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Existence of the Unique Solution to Abstract Second Order Semilinear Integrodifferential Equations

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Abstract: In this paper, a strongly damped semilinear integrodifferential equation has been considered and reformulated as an abstract second order integrodifferential equation in a Banach space. The local existence and uniqueness of a classical solution is estabilished. The continuation of classical solution, the maximal interval of the existence and the global existence of the classical solution have been also studied. Finally an application of the established results is demonstrated.

Keywords: *analytic semigroup; second order integrodifferential equation; mild solution; classical solution; contraction mapping theorem.*

Mathematics Subject Classification (2000): 34G20, 35A07, 35A35.

1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^N$ with sufficiently smooth boundary $\partial\Omega$ and $Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$ be a symmetric second order strongly elliptic differential operator in Ω. Consider the following initial boundary value problem for the strongly damped partial integrodifferential equation,

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial t^2} \quad &+ \quad (aL+bI)\left(\frac{\partial u(x,t)}{\partial t}\right) + (cL+dI)u(x,t) = h\left(x,t,u(x,t),\frac{\partial u(x,t)}{\partial t}\right) \\ &+ \quad \int_{t_0}^{t} k(t-s)g\left(x,s,u(x,s),\frac{\partial u(x,s)}{\partial s}\right)ds, \\ &\qquad (x,t)\in\Omega\times(t_0,T), \quad 0<T<\infty, \end{aligned} \tag{1}$$

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with initial conditions

$$u(x,t_0)=x_0(x), \quad \frac{\partial u(x,t_0)}{\partial t}=x_1(x), \quad x\in\Omega,$$

and the homogeneous Dirichlet boundary conditions, where $a>0$, b, c, d are constants and h and g are smooth nonlinear functions and k is a locally p-integrable function for $1<p<\infty$.

Duvaut and Lions [5], Glowinski, Lions and Tremolieres [7] have studied particular case of (1) in which $L=-\triangle$ and $k\equiv 0$, in the context of the theory of viscoelastic materials.

We may rewrite (1) with initial and homogeneous Dirichlet boundary conditions in the abstract form as the following initial value problem in the Banach space $H=L^2(\Omega)$,

$$\begin{aligned}
&\frac{d^2u(t)}{dt^2}+A\left(\frac{du(t)}{dt}\right)+Bu(t)\\
&=f\left(t,u(t),\frac{du(t)}{dt}\right)+\int_{t_0}^{t}k(t-s)g\left(s,u(s),\frac{du(s)}{ds}\right)ds, \quad t>t_0,\\
&u(t_0)=x_0, \quad u'(t_0)=x_1.
\end{aligned}\tag{2}$$

where operator A with domain $D(A)=H^2(\Omega)\bigcap H_0^1(\Omega)$ is given by

$$Au=aLu, \quad u\in D(A),$$

and the operator B is such that $D(A)=D(B)$ with $B=(cL+dI)$ for some constants c and d. The function f is defined from $R_+\times H\times H$ into H given by $f(t,u,v)=h(t,u,v)-bv$. We assume that $-A$ generates an analytic semigroup $T(t)$ in X. The nonlinear maps f and g satisfy the assumptions (F) and (G), respectively, and the kernel k satisfies (K) stated in the next section.

In this paper, we concentrate on the study of the abstract second order semilinear integrodifferential equation

$$\begin{aligned}
&u''(t)+Au'(t)=f(t,u(t),u'(t))+\int_{t_0}^{t}k(t-s)g(s,u(s),u'(s))\,ds,\\
&u(t_0)=x_0, \quad u'(t_0)=x_1,
\end{aligned}\tag{3}$$

as we can merge the term Bu in the function f so that the modified function f still satisfies the assumption (F).

Sandefur [10] has studied the second order semilinear differential equation

$$\begin{aligned}
&u''(t)+Au'(t)+Bu(t)=f(t,u(t)),\\
&u(0)=\phi, \quad u'(0)=\psi,
\end{aligned}\tag{4}$$

in a Banach space X under the assumptions that the linear operators A and B can be decomposed as $-A=A_1+A_2$ and $B=A_2A_1$, where each A_k generates a C_0-semigroup $T_k(t)$, $k=1,2$; and the function f satisfies a locally Lipschitz condition. He has established the local existence and uniqueness of a mild solution to (4), i.e., there exists a continuous function u on $[0,c]$ for some $c>0$ such that u satisfies the integral equation,

$$\begin{aligned}
u(t) \;=\; T_1(t)\phi \;+\; &\int_0^t T_1(t-\tau)T_2(\tau)(\psi-A_1\phi)d\tau\\
&+\int_0^t\int_0^\tau T_1(t-\tau)T_2(t-s)f(s,u(s))dsd\tau,
\end{aligned}$$

where $\phi \in D(A_1)$. Aviles and Sandefur [1] have studied the well-posedness of (4) under the similar conditions.

