Nonlinear Dynamics and Systems Theory, **10** (4) (2010) 317–323

Functional Differential Equations with Nonlocal Conditions in Banach Spaces

D. Bahuguna [1*], R.K. Shukla [2] and S. Saxena [2]

1. Department of Mathematics & Statistics,
I.I.T. Kanpur - 208 016, INDIA
2. Invertis Group of Institutions,
Invertis Village, Bareilly-Lucknow National Highway-24,
Bareilly (U.P.) - 243123

Received: August 4, 2009; Revised: October 25, 2010

Abstract: In this paper we consider a nonlocal initial boundary value problem for a parabolic integro-differential equation. We reformulate this problem as an abstract functional differential equation in a Banach space with a nonlocal history condition. We establish the existence, uniqueness and continuation of mild, strong and classical solutions of the abstract functional differential equation under different conditions.

Keywords: *functional differential equation; mild solution; classical solution; continuation of solution; semigroup of operator; nonlocal condition.*

Mathematics Subject Classification (2000): 34G20, 47D06, 47H10, 34K06.

1 Introduction

Consider the following parabolic integro-differential equation in a bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$:

$$\left.\begin{aligned}
&\partial_t w(t,x) + \textstyle\sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha w(t,x) = f_1(t,x)\\
&+ \left(\textstyle\int_\Omega f_2(w(t,x))dx\right) \textstyle\int_{t-\tau}^{t} k(t-s) f_3(s, w(s,x))ds,\ 0 < t \le T,\ \tau > 0,\ x \in \Omega,\\
&D^\alpha w(t,x) = 0, \quad t \ge 0,\ x \in \partial\Omega,\ |\alpha| \le m-1,\\
&g(w_0)(x) = \phi(x), \quad x \in \Omega,
\end{aligned}\right\} \qquad (1.1)$$

* Corresponding author: mailto:dhiren@iitk.ac.in

where the sought-for real-valued function w is defined on $[-\tau,T]\times\Omega$, w_0 is the restriction of w on $[-\tau,0]\times\Omega$, for all multi-indices α, with $|\alpha|\le 2m$, the functions $a_\alpha(x)$ are sufficiently smooth and are such that the corresponding partial differential operator is strongly elliptic in Ω, f_i, $i=1,2,3$, are smooth real-valued functions defined on $[0,T]\times\Omega$, $\mathbb{R}$, $[-\tau,T]\times\mathbb{R}$, respectively, for $t\in[0,T]$, $k\in L^p(0,\tau)$, $1<p<\infty$, g is a map from $C([-\tau,0];L^p(\Omega))$ into $L^p(\Omega)$ and $\phi\in L^p(\Omega)$.

A few choices of the function g, for instance, are the following:

$$g(\psi)(x)=\int_{-\tau}^{0}k_1(-s)\psi(s)(x)ds,\quad x\in\Omega,\ \psi\in C([-\tau,0];L^p(\Omega)),$$

where $k_1\in L^1(0,\tau)$ with $\int_0^\tau k_1(s)ds\neq 0$;

$$g(\psi)(x)=\sum_{i=1}^{r}c_i\psi(t_i)(x),\quad x\in\Omega,\ \psi\in C([-\tau,0];L^p(\Omega)),$$

where $-\tau\le t_1<t_2<\cdots<t_r\le 0$, $C:=\sum_{i=1}^r c_i\neq 0$; and

$$g(\psi)(x)=\sum_{i=1}^{r}c_i\int_{t_i-\epsilon_i}^{t_i}\psi(s)(x)ds,\quad x\in\Omega,\ \psi\in C([-\tau,0];L^p(\Omega)),$$

where r and c_i are as above and $\epsilon_i>0$, $i=1,2,\dots,r$.

Let $X:=L^p(\Omega)$, $1<p<\infty$. Let the linear operator $A:D(A)\subset X\to X$ be defined by

$$D(A)=W^{2m,p}(\Omega)\cap W_0^{m,p}(\Omega),\quad Au=\sum_{|\alpha|\le 2m}a_\alpha(x)D^\alpha u,\quad u\in D(A).$$

Then $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t\ge 0$, of bounded linear operators in X (cf. Theorem 7.3.5 in [14]).

