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Stability in the Models of Real World Phenomena

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Abstract: In this paper we consider several examples of real world models to illustrate the general methods of stability analysis of nonlinear systems developed recently in the Department of Processes Stability of S.P. Timoshenko Institute of Mechanics of NAS of Ukraine.

Keywords: *robot dynamics and control; neural networks on time scales; lasers; dynamic economic models; fuzzy control; scalar and vector Lyapunov functions.*

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1 Introduction

In this paper, we offer several examples of real world models to illustrate the general methods of stability analysis developed in the books [8, 16, 17].

Section 2 deals with the motion stability problem of robot motion whose mathematical model takes into account the dynamics of the environment interacting with the robot. We apply here some integral inequalities from Chapter 1 of the book [8].

In Section 3, we consider neural networks on time scales and introduce the study of the stability problem in this new direction.

In Section 4, we consider a problem of stability of regular synchronous generator of optical connected lasers.

Section 5 presents models from economics and using the method of vector Lyapunov functions proves that a market tends to some given evolution independent of initial conditions.

Finally in Section 6, we analyze a model of impulsive Takagi–Sugeno systems with application to the mathematical model in population growth under the impulsive control.

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2 Stability of a Robot Interacting with a Dynamic Environment

A dynamic robot model is described by the differential equation

$$H(q)\ddot{q} + h(q,\dot{q}) = \tau + J^T(q)F, \tag{2.1}$$

where q, $\dot{q}$, $\ddot{q} \in R^n$ are the vectors of the generalized coordinates, velocities and accelerations of the robot; $H(q)$ is the positive definite matrix of inertia moments of manipulator mechanisms; $h(q,\dot{q})$ is the n-dimensional nonlinear vector function which takes into consideration centrifugal, Coriolis and gravitational moments; $\tau = \tau(t)$ is the n-dimensional vector on input (control); $J^T(q)$ is the $n \times m$ Jacobi matrix associated with the motion velocity of control robot devices and its generalized coordinates; $F(t)$ is the n-dimensional vector of generalized forces or generalized forces and moments acting on the executive robot device due to the dynamic environment.

Under the condition when the environment does not admit eigen "motions" independent of the motion of the executive robot organs, the mathematical model of environment is described by the nonlinear vector equation

$$M(s)\ddot{s} + L(s,\dot{s}) = -F, \tag{2.2}$$

$$s = \varphi(q), \tag{2.3}$$

where s is the vector of the environment motions; $\varphi(q)$ is the vector function connecting the coordinates s and q. Note that in the case of traditional hybrid control, the environment plays the role of kinematic limitation and the relationship (2.3) becomes

$$\varphi(q) = 0. \tag{2.4}$$

Under certain assumptions the equation (2.2) may be represented as

$$M(q)\ddot{q} + L(q,\dot{q}) = -S^T(q)F, \tag{2.5}$$

where $M(q)$ is the $n \times m$ non-singular matrix; $L(q,\dot{q})$ is the nonlinear n-dimensional vector function; $S^T(q)$ is the $n \times m$ matrix of the n rank.

Thus the equation set (2.1), (2.5) represents the closed mathematical model of the robot interacting with the environment.

Let $q_p(t)$ be the n-dimensional vector of the program value of the generalized coordinates, $\dot{q}_p(t)$ be the n-dimensional vector of the program value of the generalized velocities, $F_p(t)$ be the m-dimensional vector of forces corresponding to the program values of the generalized coordinates and velocities. The program values of force $F_p(t)$ and those of functions $q_p(t)$, $\dot{q}_p(t)$, $\ddot{q}_p(t)$ cannot be arbitrary and should satisfy the relationship $F_p \equiv \Phi(q_p(t), \dot{q}_p(t), \ddot{q}_p(t))$ where $\Phi \in C(R^n \times R^n \times R^n, R^m)$. The connected system of equations (2.1), (2.5) can easily be reduced to the form

$$M(q)\ddot{q} - M(q_p) + L(q,\dot{q}) - L(q_p,\dot{q}_p) + \left[S^T(q) - S^T(q_p)\right]F_p = -S^T(q)(F - F_p). \tag{2.6}$$

The n-dimensional vector of deviations of the program trajectory from real one is designated by y. Then the equation (2.6) becomes

$$\ddot{y} + K(t,y,\dot{y}) = -M^{-1}(y+q_p)S^T(y+q_p)(F - F_p), \tag{2.7}$$

where

$$\begin{aligned} K(t,y,\dot y) \ &= M^{-1}(y+q_p)\Big\{L(y+q_p,\dot y+\dot q_p)- \\ &- L(q,\dot q_p)+\big[M(y+q_p)-M(q_p)\big]\ddot q_p+ \\ &+\big[S^T(y+q_p)-S^T(q_p)\big]F_p\Big\}. \end{aligned}$$

The equation set (2.7) is transformed to the following form

$$\frac{dx}{dt}=A(t)x+\alpha(t,x)+\beta(t,x)\mu(t),$$

where

$$x=(x_1,x_2)^T, \quad x_1=y, \quad x_2=\dot y, \quad A(t)=\begin{pmatrix} O_n & I_n \\ -\left.\dfrac{\partial K}{\partial y}\right|_{(y,\dot y)=(0,0)} & -\left.\dfrac{\partial K}{\partial \dot y}\right|_{(y,\dot y)=(0,0)} \end{pmatrix},$$

O_n and I_n are the respective zero and unit matrices of dimension n,

$$\alpha(t,x)=\begin{pmatrix} 0 \\ -\alpha_0(t,x_1,x_2)\end{pmatrix}, \qquad \alpha_0(y,\dot y,t)=o(\|y\|^2+\|\dot y\|^2)^{1/2},$$

$$\beta(t,x)=\begin{pmatrix} 0 \\ -M^{-1}(x_1+q_p)S^T(x_1+q_p)\end{pmatrix}, \qquad \mu(t)=F(t)-F_p(t).$$

Within the general statement, the problem of choosing the program forces $F_p(t)$ is associated with studying the solutions of the differential equation

$$\frac{d\mu}{dt}=Q(\mu),$$

where $Q\in C(R^m,R^m)$, $\mu(t)=F(t)-F_p(t)$, $\mu(t_0)=0$ and $Q(0)=0$.

Thus the problem of stability of the robot motion interacting with a dynamic environment results in the need to analyze the solutions of the systems of equations

$$\frac{dx}{dt}=A(t)x+\alpha(t,x)+\beta(t,x)\mu(t), \quad x(t_0)=x_0, \tag{2.8}$$

$$\frac{d\mu}{dt}=Q(\mu), \quad \mu(t_0)=\mu_0 \tag{2.9}$$

under certain assumptions of functions specifying the action of dynamic environment on the robot.

Consider the independent equation (2.9) which specifies the influence of dynamic environment on the executive organ of the robot. From (2.9) it follows that

$$\mu(t)=\mu_0+\int\limits_{t_0}^{t} Q(\mu(s))\,ds, \quad t\ge t_0. \tag{2.10}$$

The term in the equation (2.8) which specifies the action of environment on the robot is designated by $u(t,x)=\beta(t,x)\mu(t)$ for $(t,x)\in R_+\times D$, $D=\{x:\ \|x\|<H\}$, H is sufficiently small, the function $u(t,x)$ satisfies the inequality

$$\|u(t,x)\|\le p(t), \tag{2.11}$$

where $p(t)$ is the function integrable over any finite time interval. With

$$\mu(t) = F(t) - F_p(t) \tag{2.12}$$

and (2.10) representing the deviation of program value of the force $F_p(t)$ from the force $F(t)$ acting due to the dynamic environment, the action of environment on the robot may be estimated by the function $p(t)$. We introduce the designations

$$p_0 = \sup_{t\geq 0} p(t), \quad p_1 = \sup_{t\geq 0} \int_t^{t+1} p(s)\,ds, \quad p_2 = \sup_{t\geq 0} \left(\int_t^{t+1} p^2(s)\,ds \right)^{1/2}.$$

Further the following definition will be used.

Definition 2.1 Let for any $\varepsilon > 0$ the values $\Delta > 0$ and $\delta > 0$ be those for which the inequality $\|x(t)\| < \varepsilon$ occurs for solving the equation (2.8) with $t \geq 0$ if $\|x(0)\| < \delta$ and one of the following conditions is satisfied

(1) $p_0 \leq \Delta$;

(2) $p_1 \leq \Delta$;

(3) $p_2 \leq \Delta$.

Here we consider that robot motion is:

(a) *stable with limited action* of environment on the robot (case 1);

(b) *stable with limited, on the average, action* of environment on the robot (case 2);

(c) *stable with limited, on the quadratic average, action* of environment on the robot (case 3).

It is of interest to consider the action of environment on the robot when the limiting relationship $\|u(t,x)\| \to 0$, $t \to \infty$ is uniformly satisfied over x with sufficiently low values $\|x\|$. This corresponds to the choice of τ control in (2.1) when $F(t) \to F_p(t)$, $t \to \infty$.

In the case when H in the estimate of the domain D is not small ($H < \infty$) and consequently, the large neighborhood of the equilibrium state of the robot-mechanical system is considered, the estimate

$$\|u(t,x)\| \leq \lambda(t)\|x\| \tag{2.13}$$

should be taken instead of (2.11), where $\lambda(t)$ is the integrable function such that

$$\int_0^\infty \lambda(s)\,ds < +\infty. \tag{2.14}$$

Let us make the following assumptions on the equations (2.8) and (2.9):

I. The fundamental matrix $X(t)$ of solutions of the first approximations of the system (2.8) satisfies the inequality

$$\|X(t)X^{-1}(s)\| \leq Ne^{-\gamma(t-s)}, \tag{2.15}$$

where N and γ are positive constants independent of t_0. Note that the condition (2.15) guarantees the exponential stability of the zero solution of

$$\frac{dx}{dt} = A(t)\,x. \tag{2.16}$$

II. The vector function $\alpha(t,x)$ in (2.8) satisfies the following condition: for each $L > 0$ the values $D = D(L)$ and $T = T(L)$ exist, such that

$$\|\alpha(t,x)\| \le L\|x\| \tag{2.17}$$

with $\|x\| \le D$ and $t \ge T$.

III. The influence of the vector function of dynamic environment on robot satisfies the condition $\|u(t,x)\| \to 0$ with $t \to \infty$, uniformly over x with sufficiently small values $\|x\|$.

Theorem 2.1 *The equations* (2.8) *and* (2.9) *of the robot movement interacting with the environment are assumed to be those where the conditions I–III are satisfied. Then there exists $t_0 \in R_+$ such that any movement $x(t; t_0, x_0)$ of the robot simulated by the system* (2.8) *will approach to zero with $t \to \infty$ and sufficiently small values $\|x(t_0)\|$.*

Proof When the condition I is satisfied, the value L in (2.17) is defined by the formula $L = \gamma(2N)^{-1}$:

$$\|\alpha(t,x)\| \le \frac{\gamma}{2N}\,\|x\|, \quad t \ge T. \tag{2.18}$$

From the condition III it follows that $\sigma > 0$ exists such that

$$\|u(t,x)\| \le \sigma < \frac{\gamma - NL}{2N}\,\delta, \quad t \ge T. \tag{2.19}$$

For a certain $t_0 \in R_+$ with $t \ge t_0$ we have

$$x(t) = W(t,t_0)x_0 + \int_{t_0}^{t} W(t,\tau)\big[\alpha(\tau, x(\tau)) + u(\tau, x(\tau))\big]\,ds. \tag{2.20}$$

With the estimates (2.15), (2.17)–(2.19) we find from (2.20)

$$\begin{aligned}\|x(t)\| \le Ne^{-\gamma(t-t_0)}\|x_0\| + NL\int_{t_0}^{t} e^{-\gamma(t-s)}\|x(s)\|\,ds \\ + N\int_{t_0}^{t} e^{-\gamma(t-s)}\|u(s,x(s))\|\,ds.\end{aligned} \tag{2.21}$$

Let us designate $M(t) = \max\limits_{t_0 \le s \le t} \|x(s)\|$ and represent (2.21) as

$$M(t) \le N\|x_0\| + \frac{NL}{\gamma}\,M(t) + \frac{N\sigma}{\gamma} \le \frac{N\gamma}{\gamma - NL}\,\|x_0\| + \frac{N\sigma}{\alpha - NL}\,.$$

Since $2N\sigma(\beta - NL)^{-1} < \delta$, then $M(t) < \delta$ with all $t \geq t_0$ as soon as

$$\|x_0\| < \frac{\delta(\beta - NL)}{4N\beta} < \frac{\delta}{4N}\,.$$

Set $\Lambda = \lim\limits_{t\to\infty} \sup \|x(t)\|$. It is evident that $0 \leq \Lambda \leq \delta < +\infty$ and the sequence $\{t_j\}$, $j = 1, 2, \dots$ exists such that the limiting relationship $\|x(t_j)\| \to \Lambda$ is valid with $t_j \to +\infty$, $j \to +\infty$.