In [3] Bahuguna, Shukla and Singh have considered initial value problem (2) with the kernel $k \equiv 0$ and $t_0 = 0$ i.e.

$$\frac{d^2u(t)}{dt^2} + A\left(\frac{du(t)}{dt}\right) + Bu(t) = f\left(t, u(t), \frac{du(t)}{dt}\right), \quad t > 0,$$
$$u(0) = x_0, \quad u'(0) = x_1.$$

in real Banach space and used the method of semidiscretization in time to prove the existence, uniqueness and continuous dependence on initial data of a solution to this initial value problem and discussed their application to the viscoelastic models involving short and long memory effects.

Bahuguna [2] has considered the following special case of (3) with the kernel $k \equiv 0$,

$$\begin{aligned} u''(t) + Au'(t) &= f(t, u(t), u'(t)), \\ u(t_0) = x_0, \qquad & u'(t) = x_1, \end{aligned} \tag{5}$$

and established the existence, uniqueness, continuation of a solution to the maximal interval of existence, and the global existence of a strong solution and a classical solution for this special case. He has assumed that $-A$ generates an analytic semigroup $T(t)$ in X and the nonlinear map f satisfies an assumption similar to the assumption (F).

Engler, Neubrander and Sandefur [6] have proved the local existence and uniqueness of a mild solution to (5) under the assumptions that $-A$ generates an analytic semigroup $T(t)$ in X and f satisfies a condition similar to the assumption (F), where a mild solution on $[t_0, t_1)$, for some $t_1 > t_0$, to (5) is the first component of a solution $(u(t), v(t))$ of the integral equations

$$\begin{aligned} u(t) &= x_0 + (T(t - t_0) - I)(-A)^{-1}x_1 \\ &\quad + \int_{t_0}^{t} (T(t - s) - I)(-A)^{-1} f(s, u(s), v(s))ds, \quad t_0 \le t \le t_1, \\ v(t) &= T(t - t_0)x_1 + \int_{t_0}^{t} T(t - s) f(s, u(s), v(s))ds, \quad t_0 \le t \le t_1. \end{aligned}$$

Bahuguna [2] has improved the results of [6] by showing that (5) has a unique local classical solution, i.e., there exists a unique $u \in C^1([t_0, t_1); X) \cap C^2((t_0, t_1); X)$ and satisfies (5) on $[t_0, t_1)$ for some $t_1 > t_0$. Further, he has established the continuation of this solution, the maximal interval of existence and the global existence.

In [4] Bahuguna and Shukla studied the Faedo-Galerkin approximation of solutions to the initial value problem (3) in a Hilbert space. Pandey, Ujlayan and Bahuguna considerd an abstract semilinear hyperbolic integrodifferential equation in [9] and used the theory of resolvent operators to establish the existence and uniqueness of a mild solution under local Lipschitz conditions on the nonlinear maps and an integrability condition on the kernel. Under some additional conditions on the nonlinear maps they also proved the existence of a classical solution.

In this paper we show that (3) has a unique local classical solution, i.e., there exists a unique $u \in C^1([t_0, t_1); X) \cap C^2((t_0, t_1); X)$ satisfying (3) on $[t_0, t_1)$ for some $t_1 > t_0$. Further, we discuss the continuation of this solution, the maximal interval of existence and the global existence. We achieve these objectives by extending the ideas and techniques used in the proofs of Theorems 6.3.1 and 6.3.3 in Pazy [8], concerning a semilinear equation of the first order, to (3). For the global existence, we require a modified version of Lemma 4.1, stated and proved at the end of the fourth section in [2]. Finally in the last section we demonstrate an application of the results established in earlier sections.

2 Preliminaries and Assumptions

Let X be a Banach space and let $-A$ generate the analytic semigroup $T(t)$ in X. we note that if $-A$ is the infinitesimal generator of an analytic semigroup then $-(A+\alpha I)$ is invertible and generates a bounded analytic semigroup for $\alpha > 0$ large enough. This allows us to reduce the general case, in which $-A$ is the infinitesimal generator of an analytic semigroup, to the case where the semigroup is bounded and the generator is invertible. Hence, for convenience, without loss of generality, we assume that $T(t)$ is bounded, that is $\|T(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$, i.e., $-A$ is invertible. Here $\rho(-A)$ is the resolvent set of $-A$. It follows that, for $0 \leq \alpha \leq 1$, A^α can be defined as a closed linear invertible operator with its domain $D(A^\alpha)$ being dense in X. We denote by X_α the Banach space $D(A^\alpha)$ equipped with the norm

$$\|x\|_\alpha = \|A^\alpha x\|,$$

which is equivalent to the graph norm of A^α. For $0 < \alpha < \beta$, we have $X_\beta \hookrightarrow X_\alpha$ and the embedding is continuous.

