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Homoclinic Orbits for Superquadratic Hamiltonian Systems with Small Forcing Terms

A. Daouas *

High Institute for Computer Sciences and Telecommunication. Hammam Sousse, 4011, Tunisia.

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Abstract: In this paper, we prove the existence of homoclinic orbits for the second order Hamiltonian system: $\ddot{q}(t) + \nabla V(t, q(t)) = f(t)$, where $V \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), V(t,q) = -K(t,q) + W(t,q)$ is T-periodic in t, K satisfies the "pinching" condition $b_1|q|^2 \le K(t,q) \le b_2|q|^2$ and W is superquadratic at the infinity and needs not satisfy the global Ambrosetti-Rabinowitz condition. A homoclinic orbit is obtained as the limit of $2kT$-periodic solutions of a certain sequence of second order differential equations.

Keywords: *homoclinic orbit; Hamiltonian system; Mountain Pass Theorem.*

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1 Introduction

Let us consider the second order Hamiltonian system

$$\ddot{q}(t) + \nabla V(t, q(t)) = f(t), \tag{HS}$$

where $V(t,x) = -K(t,x) + W(t,x), \nabla V(t,x) = (\partial V/\partial x)(t,x),\ K,\ W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are C^1-maps, T-periodic with respect to t, $T > 0$ and $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ is continuous and bounded. We will say that a solution q of (HS) is *homoclinic* (to 0) if $q(t) \longrightarrow 0$ as $t \longrightarrow \pm\infty$. In addition, if $q \not\equiv 0$ then q is called a nontrivial homoclinic solution.

The problem of finding subharmonic and homoclinic solutions for Hamiltonian systems has been the object of many works under different assumptions on the growth

* Corresponding author: mailto:daouas_adel@yahoo.fr

of W at infinity, see [1,3-5,8,12,13] and references therein. Most of them treat the superquadratic case. They usually suppose $K(t,x)=\frac{1}{2}(L(t)x,x)$ with $L(t)$ is a symmetric matrix valued function and W satisfies the global Ambrosetti-Rabinowitz condition, that is, there exists $\mu>2$ such that

$$0<\mu W(t,x)\leq(\nabla W(t,x),x),\quad \forall\ (t,x)\in\mathbb{R}\times\mathbb{R}^n\backslash\{0\}.$$

Especially, in [13], Rabinowitz established the existence of homoclinic orbits for the Hamiltonian system (HS) under the above assumptions and $f\equiv 0$. Recently, the authors in [7] consider a more general case where K is assumed to satisfy the "pinching" condition $b_1|x|^2\leq K(t,x)\leq b_2|x|^2$ and the function f may be nonzero.

In this paper, we shall study the existence of homoclinic orbits for (HS) when W satisfies the following superquadratic condition:

$$W(t,x)/|x|^2\longrightarrow+\infty \text{ as } |\,x\,|\to\infty \ \ \textit{uniformly in}\ \ t\ \in\mathbb{R}, \tag{1}$$

and needs not satisfy the global Ambrosetti-Rabinowitz condition.

The superquadratic condition (1) was used in many recent works to study the existence of periodic and subharmonic solutions for Hamiltonian systems (see for example [6,12]). Subsequently, this condition was applied among other conditions in [9,11] to look for homoclinic orbits. Our approach is different from the last ones, in fact, similarly to [13], a homoclinic orbit will be obtained as a limit, as $k\longrightarrow\infty$, of sequence q_k of subharmonics for second order differential equations. The sequence q_k is obtained via a standard version of the Mountain Pass Theorem (Theorem 2.2 in [14]). Part of the difficulty in applying this theorem is in verifying the Palais-Smale condition. However, as it's shown in [2], a deformation lemma can be proved with the (C) condition, replacing the usual Palais-Smale condition, and it turns out that the Mountain Pass Theorem still holds true.