For $t\ge 0$, let $\mathcal{C}_t:=C([-\tau,t];X)$ be the Banach space of all continuous functions from $[-\tau,t]$ into X endowed with the supremum norm

$$\|\psi\|_t:=\sup_{-\tau\le\eta\le t}\|\psi(\eta)\|_X,\quad u\in\mathcal{C}_t,$$

where $\|.\|_X$ is the norm in X. Define the nonlinear map $F:[0,T]\times X\times\mathcal{C}_0\to X$ by

$$\begin{aligned}&F(t,u,\psi)(x)=f_1(t,x)\\&+\left(\int_\Omega f_2(u(x))dx\right)\int_{-\tau}^{0}k(-\theta)f_3(t+\theta,\psi(\theta))d\theta,\ t\in[0,T],\ u\in X,\ \psi\in\mathcal{C}_0.\end{aligned}\quad(1.2)$$

For $u\in\mathcal{C}_T$, let $u_t\in\mathcal{C}_0$ be defined by $u_t(\theta)=u(t+\theta)$, $\theta\in[-\tau,0]$. Then (1.1) can be reformulated as the following functional differential equation with a nonlocal history condition in the Banach space $X=L^p(\Omega)$:

$$\left.\begin{array}{rcl}u'(t)+Au(t)&=&F(t,u(t),u_t),\quad 0<t\le T,\\ g(u_0)&=&\phi.\end{array}\right\}\qquad(1.3)$$

If we define $\Phi\in\mathcal{C}_0$ given by $\Phi(\theta)\equiv\phi$ for all $\theta\in[-\tau,0]$ and $H:\mathcal{C}_0\to\mathcal{C}_0$ given by $H(\chi)(\theta)\equiv g(\chi)$ for all $\theta\in[-\tau,0]$ and all $\chi\in\mathcal{C}_0$, then the condition $g(\chi)=\phi$ is equivalent to the condition $H(\chi) = \Phi$. Thus we may consider the following functional differential equation with a more general nonlocal history condition:

$$\left.\begin{array}{rcl} u'(t) + Au(t) &=& F(t, u(t), u_t), \quad 0 < t \leq T, \\ H(u_0) &=& \Phi, \end{array}\right\} \tag{1.4}$$

which also includes the functional differential equation:

$$\left.\begin{array}{rcl} u'(t) + Au(t) &=& F(t, u(t), u_t), \quad 0 < t \leq T, \\ u_0 &=& \Phi, \end{array}\right\} \tag{1.5}$$

as a particular case.

The functional differential equation (1.5) has been extensively studied in literature. We refer to Kartsatos [10, 11], Kartsatos and Liu [9], Kartsatos and Parrott [12, 13].

Amraoui and Rhali [3] have used integrated semigroups to study the existence and uniqueness of integral solutions and other forms of solutions of the abstract Cauchy problem $u'(t) = Bu(t) + Lu_t$, $t > 0$, where B is a nondensely defined linear operator in a Banach space X and L is a bounded linear operator on X.

Recently, Bahuguna [4], Bahuguna, Dabas and Shukla [5], Bahuguna and Dabas [6], Bahuguna and Muslim [7, 8], Agarwal and Bahuguna [1, 2] have linear as well as nonlinear nonlocal history-valued evolution equations using the theory of semigroups and the theory of accretive operators.

Let $\psi \in \mathcal{C}_0$ such that $H(\psi) = \Phi$. The function $u \in \mathcal{C}_{\tilde{T}}$, $0 < \tilde{T} \leq T$, such that

$$u(t) = \begin{cases} \psi(t), & t \in [-\tau, 0], \\ S(t)\psi(0) + \int_0^t S(t-s)F(s, u(s), u_s)ds, & t \in [0, \tilde{T}], \end{cases} \tag{1.6}$$

is called a *mild solution* of (1.4) on $[-\tau, \tilde{T}]$. If a mild solution u of (1.4) on $[-\tau, \tilde{T}]$ is such that $u(t) \in D(A)$ for a.e. $t \in [0, \tilde{T}]$, u is differentiable a.e. on $[0, \tilde{T}]$ and

$$u'(t) + Au(t) = F(t, u(t), u_t), \quad \text{a.e. on } [0, \tilde{T}],$$

it is called a *strong solution* of (1.4) on $[-\tau, \tilde{T}]$. If a mild solution u of (1.4) on $[-\tau, \tilde{T}]$ is such that $u \in C^1((0, \tilde{T}]; X)$, $u(t) \in D(A)$ for $t \in (0, \tilde{T}]$ and satisfies

$$u'(t) + Au(t) = F(t, u(t), u_t), \quad t \in (0, \tilde{T}],$$

then it is called a *classical solution* of (1.4) on $[-\tau, \tilde{T}]$.