From the inequality (2.21) we obtain

$$\begin{aligned}\|x(t_j)\| \leq N\|x_0\|e^{-\gamma(t_j - t_0)} + NL \int\limits_{t_0}^{t_j/2} e^{-\gamma(t_j - s)}\|x(s)\|\, ds \\ + NL \int\limits_{t_j/2}^{t_j} e^{-\gamma(t_j - s)}\|x(s)\|\, ds + N \int\limits_{t_0}^{t_j/2} e^{-\gamma(t_j - s)}\|u(s, x(s))\|\, ds \\ + N \int\limits_{t_j/2}^{t_j} e^{-\gamma(t_j - s)}\|u(s, x(s))\|\, ds.\end{aligned}$$

For a given $\eta > 0$ there exists J_η such that $\|x(t_j)\| > \Lambda - \eta$ for all $j \geq J_\eta$ and $\|x(t)\| < \Lambda + \eta$ with $t \geq t_j/2$. Consequently, with $j \geq J_\eta$ we find

$$\begin{aligned}\Lambda - \eta \leq N\|x_0\|e^{-\gamma(t_j - t_0)} + \frac{NL\delta}{\gamma}\, e^{-\frac{1}{2}\beta t_j} + \frac{NL(\Lambda + \eta)}{\gamma} + \frac{N\sigma}{\gamma}\, e^{-\frac{1}{2}\beta t_j} \\ + \frac{NL}{\gamma} \max_{\frac{1}{2} t_j \leq s \leq t_j} \|u(t, x(s))\|.\end{aligned}$$

Thus $\Lambda - \eta \leq \dfrac{NL(\Lambda + \eta)}{\gamma}$ is obtained as $j \to +\infty$. Since $NL\beta^{-1} < 1/2$, we have $\Lambda < 3\eta$. It follows from arbitrariness of η that $\Lambda = 0$. With the definition of Λ, we may conclude that the motion $x(t)$ at vanishing interactions of robot with the environment tends to the equilibrium state corresponding to the zero solution of (2.8).

Further we study the motion of the robot interacting with dynamic environment under the conditions (2.13) and (2.14). For providing sufficient stability conditions the following Lemma is needed.

Lemma 2.1 *Let γ be the positive constant and the function $\lambda(t) \in C(R_+, R_+)$ be such that*

$$\int\limits_0^\infty \lambda(s)\, ds < +\infty \quad \text{or} \quad \lim_{t\to+\infty} \lambda(t) = 0.$$

Then

$$\lim_{t\to+\infty} e^{-\gamma t} \int\limits_0^t e^{\gamma s}\lambda(s)\, ds = 0.$$

Proof Let us first prove the case when $\lambda(t)$ is integrable. For the given $\varepsilon > 0$ we choose t to be large enough that

$$\int_{t/2}^{\infty} \lambda(s)\,ds < \frac{\varepsilon}{2}\,, \qquad e^{-\gamma t}\int_{0}^{t/2} \lambda(s)\,ds < \frac{\varepsilon}{2}\,.$$

Then

$$e^{-\gamma t}\int_{0}^{t/2} e^{\gamma s}\lambda(s)\,ds \le e^{-\gamma \frac{t}{2}}\int_{0}^{t/2} \lambda(s)\,ds < \frac{\varepsilon}{2}\,,$$

$$e^{-\gamma t}\int_{t/2}^{0} e^{\gamma s}\lambda(s)\,ds \le \int_{t/2}^{t} \lambda(s)\,ds \le \int_{t/2}^{\infty} \lambda(s)\,ds < \frac{\varepsilon}{2}\,.$$

Consequently,

$$e^{-\gamma t}\int_{0}^{t} e^{\gamma s}\lambda(s)\,ds < \varepsilon$$

with a sufficiently large t. Therefore

$$\lim_{t\to\infty} e^{-\gamma t}\int_{0}^{t} e^{\gamma s}\lambda(s)\,ds = 0.$$

Consider the case $\lambda(t) \to 0$ with $t \to +\infty$. If $\int_0^\infty e^{\gamma s}\lambda(s)\,ds < +\infty$ the statement of Lemma 2.1 is evident. On the other hand, using the L'Hospital rule we obtain

$$\lim_{t\to+\infty} \frac{1}{e^{\gamma t}}\int_{0}^{t} e^{\gamma s}\lambda(s)\,ds = \lim_{t\to\infty} \frac{\lambda(t)}{\gamma} = 0.$$

Lemma 2.2 *Let the function $u(t)$ be continuous and non-negative and satisfy the inequality*

$$u(t) \le c + \int_{0}^{t} \big[ku(s) + \lambda(s)\big]\,ds, \quad t \ge 0,$$

where c and k are non-negative constants. Then

$$u(t) \le ckt + \int_{0}^{t} e^{k(t-s)}\lambda(s)\,ds, \quad t \ge 0.$$

Proof of this lemma follows by the standard method developed in the theory of integral inequalities.

Theorem 2.2 *The equations* (2.8), (2.9) *of robot movement interacting with the environment are supposed to be such that*

(1) *the condition I is satisfied;*

(2) *for any* $\varepsilon > 0$ *there exists* $L = L(\varepsilon) > 0$ *such that* $\|\alpha(t,x)\| \le L(\varepsilon)\|x\|$ *with* $\|x\| \le \delta$, $t \ge 0$*;*

(3) *the vector function* $u(t,x) = \beta(t,x)\mu(t)$ *satisfies the estimate*

$$\|u(t,x)\| \le \sigma < \frac{\gamma - NL}{2N}\,\delta, \quad t \ge 0.$$

Then any robot motion beginning in the domain $\{x \in R^{2n} \colon \|x(0)\| < \delta/(2N)\}$ *will remain in the domain* $\{x \in R^{2n} \colon \|x\| < \delta/2\}$ *for all* $t \ge 0$.

Proof From the inequality (2.21) we obtain

$$u(t) \le N\|x_0\| + \int_0^t \big[NL(\varepsilon)u(s) + Ne^{\gamma s}\|u(s,x(s))\|\big]\,ds, \tag{2.22}$$

where $u(t) = e^{\gamma t}\|x(t)\|$. Applying Lemma 2.2 to (2.22) we find

$$u(t) \le e^{NLt}\left[N\|x_0\| + \int_0^t Ne^{\gamma s}\|u(s,x(s))\|e^{-NLs}\,ds\right]$$

and consequently,

$$\|x(t)\| \le N\|x_0\|e^{-(\gamma-NL)t} + Ne^{-(\gamma-NL)t}\int_0^t \|u(s,x(s))\|e^{(\gamma-NL)s}\,ds. \tag{2.23}$$

From the inequality $\|x_0\| \le \delta/(2N)$ it follows that the first summand in (2.23) will be smaller than $\delta/2$ for all $t \ge 0$. From condition 3 of Theorem 2.2 it follows that

$$\frac{\delta(\gamma - NL)}{2}\,e^{-(\gamma-NL)t}\int_0^t e^{(\gamma-NL)s}\,ds \le \frac{\delta}{2}\,. \tag{2.24}$$

Consequently, from (2.24) we obtain

$$\begin{aligned}\|x(t)\| &\le N\|x_0\|e^{-(\gamma-NL)t} + \frac{\delta}{2}\left(1 - e^{-(\gamma-NL)t}\right)\\ &\le \frac{\delta}{2}\,e^{-(\gamma-NL)t} + \frac{\delta}{2}\left(1 - e^{-(\gamma-NL)t}\right) = \frac{\delta}{2}\end{aligned}$$

for all $t \ge 0$.

The proof is complete.

Remark 2.1 From Theorem 2.2 it follows that if $\|u(t,x)\| \to 0$ or $\int_0^\infty \|u(s,x(s))\|\,ds < \infty$, the robot motion tends to the equilibrium state as $t \to +\infty$.

Case A. Consider the interactions of the robot with dynamic environment when functions $\beta(t,x)\mu(t)$ satisfy the estimate

$$\|\beta(t,x)\mu(t)\| \le \lambda(t) \tag{2.25}$$

for $\|x\| \le r$, $r > 0$, $t \ge 0$ and

$$G(t) = \int_t^{t+1} \lambda(s)\,ds \to 0 \tag{2.26}$$

as $t \to \infty$.

It is evident that the condition (2.26) will be satisfied if $\lambda(t) \to 0$ with $t \to +\infty$ or $\int_0^\infty \lambda(s)\,ds < +\infty$. It is shown (see [26]) that the function $\lambda(t)$ may be determined as follows:

$$\lambda(t) = \begin{cases} 1 & \text{with} \quad t = 3n, \\ 0 & \text{with} \quad 3n + \frac{1}{n} \le t \le 3(n+1) - \frac{1}{n+1}, \\ 0 & \text{with} \quad 0 \le t \le 2. \end{cases}$$

The robot motion under the conditions (2.25) and (2.26) is described by the following statement.

Theorem 2.3 *Let us assume that the equations* (2.8) , (2.9) *of the perturbed motion of the robot interacting with environment are such that*

(1) *for the equations of the first approximation* (2.16) *the condition I is satisfied;*

(2) *for the vector function* $\alpha(t,x)$ *nonlinearities with any* $L > 0$, $\delta = \delta(L) > 0$ *and* $\tau = \tau(L) > 0$ *exist such that* $\|\alpha(t,x)\| \le L\|x\|$ *with* $\|x\| \le \delta$ *and* $t \ge \tau$*;*

(3) *for the arbitrary solution* $\mu(t)$ *of the relationships* (2.10) *and* (2.12) *which determine the quality of unsteady response to the robot interaction with environment, the conditions* (2.25) *and* (2.26) *are satisfied.*

Then the time $\tau^* \ge 0$ *and the domain* $S_\delta = \{x \in R^{2n} : \|x\| < \delta,\ \delta > 0\}$ *will be found, such that the robot motion starting in the domain* S_δ *at any time moment* $t_0 \ge \tau^*$*, will approach the equilibrium state, i.e.* $\|x(t)\| \to 0$ *at* $t \to +\infty$*.*

Proof When the condition (1) is satisfied the Cauchy matrix $W(t,s)$ of the linear approximation (2.16) of the system (2.8) satisfies the condition $\|W(t,t_0)\| \le Ne^{-\gamma(t-t_0)}$ at all $t \ge t_0$. Let $0 < L < \min\{(\gamma/N), r\}$. By the condition (2), $\tau(L)$ and $\delta(L)$ can be chosen such that $\tau(L) \ge 1$ and $\delta(L) \le L$. Besides, let $\tau^* \ge \tau(L)$ be such that with $t \ge \tau^*$ the estimate

$$\int_1^t e^{-(\gamma - NL)(t-s)}\lambda(s)\,ds < \frac{\delta(L)}{2N} = \delta_1 \tag{2.27}$$

is valid.

It is easy to show that for all $t \ge t_0 \ge 1$ the inequality

$$\int_{t_0}^t \lambda(s)\,ds \le \int_{t_0-1}^t G(s)\,ds$$

is satisfied. Thus for any $k > 0$ the estimate

$$\int_{t_0}^{t} e^{ks}\lambda(s)\,ds \le \int_{t_0-1}^{t} e^{k(s+1)}\left[\int_{s}^{s+1}\lambda(u)\,du\right]ds = \int_{t_0-1}^{t} e^{k(s+1)}G(s)\,ds \tag{2.28}$$

is valid

With (2.28) we obtain

$$e^{-kt}\int_{t_0}^{t} e^{ks}\lambda(s)\,ds \le e^{-kt}\int_{t_0-1}^{t} e^{k(s+1)}G(s)\,ds. \tag{2.29}$$

Applying the L'Hospital rule to the expression in the right side of the inequality (2.29) we can find

$$\lim_{t\to\infty} e^{-kt}\int_{t_0-1}^{t} e^{k(s+1)}G(s)\,ds = 0$$

whence it follows that the inequality (2.27) is justified. Then let $t_0 \ge \tau^*$ and $\|x(t_0)\| < \delta_1 = \dfrac{\delta(L)}{2N}$. From the equality (2.20) we obtain

$$\|x(t)\| \le N\delta_1 e^{-\gamma(t-t_0)} + N\int_{t_0}^{t} e^{-\gamma(t-s)}\big[L\|x(s)\| + \lambda(s)\big]\,ds,$$

thus

$$\|x(t)\|e^{\gamma t} \le N\delta_1 e^{\gamma t_0} + \int_{t_0}^{t}\big[NL\|x(s)\|e^{\gamma s} + Ne^{\gamma s}\lambda(s)\big]\,ds. \tag{2.30}$$

Let us designate $\|x(t)\|e^{\gamma t} = w(t)$ and use Lemma 2.2 for the inequality (2.30). It is easy to see that

$$w(t) \le N\delta_1 e^{\gamma t_0} e^{NL(t-t_0)} + \int_{t_0}^{t} e^{NL(t-s)} Ne^{\gamma s}\lambda(s)\,ds,$$

or

$$\|x(t)\| \le N\delta_1 e^{-(\gamma-NL)(t-t_0)} + N\int_{t_0}^{t} e^{-(\gamma-NL)(t-s)}\lambda(s)\,ds.$$

Then by the condition (2.27) we find

$$\|x(t)\| \le N\delta_1 + N\int_{t_0}^{t} e^{-(\gamma-NL)(t-s)}\lambda(s)\,ds < \frac{\delta}{2} + N\delta_1 = \delta.$$

Thus, $\|x(t)\| < \delta$ for all $t \ge t_0$ and the limiting relationship $\|x(t)\| \to 0$ is satisfied for $t \to +\infty$. The statement is proved.