We make the following assumptions :

(H_1) *there exist* $a_1,a_2>0$ *such that*

$$a_1|x|^2\leq K(t,x)\leq a_2|x|^2,\ \forall\ (t,x)\in\mathbb{R}\times\mathbb{R}^n,$$

(H_2) $K(t,x)\leq(x,\nabla K(t,x))\leq 2K(t,x),\forall\ (t,x)\in\mathbb{R}\times\mathbb{R}^n,$

(H_3) $W(t,0)\equiv 0$ *and* $\nabla W(t,x)=o(|x|)$ *as* $x\longrightarrow 0$ *uniformly in* t,

(H_4) *there exist constants* $d_1>0$ *and* $r>2$ *such that*

$$W(t,x)\leq d_1|x|^r,\ \forall\ (t,x)\in\mathbb{R}\times\mathbb{R}^n,$$

(H_5) *there exist constants* $d_2>0,\mu>1,\mu>r-2$ *and* $\beta\in L^1(\mathbb{R},\mathbb{R}_+)$ *such that*

$$(\nabla W(t,x),x)-2W(t,x)\geq d_2|x|^\mu-\beta(t),\ \forall\ (t,x)\in\mathbb{R}\times\mathbb{R}^n.$$

Here $(.,.)$ denotes the standard inner product in $\mathbb{R}^n$ and $|\ .\ |$ is the induced norm.

For each $k\in\mathbb{N}$, let $E_k=W^{1,2}_{2kT}(\mathbb{R},\mathbb{R}^n)$, the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm

$$||q||_{E_k}=\Big(\int_{-kT}^{kT}(|\dot q(t)|^2+|q(t)|^2)dt\Big)^{\frac{1}{2}}.$$

Furthermore, let $L^{\infty}_{2kT}(\mathbb{R}, \mathbb{R}^n)$ denote the space of $2kT$-periodic essentially bounded (measurable) functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$||q||_{L^{\infty}_{2kT}} = ess\ \sup\{|q(t)|; t \in [-kT, kT]\}.$$

The following result was proved by Rabinowitz in [13].

Proposition 1.1 *There is a positive constant C such that for each $k \in \mathbb{N}$, and $q \in E_k$ the following inequality holds:*

$$||q||_{L^{\infty}_{2kT}} \le C||q||_{E_k}. \tag{2}$$

Set $b_1 := \min\{1, 2a_1\},\ b_2 := \max\{1, 2a_2\}$ and suppose that

(H_6) $\quad 2d_1 < b_1,\ f \in L^2(\mathbb{R}, \mathbb{R}^n) \cap L^{\gamma}(\mathbb{R}, \mathbb{R}^n)$ *and* $||f||_{L^2} < \dfrac{b_1 - 2d_1}{2C}$, *where* $\dfrac{1}{\gamma} + \dfrac{1}{\mu} = 1.$

Our main result is the following :

Theorem 1.1 *Suppose $(H_1) - (H_6)$ and (1) are satisfied then the system (HS) possesses a nontrivial homoclinic solution $q \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ such that $\dot{q}(t) \longrightarrow 0$ as $t \longrightarrow \pm\infty$.*

Remark 1.1 Consider the functions

$$K(t, x) = (1 + \frac{1}{1 + x^2})x^2, \quad W(t, x) = h(t)|x|^2 \ln(1 + |x|^2),$$

where h is positive, continuous and T-periodic function. A straightforward computation shows that W satisfies the assumptions $(H_3) - (H_5)$ of Theorem 1.1 but does not satisfy the global Ambrosetti–Rabinowitz condition essentially. Moreover, $K(t, x)$ satisfies the assumptions (H_1) and (H_2) but can not be written in the form $1/2(L(t)x, x)$. Hence, Theorem 1.1 extends the results in [7,13] mainly. Furthermore, contrary to [7,13], the conditions of our result permit to W to change sign near the origin. Theorem 1.1 is also related to those in [9,11,15], where $K(t, x)$ has the form $1/2(L(t)x, x)$ without periodicity assumption on V and $f \equiv 0$.

2 Proof of Theorem 1.1

For each $k \in \mathbb{N}$, let $L^2_{2kT}(\mathbb{R}, \mathbb{R}^n)$ denote the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm $||q||_{L^2_{2kT}} = (\int_{-kT}^{kT} |q(t)|^2 dt)^{1/2}$. Let $f_k : \mathbb{R} \longrightarrow \mathbb{R}^n$ be the $2kT$-periodic extension of the restriction of f to the interval $[-kT, kT]$ and $\eta_k : E_k \longrightarrow [0, +\infty[$ given by

$$\eta_k(q) = \Big(\int_{-kT}^{kT} \Big[|\dot{q}(t)|^2 + 2K(t, q(t))\Big] dt\Big)^{1/2}.$$