We consider the problem

$$\begin{aligned} &u''(t) + Au'(t) = f(t, u(t), u'(t)) + \int_{t_0}^{t} k(t-s) g(t, u(t), u'(t)) ds, \quad t > t_0, \\ &u(t_0) = x_0, \quad u'(t_0) = x_1. \end{aligned} \tag{6}$$

On the kernel k we assume the following condition.

(K) The kernel $k \in L^p_{loc}(0, \infty)$ for some $1 < p < \infty$ is locally Hölder continuous on $(0, \infty)$ i.e.,

$$|k(t) - k(s)| \leq L_k |t-s|^\mu \quad \text{for} \quad s, t \in (0, \infty) \quad \text{and} \quad 0 < \mu < 1.$$

The nonlinear functions f and g satisfy the following assumptions on an open subset U of $R_+ \times X_1 \times X_\alpha$.

Assumption (F): A function f is said to satisfy the assumption (F) if for every $(t, x, \tilde{x}) \in U$ there exists a neighborhood $V \subset U$ and constant $L_f \geq 0$, $0 < \vartheta \leq 1$, such that

$$\|f(t, x_1, \tilde{x}_1) - f(t, x_2, \tilde{x}_2)\| \leq L_f[|t_1 - t_2|^\vartheta + \|x_1 - x_2\|_1 + \|\tilde{x}_1 - \tilde{x}_2\|_\alpha], \tag{7}$$

for all $(t_i, x_i, \tilde{x}_i) \in V$.

Assumption (G): A function g is said to satisfy the assumption (G) if for every $(t, x, \tilde{x}) \in U$ there exists a neighborhood $V \subset U$ and a nonnegative function $L_g \in L^q_{loc}(0, \infty)$ where $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$\|g(t, x_1, \tilde{x}_1) - g(t, x_2, \tilde{x}_2)\| \leq L_g(t)[\|x_1 - x_2\|_1 + \|\tilde{x}_1 - \tilde{x}_2\|_\alpha], \tag{8}$$

for all $(t, x_i, \tilde{x}_i) \in V$.

Definition 2.1 *By a* local classical solution *to (6) we mean a function* $u \in C^1([t_0, t_1); X) \cap C^2((t_0, t_1); X)$ *satisfying (6) on* $[t_0, t_1)$ *for some* $t_1 > t_0$.

Definition 2.2 *By a* local mild solution *to (6) we mean the first component of a solution* (u, v) *to the pair of integral equations*

$$\begin{aligned} u(t) &= x_0 + (T(t-t_0) - I)(-A)^{-1}x_1 + \int_{t_0}^{t} (T(t-s) - I)(-A)^{-1}[f(s, u(s), v(s)) \\ &\quad + \int_{t_0}^{s} k(s-\tau) g(\tau, u(\tau), v(\tau)) d\tau] ds, \quad t_0 \le t \le t_1, \\ v(t) &= T(t-t_0)x_1 + \int_{t_0}^{t} T(t-s)[f(s, u(s), v(s)) \\ &\quad + \int_{t_0}^{s} k(s-\tau) g(\tau, u(\tau), v(\tau)) d\tau] ds, \quad t_0 \le t \le t_1, \end{aligned} \tag{9}$$

on $[t_0, t_1)$ *for some* $t_1 > t_0$.

3 Local Existence of Solution

As we have already pointed out, without loss of generality, the semigroup generated by $-A$, can be assumed to be bounded and A is invertible. Under these conditions imposed on A we prove the following local existence and uniqueness theorem.