Case B. Three conditions for the vector function $\beta(t,x)\mu(t)$, $\mu(t) = F(t) - F_p(t)$ will be taken into consideration which specify the robot interacting with the dynamic environment. The following estimate of the function of transient process in (2.8) is needed.

Lemma 2.3 *Let the conditions be satisfied for the equations of perturbed motion* (2.8)*:*

(1) *the Cauchy matrix $W(t,t_0)$ of the equations of the first approximation* (2.16) *satisfies the condition I;*

(2) *for the vector function of nonlinearities $\alpha(t,x)$ with each $L > 0$, a certain value $H = H(L) > 0$ exists such that $\|\alpha(t,x)\| \le L\|x\|$ for $\|x\| \le H$ and $t \ge 0$;*

(3) *for any function $\mu(t)$, satisfying the relationships* (2.10) *and* (2.12) *the estimation holds for all $\|x\| < H$ and $t \ge 0$.*

Then for sufficiently small initial perturbations $x_0 = x(0)$ and $\mu(0) = F(0) - F_p(0)$ the transient process in (2.8) *satisfies the estimate*

$$\|x(t)\| \le N(\Phi_1(t) + \Phi_2(t)), \tag{2.31}$$

where

$$\Phi_1(t) = e^{-\varkappa t}\|x_0\|, \quad x_0 = x(0),$$
$$\Phi_2(t) = e^{-\varkappa t}\int\limits_0^t e^{\varkappa s}p(s)\,ds, \quad \varkappa = \beta - NL.$$

The estimate (2.31) follows from Lemma 6.1 of Barbashin [1], p. 185, where the function $\Phi_2(t)$ is shown to satisfy one of the following inequalities for all $t \ge 0$

$$\Phi_2(t) \le p_0\varkappa^{-1}, \quad \Phi_2(t) \le p_1 e^{\varkappa}(1-e^{-\varkappa})^{-1}, \quad \Phi_2(t) \le p_2(1-e^{-\varkappa})^{-1}\left(\frac{e^{2\varkappa}-1}{2\varkappa}\right)^{1/2}.$$

The sufficient conditions which provide the stability of motion of the robot interacting with the environment are given in the following statement.

Theorem 2.4 *The equations of perturbed motion of the robot interacting with the environment are supposed to be such that*

(1) *for the equations of the first approximation* (2.16) *the condition I is satisfied;*

(2) *for the vector function of nonlinearities $\alpha(t,x)$ with any $L > 0$, $\delta = \delta(L) > 0$ exists such that $\|\alpha(t,x)\| \le L\|x\|$ with $\|x\| \le H$ and $t \ge 0$;*

(3) *for any function $\mu(t)$ which satisfies the relationships* (2.10) *and* (2.12) *for all $\|x\| \le H$ and $t \ge 0$ the estimate* (2.11) *and one of the inequalities are satisfied*

$$p_0 < \frac{\Delta}{2N}\,\varkappa, \tag{2.32}$$

$$p_1 < \frac{\Delta}{2N}\,e^{-\varkappa}\big(1-e^{-\varkappa}\big), \tag{2.33}$$

$$p_2 < \frac{\Delta}{2N}\left(\frac{2\varkappa}{e^{2\varkappa}-1}\right)^{1/2}\big(1-e^{-\varkappa}\big). \tag{2.34}$$

Then under any initial condition

$$x_0 = x(0), \quad \mu(0) = F(0) - F_p(0) \tag{2.35}$$

for which $\|x_0\| < \Delta(2N)^{-1}$*, the transient process of the system* (2.8) *satisfies the estimate*

$$\|x(t)\| \le N(\Phi_1(t) + \Phi_2(t)) \tag{2.36}$$

for all $t \ge 0$ *and* $\|x(t)\| < \Delta$*.*

The ***Proof*** of Theorem 2.4 is based on the estimate of the transient process (2.31). Under the initial conditions (2.35) when $\|x_0\| < \Delta(2N)^{-1}$, the estimate $\Phi_1(t) < \Delta(2N)^{-1}$ is valid for the function $\Phi_1(t)$ for all $t \ge 0$. When the conditions (2.32)–(2.34) are satisfied the estimate $\Phi_2(t) < \Delta(2N)^{-1}$ is valid for the function $\Phi_2(t)$. Therefore it follows from (2.36) that $\|x(t)\| < \Delta$ for all $t \ge 0$. The proof of Theorem is complete.

Next we will show that the motion of the robot interacting with the environment medium may be dissipative under appropriate limitation on the initial state x_0 and the function $\mu(t) = F(t) - F_p(t)$.

Theorem 2.5 *Let us suppose that for the equation* (2.8) *of perturbed motion of robot interacting with environment*

(1) *the conditions (1)–(2) of the Theorem 2.4 hold;*

(2) *the inequalities*

$$p_0 < \rho \frac{\Delta}{N} \varkappa, \tag{2.37}$$

$$p_1 < \rho \frac{\Delta}{N} e^{-\varkappa}(1 - e^{-\varkappa}), \tag{2.38}$$

$$p_2 < \rho \frac{\Delta}{N} \left(\frac{2\varkappa}{e^{2\varkappa} - 1}\right)^{1/2} (1 - e^{-\varkappa}), \tag{2.39}$$

are satisfied in Δ*-neighborhood of the state* $x = 0$*, i.e. with all* $(x \ne 0) \in \{x : \|x\| < \Delta\}$ *where* $0 < \rho < 1,\ 0 < \delta < \Delta(2N)^{-1}$*.*

Then the positive number $\tau \in R_+$ *exists such that for* $t > \tau$ *and* $\|x_0\| < \delta$ *the transient process in* (2.8) *satisfies the estimate* $\|x(t)\| < \delta$ *for all* $t \ge \tau$*.*

Proof Consider the estimate (2.36). Then choose $\tau > \frac{1}{\varkappa} \ln N(1 - p)^{-1}$ and the estimate for the functions $\Phi_1(t)$ and $\Phi_2(t)$ can be obtained. For $t > \tau$ we have $\Phi_1(t) = e^{-\varkappa t}\|x_0\| < (1 - p)\delta N^{-1}$ for all $t \ge \tau$. When at least one of the conditions (2.37)–(2.39) is satisfied, $\Phi_2(t) < \rho\delta N^{-1}$ is obtained for all $t \ge \tau$. It follows from the estimate (2.36) that the transient process in the system will be damping, i.e. $\|x(t)\| < \delta$ for all $t > \tau$.

Further the equations of the perturbed motion (2.8) will be considered under the following assumptions:

I′. The matrix $A(t)$, the vector function of nonlinearity $\alpha(t, x)$ and the vector function $\beta(t, x)\mu(t)$ where $\mu(t) = F(t) - F_p(t)$ are continuous and periodic with respect to t. The period of these functions are supposed to be common, for example, unity.

II′. As above, the assumption I is preserved for the case of periodic matrix $A(t)$, i.e.

$$\|W(t,s)\| \le Ne^{-\gamma(t-s)}, \tag{2.40}$$

where $W(t,s) = X(t)X^{-1}(s)$.

III′. The vector function of nonlinearities $\alpha(t,x)$ in the domain $\|x\| < H$, $t \ge 0$ satisfies the Lipschitz condition

$$\|\alpha(t,x) - \alpha(t,y)\| \le K\|x-y\|. \tag{2.41}$$

IV′. The constants N, γ, K in the inequalities (2.40), (2.41) satisfy the inequality $\varkappa^* = \gamma - NK > 0$.

Theorem 2.6 *Suppose that for the equations of perturbed motion* (2.8) *for the robot interacting with the environment, the conditions I′–IV′ are satisfied. Besides, one of the conditions* (2.37)–(2.39) *is satisfied. Then in the domain $\|x\| < H(2N)^{-1}$ the periodic robot motion $z(t)$ is possible, and for any other motion $x(t)$ of the robot, which is started in the domain $\|x(0)\| \le H(2N)^{-1}$, the limiting relationship $\|x(t) - z(t)\| \to 0$ with $t \to +\infty$ is valid, i.e. the periodic robot motion is asymptotically stable.*

The ***Proof*** of this theorem is based on the principle of contracting mappings and Theorem 6.4 in Barbashin [1].

3 Stability Analysis of Neural Networks on Time Scales

In this section we consider stability of a *neural network on time scale* [6] the dynamics of which is described by equation of the type

$$x^{\Delta}(t) = -Bx(t) + TS(x(t)) + J, \quad t \in [0,+\infty), \tag{3.1}$$

whose solution $x(t;t_0,x_0)$ for $t = t_0$ takes the value x_0, i.e.

$$x(t_0;t_0,x_0) = x_0, \quad t_0 \in [0,+\infty), \quad x_0 \in \mathbb{R}^n, \tag{3.2}$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is an arbitrary time scale, $0 \in \mathbb{T}$, $\sup \mathbb{T} = +\infty$. In system (3.1) $x^{\Delta}(t)$ is a Δ-derivative on time scale $\mathbb{T}$, $x \in \mathbb{R}^n$ characterizes the state of neurons, $T = \{t_{ij}\} \in \mathbb{R}^{n\times n}$, the components t_{ij} describe the interaction between the i-th and j-th neurons, $S\colon \mathbb{R}^n \to \mathbb{R}^n$, $S(x) = (s_1(x_1), s_2(x_2), \dots, s_n(x_n))^{\mathrm{T}}$, the function s_i describes response of the i-th neuron, $B \in \mathbb{R}^{n\times n}$, $B = \operatorname{diag}\{b_i\}$, $b_i > 0$, $i = 1,2,\dots,n$, $J \in \mathbb{R}^n$ is a constant external input vector.

If $\mathbb{T} = \mathbb{R}$, then $x^{\Delta} = d/dt$ and the initial problem (3.1)–(3.2) is equivalent to the initial problem for a continuous *Hopfild type neural network*

$$\frac{dx(t)}{dt} = -Bx(t) + TS(x(t)) + J, \quad t \ge 0, \tag{3.3}$$

$$x(t_0;t_0,x_0) = x_0, \quad t_0 \ge 0, \quad x_0 \in \mathbb{R}^n. \tag{3.4}$$

If $\mathbb{T} = \mathbb{N}_0$, then $x^{\Delta}(k) = x(k+1) - x(k) = \Delta x(k)$ and the initial problem (3.1)–(3.2) is equivalent to

$$\Delta x(k) = -Bx(k) + TS(x(k)) + J, \quad t \in \mathbb{N}_0, \tag{3.5}$$

$$x(k_0;k_0,x_0) = x_0, \quad k_0 \in \mathbb{N}_0, \quad x_0 \in \mathbb{R}^n. \tag{3.6}$$

We assume relative to system (3.1) the following.

S_1. The vector-function $f(x) = -Bx + TS(x) + J$ is regressive.

S_2. There exist positive constants $M_i > 0$, $i = 1, 2, \dots, n$, such that $|s_i(u)| \le M_i$ for all $u \in \mathbb{R}$.

S_3. There exist positive constants $L_i > 0$, $i = 1, 2, \dots, n$, such that $|s_i(u) - s_i(v)| \le L_i|u - v|$ for all $u, v \in \mathbb{R}$.

S_4. Granularity function of the time scale $\mathbb{T}$ $0 < \mu(t) \in \mathcal{M}$ for all $t \in [0, +\infty)$, where $\mathcal{M} \subset \mathbb{R}$ is a compact set.

We recall that the matrix $A \in \mathbb{R}^{n\times n}$ is called *M-matrix* if its all non-diagonal elements are non-positive and all principle minors are positive.

We denote by $r = \Big(\sum_{i=1}^n \big(\sum_{j=1}^n M_j|T_{ij}| + |J_i|\big)^2/b_i^2\Big)^{1/2}$ and $\Lambda = \operatorname{diag}\{L_i\} \in \mathbb{R}^{n\times n}$ and prove the following assertion.