By (H_1) we get

$$b_1||q||^2_{E_k} \le \eta^2_k(q) \le b_2||q||^2_{E_k}. \tag{3}$$

Let $I_k : E_k \longrightarrow \mathbb{R}$, be defined by

$$\begin{aligned} I_k(q) &= \int_{-kT}^{kT} \Big[\frac{1}{2}|\dot{q}(t)|^2 - V(t, q(t))\Big] dt + \int_{-kT}^{kT} (f_k(t), q(t))dt \\ &= \frac{1}{2}\eta^2_k(q) - \int_{-kT}^{kT} W(t, q(t))dt + \int_{-kT}^{kT} (f_k(t), q(t))dt. \end{aligned} \tag{4}$$

Then $I_k \in C^1(E_k, \mathbb{R})$ and it's easy to show that

$$I'_k(q)v = \int_{-kT}^{kT} \Big[(\dot{q}(t), \dot{v}(t)) - (\nabla V(t, q(t)), v(t))\Big] dt + \int_{-kT}^{kT} (f_k(t), v(t))dt.$$

By (H_2), we get

$$I'_k(q)q \leq \eta_k^2(q) - \int_{-kT}^{kT} (\nabla W(t, q(t)), q(t))dt + \int_{-kT}^{kT} (f_k(t), q(t))dt. \tag{5}$$

Moreover, it is well known that critical points of I_k are classical $2kT$-periodic solutions of the second order Hamiltonian system

$$\ddot{q}(t) + \nabla V(t, q(t)) = f_k(t). \tag{HS_k}$$

Lemma 2.1 *If V and f satisfy $(H_1) - (H_6)$ and (1), then for all $k \in \mathbb{N}$ the system (HS_k) possesses a $2kT$-periodic solution.*

Proof It suffices to prove that the functional I_k satisfies all the assumptions of the Mountain Pass Theorem (Theorem 2.2 in [14]) with the (C) condition replacing the usual Palais-Smale condition. This will be done by a sequence of lemmas. □

Lemma 2.2 *I_k satisfies the (C) condition, i.e., for every constant c and sequence $\{u_n\} \subset E_k$, $\{u_n\}$ has a convergent subsequence if $I_k(u_n) \longrightarrow c$ and $(1+||u_n||)I'_k(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Proof Assume that $\{u_n\} \subset E_k$ is a (C) sequence of I_k, that is, $I_k(u_n)$ is bounded and $(1 + ||u_n||)||I'_k(u_n)|| \longrightarrow 0$ as $n \longrightarrow \infty$. Then there exists $M_k > 0$ such that

$$\begin{aligned} M_k &\geq 2I_k(u_n) - I'_k(u_n)u_n \\ &\geq \int_{-kT}^{kT} \Big[(\nabla W(t, u_n(t)), u_n(t)) - 2W(t, u_n(t))\Big] dt + \int_{-kT}^{kT} (f_k(t), u_n(t))dt. \end{aligned}$$

So, by (H_5), we get

$$M_k \geq d_2 \int_{-kT}^{kT} |u_n(t)|^\mu dt - \int_{-kT}^{kT} \beta(t)dt + \int_{-kT}^{kT} (f_k(t), u_n(t))dt.$$

Then, by Hölder inequality

$$d_2||u_n||^\mu_{L^\mu_{2kT}} \leq M_k + \int_{-kT}^{kT} \beta(t)dt + ||f_k||_{L^\gamma_{2kT}} ||u_n||_{L^\mu_{2kT}},$$

where γ is the conjugate exponent of μ. Since $\mu > 1$, there exists a constant C_k such that

$$||u_n||_{L^\mu_{2kT}} \leq C_k. \tag{6}$$

On the other hand, by (3), (4) and (H_4), one has

$$\begin{aligned} b_1||u_n||^2_{E_k} &\leq 2I_k(u_n) + 2d_1 \int_{-kT}^{kT} |u_n(t)|^r dt - 2\int_{-kT}^{kT} (f_k(t), u_n(t))dt \\ &\leq 2I_k(u_n) + 2d_1 \int_{-kT}^{kT} |u_n(t)|^r dt + 2C_k||f_k||_{L^\gamma_{2kT}}. \end{aligned} \tag{7}$$

If $\mu \geq r$, by Hölder inequality

$$\int_{-kT}^{kT} |u_n(t)|^r dt \leq (2kT)^{\frac{\mu-r}{\mu}} \Big(\int_{-kT}^{kT} |u_n(t)|^\mu dt \Big)^{\frac{r}{\mu}}.$$