Theorem 3.1 *Suppose that* $-A$ *generates the analytic semigroup* $T(t)$ *such that* $\|T(t)\| \le M$ *and* $0 \in \rho(-A)$. *If the maps* f *and* g *satisfy assumptions (F) and (G), respectively, and the kernel* k *satisfies (K) then (6) has a unique local classical solution.*

Proof Fix (t_0, x_0, x_1) in U and choose $t_1' > t_0$ and $\delta > 0$ such that (7), with some fixed constant $L_f > 0$, $0 < \vartheta \le 1$ and (8) with the nonnegative function $L_g(t)$ hold on the set

$$V = \{(t, x, \tilde{x}) \in U \mid t_0 \le t \le t_1', \|x - x_0\|_1 + \|\tilde{x} - x_1\|_\alpha \le \delta\}.$$

Let

$$B_f = \max_{t_0 \le t \le t_1'} \|f(t, x_0, x_1)\|,$$

$$B_g = \max_{t_0 \le t \le t_1'} \|g(t, x_0, x_1)\|$$

and

$$C(\delta) = [L_f + \|k\|_{L^p(t_0, t_1')} \|L_g\|_{L^q(t_0, t_1')}]\delta + B_f + B_g \|k\|_{L^p(t_0, t_1')} (t_1' - t_0)^{\frac{1}{q}}.$$

Choose $t_1 > t_0$ such that

$$\|T(t-t_0)x_1 - x_1\| + \|T(t-t_0)A^\alpha x_1 - A^\alpha x_1\| \le \frac{\delta}{3}$$

and

$$t_1 - t_0 < \min\left\{t_1' - t_0, \frac{\delta}{3}(M+1)^{-1}C(\delta)^{-1}, [\frac{\delta}{2}C_\alpha^{-1}(1-\alpha)C(\delta)^{-1}]^{\frac{1}{1-\alpha}}\right\},$$

where C_α is a positive constant depending on α and satisfying

$$\|A^\alpha T(t)\| \leq C_\alpha t^{-\alpha} \quad \text{for} \quad t > 0. \tag{10}$$

Let $Y = C([t_0, t_1]; X \times X)$. Then $y \in Y$ is of the form $y = (y_1, y_2)$, $y_i \in C([t_0, t_1]; X)$, $i = 1, 2$. Y, endowed with the supremum norm,

$$\|(y_1, y_2)\|_Y = \sup_{t_0 \leq t \leq t_1} [\|y_1(t)\| + \|y_2(t)\|]$$

is a Banach space. We define a map F on Y by $Fy = F(y_1, y_2) := (\hat{y}_1, \hat{y}_2)$ with

$$\begin{aligned} \hat{y_1}(t) &= Ax_0 - (T(t-t_0) - I)x_1 - \int_{t_0}^t (T(t-s) - I)F_y(s)ds, \\ \hat{y_2}(t) &= T(t-t_0)A^\alpha x_1 + \int_{t_0}^t T(t-s)A^\alpha F_y(s)ds, \end{aligned} \tag{11}$$

where

$$F_y(t) = f(t, A^{-1}y_1(t), A^{-\alpha}y_2(t)) + \int_{t_0}^t k(t-\tau)g(\tau, A^{-1}y_1(\tau), A^{-\alpha}y_2(\tau))d\tau,$$

for $t \in [t_0, t]$.

For every $y \in Y$, $Fy(t_0) = (Ax_0, A^\alpha x_1)$, and the assumptions (F) and (G) on f and g, respectively, and (K) on the kernel k imply that $F : Y \to Y$. Let S be a nonempty closed and bounded set given by

$$S = \{y \in Y \mid y = (y_1, y_2), y_1(t_0) = Ax_0, y_2(t_0) = A^\alpha x_1, \|y_1(t) - Ax_0\| + \|y_2(t) - A^\alpha x_1\| \leq \delta\}.$$

Let $y = (y_1, y_2)$ be any element of S. We have from (11)

$$\begin{aligned} \|\hat{y}_1(t) - Ax_0\| &+ \|\hat{y}_2(t) - A^\alpha x_1\| \\ &\leq \|(T(t-t_0) - I)x_1\| + \int_{t_0}^t \|T(t-s) - I\| \|F_y(s)\| ds \\ &\quad + \|(T(t-t_0) - I)A^\alpha x_1\| + \int_{t_0}^t \|A^\alpha T(t-s)\| \|F_y(s)\| ds. \end{aligned} \tag{12}$$

To find the estimate for $F_y(s)$, we add and subtract $f(s, x_0, x_1)$ and $g(s, x_0, x_1)$ and using (F), (G) and (K), we get

$$\begin{aligned} \|F_y(s)\| &\leq \|f(s, A^{-1}y_1(s), A^{-\alpha}y_2(s)) - f(s, x_0, x_1)\| + B_f \\ &\quad + \int_{t_0}^s |k(s-\tau)| [\|g(\tau, A^{-1}y_1(\tau), A^{-\alpha}y_2(\tau)) - g(\tau, x_0, x_1)\| + B_g] d\tau \\ &\leq [L_f + \|k\|_{L^p(t_0, t_1')} \|L_g\|_{L^q(t_0, t_1')}]\delta + B_f + B_g \|k\|_{L^p(t_0, t_1')} (t_1' - t_0)^{\frac{1}{q}} \\ &\leq C(\delta). \end{aligned} \tag{13}$$