We first establish the existence of a *mild solution* $u \in \mathcal{C}_{\tilde{T}}$ of (1.4) for some $0 < \tilde{T} \leq T$ and its continuation to either on the whole of $[-\tau, T]$ or show that there exists the maximal interval $[-\tau, t_{max})$, $0 < t_{max} \leq T$ such that u is a mild solution of (1.4) on every subinterval $[-\tau, \tilde{T}]$, $0 < \tilde{T} < t_{max}$, under the assumptions that there exists a $\psi \in \mathcal{C}_0$ such that $H(\psi) = \Phi$ and $-A$ is the infinitesimal generator of a C_0-semigroup $S(t),\ t \geq 0$, of bounded linear operators in X. In the later case, since $t_{max} \leq T < \infty$, we obtain that

$$\lim_{t \to t_{max}-} \|u(t)\|_X = \infty.$$

Under the additional assumption of Lipschitz continuity on ψ on $[-\tau, 0]$, we show that the mild solution u is a strong solution of (1.4) on the interval of existence and it is Lipschitz continuous. Under further additional assumption that $S(t)$ is analytic, we show that u is a classical solution of (1.4) on the interval of existence. We also show that u is unique if and only if ψ satisfying $H(\psi) = \Phi$ is unique.

2 Local Existence of Mild Solutions

We first prove the following result establishing the local existence and uniqueness of a mild solution of (1.4).

Theorem 2.1 *Suppose that $-A$ is the infinitesimal generator of a C_0-semigroup $S(t),\ t \geq 0$ of bounded linear operators in X. Let $H : \mathcal{C}_0 \to \mathcal{C}_0$ be such that there exists a function $\psi \in \mathcal{C}_0$ such that $H(\psi) = \Phi$. Let $F : [0,T] \times X \times \mathcal{C}_0 \to X$ satisfy a Lipschitz-like condition*

$$\|F(t_1,u_1,\phi_1) - F(t_2,u_2,\phi_2)\|_X \leq L_F(r)[|t_1-t_2| + \|u_1-u_2\|_X + \|\phi_1-\phi_2\|_{\mathcal{C}_0}],$$

for all $t_i \in [0,T],\ u_i \in B_r(X,\psi(0)),\ \phi_i \in B_r(\mathcal{C}_0,\psi)\ i = 1,2$, where $L_F : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function. Then there exists a mild solution u of (1.4) on $[-\tau,T_0]$ for some $0 < T_0 \leq T$. Here $B_r(Z,z_0) := \{z \in Z : \|z-z_0\|_Z \leq r\}$ for any Banach space $(Z,\|.\|_Z)$, $z_0 \in Z$ and $r > 0$. Moreover, the mild solution u is unique if and only if ψ is unique.

Proof Let $R > 0$ be fixed. Let $M \geq 1$ and $\omega \geq 0$ be such that $\|S(t)\|_{B(X)} \leq Me^{\omega t}$ for $t \geq 0$. Here $B(X)$ is the space of all bounded linear operators on X. Choose $0 < T_0 \leq T$ be such that

$$\begin{aligned} T_0 L_F(R) &\leq 3/8, \\ \sup_{0\leq t\leq T_0} \|(S(t)-I)\psi(0)\|_X &\leq R/2, \\ T_0 M_0 &\leq R/2, \end{aligned}$$

where

$$M_0 := T + 2\|\psi\|_0 + 2MRL_F(R)e^{\omega T} + \|F(0,\psi(0),\psi)\|_X.$$

Define a map $\mathcal{F} : \mathcal{C}_{T_0} \to \mathcal{C}_{T_0}$ by

$$\mathcal{F}w(t) = \begin{cases} \psi(t), & t \in [-\tau,0], \\ S(t)\psi(0) + \int_0^t S(t-s)F(s,w(s),w_s)ds, & t \in [0,T_0], \end{cases} \qquad w \in \mathcal{C}_{T_0}. \tag{2.7}$$