Combining the above with (6) and (7), we obtain that $||u_n||_{E_k}$ is bounded.
If $\mu < r$, by (2), we have

$$\begin{aligned}\int_{-kT}^{kT} |u_n(t)|^r dt &= \int_{-kT}^{kT} |u_n(t)|^{r-\mu} |u_n(t)|^\mu dt \\ &\leq ||u_n||^{r-\mu}_{L^\infty_{2kT}} \int_{-kT}^{kT} |u_n(t)|^\mu dt \\ &\leq C^{r-\mu} ||u_n||^{r-\mu}_{E_k} \int_{-kT}^{kT} |u_n(t)|^\mu dt. \end{aligned} \tag{8}$$

Hence, by (6) and (8) there exists a constant C'_k such that

$$b_1 ||u_n||^2_{E_k} \leq 2I_k(u_n) + C'_k ||u_n||^{r-\mu}_{E_k} + 2\ C_k ||f_k||_{L^\gamma_{2kT}}.$$

Since $r - \mu < 2$ and $I_k(u_n)$ is bounded, then $||u_n||_{E_k}$ will be bounded too.
In a similar way to Proposition B.35 in [14], we can prove that $\{u_n\}$ has a convergent subsequence. Hence I_k satisfies the (C) condition. $\Box$

Lemma 2.3 *The functional I_k satisfies the condition (I_1) of the Mountain Pass Theorem.*

Proof Let $q \in E_k$, such that $0 < ||q||_{L^\infty_{2kT}} \leq 1$. By (H_4) we have

$$\int_{-kT}^{kT} W(t, q(t)) dt \leq d_1 \int_{-kT}^{kT} |q(t)|^2 dt \leq d_1 ||q||^2_{E_k}. \tag{9}$$

Then, by (3), (4), (9) and (H_6) it follows that

$$\begin{aligned} I_k(q) &\geq \frac{b_1}{2} ||q||^2_{E_k} - d_1 ||q||^2_{E_k} - ||f_k||_{L^2_{2kT}} ||q||_{L^2_{2kT}} \\ &\geq \frac{b_1}{2} ||q||^2_{E_k} - d_1 ||q||^2_{E_k} - ||f||_{L^2} ||q||_{E_k} \\ &\geq \frac{1}{2} (b_1 - 2d_1 - 2C||f||_{L^2}) ||q||^2_{E_k} + C||f||_{L^2} \Big(||q||^2_{E_k} - \frac{||q||_{E_k}}{C} \Big). \end{aligned}$$

Set

$$\rho = \frac{1}{C}, \quad \alpha = \frac{b_1 - 2d_1 - 2C||f||_{L^2}}{2C^2}.$$

By (2), if $||q||_{E_k} = \rho$, then $0 < ||q||_{L^\infty} \leq 1$ and $I_k(q) \geq \alpha$. $\Box$

Lemma 2.4 *Under the assumption (1), I_k satisfies the condition (I_2) of the Mountain Pass Theorem.*

Proof Let $q \in E_1, q \not\equiv 0$ such that $q(T) = q(-T) = 0$ and $A > \dfrac{b_2 ||q||^2_{E_1}}{2||q||^2_{L^2_{2T}}}$. By (1), there exists $B > 0$ such that for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $W(t,x) \geq A|x|^2 - B$.
Hence, for all $\zeta \in \mathbb{R}$ the following inequality holds :

$$I_1(\zeta q) \leq \frac{b_2}{2}\zeta^2 ||q||^2_{E_1} - A\zeta^2 ||q||^2_{L^2_{2T}} + |\zeta| ||f_1||_{L^2_{2T}} ||q||_{L^2_{2T}} + 2TB. \tag{10}$$

Then by (10) and the choice of A there exists $\zeta \in \mathbb{R}$ satisfying $||\zeta q||_{E_1} > \rho$ and $I_1(\zeta q) < 0$. For $k > 1$, set $e_1(t) = \zeta q(t)$ and

$$e_k(t) = \begin{cases} e_1(t) & \text{for} \quad |t| \leq T, \\ 0 & \text{for} \quad T < |t| \leq kT. \end{cases} \tag{11}$$

Then $e_k \in E_k, ||e_k||_{E_k} = ||e_1||_{E_1} > \rho$ and $I_k(e_k) = I_1(e_1) < 0$ for every $k \in \mathbb{N}$. $\square$

For our setting, clearly $I_k(0) = 0$, so, by applying the Mountain Pass Theorem, I_k possesses a critical value $c_k \geq \alpha$. Hence, for every $k \in \mathbb{N}$, there is $q_k \in E_k$ such that

$$I_k(q_k) = c_k, \qquad I'_k(q_k) = 0. \tag{12}$$

This completes the proof of Lemma 2.4.