Using the estimate (13) and the fact that $\|T(t)\| \leq M$ together with (10) and (12), we get

$$\begin{aligned} \|\hat{y}_1(t) - Ax_0\| + \|\hat{y}_2(t) - A^\alpha x_1\| &\leq \frac{\delta}{3} + (M+1)C(\delta)(t-t_0) + \frac{C_\alpha C(\delta)(t-t_0)^{1-\alpha}}{1-\alpha} \\ &\leq \delta. \end{aligned}$$

Hence, $F : S \to S$. Now, we show that F is a contraction on S. Let (y_1, y_2) and (z_1, z_2) be any two points of S. From (11) we have

$$\begin{aligned}\|\hat{y_1}(t) - \hat{z_1}(t)\| + \|\hat{y_2}(t) - \hat{z_2}(t)\| \quad &\leq \quad \int_{t_0}^{t} \|T(t-s) - I\|\|F_y(s) - F_z(s)\|ds \\ &\quad + \int_{t_0}^{t} \|T(t-s)A^{\alpha}\|\|F_y(s) - F_z(s)\|ds. \qquad (14)\end{aligned}$$

Using (F), (G) and (K), we get

$$\begin{aligned}&\|F_y(s) - F_z(s)\| \\ &\leq \|f(s, A^{-1}y_1(s), A^{-\alpha}y_2(s)) - f(s, A^{-1}z_1(s), A^{-\alpha}z_2(s))\| \\ &\quad + \int_{t_0}^{s} |a(s-\tau|\|g(\tau, A^{-1}y_1(\tau), A^{-\alpha}y_2(\tau)) - g(\tau, A^{-1}z_1(\tau), A^{-\alpha}z_2(\tau))\|d\tau \\ &\leq [L_f + \|k\|_{L^p(t_0,t_1')}\|L_g\|_{L^q(t_0,t_1')}]\|(y_1, y_2) - (z_1, z_2)\|_Y \\ &\leq \frac{C(\delta)}{\delta}\|(y_1, y_2) - (z_1, z_2)\|_Y. \qquad (15)\end{aligned}$$

Using (15) in (14), we get

$$\begin{aligned}\|\hat{y_1}(t) - \hat{z_1}(t)\| \quad + \quad &\|\hat{y_2}(t) - \hat{z_2}(t)\| \\ \leq \quad &\left[\frac{(M+1)C(\delta)(t-t_0)}{\delta} + \frac{C_\alpha C(\delta)(t-t_0)^{1-\alpha}}{1-\alpha}\right]\|(y_1, y_2) - (z_1, z_2)\|_Y \\ \leq \quad &\frac{2}{3}\|(y_1, y_2) - (z_1, z_2)\|_Y.\end{aligned}$$

Taking supremum over $[t_0, t_1]$, we have

$$\|(\hat{y_1}, \hat{y_2}) - (\hat{z_1}, \hat{z_2})\|_Y \leq \frac{2}{3}\|(y_1, y_2) - (z_1, z_2)\|_Y.$$

Thus, F is a contraction on S. Therefore, it has a unique fixed point in S. Let $\bar{y} = (\bar{y_1}, \bar{y_2}) \in S$ be that fixed point of F. Then

$$\begin{aligned}\bar{y_1}(t) \quad &= \quad Ax_0 - (T(t-t_0) - I)x_1 - \int_{t_0}^{t}(T(t-s) - I)F_{\bar{y}}(s)ds, \\ \bar{y_2}(t) \quad &= \quad T(t-t_0)A^{\alpha}x_1 + \int_{t_0}^{t} T(t-s)A^{\alpha}F_{\bar{y}}(s)ds, \qquad (16)\end{aligned}$$

where

$$F_{\bar{y}}(t) = f(t, A^{-1}\bar{y_1}(t), A^{-\alpha}\bar{y_2}(t)) + \int_{t_0}^{t} k(t-\tau)g(\tau, A^{-1}\bar{y_1}(\tau), A^{-\alpha}\bar{y_2}(\tau))d\tau.$$