Theorem 3.1 *If for system* (3.1) *conditions* S_1–S_4 *are satisfied then there exists an equilibrium state* $x(t) = x^*$ *of system* (3.1) *and moreover,* $\|x^*\| \le r$. *Besides, if the matrix* $B\Lambda^{-1} - |T|$ *is an M-matrix, this equilibrium state is unique.*

Proof For the state $x(t) = x^*$ to be the equilibrium state of system (3.1) it is necessary and sufficient that

$$-Bx^* + TS(x^*) + J = 0$$

or

$$x^* = B^{-1}(TS(x^*) + J).$$

Consider the mapping $h\colon \mathbb{R}^n \to \mathbb{R}^n$, $h(x) = (h_1(x), h_2(x), \dots, h_n(x))^{\mathrm{T}}$,

$$h_i(x) = \frac{1}{b_i}\left(\sum_{j=1}^n T_{ij}s_j(x_j) + J_i\right), \quad i = 1, 2, \dots, n.$$

Since

$$\|h(x)\| \le \left(\sum_{i=1}^n \frac{1}{b_i^2}\left(\sum_{j=1}^n M_j|T_{ij}| + |J_i|\right)^2\right)^{1/2} = r,$$

the continuous mapping h carries the convex compact set $B_r = \{x \in \mathbb{R}^n : \ \|x\| \le r\}$ onto itself. The Schauder principle implies that the mapping h possesses a fixed point x^* which is the equilibrium state of system (3.1).

Besides, if the matrix $B\Lambda^{-1} - |T|$ is an M-matrix, the mapping $H\colon \mathbb{R}^n \to \mathbb{R}^n$

$$H(x) = -Bx + TS(x) + J$$

is a homeomorphism (see [28]). This implies uniqueness of the equilibrium state of system (3.1). The theorem is proved.

Let x^* be the equilibrium state of system (3.1). We perform the change of variables $y(t) = x(t) - x^*$ and rewrite the initial problem (3.1)–(3.2) as

$$y^{\triangle}(t) = -By(t) + TG(y(t)), \quad t \in [0, +\infty), \tag{3.7}$$

$$y(t_0; t_0, y_0) = y_0, \quad t_0 \in [0, +\infty), \quad y_0 \in \mathbb{R}^n, \tag{3.8}$$

where $y \in \mathbb{R}^n$, $G\colon \mathbb{R}^n \to \mathbb{R}^n$, $G(y) = (g_1(y_1), g_2(y_2), \dots, g_n(y_n))^{\mathrm{T}}$, $G(y(t)) = S(y(t) + x^*) - S(x^*)$.

If for system (3.1) assumptions S_1–S_4 are valid, then for system (3.7) the following assertions hold true.

G_1. The vector-function $g(y) = -By + TG(y)$ is regressive.

G_2. For all $u \in \mathbb{R}$ $|g_i(u)| \le 2M_i$, $i = 1, 2, \dots, n$.

G_3. For all $u, v \in \mathbb{R}$ $|g_i(u) - g_i(v)| \le L_i|u - v|$, $i = 1, 2, \dots, n$.

G_4. $G(0) = 0$.

Note that under conditions G_1–G_4 there exists a unique solution of problem (3.7)–(3.8).

Designate by $\underline{b} = \min\{b_i\}$, $\overline{b} = \max\{b_i\}$, $L = \max\{L_i\}$.

Theorem 3.2 *For system* (3.1) *assume that assumptions* S_1–S_4 *are valid on time scale* $\mathbb{T}$ *and there exists a constant* $\mu^* \in \mathcal{M}$ *such that* $\mu(t) \le \mu^*$ *for all* $t \in [0, +\infty)$. *If the inequality*

$$2\underline{b} - 2L\|T\| - \mu^*(\overline{b} + L\|T\|)^2 \ge 0,$$

is satisfied, the equilibrium state $x(t) = x^*$ *of system* (3.1) *is uniformly asymptotically stable.*

Proof It is clear that the behavior of solution $x(t)$ of system (3.1) in the neighborhood of the equilibrium state x^* is equivalent to the behavior of solution $y(t)$ of system (3.7) in the neighborhood of zero. For the proof we shall apply the Lyapunov function $V(y) = y^{\mathrm{T}}y$. If $y(t)$ is Δ-differentiable at the point $t \in \mathbb{T}^k$, for the derivative of function $V(y(t))$ we have the expression

$$\begin{aligned} V^{\Delta}(y(t)) &= (y^{\mathrm{T}}(t)\, y(t))^{\Delta} = y^{\mathrm{T}}(t)\, y^{\Delta}(t) + [y^{\mathrm{T}}(t)]^{\Delta}\, y(\sigma(t)) \\ &= y^{\mathrm{T}}(t)\, y^{\Delta}(t) + [y^{\Delta}(t)]^{\mathrm{T}}[y(t) + \mu(t) y^{\Delta}(t)]. \end{aligned}$$

The derivative of V along solutions of system (3.7) is given by

$$\begin{aligned} V^{\Delta}(y(t))|_{(3.7)} &= 2y^{\mathrm{T}}(t)\, y^{\Delta}(t) + \mu(t)[y^{\Delta}(t)]^{\mathrm{T}} y^{\Delta}(t) \\ &= 2y^{\mathrm{T}}(t)[-By(t) + TG(y(t))] + \mu(t)\| - By(t) + TG(y(t))\|^2 \\ &\le -2\lambda_m(B)\|y(t)\|^2 + 2\|y(t)\|\|T\|\|G(y(t))\| + \mu^*(\|B\|\|y(t)\| + \|T\|\|G(y(t))\|)^2 \\ &= -2\underline{b}\,\|y(t)\|^2 + 2\|T\|\|G(y(t))\|\|y(t)\| + \mu^*(\overline{b}\,\|y(t)\| + \|T\|\|G(y(t))\|)^2. \end{aligned}$$

We shall estimate separately the term $\|G(y(t))\|$:

$$\|G(y(t))\| \le \left(\sum_{i=1}^{n} L_i^2 y_i^2(t)\right)^{1/2} \le \max_i\{L_i\}\left(\sum_{i=1}^{n} y_i^2(t)\right)^{1/2} = L\,\|y(t)\|.$$

As a result we have

$$\begin{aligned} V^{\Delta}(y(t))|_{(3.7)} &\le -2\,\underline{b}\,\|y(t)\|^2 + 2L\|T\|\|y(t)\|^2 + \mu^*\left(\overline{b}\,\|y(t)\| + L\|T\|\|y(t)\|\right)^2 \\ &= -\left(2\underline{b} - 2L\|T\| - \mu^*(\overline{b} + L\|T\|)^2\right)\|y(t)\|^2. \end{aligned}$$

Therefore, the equilibrium state $y(t) = 0$ of system (3.7) is uniformly asymptotically stable. This is equivalent to the uniform asymptotic stability of the equilibrium state $x(t) = x^*$ of system (3.1).

Lemma 3.1 *Assume that* $g_i \in C^2(\mathbb{R})$, $g_i(0) = 0$, $i = 1, 2, \dots, n$, *and constants* $K_i > 0$, $i = 1, 2, \dots, n$, *exist so that* $|g_i''(u)| \le K_i$ *for all* $u \in \mathbb{R}$. *Then the vector-function* $G(y)$ *can be represented as*

$$G(y) = Hy + G_2(y),$$

where $H = \operatorname{diag}\{g_i'(0)\} \in \mathbb{R}^{n\times n}$, $G_2\colon \mathbb{R}^n \to \mathbb{R}^n$ *and the estimate*

$$\|G_2(y)\| \le K\|y\|^2, \tag{3.9}$$

holds true, where $K = \max_i\{K_i\}/2$.

Proof We decompose functions $g_i(y_i)$ by the Maclaurin formula

$$g_i(y_i) = g_i'(0)y_i + 1/2\, g_i''(\theta_i y_i)y_i^2, \quad \theta_i \in (0,1).$$

Then

$$G(y) = \begin{pmatrix} g_1'(0)y_1 + 1/2\, g_1''(\theta_1 y_1)y_1^2 \\ g_2'(0)y_2 + 1/2\, g_2''(\theta_2 y_2)y_2^2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ g_n'(0)y_n + 1/2\, g_n''(\theta_n y_n)y_n^2 \end{pmatrix} = Hy + G_2(y),$$

where $G_2(y) = \frac{1}{2}\operatorname{diag}\{g_i''(\theta_i y_i)\}z$, $z = (y_1^2, y_2^2, \dots, y_n^2)^{\mathrm{T}}$.

$$\|G_2(y)\| = \frac{1}{2}\left(\sum_{i=1}^{n}(g_i''(\theta_i y_i))^2 y_i^4\right)^{1/2} \le K\left(\sum_{i=1}^{n} y_i^4\right)^{1/2}$$
$$\le K\left(\sum_{i=1}^{n} y_i^4 + \sum_{k\ne j} y_k^2 y_j^2\right)^{1/2} = K\sum_{i=1}^{n} y_i^2 = K\|y\|^2.$$

Theorem 3.3 *Let the following conditions be satisfied*

(1) *for system* (3.1) *on time scale* $\mathbb{T}$ *assumptions* S_1–S_4 *are valid;*

(2) *functions* $s_i \in C^2(\mathbb{R})$ *and there exist constants* $K_i > 0$ *such that* $|s_i''(u)| \le K_i$ *for all* $u \in \mathbb{R}$, $i = 1, 2, \dots, n$;

(3) *there exists a constant* $\mu^* \in \mathcal{M}$ *such that* $\mu(t) \le \mu^*$ *for all* $t \in [0, +\infty)$;

(4) *there exists a positive definite symmetric matrix* $P \in \mathbb{R}^{n\times n}$ *such that the inequality* $\lambda_M(PB_1 + B_1^T P) + \mu^*\|P\|\|B_1\|^2 < 0$ *holds true, where* $B_1 = -B + TH$, $H = \operatorname{diag}\{s_i'(0)\} \in \mathbb{R}^{n\times n}$.

Then the equilibrium state $x(t) = x^*$ *of system* (3.1) *is uniformly asymptotically stable.*

Proof We use the function $V(y) = y^{\mathrm{T}}Py$. For the derivative of function V along solutions of system (3.7) we have

$$
\begin{aligned}
V^{\Delta}(y(t))|_{(3.7)} &= y^{\mathrm{T}}(t)Py^{\Delta}(t) + [y^{\mathrm{T}}(t)]^{\Delta}Py(\sigma(t)) \\
&= y^{\mathrm{T}}(t)Py^{\Delta}(t) + [y^{\mathrm{T}}(t)]^{\Delta}Py(t) + \mu(t)y^{\Delta}(t)^{\mathrm{T}}Py^{\Delta}(t) \\
&= y^{\mathrm{T}}(t)P\big[B_1y(t) + TG_2(y(t))\big] + \big[B_1y(t) + TG_2(y(t))\big]^{\mathrm{T}}Py(t) \\
&\quad + \mu(t)\big[B_1y(t) + TG_2(y(t))\big]^{\mathrm{T}}P\big[B_1y(t) + TG_2(y(t))\big] \\
&\le y^{\mathrm{T}}(t)\big[PB_1 + B_1^{\mathrm{T}}P\big]y(t) + 2y^{\mathrm{T}}(t)PTG_2(y(t)) + \mu(t)\|P\|\|B_1y(t) + TG_2(y(t))\|^2 \\
&\le \big(\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu(t)\|P\|\|B_1\|^2\big)\|y(t)\|^2 + 2\|P\|\|T\|\|G_2(y(t))\|\|y(t)\| \\
&\quad + \mu(t)\|P\|\|G_2(y(t))\|^2\|T\|^2 + 2\mu(t)\|P\|\|B_1\|\|T\|\|G_2(y(t))\|\|y(t)\|.
\end{aligned}
$$

Using inequality (3.9) and condition (3) of Theorem 3.3 we get

$$
\begin{aligned}
V^{\Delta}(y(t))|_{(3.7)} &\le \big(\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu^*\|P\|\|B_1\|^2\big)\|y(t)\|^2 \\
&\quad + 2K\|P\|\|T\|\|y(t)\|^3 + 2\mu^*K\|P\|\|B_1\|\|T\|\|y(t)\|^3 + \mu^*K^2\|P\|\|T\|^2\|y(t)\|^4.
\end{aligned}
$$

Designate

$$
\begin{aligned}
a &= -\big(\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu^*\|B_1\|\|P\|^2\big) > 0, \\
\psi(\|y\|) &= a\|y\|^2, \\
m(\psi) &= 2a^{-\frac{1}{3}}K\|P\|\|T\|\,(1 + \mu^*\|B_1\|)\,\psi^{\frac{1}{3}} + \mu^*a^{-2}K^2\|P\|\|T\|^2\psi.
\end{aligned}
$$

For the derivative of function V along solutions of system (3.7) we obtain the inequality

$$
V^{\Delta}(y(t))|_{(3.7)} \le -\psi(\|y\|) + m(\psi(\|y\|)).
$$

Since the function $\psi \in K$-class, $\lim\limits_{\psi\to 0} m(\psi) = 0$ and therefore, the equilibrium state $y(t) = 0$ of system (3.7) is uniformly asymptotically stable. This is equivalent to the uniform asymptotic stability of the equilibrium state $x(t) = x^*$ of system (3.1).