Lemma 2.5 *Let $(q_k)_{k\in\mathbb{N}}$ be the sequence given by (12). Then there exists a subsequence $(q_{k_j})_{j\in\mathbb{N}}$ convergent to a certain function q_0 in $C^1_{loc}(\mathbb{R}, \mathbb{R}^n)$.*

Proof First of all we show that the sequences $\{c_k\}_{k\in\mathbb{N}}$ and $\{||q_k||_{E_k}\}_{k\in\mathbb{N}}$ are bounded. For every $k \in \mathbb{N}$, let $g_k : [0,1] \longrightarrow E_k$ be a curve given by $g_k(s) = se_k$, where e_k is defined by (11). Then $g_k \in \Gamma_k$ and $I_k(g_k(s)) = I_1(g_1(s))$ for all $k \in \mathbb{N}$ and $s \in [0,1]$. Therefore, by the Mountain Pass Theorem,

$$c_k \leq \max_{s\in[0,1]} I_1(g_1(s)) \equiv M_0 \tag{13}$$

independent of $k \in \mathbb{N}$. As $I'_k(q_k) = 0$, we receive from (4), (5) and (H_5) that

$$\begin{aligned} 2c_k &= 2I_k(q_k) - I'_k(q_k)q_k \\ &\geq \int_{-kT}^{kT} \Big[(\nabla W(t, q_k(t)), q_k(t)) - 2W(t, q_k(t))\Big]dt + \int_{-kT}^{kT} (f_k(t), q_k(t))dt \\ &\geq d_2 \int_{-kT}^{kT} |q_k(t)|^\mu dt - \int_{-kT}^{kT} \beta(t)dt + \int_{-kT}^{kT} (f_k(t), q_k(t))dt. \end{aligned} \tag{14}$$

By Hölder inequality, (13) and (14) we get

$$d_2 ||q_k||^\mu_{L^\mu_{2kT}} \leq 2M_0 + \beta_0 + \alpha_0 ||q_k||_{L^\mu_{2kT}},$$

where $\alpha_0 = ||f||_{L^\gamma_{\mathbb{R}}}$ and $\beta_0 = \displaystyle\int_{-\infty}^{+\infty} \beta(t)dt$. Since $\mu > 1$ and all the constants in the above inequality are independent of k, then there exists a constant L such that

$$||q_k||_{L^\mu_{2kT}} \leq L. \tag{15}$$

On the other hand, by (3), (4) and (H_4), one has

$$b_1||q_k||^2_{E_k} \le 2M_0 + 2d_1\int_{-kT}^{kT}|q_k(t)|^r dt - 2\int_{-kT}^{kT}(f_k(t),q_k(t))dt. \tag{16}$$

If $r \ge \mu$, by (1), (15) and Hölder inequality we obtain

$$\begin{aligned} b_1||q_k||^2_{E_k} &\le 2M_0 + 2d_1||q_k||^{r-\mu}_{L^\infty_{2kT}}\int_{-kT}^{kT}|q_k(t)|^\mu dt - 2\int_{-kT}^{kT}(f_k(t),q_k(t))dt \\ &\le 2M_0 + 2cL^\mu||q_k||^{r-\mu}_{E_k} + 2\alpha_0 L. \end{aligned} \tag{17}$$

Since $r-\mu < 2$ and all coefficients of (17) are independent of k, we see that there is $M_1 > 0$ independent of k such that

$$||q_k||_{E_k} \le M_1. \tag{18}$$

If $r < \mu$, we have

$$\begin{aligned} \int_{-kT}^{kT}|q_k(t)|^r dt &= \int_{\{t\in[-kT,kT];|q_k(t)|\le 1\}}|q_k(t)|^r dt + \int_{\{t\in[-kT,kT];|q_k(t)|>1\}}|q_k(t)|^r dt \\ &\le \int_{\{t\in[-kT,kT];|q_k(t)|\le 1\}}|q_k(t)|^2 dt + \int_{\{t\in[-kT,kT];|q_k(t)|>1\}}|q_k(t)|^\mu dt \\ &\le \int_{-kT}^{kT}|q_k(t)|^2 dt + \int_{-kT}^{kT}|q_k(t)|^\mu dt. \end{aligned} \tag{19}$$

By (16) and (19) we get

$$b_1||q_k||^2_{E_k} \le 2M_0 + 2d_1||q_k||^2_{E_k} + 2d_1L^\mu + 2\alpha_0 L.$$

Hence

$$(b_1 - 2d_1)||q_k||^2_{E_k} \le 2M_0 + 2d_1L^\mu + 2\alpha_0 L.$$

Since $b_1 > 2d_1$, (18) remains true.