We note that $(u, v) = (A^{-1}\bar{y_1}, A^{-\alpha}\bar{y_2})$ is the unique solution of the integral equations (9) on $[t_0, t_1]$. We can easily check that the assumption (F) and the continuity of $\bar{y_1}$ and $\bar{y_2}$ on $[t_0, t_1]$ imply that the map $t \mapsto F_{\bar{y}}(t)$ is continuous and hence bounded on $[t_0, t_1]$. Let $\|F_{\bar{y}}(t)\| \leq N$ for $t_0 \leq t \leq t_1$. We will now show that $t \mapsto F_{\bar{y}}(t)$ is locally Hölder continuous on $(t_0, t_1]$. For this we first show that $\bar{y_1}$ and $\bar{y_2}$ are locally Hölder continuous on $(t_0, t_1]$. From Theorem 2.6.13 in Pazy [8], for every $0 < \beta < 1 - \alpha$ and every $0 < h < 1$, we have

$$\|(T(h) - I)A^{\alpha}T(t-s)\| \leq C_{\beta}h^{\beta}\|A^{\alpha+\beta}T(t-s)\| \leq Ch^{\beta}(t-s)^{-(\alpha+\beta)}. \tag{17}$$

Now

$$\begin{aligned} \|\bar{y}_2(t+h) - \bar{y}_2(t)\| &\leq \|(T(h)-I)A^{\alpha}T(t-t_0)x_1\| + \int_{t_0}^{t}\|(T(h)-I)A^{\alpha}T(t-s)F_{\bar{y}}(s)\|ds \\ &\quad + \int_{t}^{t+h}\|A^{\alpha}T(t+h-s)F_{\bar{y}}(s)\|ds := I_1 + I_2 + I_3 \text{ (respectively).} \end{aligned}$$

We use (17) to get

$$\begin{aligned} I_1 &\leq C(t-t_0)^{-(\alpha+\beta)}h^{\beta} \leq M_1 h^{\beta}, \\ I_2 &\leq NCh^{\beta}\int_{t_0}^{t}(t-s)^{-(\alpha+\beta)}ds = \frac{NCh^{\beta}(t-t_0)^{1-(\alpha+\beta)}}{1-(\alpha+\beta)} \leq M_2 h^{\beta}, \\ I_3 &\leq NC_{\alpha}\int_{t}^{t+h}(t+h-s)^{-\alpha} = \frac{NC_{\alpha}h^{1-\alpha}}{1-\alpha} \leq M_3 h^{\beta}. \end{aligned}$$

Here M_1 depends on t and increases to infinity as $t \downarrow t_0$, while M_2 and M_3 can be chosen independent of t. From the above estimates, it follows that there exists a positive constant C such that for every $t_0' > t_0$,

$$\|\bar{y}_2(t) - \bar{y}_2(s)\| \leq C|t-s|^{\beta}, \quad \text{for} \quad t_0 < t_0' \leq t, s \leq t_1.$$

Similar result holds for $\bar{y}_1$ (if we take $\alpha = 0$ in the above consideration). For $s, t \in (t_0, t_1]$ with $t > s$ we have

$$\begin{aligned} \|F_{\bar{y}}(t) - F_{\bar{y}}(s)\| &\leq \|f(t, A^{-1}\bar{y}_1(t), A^{-\alpha}\bar{y}_2(t)) - f(s, A^{-1}\bar{y}_1(s), A^{-\alpha}\bar{y}_2(s))\| \\ &\quad + \int_{t_0}^{s}|k(t-\tau) - a(s-\tau)|\|g(\tau, A^{-1}\bar{y}_1(\tau), A^{-\alpha}\bar{y}_2(\tau))\|d\tau \\ &\quad + \int_{s}^{t}|k(t-\tau)|\|g(\tau, A^{-1}\bar{y}_1(\tau), A^{-\alpha}\bar{y}_2(\tau))\|d\tau. \end{aligned}$$

Since k is Hölder continuous with the exponent μ, we have

$$\int_{t_0}^{s}|k(t-\tau) - k(s-\tau)|\|g(\tau, A^{-1}\bar{y}_1(\tau), A^{-\alpha}\bar{y}_2(\tau))\|d\tau \leq N(t_1 - t_0)|t-s|^{\mu}, \tag{18}$$

and

$$\int_{s}^{t}|k(t-\tau)|\|g(\tau, A^{-1}\bar{y}_1(\tau), A^{-\alpha}\bar{y}_2(\tau))\|d\tau \leq Nk_0(t_1 - t_0)^{\alpha}|t-s|^{1-\alpha}, \tag{19}$$

where $k_0 = \max_{t_0 \leq t \leq t_1}|k(t)|$. The local Hölder continuity of $F_{\bar{y}}(t)$ on $(t_0, t_1]$ follows from the assumption (F), and the local Hölder continuity of $\bar{y_1}$ and $\bar{y_2}$ on $(t_0, t_1]$ and from estimates (18) and (19).