We define the function

$$
\beta_k(t) = \begin{cases} \mu^{-1}(t)\log|1 + \mu(t)k(t)|, & \text{if} \quad \mu(t) > 0, \\ k(t), & \text{if} \quad \mu(t) = 0, \end{cases}
$$

where $k \in \mathcal{R},\ t \in [0, +\infty)$.

Theorem 3.4 *Let the following conditions be satisfied*

(1) *for system* (3.1) *assumptions* $\mathrm{S}_1 - \mathrm{S}_3$ *hold true.*

(2) *functions* $s_i \in C^2(\mathbb{R})$ *and there exist constants* $K_i > 0$ *such that* $|s_i''(u)| \le K_i$ *for all* $u \in \mathbb{R},\ i = 1, 2, \dots, n$.

(3) *there exist a positive definite symmetric matrix* $P \in \mathbb{R}^{n\times n}$ *and a constant* $M > 0$ *such that* $|1 + \mu(t)A(t)| \ge M$ *for all* $t \in [0, +\infty)$, *where* $B_1 = -B + TH,\ H = \operatorname{diag}\{s_i'(0)\} \in \mathbb{R}^{n\times n},\ A(t) = \lambda_M(PB_1 + B_1^TP) + \mu(t)\|P\|\|B_1\|^2$.

Then, if

(a) $\lim_{t\to+\infty}\sup\beta_A(t) = q < 0$, *the equilibrium state* $x(t) = x^*$ *of system* (3.1) *is exponentially stable;*

(b) $\sup\{\beta_A(t)\colon\ t\in[0,+\infty)\} = \overline{q} < 0$, *the equilibrium state* $x(t) = x^*$ *of system* (3.1) *is uniformly exponentially stable.*

Proof We shall apply function $V(y) = y^{\mathrm{T}}Py$ and for the derivative of function V along solutions of system (3.7) we shall use the expression obtained in the previous theorem

$$
\begin{aligned}
V^{\Delta}(y(t))|_{(3.7)} \le \big(&\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu(t)\|P\|\|B_1\|^2\big)\|y(t)\|^2 \\
&+ 2\|P\|\|T\|\|G_2(y(t))\|\|y(t)\| \\
&+ 2\mu(t)\|P\|\|B_1\|\|T\|\|G_2(y(t))\|\|y(t)\| \quad + \mu(t)\|P\|\|G(y(t))\|^2\|T\|^2 \\
\le \big(&\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu(t)\|P\|\|B_1\|^2\big)\|y(t)\|^2 + \big(2K\|P\|\|T\|\|y(t)\| \\
&+ 2\mu(t)K\|P\|\|B_1\|\|T\|\|y(t)\| + \mu(t)K^2\|P\|\|T\|^2|y(t)\|^2\big)\|y(t)\|^2 \\
= A&(t)\|y(t)\|^2 + \Phi(t, V(y)),
\end{aligned}
$$

where

$$
\Phi(t,V) = \big(2K\|P\|\|T\|(1+\mu(t)\|B_1\|)\sqrt{V} + \mu(t)K^2\|P\|\|T\|^2V\big)V.
$$

Consider the set $\mathcal{T} = \{t\in[0,+\infty):\ \mu(t)\neq 0\}$. If there exists $\sup\mathcal{T} < +\infty$ then there exists $t_1\in[0,+\infty)$ such that $\mu(t) = 0$ for all $t\in[t_1,+\infty)$. If the set $\mathcal{T}$ is not bounded, the condition $\lim_{t\to+\infty}\sup\beta_A(t) = q < 0$ implies that there exists a sufficiently large $t_2\in[0,+\infty)\cap\mathcal{T}$ such that for all $t\in[t_2,+\infty)\cap\mathcal{T}$ inequality $\beta_A(t) < 0$ holds true. This yields for all $t\in[t_2,+\infty)\cap\mathcal{T}$ the inequality

$$
\log\big|1 + \mu(t)(\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu(t)\|P\|\|B_1\|^2)\big| < 0.
$$

Then

$$
\begin{gathered}
\mu(t)(\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu(t)\|P\|\|B_1\|^2) - 1 < 1, \\
\|P\|\|B_1\|^2\mu^2(t) + \lambda_M(PB_1 + B_1^{\mathrm{T}}P)\mu(t) - 2 \le 0.
\end{gathered}
$$

Since $D = \lambda_M(PB_1 + B_1^{\mathrm{T}}P)^2 + 8\|P\|\|B_1\|^2 \ge 0$, we obtain the estimate $\mu(t)\le\mu_1$ for all $t\in[t_2,+\infty)\cap\mathcal{T}$, where $\mu_1 = (-\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \sqrt{D})/2\|P\|\|B_1\|^2 \ge 0$. Hence, one can conclude that $\mu(t)\le\mu_1$ for all $t\in[t_3,+\infty)$, $t_3 = \max\{t_1,t_2\}$. If $t\in[0,\rho(t_3)]\cap\mathbb{T}$ then $\mu(t)\le t_3$. This implies the estimate $\mu(t)\le\mu^* = \max\{\mu_1,t_3\}$ for all $t\in[0,+\infty)$. Since

$$
\begin{aligned}
\frac{\Phi(t,V)}{V} &= 2K\|P\|\|T\|(1+\mu(t)\|B_1\|)\sqrt{V} + \mu(t)K^2\|P\|\|T\|^2V \\
&\le 2\|P\|K\|T\|(1+\mu^*\|B_1\|)\sqrt{V} + \mu^*K^2\|P\|\|T\|^2V,
\end{aligned}
$$

we get $\Phi(t,V)/V\to 0$ for $V\to 0$ uniformly in t. According to Theorem 2 from the paper [20] we conclude that the equilibrium state $y(t) = 0$ of system (3.7) is exponentially stable. This is equivalent to the exponential stability of the equilibrium state $x(t) = x^*$ of system (3.1).

Now we shall prove the second part of the theorem. Condition $\sup\{\beta_A : t \in [0,+\infty)\} = \overline{q} < 0$ for $t \in \mathcal{T}$ implies

$$\log|1+\mu(t)(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\|B_1\|^2)| \le \mu(t)\overline{q} < 0$$

for all $t \in \mathcal{T}$. Hence, we get

$$\mu(t) \le \frac{-\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\sqrt{D}}{2\|P\|\|B_1\|^2} = \mu^*, \quad \mu^* \ge 0, \quad t \in \mathcal{T}.$$

That is $\mu(t) \le \mu^*$ for all $t \in [0,+\infty)$. Then, similar to the above, we have $\Phi(t,V)/V \to 0$ for $V \to 0$ uniformly in t.

Therefore, all conditions of Theorem 2 from the paper [20] are satisfied and the equilibrium state $y = 0$ of system (3.7) is uniformly exponentially stable. This is equivalent to the uniform exponential stability of the equilibrium state $x(t) = x^*$ of system (3.1).

Remark 3.1 Consider the scale $\mathbb{T} = \mathbb{N}_0$ ($\mu(t) \equiv 1$). In this case system of equations (3.1) is equivalent to system (3.5) and the condition of uniform asymptotic stability of the equilibrium state of system (3.1) established in Theorem 3.2 for $\mu^* = 1$ becomes

$$2\underline{b} - 2L\|T\| - (\overline{b} + L\|T\|)^2 \ge 0.$$

This result coincides completely with the following result for discrete system (3.5).

Theorem 3.5 *For neural discrete system* (3.5) *let assumptions* $\mathrm{S}_2 - \mathrm{S}_3$ *be satisfied. Then the equilibrium state* $x(t) = x^*$ *of system* (3.5) *is uniformly asymptotically stable, provided that*

$$2\underline{b} - 2L\|T\| - (\overline{b} + L\|T\|)^2 \ge 0.$$

Proof Consider function $y(k) = x(k) - x^*$ and rewrite equations (3.5) as

$$y(k+1) = (-B+I)y(k) + TG(x(k)), \quad k \in \mathbb{N}_0, \tag{3.10}$$

where I is an identity $n \times n$-matrix and for the first difference of function $V(y) = y^{\mathrm{T}}y$ we get the estimate

$$\begin{aligned}
\Delta V(y(k))|_{(3.10)} &= y^{\mathrm{T}}(k+1)y(k+1) - y^{\mathrm{T}}(k)y(k) \\
&= [(-B+I)y(k) + TG(y(k))]^{\mathrm{T}}[(-B+I)y(k) + TG(y(k))] - y^{\mathrm{T}}(k)y(k) \\
&= y^{\mathrm{T}}(k)B^{\mathrm{T}}By(k) - 2y^{\mathrm{T}}(k)B^{\mathrm{T}}y(k) - 2y(k)^{\mathrm{T}}BTG(y(k)) \\
&\quad + 2y^{\mathrm{T}}(k)TG(y(k)) + G^{\mathrm{T}}(y(k))T^{\mathrm{T}}TG(y(k)) \\
&\le \|B\|^2\|y(k)\|^2 - 2\lambda_m(B)\|y(k)\|^2 + 2L\|B\|\|T\|\|y(k)\|^2 \\
&\quad + 2L\|T\|\|y(k)\|^2 + \|T\|^2\|G(y(k))\|^2 \\
&\le \left[\overline{b}^2 - 2\underline{b} + 2L\overline{b}\|T\| + 2L\|T\| + \|T\|^2L^2\right]\|(y(k))\|^2 \\
&= -\left[2\underline{b} - 2L\|T\| - (\overline{b} + L\|T\|)^2\right]\|(y(k))\|^2 \le 0.
\end{aligned}$$

This yields the assertion of the theorem.

Theorem 3.6 *Let assumption* S_3 *be fulfilled. If for every fixed* $t \in \mathbb{T}$ *the matrix* $C = (I - \mu(t)B)\Lambda^{-1} - \mu(t)|T|$ *is an* M*-matrix, the function* $f(x) = -Bx + TS(x) + J$ *is regressive.*

Proof We fix $t \in \mathbb{T}$ and consider the mapping $R\colon \mathbb{R}^n \to \mathbb{R}^n$ given by the formula

$$R(x) = x + \mu(t)f(t,x) = (I - \mu(t)B)x + \mu(t)TS(x) + \mu(t)J.$$

Designate by $\widetilde{B} = (I - \mu(t)B)$, $\widetilde{T} = \mu(t)T$ and $\widetilde{J} = \mu(t)J$. Then we get

$$R(x) = \widetilde{B}x + \widetilde{T}S(x) + \widetilde{J}.$$

Since the matrix $C = \widetilde{B}\Lambda^{-1} - |\widetilde{T}|$ is an M-matrix, the mapping $R\colon \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism (see [28]). Hence follows the reversibility of the mapping $R(x)$ which is equivalent to the reversibility of the operator $I + \mu(t)f(\cdot)\colon \mathbb{R}^n \to \mathbb{R}^n$.

Example 3.1 On the time scale

$$\mathbb{P}_{1,b} = \bigcup_{j=0}^{\infty} [j(1+b), j(1+b)+1], \quad b > 0,$$

we consider a neural network

$$\begin{aligned} x_1^{\Delta} &= -b_1 x_1 + t_{11}s_1(x_2) + t_{12}s_2(x_2) + i_1,\\ x_2^{\Delta} &= -b_2 x_1 + t_{21}s_1(x_1) + t_{22}s_2(x_2) + i_2, \end{aligned} \tag{3.11}$$

where $x_1, x_2 \in \mathbb{R}$,

$$b_1 = b_2 = 1, \quad T = \begin{pmatrix} 0.1 & -0.5\\ 0.5 & 0.1 \end{pmatrix}, \qquad s_1(u) = s_2(u) = \tanh u.$$

For the time scale $\mathbb{P}_{1,b}$ the granularity function

$$\mu(t) = \begin{cases} 0, & t \in \bigcup_{j=0}^{\infty} [j(1+b), j(1+b)+1)\,,\\ b, & t \in \bigcup_{j=0}^{\infty} \{j(1+b)+1\}\,. \end{cases}$$

We take matrix $P = \operatorname{diag}\{0.5, 0.5\}$ and write out all the functions and constants mentioned in the conditions of Theorem 3.4

$$M_1 = M_2 = L_1 = L_2 = 1, \quad \Lambda = \begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix},$$

$$K_1 = K_2 = 8\left|e^{\frac{2+\sqrt{3}}{2}} - e^{-\frac{2+\sqrt{3}}{2}}\right| \Big/ \left(e^{\frac{2+\sqrt{3}}{2}} + e^{-\frac{2+\sqrt{3}}{2}}\right)^3,$$

$$H = \begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -0.9 & -0.5\\ 0.5 & -0.9 \end{pmatrix}, \quad C = \begin{pmatrix} 1-1.1b & -0.5b\\ -0.5b & 1-1.1b \end{pmatrix},$$

$$\lambda_M(PB_1 + B_1^{\mathrm{T}}P) = -0.9, \quad A(t) = -0.9 + 0.53\,b, \quad \|B_1\|^2 = 1.06,$$

$$\beta_A(t) = \begin{cases} b^{-1}\log|1 + b(-0.9 + 0.53\,b)|, & t \in \bigcup\limits_{j=0}^{\infty} \{j(1+b)+1\}\,,\\ -0.9 + 0.53\,b, & t \in \bigcup\limits_{j=0}^{\infty} [j(1+b), j(1+b)+1)\,. \end{cases}$$