Now, we observe that the sequences $\{q_k\}_{k\in\mathbb{N}}$, $\{\dot{q}_k\}_{k\in\mathbb{N}}$ and $\{\ddot{q}_k\}_{k\in\mathbb{N}}$ are uniformly bounded. By (2) and (18),

$$||q_k||_{L^\infty_{2kT}} \le CM_1 \equiv M_2 \tag{20}$$

for every $k \in \mathbb{N}$. Since q_k satisfies (HS_k), if $t \in [-kT,kT]$ we have

$$|\ddot{q}_k(t)| \le |f_k(t)| + |\nabla V(t,q_k(t))| \le \sup_{t\in\mathbb{R}}|f(t)| + |\nabla V(t,q_k(t))|,$$

so, by (20), there exists $M_3 > 0$ independent of k such that

$$||\ddot{q}_k||_{L^\infty_{2kT}} \le M_3. \tag{21}$$

From the Mean Value Theorem it follows that for every $k \in \mathbb{N}$ and $t \in \mathbb{R}$ there exists $\tau_k \in [t-1,t]$ such that

$$\dot{q}_k(\tau_k) = \int_{t-1}^{t}\dot{q}_k(s)ds = q_k(t) - q_k(t-1).$$

Combining the above with (20) and (21) we obtain

$$\begin{aligned}|\dot{q}_k(t)| &= |\int_{\tau_k}^{t}\ddot{q}_k(s)ds + \dot{q}_k(\tau_k)| \\ &\le \int_{t-1}^{t}|\ddot{q}_k(s)|ds + |q_k(t) - q_k(t-1)| \le M_3 + 2M_2 \equiv M_4,\end{aligned}$$

and hence for every $k \in \mathbb{N}$

$$||\dot{q}_k||_{L^{\infty}_{2kT}} \le M_4. \tag{22}$$

To finish the proof it is sufficient to note that the sequences $\{q_k\}_{k\in\mathbb{N}}$ and $\{\dot{q}_k\}_{k\in\mathbb{N}}$ are equicontinuous. Indeed, for every $k \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$, we have by (22)

$$|q_k(t_1) - q_k(t_2)| = |\int_{t_1}^{t_2}\dot{q}_k(s)ds| \le \int_{t_1}^{t_2}|\dot{q}_k(s)|ds \le M_4|t_1 - t_2|,$$

and similarly, by (21), we have

$$|\dot{q}_k(t_1) - \dot{q}_k(t_2)| \le M_3|t_1 - t_2|.$$

Applying now the Arzelà-Ascoli theorem, we receive the claim. □

Lemma 2.6 *Let $q_0 : \mathbb{R} \longrightarrow \mathbb{R}^n$ be the function given by Lemma 2.5. Then q_0 is the desired homoclinic solution of (HS).*

Proof The proof of this lemma is based on the two following facts.
Fact 1 *Let $q : \mathbb{R} \longrightarrow \mathbb{R}^n$ be a continuous map. If $\dot{q} : \mathbb{R} \longrightarrow \mathbb{R}^n$ is continuous at t_0 then*

$$\lim_{t\longrightarrow t_0}\frac{q(t) - q(t_0)}{t - t_0} = \dot{q}(t_0).$$

Fact 2 *Let $q : \mathbb{R} \longrightarrow \mathbb{R}^n$ be a continuous map such that $\dot{q}$ is locally square integrable. Then, for all $t \in \mathbb{R}$, we have*

$$|q(t)| \le \sqrt{2}\Big(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}}(|q(s)|^2 + |\dot{q}(s)|^2)ds\Big)^{\frac{1}{2}}. \tag{23}$$

The proofs of these facts are elementary and can be found in [7, p 385].