Consider the inhomogeneous initial value problem

$$\frac{dv(t)}{dt} + Av(t) = F_{\bar{y}}(t), \quad v(t_0) = x_1. \tag{20}$$

By the corollary 4.3.3 in [8], (20) has a unique solution $v \in C^1((t_0, t_1]; X)$ given by

$$v(t) = T(t - t_0)x_1 + \int_{t_0}^{t} T(t - s)F_{\bar{y}}(s)ds, \tag{21}$$

for $t > t_0$. Each term on the right hand side belongs to $D(A)$ and hence belongs to $D(A^\alpha)$ since $D(A) \subset D(A^\alpha)$, $0 \le \alpha \le 1$. Operating on both sides of (21) with A^α, we find that

$$A^\alpha v(t) = T(t - t_0)A^\alpha x_1 + \int_{t_0}^{t} T(t - s)A^\alpha F_{\bar{y}}(s)ds. \tag{22}$$

By (16), the right hand side of (22) equals to $\bar{y_2}(t)$ and therefore $A^\alpha v(t) = \bar{y_2}(t)$, i.e., $v(t) = A^{-\alpha}\bar{y_2}(t)$. Let $u(t) = A^{-1}\bar{y_1}(t)$, then we have $u(t) = x_0 + \int_{t_0}^{t} v(s)ds$ which yields $u(t) \in C^1([t_0, t_1); X) \cap C^2((t_0, t_1); X)$. Thus, u satisfies (6) on $[t_0, t_1)$. $\square$

4 Global Existence of Solutions

In this section we will prove, under additional growth conditions on the nonlinear map f and g, the following global existence result.

Theorem 4.1 *Let $0 \in D(-A)$ and $-A$ be the infinitesimal generator of an analytic semigroup $T(t)$ such that $\|T(t)\| \le M$ for $t \ge 0$. Let $f, g : [0, \infty) \times X_1 \times X_\alpha \mapsto X$ satisfy the assumptions (F) and (G) respectively and let k satisfy (K). If there exist a nondecreasing function $a_f : [t_0, \infty) \mapsto R_+$ and a nonnegative function $a_g \in L^q_{loc}(0, \infty)$, where q is the same as before, such that*

$$\begin{aligned} \|f(t, x, \tilde{x})\| &\le a_f(t)[1 + \|x\|_1 + \|\tilde{x}\|_\alpha], \quad \textit{for} \quad t \ge t_0, (x, \tilde{x}) \in X_1 \times X_\alpha, \\ \|g(t, x, \tilde{x})\| &\le a_g(t)[1 + \|x\|_1 + \|\tilde{x}\|_\alpha], \quad \textit{for} \quad t \ge t_0, (x, \tilde{x}) \in X_1 \times X_\alpha, \end{aligned}$$

then for each $(x_0, x_1) \in X_1 \times X_\alpha$, (6) has a unique classical solution u which exists for all $t \ge t_0$.

Proof Let $[t_0, T)$ be the maximal interval of existence for the solution u to (6) guaranteed by Theorem (3.1). It suffices to prove that $[\|u(t)\|_1 + \|v(t)\|_\alpha] \le C$ on $[t_0, T)$ for some fixed constant $C \ge 0$ independent of t.

Now, since $u(t)$ is a solution of (6) on $[t_0, T)$, it is also a mild solution to (6) therefore from (16), we have

$$\begin{aligned} Au(t) &= Ax_0 - (T(t - t_0) - I)x_1 - \int_{t_0}^{t} (T(t - s) - I)\bar{F}(s)ds, \\ A^\alpha u'(t) &= T(t - t_0)A^\alpha x_1 + \int_{t_0}^{t} T(t - s)A^\alpha \bar{F}(s)ds, \end{aligned} \tag{23}$$

where

$$\bar{F}(t) = f(t, u(t), u'(t)) + \int_{t_0}^{t} k(t - \tau)g(\tau, u(\tau), u'(\tau))d\tau.$$

From (23), we have

$$\begin{aligned}[1+\|u(\eta)\|_1+\|u'(\eta)\|_\alpha] &= [1+\|Au(\eta)\|+\|A^\alpha u'(\eta)\|] \\ &\leq 1+\|Ax_0\|+(M+1)\|x_1\|+(M+1)\int_{t_0}^{\eta}\|\bar{F}(s)\|ds \\ &\quad +M\|x_1\|_\alpha+\int_{t_0}^{\eta}C_\alpha(\eta-s)^{-\alpha}\|\bar{F}(s)\|ds. \qquad (24)\end{aligned}$$