The regressivity condition has the inequalities

$$\begin{cases} 1 - 1.1\,b > 0,\\ (1-1.1b)^2 - 0.25\,b^2 > 0, \end{cases}$$

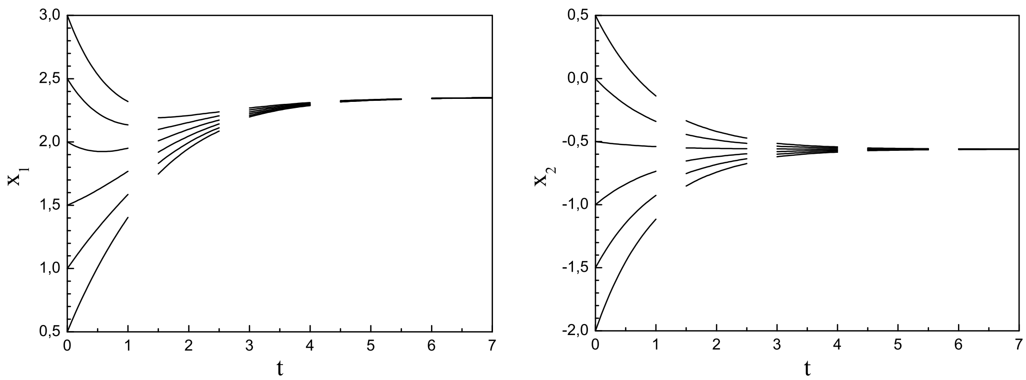

Figure 1. Dependence of functions $x_1(t)$ and $x_2(t)$ on time t obtained by numerical solution of system of equations (3.11). The first figure is drawn for the initial values: $x_2(0) = 1$ and $x_1(0) = 0.5;\ 1;\ 1.5;\ 2;\ 2.5;\ 3$. The second figure is drawn for the initial values: $x_1(0) = 2$ and $x_2(0) = -2;\ -1.5;\ -1;\ -0.5;\ 0;\ 0.5$.

which yields $b < 0.625$.

Since $1 + b(-0.9 + 0.53\,b) \geq 1 + b_0(-0.9 + 0.53\,b_0)$, $b_0 = 0.9/(2 \cdot 0.53)$ for any b, we can take for the constant M the following value: $M = 1 + b_0(-0.9 + 0.53\,b_0) \simeq 0.61$.

For $b < 1.69$ the system of inequalities

$$\begin{cases} M \leq |1 + b(-0.9 + 0.53\,b)| < 1, \\ -0.9 + 0.53\,b < 0 \end{cases}$$

is satisfied. This implies that

$$\sup_t \beta_A(t) = \max\{b^{-1} \log |1 + b(-0.9 + 0.53\,b)|, -0.9 + 0.53\} < 0.$$

Since the matrix

$$B\Lambda^{-1} - |T| = \begin{pmatrix} 0.9 & -0.5 \\ -0.5 & 0.9 \end{pmatrix}$$

is an M-matrix, for $0 < b < 0.625$ system (3.11) possesses a unique equilibrium state for any $i_1, i_2 \in \mathbb{R}$ and this equilibrium state is uniformly exponentially stable.

We shall consider a model example for this problem. We take the following values of the constants: $i_1 = 2$, $i_2 = -1$, $b = 0.5$. The result of numerical solution of system (3.11) is shown on Figures 1. It is seen from the Figures 1, for arbitrary chosen initial conditions $x_1(0) = 0.5 \div 3$ and $x_2(0) = -2 \div 0.5$, the functions $x_1(t)$ and $x_2(t)$ approach asymptotically with time t to the equilibrium state $x_1^* \simeq 2.35$, $x_2^* \simeq -0.56$.

4 Stability of Regular Synchronous Generation of Optically Coupled Lasers

This section deals with the stability with respect to linear approximation of some periodic solutions to a system of nonlinear differential equations. This system describes some experimental realization of a "chaotic" CO_2-laser with a 100 per cent depth-modulated periodic pumping by alternate current (see [10, 11]).

Variation of the factor of strengthening g and amplitude E of a synchronized field of *two optically coupled lasers* is described by the simplest model

$$
\begin{aligned}
\tau\dot{g} &= g_0(t) - g(1+E^2), \\
\dot{E} &= (g - \widetilde{g}_{\rm th})E/2,
\end{aligned} \tag{4.1}
$$

where τ is an efficient time of relaxation of the active medium ($\tau \gg 1$), $g_0(t) = A(1+\sin\omega t)$ is a $(2\pi/\omega)$-periodic pumping, $\widetilde{g}_{\rm th} = g_{\rm th} + 2M(1-\sqrt{1-(\Delta/M)^2})$ is a threshold coefficient of strengthening. Here $g_{\rm th}$ means threshold strengthening, M is a real positive coupling factor, Δ is a value of resonance eigenfrequency detuning (further on — detuning). For the problems considered below the difference of real medium kinetics of CO_2-from the model one is not of essential importance (see [11]).

The mode of phase synchronization, for which the field amplitudes of both lasers are equal at any moment and the phase is constant and depends on detuning, is realized under the condition $|\Delta| < M$. Moreover, the dynamics of two coupled lasers coincides with the dynamics of one equivalent laser whose threshold grows with the growth of detuning. In the mode of synchronous generation (for a fixed M) the growth of detuning corresponds to lessening of the parameter $A/\widetilde{g}_{\rm th}$ for the equivalent laser. Due to the complex bifurcation diagram of the laser with periodic pumping this results in generation of both chaotic and regular signals.

Designate by $(g_T(t), E_T(t))^T$, $t \in [t_0', \infty) = T_0$, $t_0' \ge 0$, T-periodic solution of system of equations (4.1) with the initial condition

$$
g(t_0') = g_0', \quad E(t_0') = E_0' \tag{4.2}
$$

and define variables y_1 and y_2 of the perturbed motion of system (4.1) as

$$
y_1 = g - g_T(t), \quad y_2 = E - E_T(t).
$$

Then the perturbed equations of motion(4.1) are

$$
\begin{aligned}
\tau\dot{y}_1 &= -(1+E_T^2(t))y_1 - 2g_T(t)E_T(t)y_2 - 2E_T(t)y_1y_2 - g_T(t)y_2^2 - y_1y_2^2, \\
2\dot{y}_2 &= E_T(t)y_1 + (g_T(t) - \widetilde{g}_{\rm th})y_2 + y_1y_2.
\end{aligned} \tag{4.3}
$$

For the linear approximation of system (4.3) (designated as (4.3$'$)) we construct an auxiliary matrix-valued function [12, 13]

$$
U(t, y_1, y_2) = \begin{bmatrix} p_{20}y_1^2 & p_{11}(t)y_1y_2 \\ p_{11}(t)y_1y_2 & p_{02}y_2^2 \end{bmatrix},
$$

where p_{20} and p_{02} are finite positive constants, $p_{11}(t) \in C^1(\mathbb{R}, \mathbb{R})$, and a scalar Lyapunov function

$$
v(t, y, \eta) = \eta^T U(t, y_1, y_2)\eta, \tag{4.4}
$$

where $y = (y_1, y_2)^T$ and $\eta = (\eta_1, \eta_2)^T > 0$.

Total time derivative of function (4.4) found by virtue of linear approximation of system (4.3) is

$$
\begin{aligned}
\frac{dv}{dt}\bigg|_{(4.3')} &= (-2\eta_1^2 p_{20}(1+E_T^2(t))/\tau + \eta_1\eta_2 p_{11T}(t)E_T(t))y_1^2 \\
&\quad + (\eta_2^2 p_{02}(g_T(t) - \widetilde{g}_{\rm th}) - 4\eta_1\eta_2 p_{11T}(t)g_T(t)E_T(t)/\tau)y_2^2 \\
&= s_{20}(t)y_1^2 + s_{02}(t)y_2^2,
\end{aligned}
$$

if $p_{11T}(t)$ is assumed to be a T-periodic solution of the linear differential equation

$$\begin{aligned}\dot{p}_{11} = ((1+E_T^2(t))/\tau - g_T(t) + \widetilde{g}_{\text{th}})p_{11} \\ + (2\eta_1 p_{20} g_T(t)/(\tau\eta_2) - \eta_2 p_{02}/(2\eta_1))E_T(t).\end{aligned} \tag{4.5}$$

Conditions of *uniform asymptotic stability of T-periodic solution* of system of equations (4.3) (noncritical case) are established in the form of a system of inequalities

$$\begin{aligned}&p_{20}p_{02} - p_{11T}^2(t) > 0,\\ &s_{20}(t) < 0,\\ &s_{02}(t) < 0 \quad \text{for all} \quad t \in [t', t'+T], \quad t' \in T_0.\end{aligned} \tag{4.6}$$

Thus, the problem on asymptotic stability of some signals of the equivalent CO_2-laser is reduced to the problem of finding T-periodic solutions to nonlinear non-stationary initial problem (4.1)–(4.2) and linear inhomogeneous equation (4.5) with periodic coefficients and the initial condition

$$p_{11}(t_0') = p_{110}'. \tag{4.7}$$

This, in its turn, involves preliminary study of the problem on the domain where equations (4.1) and (4.5) form T-system (see [25]) and establishing existence conditions for the corresponding T-periodic solutions passing through the point (g_0', E_0', p_{110}') at the initial instant t_0'.

We set

$$T = k(2\pi/\omega), \tag{4.8}$$

where k is a positive integer, and define the domain $D \subset \mathbb{R}^3$ which singles out T-system, by the inequalities

$$D: \quad |g| \le g_{\max}, \quad |E| \le E_{\max}, \quad |p_{11}| \le p_{11\,\max}. \tag{4.9}$$

We introduce the vector $M = (M_1, M_2)^T$ and the scalar M_3 which bounds for all $t \in T_0$ and $(g, E, p_{11}) \in D$ the absolute values of the corresponding right-hand sides of equations (4.1) and (4.5) (further on f_1, f_2 and f_3):

$$\begin{aligned}&M_1 = (2A + g_{\max}(1+E_{\max}^2))/\tau,\\ &M_2 = (g_{\max} + \widetilde{g}_{\text{th}})E_{\max}/2,\\ &M_3 = ((1+E_{\max}^2)/\tau + g_{\max} + \widetilde{g}_{\text{th}})p_{11\,\max} + (2\eta_1 p_{20} g_{\max}/(\tau\eta_2) + \eta_2 p_{02}/(2\eta_1))E_{\max}.\end{aligned} \tag{4.10}$$

Continuous vector function $f = (f_1, f_2)^T$ periodic in t with the period T satisfies in $T_0 \times [-g_{\max}, g_{\max}] \times [-E_{\max}, E_{\max}]$ the Lipschitz condition with the matrix

$$K = \begin{bmatrix} (1+E_{\max}^2)/\tau & 2g_{\max}E_{\max}/\tau \\ E_{\max}/2 & (g_{\max} + \widetilde{g}_{\text{th}})/2 \end{bmatrix},$$

and the scalar continuous periodic function f_3 in $T_0 \times [-p_{11\,\max}, p_{11\,\max}]$ with the constant

$$K_3 = (1+E_{\max}^2)/\tau + g_{\max} + \widetilde{g}_{\text{th}}.$$

Following the definition of T-system and relating with vector-function $(f^T, f_3)^T$ and domain D the nonempty set D_f of points $\mathbb{R}^3$ contained in D together with its $\frac{T}{2}(M^T, M_3)^T$-neighborhood the conditions defining T-system are obtained in the form of a system of inequalities

$$2g_{\max} - TM_1 > 0, \quad 2E_{\max} - TM_2 > 0, \quad 2p_{11\,\max} - TM_3 > 0,$$
$$\frac{T}{\pi}\frac{K_{11} + K_{22} + \sqrt{(K_{11} - K_{22})^2 + 4K_{12}K_{21}}}{2} < 1, \quad \frac{T}{\pi}K_3 < 1.$$

Moreover, it is also assumed that the initial value (g_0', E_0', p_{110}') belongs to D_f.