First, we show that q_0 is a solution of (HS). By Lemma 2.1 and Lemma 2.5, we have $q_{k_j} \longrightarrow q_0$ in $C^1_{loc}(\mathbb{R}, \mathbb{R}^n)$, as $j \longrightarrow \infty$, and

$$\ddot{q}_{k_j}(t) + \nabla V(t, q_{k_j}(t)) = f_{k_j}(t)$$

for every $j \in \mathbb{N}$, and $t \in [-k_jT, k_jT]$. Take $a, b \in \mathbb{R}$ with $a < b$. There exists $j_0 \in \mathbb{N}$ such that for all $j > j_0$ and $t \in [a, b]$, we have

$$\ddot{q}_{k_j}(t) = -\nabla V(t, q_{k_j}(t)) + f(t).$$

Hence, $\ddot{q}_{k_j}$ is continuous in $[a, b]$ and $\ddot{q}_{k_j}(t) \longrightarrow -\nabla V(t, q_0(t)) + f(t)$ uniformly on $[a, b]$. Fact 1 implies that $\ddot{q}_{k_j}$ is a classical derivative of $\dot{q}_{k_j}$ in (a, b) for all $j > j_0$. Moreover, since $\dot{q}_{k_j} \longrightarrow \dot{q}_0$ uniformly on $[a, b]$, we obtain

$$\ddot{q}_0(t) = -\nabla V(t, q_0(t)) + f(t)$$

for every $t \in (a,b)$. Since a and b are arbitrary, we conclude that q_0 satisfies (HS).

Now we prove that $q_0(t) \longrightarrow 0$, as $|t| \longrightarrow \infty$. First of all remark that for all $l \in \mathbb{N}$ there exists $j_0 \in \mathbb{N}$ such that for all $j > j_0$, we have

$$\int_{-lT}^{lT} (|q_{k_j}(t)|^2 + |\dot{q}_{k_j}(t)|^2)dt \leq ||q_{k_j}||^2_{E_{k_j}} \leq M_1^2.$$

By Lemma 2.5, we get

$$\int_{-lT}^{lT} (|q_0(t)|^2 + |\dot{q}_0(t)|^2)dt \leq M_1^2.$$

Letting $l \longrightarrow \infty$, we obtain $\int\limits_{-\infty}^{\infty} (|q_0(t)|^2 + |\dot{q}_0(t)|^2)dt \leq M_1^2$, and so

$$\int_{|t|\geq r} (|q_0(t)|^2 + |\dot{q}_0(t)|^2)dt \longrightarrow 0, \tag{24}$$

as $r \longrightarrow \infty$. Combining (23) and (24), we receive our claim.

In the next step we show that $\dot{q}_0(t) \longrightarrow 0$, as $|t| \longrightarrow \infty$. To do this, applying (23), we obtain

$$|\dot{q}_0(t)| \leq \sqrt{2}\Big(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|\dot{q}_0(s)|^2 + |\ddot{q}_0(s)|^2)ds\Big)^{\frac{1}{2}}.$$

From (24), we get

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\dot{q}_0(s)|^2 ds \longrightarrow 0,$$

as $|t| \longrightarrow \infty$. Hence, it suffices to prove that

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{q}_0(s)|^2 ds \longrightarrow 0, \tag{25}$$

as $|t| \longrightarrow \infty$. Since q_0 is a solution of (HS), we obtain

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{q}_0(s)|^2 ds = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\nabla V(s, q_0(s))|^2 ds + \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |f(s)|^2 ds$$
$$-2\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (\nabla V(s, q_0(s)), f(s))ds,$$

and then

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{q}_0(s)|^2 ds \leq \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\nabla V(s, q_0(s))|^2 ds + \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |f(s)|^2 ds$$
$$+2\Big(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\nabla V(s, q_0(s)|^2 ds\Big)^{\frac{1}{2}}\Big(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |f(s)|^2 ds\Big)^{\frac{1}{2}}. \tag{26}$$

By (H_6), we have

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |f(s)|^2 ds \longrightarrow 0, \tag{27}$$

as $|t| \longrightarrow \infty$. On the other hand, since $\nabla V(t, 0) = 0$ for all $t \in \mathbb{R}$ and $q_0(t) \longrightarrow 0$, as $|t| \longrightarrow \infty$, (25) follows from (26) and (27).

Finally, it remains to show that q_0 is nontrivial. Obviously, this will be the case when $f \not\equiv 0$, otherwise, using (H_3), the proof is the same as in [13]. $\square$

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