Let $\tilde{\psi} \in \mathcal{C}_T$ be defined by

$$\begin{cases} \tilde{\psi}(t) = \psi(t), & t \in [-\tau,0], \\ \psi(0), & t \in [0,T]. \end{cases}$$

Then from the choice of T_0 it follows that $\mathcal{F}$ maps $B_R(\mathcal{C}_{T_0},\tilde{\psi})$ into itself. Here and subsequently, any function in $\mathcal{C}_T$ is also in $\mathcal{C}_{\tilde{T}}$, $0 \leq \tilde{T} \leq T$, as its restriction on the subinterval. Also, for $w_i \in B_R(\mathcal{C}_{T_0},\tilde{\psi})$, $i = 1,2$, we have

$$\|\mathcal{F}w_1(t) - \mathcal{F}w_2(t)\|_X \leq 2T_0L_F(R)\|w_1-w_2\|_{T_0}.$$

Since $T_0L_F(R) \leq 3/8$, $\mathcal{F}$ is a strict contraction on $B_R(\mathcal{C}_{T_0},\tilde{\psi})$ and hence has a unique fixed point $u \in B_R(\mathcal{C}_{T_0},\tilde{\psi})$. Clearly u is a mild solution of (1.4) on $[-\tau,T_0]$. It can be shown that if $\psi \in \mathcal{C}_0$ satisfying $H(\psi) = \Phi$ is unique then the mild solution $u \in \mathcal{C}_{T_0}$ is unique. If there are two different ψ_1 and ψ_2 in $\mathcal{C}_0$ satisfying $H(\psi_1) = H(\psi_2) = \Phi$, then the corresponding mild solutions differ on $[-\tau,0]$. This completes the proof of Theorem 2.1. $\square$

3 Continuation of Solutions

Theorem 3.1 *Assume the hypotheses of Theorem 2.1. Then the local mild solution u of (1.4) on $[-\tau, T_0]$, $0 < T_0 \le T$, can be continued either on the whole interval $[-\tau, T]$ or on the maximal interval $[-\tau, t_{max})$ of existence and since in the later case $t_{\max} \le T < \infty$, we have*

$$\lim_{t \to t_{max}-} \|u(t)\|_X = \infty.$$

Proof Assume that $T_0 < T$. Consider the functional differential equation

$$\left.\begin{array}{rcl} v'(t) + Av(t) &=& G(t, v(t), v_t), \quad 0 < t \le T - T_0, \\ \tilde{H}(v_0) &=& \tilde{\Phi}, \end{array}\right\} \tag{3.8}$$

where $G : [0, T - T_0] \times X \times C([-\tau, 0]; X) \to X$ is defined by $G(t, u, \chi) = F(t + T_0, u, \chi)$, $\tilde{H} : \mathcal{C}_0 \to \mathcal{C}_0$ given by $\tilde{H}\chi = \chi$ for $\chi \in \mathcal{C}_0$ and $\tilde{\Phi}(\theta) = u(T_0 + \theta)$ for $\theta \in [-\tau, 0]$. Since all the hypotheses of Theorem 2.1 are satisfied for problem (3.8), we have the existence of a mild solution $w \in \mathcal{C}_{T_1}$, $0 < T_1 \le T - T_0$ of (3.8). This mild solution w is unique as $\tilde{H}$ in (3.8) is the identity map on $\mathcal{C}_0$. We define

$$\bar{u}(t) = \begin{cases} u(t), & t \in [-\tau, T_0], \\ w(t - T_0), & t \in [T_0, T_0 + T_1]. \end{cases} \tag{3.9}$$

Then $\bar{u}$ is a mild solution of (1.4) on $[-\tau, T_0 + T_1]$. Continuing this way, we get the existence of a mild solution u either on the whole interval $[-\tau, T]$ or on the maximal interval $[-\tau, t_{max})$ of existence. In the later case we may use the arguments similar to those in the proof of Theorem 6.2.2 in [14] (pp. 193–194) to conclude that $\lim_{t \to t_{max}-} \|u(t)\|_X = \infty$. This completes the proof of Theorem 3.1. $\square$