The immediate construction of the desired T-periodic solutions is achieved, for example, by the method of trigonometric collocations by a numerical-analytical scheme. To this end, we assume that the values of functions $f_j(t, g, E, p_{11})$, $j = 1, 2, 3$, calculated basing on the m-th approximation to the desired periodic solution coincide in $N = 2r+1$ collocation points $t_i = i\frac{T}{N}$, $i = 0, 1, \dots, 2r$, with the values of the trigonometric polynomials

$$f_j^m = \alpha_{j0}^m + \sum_{l=1}^{r}(\alpha_{jl}^m \cos l\Omega t + \beta_{jl}^m \sin l\Omega t), \tag{4.11}$$

where $\Omega = 2\pi/T$. Then the vectors of the coefficients

$$f_j^{m\Gamma} = (\alpha_{j0}^m, \alpha_{j1}^m, \beta_{j1}^m, \dots, \alpha_{jr}^m, \beta_{jr}^m)^T \tag{4.12}$$

of trigonometric polynomials (4.11) are expressed via the respective vectors of values of these polynomials

$$f_j^{mM} = (f_j(t_i, g^m(t_i), E^m(t_i), p_{11}^m(t_i)))_{i=0}^{2r}$$

with the help of the matrix

$$\Gamma = [\Gamma_{pq}]_{p,q=1}^N,$$

where

$$\Gamma_{pq} = \begin{cases} \frac{1}{N}, & p = 1, \\ \frac{2}{N}\cos\left(p(q-1)\frac{\pi}{N}\right), & p = 2, 4, \dots, 2r, \\ \frac{2}{N}\sin\left((p-1)(q-1)\frac{\pi}{N}\right), & p = 3, 5, \dots, N, \end{cases}$$

and

$$f_j^{m\Gamma} = \Gamma f_j^{mM}.$$

By introducing into consideration the $N \times N$-two-diagonal matrix

$$\mu^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -\frac{1}{\Omega} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{\Omega} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2\Omega} & \dots & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\Omega} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -\frac{1}{r\Omega} \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{1}{r\Omega} & 0 \end{bmatrix}$$

and N-dimensional vectors

$$z_j^{m\Gamma} = \left(\alpha_{j0}^{m\,\prime} + \sum_{l=1}^{r}(\alpha_{jl}^{m\,\prime} \cos l\Omega t_0' + \beta_{jl}^{m\,\prime} \sin l\Omega t_0'), 0, \dots, 0\right)^{\mathrm{T}},$$

where

$$(\alpha_{j0}^{m\prime}, \alpha_{j1}^{m\prime}, \beta_{j1}^{m\prime}, \dots, \alpha_{jr}^{m\prime}, \beta_{jr}^{m\prime})^T = \mu^1 f_j^{m\Gamma},$$

we obtain the vectors of the coefficients of $(m+1)$-th "trigonometric" approximation to the desired T-periodic solution in the form

$$\begin{aligned} g^{m+1,\Gamma} &= g^{0\Gamma} + \mu^1 f_1^{m\Gamma} - z_1^{m\Gamma}, \\ E^{m+1,\Gamma} &= E^{0\Gamma} + \mu^1 f_2^{m\Gamma} - z_2^{m\Gamma}, \\ p_{11}^{m+1,\Gamma} &= p_{11}^{0\Gamma} + \mu^1 f_3^{m\Gamma} - z_3^{m\Gamma}, \end{aligned}$$

where $g^{0\Gamma}$, $E^{0\Gamma}$ and $p_{11}^{0\Gamma}$ are the vectors of the coefficients of appropriate zero approximations.

The form of the zero approximation $(g^0(t), E^0(t))^{\mathrm{T}}$ and the vector of the initial values at the collocation points and the initial vector of the coefficients of the right-hand sides f_1, f_2 of equations (4.1) respectively are taken based on solution of system (4.1) linearized by the equation for g

$$\begin{aligned} g^0(t) &= C_g e^{-\frac{t}{\tau}} + \frac{A}{1+\omega^2\tau^2}(\sin\omega t - \omega\tau\cos\omega t) + A, \qquad && g^0(t_0') = g_0', \\ E^0(t) &= C_E \exp\bigg\{\frac{1}{2}\bigg(- C_g\tau e^{-\frac{t}{\tau}} + (A - \widetilde{g}_{\mathrm{th}})t \\ &\quad - \frac{A}{1+\omega^2\tau^2}\bigg(\frac{1}{\omega}\cos\omega t + \tau\sin\omega t\bigg)\bigg)\bigg\}, && E^0(t_0') = E_0', \end{aligned}$$

where constants C_g and C_E are defined univalently. We take solution of the corresponding homogeneous initial problem (4.5), (4.7) as $p_{11}^0(t)$, assuming T-periodic functions to be known

$$g_T(t) \approx g^m(t) = \sum_{j=-r}^{r} g_j^m e^{i\Omega jt}, \quad E_T(t) \approx E^m(t) = \sum_{j=-r}^{r} E_j^m e^{i\Omega jt},$$

where $g_j^m = (\alpha_{gj}^m - i\beta_{gj}^m)/2$, $g_{-j}^m = \overline{g_j^m}$, $E_j^m = (\alpha_{Ej}^m - i\beta_{Ej}^m)/2$, $E_{-j}^m = \overline{E_j^m}$, and $\alpha_{gj}^m, \beta_{gj}^m$ and $\alpha_{Ej}^m, \beta_{Ej}^m$ stand for coefficients (4.12) of the corresponding trigonometric series (4.11). Then

$$\begin{aligned} p_{11}^0(t) = C_{p_{11}} \exp\bigg\{ &\bigg(\bigg(1 + \sum_j E_j^m E_{-j}^m\bigg)t + \sum_j \sum_{s\neq -j} \frac{E_j^m E_s^m}{i\Omega(j+s)} e^{i\Omega(j+s)t}\bigg)/\tau \\ &+ (\widetilde{g}_{\mathrm{th}} - g_0^m)t - \sum_{j\neq 0} \frac{g_j^m}{i\Omega j} e^{i\Omega jt}\bigg\}, \quad p_{11}^0(t_0') = p_{110}'. \end{aligned}$$

The control of convergence of the described iteration process of finding T-periodic solution is performed by comparing with a pre-given accuracy ε_1 the difference between the vectors of coefficients of the m-th and $(m+1)$-th trigonometric approximations for $g_T(t), E_T(t), p_{11T}(t)$ with zero-vector, and by comparing with a pre-given accuracy ε_2 the mean values of functions $f_j(t, g^m(t), E^m(t), p_{11}^m(t))$, taken over a period, with zero. The latter condition is necessary and sufficient (see [25]) for the existence of periodic solutions of the period T passing through the point $(g_0', E_0', p_{110}') \in D_f$ for $t = t_0'$ and is an indicator of a good choice of the values k (see (4.8)), $g_{\max}, E_{\max}, p_{11\max}$ (see (4.9)), $t_0', g_0', E_0', p_{110}'$ (see (4.2), (4.7)), $p_{20}, p_{02}, \eta_1, \eta_2$ (see (4.10)) and the parameter values of the system under consideration.

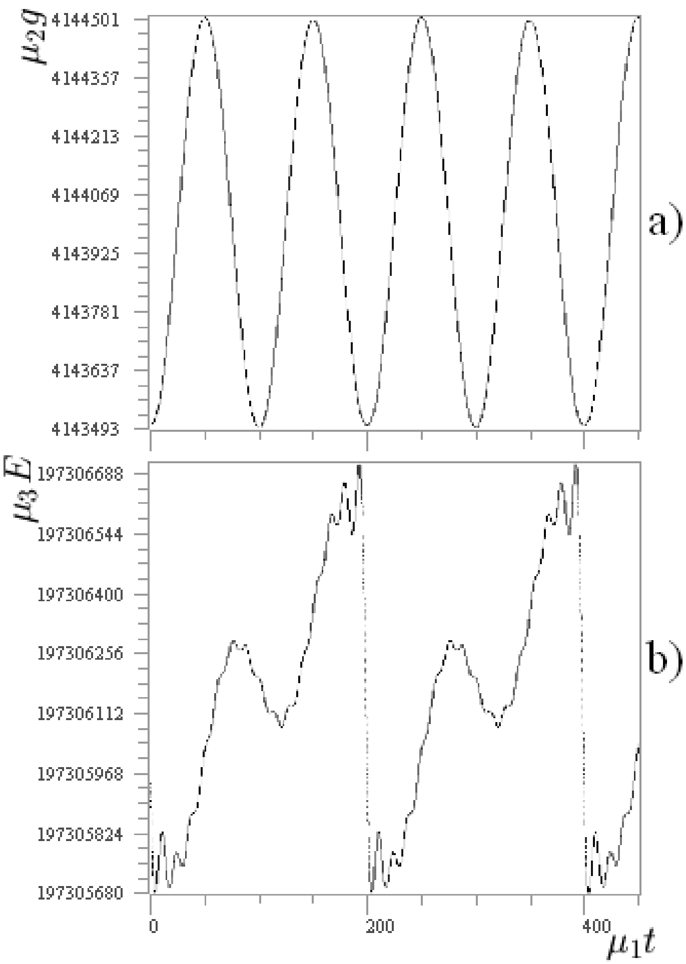

Figure 2. Graphs of $(4\pi/\omega)$-periodic functions $g_T(t)$ and $E_T(t)$.

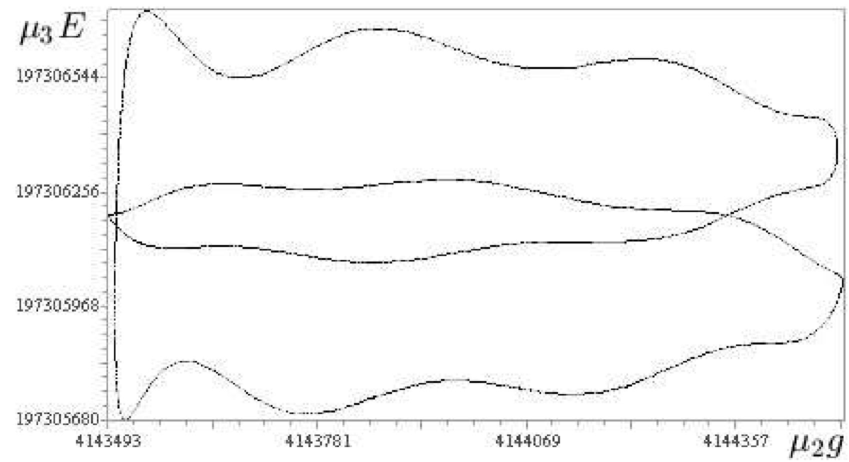

Figure 3. Phase trajectory corresponding to $(4\pi/\omega)$-periodic solution of $(g_T(t), E_T(t))^{\mathrm{T}}$.

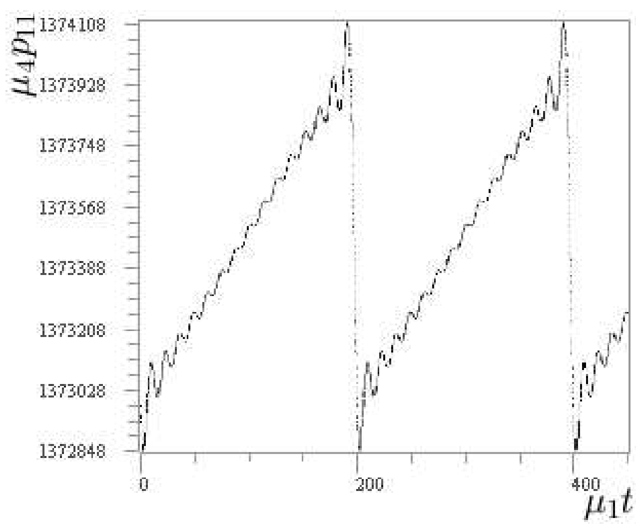

Figure 4. Graph of $(4\pi/\omega)$-periodic function $p_{11T}(t)$.

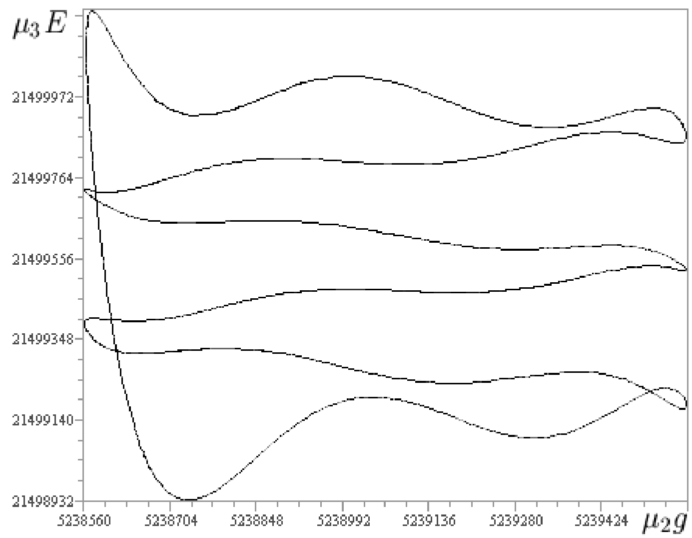

Figure 5. Phase trajectory corresponding to $(6\pi/\omega)$-periodic solution of $(g_T(t), E_T(t))^{\mathrm{T}}$.