The assumptions on f, g and k imply that

$$\begin{aligned}\|\bar{F}(s)\| &\leq \|f(t,u(t),u'(t))\|+\int_{t_0}^{s}|k(s-\tau)|\|g(\tau,u(\tau),u'(\tau))\|d\tau \\ &\leq (a_f(T)+\|k\|_{L^p(t_0,T)}\|a_g\|_{L^q(t_0,T)})\sup_{t_0\leq\tau\leq s}[1+\|u(\tau)\|_1+\|u'(\tau)\|_\alpha]. \quad (25)\end{aligned}$$

Using (25) in (24), we get

$$\begin{aligned}[1+\|u(\eta)\|_1+\|u'(\eta)\|_\alpha] &\leq C_1+C_2\int_{t_0}^{\eta}\sup_{t_0\leq\tau\leq s}[1+\|u(\tau)\|_1+\|u'(\tau)\|_\alpha]ds \\ &\quad C_3\int_{t_0}^{\eta}(\eta-s)^{-\alpha}\sup_{t_0\leq\tau\leq s}[1+\|u(\tau)\|_1+\|u'(\tau)\|_\alpha]ds.\end{aligned}$$

Hence, we have

$$\begin{aligned}\sup_{t_0\leq\eta\leq t}[1+\|u(\eta)\|_1+\|u'(\eta)\|_\alpha] &\leq C_1+C_2\int_{t_0}^{t}\sup_{t_0\leq\tau\leq s}[1+\|u(\tau)\|_1+\|u'(\tau)\|_\alpha]ds \\ &\quad C_3\int_{t_0}^{t}(t-s)^{-\alpha}\sup_{t_0\leq\tau\leq s}[1+\|u(\tau)\|_1+\|u'(\tau)\|_\alpha]ds.\end{aligned}$$

Using Lemma 4.1 in [2], we obtain $\sup_{t_0\leq\eta\leq t}[1+\|u(\eta)\|_1+\|u'(\eta)\|_\alpha]\leq C$. □

5 Example

Let $\Omega=(0,1)$ and $H=L^2(\Omega)$. Consider the following initial boundary value problem

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^3 u(x,t)}{\partial x^2\partial t} &= F\left(x,t,u(x,t),\frac{\partial^2 u(x,t)}{\partial x^2},\frac{\partial u(x,t)}{\partial t}\right) \\ &+ \int_{t_0}^{t}k(t-s)G\left(x,s,u(x,s),\frac{\partial u(x,s)}{\partial s}\right)ds, \\ &(x,t)\in\Omega\times(t_0,T),\quad 0<T<\infty \qquad (26)\end{aligned}$$

with the initial conditions

$$u(x,t_0)=x_0(x),\quad \frac{\partial u(x,t_0)}{\partial t}=x_1(x),\quad x\in\Omega,$$

and the boundary conditions

$$u(0,t)=u(1,t)=0,\quad t\in(t_0,T),\quad 0<T<\infty$$

F and G are sufficiently smooth nonlinear functions and k is a locally p-integrable function for $1 < p < \infty$.

We define the operator A with domain $D(A) = H^2(\Omega) \bigcap H_0^1(\Omega)$ as follows

$$Au = -\frac{\partial^2 u}{\partial x^2}, \quad u \in D(A).$$

Here clearly the operator A is self-adjoint with the compact resolvent and is the infinitesimal generator of an analytic semigroup $T(t)$. Now we take $\alpha = 1/2$, $D(A^{1/2})$ is the Banach space endowed with the norm

$$\|x\|_{1/2} = \|A^{1/2}x\|, \quad x \in D(A^{1/2}).$$

Using the above definition of the operator A the equation (26) can be reformulated as the following abstract equation in H

$$\begin{aligned} &u''(t) + Au'(t) = f(t, u(t), u'(t)) + \int_{t_0}^{t} k(t-s) g(s, u(s), u'(s))\, ds, \\ &u(t_0) = x_0, \quad u'(t_0) = x_1, \end{aligned} \tag{27}$$

where $u(t)(x) = u(x,t)$, the function f is defined from $[t_0, T] \times D(A) \times D(A^{1/2})$ into H such that

$$f(t, u(t), u'(t))(x) = F\left(x, t, u(x,t), \frac{\partial^2 u(x,t)}{\partial x^2}, \frac{\partial u(x,t)}{\partial t}\right)$$

and g is defined from $[t_0, T] \times D(A) \times D(A^{1/2})$ into H such that

$$g(t, u(t), u'(t))(x) = G\left(x, t, u(x,t), \frac{\partial u(x,t)}{\partial t}\right).$$

It can be varified that the assumptions in earlier sections for (27) are satisfied and hence the existence of a unique classical solution is guarenteed.

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