4 Regularity of Solutions

Theorem 4.1 *Assume the hypotheses of Theorem 2.1. If, in addition, ψ is Lipschitz continuous on $[-\tau, 0]$ and $\psi(0) \in D(A)$, then u is Lipschitz continuous on every compact subinterval of existence. If, in addition, X is reflexive, then u is a strong solution of (1.4) on the interval of existence and this strong solution is a classical solution of (1.4) provided $S(t)$ is an analytic semigroup.*

Proof We shall prove the result for the first case when the mild solution u exists on the whole interval. The proof can be modified easily for the second case.

We need to show the Lipschitz continuity of u only on $[0, T]$. In what follows, C_i's are positive constants depending only on R, T and $\|\phi\|_0$. Let $t \in [0, T]$ and $h \ge 0$. Then

$$\begin{aligned} \|u(t+h) - u(t)\|_X \;\le\; & \|(S(h) - I)S(t)\psi(0)\|_X \\ & + \int_{-h}^{0} \|S(t-s) f(s+h, u(s+h), u_{s+h})\|_X ds \\ & + \int_0^t \|s(t-s)[f(s+h, u(s+h), u_{s+h}) - f(s, u(s), u_s)]\|_X \, ds \\ \le\; & C_1 \left[h + \int_0^t [\|u(s+h) - u(s)\|_X + \|u_{s+h} - u_s\|_{\mathcal{C}_0}] ds \right] \\ \le\; & 2C_1 \left[h + \int_0^t \sup_{-\tau \le \theta \le 0} \|u(s+h+\theta) - u(s+\theta)\|_X \right] ds, \end{aligned} \tag{4.10}$$

For the case when $-\tau \le t < 0$ and $0 \le t+h$ (clearly, $t+h \le h$ in this case), we have

$$\begin{aligned} \|u(t+h)-u(t)\|_X &\le \|(S(t+h)-I)\psi(0)\|_X + \|\psi(t)-\psi(0)\|_X \\ &\quad + \int_0^h \|S(t+h-s)f(s,u(s),u_s)\|_X\, ds \\ &\le C_2 h. \end{aligned} \tag{4.11}$$

Combining inequalities (4.10) and (4.11), we have for $-\tau \le \bar{t} \le t$,

$$\|u(\bar{t}+h)-u(\bar{t})\|_X \le C_3\left[h + \int_0^t \sup_{-\tau\le\theta\le 0}\|u(s+h+\theta)-u(s+\theta)\|_X ds\right]. \tag{4.12}$$

Putting $\bar{t} = t+\bar{\theta}$, $-t-\tau \le \bar{\theta} \le 0$, in (4.12), and taking supremum over $\bar{\theta}$ on $[-\tau, 0]$, we get

$$\begin{aligned} &\sup_{-\tau\le\theta\le 0}\|u(t+h+\theta)-u(t+\theta)\|_X \\ &\le 2C_3\left[h + \int_0^t \sup_{-\tau\le\theta\le 0}\|u(s+h+\theta)-u(s+\theta)\|_X ds\right]. \end{aligned} \tag{4.13}$$

Applying Gronwall's inequality in (4.13), we obtain

$$\|u(t+h)-u(t)\|_X \le \sup_{-\tau\le\theta\le 0}\|u(t+h+\theta)-u(t+\theta)\|_X \le C_4 h.$$

Thus, u is Lipschitz continuous on $[-\tau, T]$.

The function $\bar{F} : [0,T] \to X$ given by $\bar{F}(t) = F(t, u(t), u_t)$, is Lipschitz continuous and therefore differentiable a.e. on $[0,T]$ and $\bar{F}\,'$ is in $L^1((0,T);X)$. Consider the Cauchy problem

$$\begin{cases} v'(t) + Av(t) = \bar{F}(t), \ t \in (0,T], \\ v(0) = u(0), \end{cases} \tag{4.14}$$

