)ds$. In view of inequalities (2.4) and (2.6), we get

$$x^2(t) + y^2(t) + z^2(t) \le D_4^{-1}V(x,y,z) \le K,$$

where $K = K_1D_4^{-1}$. Aforementioned inequality implies that

$$|x(t)| \le \sqrt{K}, \quad |y(t)| \le \sqrt{K}, \quad |z(t)| \le \sqrt{K}$$

for all $t \ge t_0 \ge 0$. Hence,

$$|x(t)| \le \sqrt{K}, \quad |x'(t)| \le \sqrt{K}, \quad |x''(t)| \le \sqrt{K}$$

for all $t \ge t_0 \ge 0$. Thus, the proof of theorem is now complete. □

Example 2.1 We consider nonlinear third order scalar differential equation:

$$x''' + (x' \sin x + (x')^2 + 4)x'' + (x')^3 + x' + x + \frac{x}{1+x^2} = \frac{1}{1+t^2+x^2+(x')^2+(x'')^2}. \quad (2.7)$$

Now, it can be seen that differential equation (2.7) has the form of (1.2), and its equivalent system is

$$x' = y,$$
$$y' = z,$$
$$z' = -\{(\sin x)y + y^2 + 4\}z - y^3 - y - x - \frac{x}{1+x^2} + \frac{1}{1+t^2+x^2+y^2+z^2}. \quad (2.8)$$

Clearly, by comparing (2.8) with (1.3) and taking into account the assumptions of the theorem, it follows:

$$f(x,y) = x + \frac{x}{1+x^2} + y + y^3,$$
$$\frac{f(x,0)}{x} = 1 + \frac{1}{1+x^2} \geq 1 = c,$$
$$f_x(x,y) = 1 + \frac{1-x^2}{(1+x^2)^2},$$
$$f_y(x,y) = 1 + 3y^2 \geq 1 = b;$$
$$\psi(x,y) = (\sin x)y + y^2 + 4 \geq -|\sin x| \ \ |y| + y^2 + 4$$
$$\geq - \ |y| + y^2 + 4 = \left(|y| - \frac{1}{2}\right)^2 + \frac{15}{4} > 3 = a,$$
$$\psi_x(x,y) = (\cos x)y;$$
$$a[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v)v dv]y = 3[y + y^3 - \int_0^y (\cos x)v^2 dv] \ y$$
$$= 3[y + y^3 - (\cos x)\frac{y^3}{3}] \ y$$
$$= 3[y^2 + y^4 - (\cos x)\frac{y^4}{3}];$$
$$y\int_0^y f_x(x,v)dv = y\int_0^y \left[1 + \frac{1-x^2}{(1+x^2)^2}\right] dv$$
$$= \left[1 + \frac{1-x^2}{(1+x^2)^2}\right] y^2$$
$$= y^2 + \frac{y^2}{(1+x^2)^2} - \frac{x^2y^2}{(1+x^2)^2},$$

Now, we observe

$$3\big[y^2 + y^4 - (\cos x)\frac{y^4}{3}\big] \geq y^2 + \frac{y^2}{(1+x^2)^2} - \frac{x^2y^2}{(1+x^2)^2}.$$

That is,

$$a[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v)v dv]y \geq y \int_0^y f_x(x,v)dv.$$

Finally, we have

$$p(t,x,y,z) = \frac{1}{1+t^2+x^2+y^2+z^2} \leq \frac{1}{1+t^2}$$

and

$$\int_0^\infty q(s)ds = \int_0^\infty \frac{1}{1+s^2}ds = \frac{\pi}{2} < \infty,$$

that is, $q \in L^1(0,\infty)$.

Hence, the above whole discussion shows that all the conditions of the theorem hold. Thus, one can conclude that all solutions of equation (2.7) are bounded.

References

[1] Aleksandrov, A.Yu. and Platonov, A.V. Conditions of ultimate boundedness of solutions for a class of nonlinear systems. *Nonlinear Dynamics and Systems Theory* **8**(2) (2008) 109–122.

[2] Barbashin, E.A. *The Lyapunov Function.* Moscow, Nauka, 1970.

[3] Barbashin, E.A. and Tabueva, V.A. Theorem on the stability of the solution of a third order differential equation with a discontinuous characteristic. *Prikl. Mat. Meh.* **27** (1963) 664–671 (Russian); translated as *J. Appl. Math. Mech.* **27** (1963) 1005–1018.

[4] Barbashin, E.A. and Tabueva, V.A. Theorems on the asymptotic stability of solutions of certain third order differential equations with discontinuous characteristics. *Prikl. Mat. Meh.* **28** (1964) 523–528 [Russian]; translated as *J. Appl. Math. Mech.* **28** (1964) 643–649.

[5] Omeike, M.O. Further results on global stability of third-order nonlinear differential equations. *Nonlinear Analysis: Theory, Methods & Applications.* **67**(12) (2007) 3394–3400.

[6] Qian, C. On global stability of third-order nonlinear differential equations. *Nonlinear Anal. Ser. A: Theory Methods.* **42**(4) (2000) 651–661.

[7] Tunç, C. Global stability of solutions of certain third-order nonlinear differential equations. *Panamer. Math. J.* **14**(4) (2004) 31–35.

[8] Tunç, C. On the asymptotic behavior of solutions of certain third-order nonlinear differential equations. *J. Appl. Math. Stoch. Anal.* (1) (2005) 29–35.

[9] Tunç, C. Uniform ultimate boundedness of the solutions of third-order nonlinear differential equations. *Kuwait J. Sci. Engrg.* **32**(1) (2005) 39–48.

[10] Tunc, C. Some new stability and boundedness results on the solutions of the nonlinear vector differential equations of second order. *Iran. J. Sci. Technol. Trans. A Sci.* **30**(2) (2006) 213–221.

[11] Tunç, C. A new boundedness theorem for a class of second order differential equations. *Arab. J. Sci. Eng. Sect. A Sci.* **33**(1) (2008) 83–92.

[12] Tunç, C. and Ateş, M. Stability and boundedness results for solutions of certain third order nonlinear vector differential equations. *Nonlinear Dynam.* **45**(3–4) (2006) 273–281.