By Corollary 2.10 on page 109 in Pazy [14], there exists a unique strong solution v of (4.14) on $[0,T]$. Clearly, $\bar{v}$ defined by

$$\bar{v}(t) = \begin{cases} u(t), & t \in [-\tau, 0], \\ v(t), & t \in [0,T], \end{cases}$$

is a strong solution of (1.4) on $[-\tau, T]$. But this strong solution is also a mild solution of (1.4) and $\bar{v} \in \mathcal{W}(\psi, T) := \{\Psi \in \mathcal{C}_T : \Psi = \psi \text{ on } [-\tau, 0]\}$. By the uniqueness of such a function in $\mathcal{W}(\psi, T)$, we get $\bar{v}(t) = u(t)$ on $[-\tau, T]$. Thus u is a strong solution of (1.4). If $S(t)$ is analytic semigroup in X then we may use Corollary 3.3 on page 113 in Pazy [14] to obtain that u is a classical solution of (1.4). Clearly, if $\psi \in \mathcal{C}_T$ satisfying $h(\psi) = \Phi$ on $[-\tau, 0]$ is unique on $[-\tau, 0]$, then u is unique. If there are two ψ and $\tilde{\psi}$ in $\mathcal{C}_T$ satisfying $h(\psi) = h(\tilde{\psi}) = \Phi$ on $[-\tau, 0]$, with $\psi \ne \tilde{\psi}$ on $[-\tau, 0]$, then $\mathcal{W}(\psi, T) \cap \mathcal{W}(\tilde{\psi}, T) = \emptyset$ and hence the corresponding solutions u and $\tilde{u}$ of (1.4) belonging to $\mathcal{W}(\psi, T)$ and $\mathcal{W}(\tilde{\psi}, T)$, respectively, are different. This completes the proof of Theorem 4.1. □

References

[1] Agarwal, S. and Bahuguna, D. Existence and uniqueness of strong solutions to nonlinear nonlocal functional differential equations. *Elec. J. Diff. Equations.* (to appear)

[2] Agarawal, S. and Bahuguna, D. Method of semidiscretization in time to nonlinear retarded differential equations with nonlocal history conditions. *Int. J. Math. & Math. Sci.* (to appear)

[3] Amraoui, S. and Rhali, S.L. Retarded functional differential equations with nondense domain operators. Numerical methods for partial differential equations (Marrakech, 1998). *Numer. Algorithms* **21** (1-4, 1-8) (1999).

[4] Bahuguna, D. Existence, uniqueness and regularity of solutions to semilinear nonlocal functional differential equations. *Nonlinear Anal.* **57** (7-8) (2004) 1021–1028.

[5] Bahuguna, D., Dabas, J. and Shukla, R.K. Method of lines to hyperbolic integro-differential equations in $\mathbb{R}^n$. *Nonlinear Dynamics & Systems Theory* **8** (4) (2008) 317–328.

[6] Bahuguna, D. and Dabas, J. Existence and uniqueness of a solution to a semilinear partial delay differential equation with an integral condition. *Nonlinear Dynamics & Systems Theory* **8** (1) (2008) 7–19.

[7] Bahuguna, D. and Muslim, M. Approximation of solutions to a class of second order history-valued delay differential equations. *Nonlinear Dynamics & Systems Theory* **8** (3) (2008) 237–254.

[8] Bahuguna, D. and Muslim, M. A study of nonlocal history-valued retarded differential equations using analytic semigroups. *Nonlinear Dynamics & Systems Theory* **6** (1) (2006) 63–75.

[9] Kartsatos, A.G. and Liu, X. On the construction and the convergence of the method of lines for quasi-nonlinear functional evolutions in general Banach spaces. *Nonlinear Anal.* **29** (4) (1997) 385–414.

[10] Kartsatos, A.G. On the construction of methods of lines for functional evolutions in general Banach spaces. *Nonlinear Anal.* **25** (12) (1995) 1321–1331.

[11] Kartsatos, A.G. On the method of steps for time-dependent delay equations in general Banach spaces. *Panamer. Math. J.* **1** (2) (1991) 67–73.

[12] Kartsatos, A.G. and Parrott, M.E. Functional evolution equations involving time dependent maximal monotone operators in Banach spaces. *Nonlinear Anal.* **8** (7) (1984) 817–833.

[13] Kartsatos, A.G. and Parrott, M.E. Functional evolutions equation involving time dependent maximal monotone operators in Banach spaces. *Nonlinear Anal.* **8** (1984) (7) 817–833.

[14] Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer-Verlag, 1983.