For the values $\omega = 40.96241$, $A = 0.39856$, $g_{\text{th}} = 0.4$, $M = 0.1$, $\tau = 400$, $\Delta = 0.001$, $t'_0 = 0$, $g_{\max} = 4.77321$, $g'_0 = 0.40001$, $E_{\max} = 9.27539$, $E'_0 = 0.03817$ with the use of $N = 31$ collocation points during 5 iterations ($\varepsilon_1 = 10^{-25}$, $\varepsilon_2 = 10^{-5}$) a periodic solution was constructed for the initial problem (4.1), (4.2) with duplication of the period ($k = 2$). The corresponding graphs are shown in Figures 2 and 3. Uniform asymptotic stability of the corresponding zero solution of system (4.3) is established during 4 iterations by constructing with the same accuracy the periodic function $p_{11T}(t)$ (Figure 4) satisfying conditions (4.6). Here $\eta_1 = 8.01158$, $\eta_2 = 4.38394$, $p_{20} = 2.97746$, $p_{02} = 0.14038$, $p_{11\max} = 4.65370$, $p'_{110} = 0.42974$ and $\mu_1 = 651$, $\mu_2 = 10358490$, $\mu_3 = 5168882277$, $\mu_4 = 3194900$ are the scale multiplies.

Uniform asymptotically stable signal with triple period with respect to the pumping period (Figure 5) is investigated in the same way for the parameters changed as compared with the previous example $\omega = 52.116990$, $A = 0.399742$, $g_{\max} = 0.904412$, $g'_0 = 0.399723$, $E_{\max} = 8.306199$, $E'_0 = 0.002538$, $\varepsilon_2 = 10^{-6}$, $\eta_1 = 8.776919$, $\eta_2 = 0.385523$, $p_{20} = 2.070760$, $p_{02} = 8.392944$, $p_{11\max} = 8.833734$, $p'_{110} = 0.083020$, $\mu_2 = 13105471$, $\mu_3 = 8472130609$.

The considered examples demonstrate the possibility of parallel solution of some problems on the spectrum and structure of collective mode as well as their stability and competition between the mode of composed resonator. The method of constructing an auxiliary function pointed out in the context of matrix-valued Lyapunov functions allows to calculate stability domains of some periodic signals of coupled lasers with periodic pumping in the regime of synchronous generation.

5 Models of World Dynamics and Sustainable Development

The *Forrester model of world dynamics* (see [5, 23]) is constructed in terms of the approach developed in the investigation of complex systems with nonlinear feedbacks. In the modeling of world dynamics the following global processes are taken into account:

(i) quick growth of the world population;

(ii) industrialization and the related production growth;

(iii) restricted food resources;

(vi) growth of industrial wastes;

(v) shortage of natural resources.

The main variables in the Forrester model are:

(1) population P (further on the designation X_1 is used);

(2) capital stocks K (X_2);

(3) stock ratio in agricultural industry X (X_3);

(4) level of environmental pollution Z (X_4);

(5) quantity of nonrenewable natural resources R (X_5).

Factors through which the variables $X_1, \dots, X_5$, effect one another are:

- relative number of population P_p (population normed to its number in 1970);
- specific stocks K_p;
- level of living standard C;
- relative level of meals F;
- normed value of specific stocks in agricultural industry X_p;
- relative pollution Z_s;
- ratio of the resources left R_R.

In addition to the enumerated factors Forrester also considers the notion of "quality of living" Q. This factor depends on the variables P_p, C, F and Z_s: $Q = Q_C Q_F Q_P Q_Z$.

For the variables P, K, X, Z, R interpreted as system equations, the equations of the type

$$\frac{dy}{dt} = y^+ - y^-, \tag{5.1}$$

are written, where y^+ is a positive rate of velocity growth of the variable; y^- is a negative rate of velocity diminishing of the variable y. In a simplified form the *world dynamics equations* are

$$\begin{aligned} &\frac{dP}{dt} = P(B - D), \qquad \frac{dZ}{dt} = Z_+ - T_Z^{-1} Z, \\ &\frac{dK}{dt} = K_+ - T_K^{-1} K, \quad \frac{dR}{dt} = -R_-, \\ &\frac{dX}{dt} = X_+ - T_X^{-1} X, \end{aligned} \tag{5.2}$$

where B is a birth rate, D is a death rate, K_+ is a velocity of capital stocks production, X_+ is an increment of the ratio of agricultural industry stocks, Z_+ is a velocity of pollution generation, T_Z is a characteristic time of natural decay of pollutants, and R_- is a velocity of resource consumption.

Mathematical analysis of model (5.2) reveals the existence of stationary and quasi-stationary solutions which are interpreted as a "global equilibrium" and a "stable society".

Let a "nation" N (a totality of international organizations) form the public opinion about global processes occurring on a certain level of the system. The measure of the change of the public opinion $\chi(t)$ will be modeled on each system by the equation (see [18])

$$\frac{d^2\chi}{dt^2} + m^2\chi = 0, \quad \chi'(t_0) = \chi_0', \quad \chi(t_0) = \chi_0. \tag{5.3}$$

Here the value m is a function of variables (1)–(5) at times $t = t_0$. Moreover, for the system levels the equations of (5.1) type are written

$$\frac{dy}{dt} = y^+ - y^- + b(t), \tag{5.4}$$

where the "discontent" function $b(t)$ is as follows

$$b(t) = ge^{\pm\alpha|\chi(t)|}, \quad \alpha = \text{const} > 0. \tag{5.5}$$

Here g is a factor of "discontent" reflecting the change of the "level of living standard" of the countries involved into world dynamics. Correlation (5.5) models the increase (decrease) of discontent of the current global processes depending on changes of the measure of the public opinion.

Thus, the Forrester model (5.1)–(5.2) is generalized by the equations

$$\begin{aligned}
\frac{dX_1}{dt} &= X_1(B - D) + g_1 e^{\pm\alpha|\chi(t)|},\\
\frac{dX_2}{dt} &= K_+ - T_K^{-1} X_2 + g_2 e^{\pm\alpha|\chi(t)|},\\
\frac{dX_3}{dt} &= X_+ - T_X^{-1} X_3 + g_3 e^{\pm\alpha|\chi(t)|},\\
\frac{dX_4}{dt} &= Z_+ - T_Z^{-1} X_4 + g_4 e^{\pm\alpha|\chi(t)|},\\
\frac{dX_5}{dt} &= -R_- + g_5 e^{\pm\alpha|\chi(t)|},\\
\frac{d^2\chi}{dt^2} &+ m^2\chi = 0,
\end{aligned} \tag{5.6}$$

where $g_1, \dots, g_5$ are the discontent factors on the corresponding level of the system.

It is proposed to describe *general nonlinear model of world dynamics* by a system of differential equations of the type

$$\frac{dX_i}{dt} = W_i(X) + g_i e^{\pm\alpha|\chi(t)|}, \tag{5.7}$$

$$\frac{d^2\chi}{dt^2} + m^2\chi = 0, \quad i = 1, 2, \dots, N. \tag{5.8}$$

Here $X = (X_1, \dots, X_5, \dots, X_N) \subseteq S(H)$, where $X_1, \dots, X_5$ are the Forrester variables and $X_{5+1}, \dots, X_n$ are some other variables involved into the world dynamics equations, $W_i\colon S(H) \to R_+^N$ is a vector-function with the components describing the variation of parameters on the appropriate system level. It is assumed that the solution $(X^T(t), \chi(t))^T$ of system of coupled equations (5.7)–(5.8) exists for all $t \geq t_0$ with the initial conditions $(X_0^T, \chi_0', \chi_0)^{\mathrm{T}} \in \mathrm{int}(R_+^N,\, R \times R)$.

Assume that the system of nonlinear equations

$$W_1(X) + g_1 e^{\pm\alpha|\chi(t)|} = 0,$$
$$\dots\dots\dots\dots\dots$$
$$W_N(X) + g_N e^{\pm\alpha|\chi(t)|} = 0$$

possesses a quasistationary solution $X_n(t) = (X_{1n}(t),\, \dots,\, X_{nN}(t))^T$ for any bounded function $\chi(t)$ being a solution of equation (5.8). Moreover, the Lyapunov substitution

$$Y(t) = X(t) - X_n(t)$$

brings system of equations (5.7) to the form

$$\frac{dY}{dt} = Y(t, Y), \tag{5.9}$$

where $Y(t,Y) = W(Y + X_n(t)) + Ge^{\pm|\chi(t)|} - (W(X_n(t)) + Ge^{\pm|\chi(t)|})$. It is clear that $Y(t,0) = 0$ for all $t \geq 0$. System (5.9) is a system of perturbed equations of world dynamics.

The problem of sustainable development is associated with the analysis of solution $Y = 0$ of equation(5.9). The stability analysis of solutions will be carried out with respect to two measures H_0 and H taking the values from the sets

$$\Phi = \{H \in C(R_+ \times R^N,\, R_+)\colon \inf_{(t,Y)} H(t,Y) = 0\};$$
$$\Phi_0 = \{H \in \{\Phi\colon \inf_{Y} H(t,Y) = 0 \;\text{ for every }\; t \in R_+\}.$$

We need the following definition.

Definition 5.1 The world dynamics (5.7)–(5.8) has *sustainable development with respect to two measures* if for every $\varepsilon > 0$ and $t_0 \in R_+$ there exists a positive function $\delta(t_0, \varepsilon) > 0$ continuous in t_0 for every ε such that the condition $H_0(t_0, Y_0) < \delta$ implies the estimate $H(t, Y(t)) < \varepsilon$ for all $t \geq t_0$ for any bounded solution $\chi(t)$ of equation (5.8).

Note that if system (5.7) having no zero solution ($W(0, \chi(t)) \neq 0$ for $X = 0$) and has the nominal solution $X_n(t)$ then the measures H_0 and H can be taken as follows: $H(t,X) = H_0(t,X) = \|X - X_n(t)\|$, where $\|\cdot\|$ is an Euclidean norm of the vector X. If it is of interest to study stability of the development in the Forrester variables, the measures H_0 and H are taken as: $H(t,X) = \|X - X_n(t)\|_s$, $1 \leq s \leq 5$, and $H_0(t,X) = \|X - X_n(t)\|$. This corresponds to stability analysis of system (5.7) in two measures with respect to a part of variables.

For system (5.9) assume that the elements $u_{ij}(t,Y)$ of the matrix-valued function

$$U(t,Y) = [u_{ij}(t,Y)], \quad i,j = 1, \dots, m, \quad m < N,$$

are constructed, where $u_{ii} \in C(R_+ \times R^N,\, R_+)$ and $u_{ij} \in C(R_+ \times\, R^N,\, R)$ for $(i \neq j) \in [1, m]$. The function

$$V(t, Y, w) = w^T U(t, Y) w, \quad w \in R^m, \tag{5.10}$$

is considered together with the function

$$D^+ V(t, Y, w) = w^T D^+ U(t, Y) w, \tag{5.11}$$

where $D^+U(t, Y)$ is the upper right Dini derivative calculated element-wise for the matrix-valued function $U(t, Y)$.

Conditions of the sustainable development in two measures (H_0, H) are established in the following result.

Theorem 5.1 *Let the functions in equations of global dynamics* (5.7)–(5.8) *be defined and continuous in the domain of values* $(t, Y, \chi) \in R_+ \times \mathcal{S} \times D$. *If, moreover,*

(1) *measures* H_0 *and* H *are of class* Φ*;*

(2) *function* (5.10) *satisfies the condition* $V(t, Y, w) \in C(R_+ \times\, \mathcal{S} \times R^m,\, R_+)$ *and is locally Lipchitz in* Y*;*

(3) *function* $V(t, Y, w)$ *satisfies the estimates*

(a) $a(H(t, Y)) \leq V(t, Y, w) \leq b(t, H_0(t, Y))$ *for all* $(t, Y, w) \in S(h, H) \times R^m$ *or*

(b) $a(H(t, Y)) \leq V(t, Y, w) \leq c(H_0(t, Y))$

where $a, c \in K$*–class and* $b \in CK$*–class of comparison functions;*

(4) *there exists a matrix-valued function* $\Theta(Y, w)$*,* $\Theta \in C(R^N \times R^m,\; R^{m \times m})$ *and* $\Theta(0, w) = 0$ *for all* $(w \neq 0) \in R^m$ *such that*

$$D^+ V(t, Y, w) \leq \mathrm{e}^T \widehat{\Theta}(Y, w) \mathrm{e}$$

for all $(t, Y, w) \in \mathcal{S} \times R^m$*, where* $\mathrm{e} = (1, 1, \ldots, 1)^T \in R^m$*,* $\mathcal{S} \subset (R^N \times R_+)$*,* $\widehat{\Theta}(Y, w) = \frac{1}{2}\left(\Theta(Y, w) + \Theta^T(Y, w)\right)$ *for any bounded solution* $\chi(t)$ *of equation* (5.8).

Then

(a) *world dynamics* (5.7)–(5.8) *has sustainable development with respect to two measures if the matrix* $\widehat{\Theta}(Y, w)$ *is negative semi-definite, the measure* H *is continuous with respect to the measure* H_0 *and condition (3)(a) is satisfied;*

(b) *world dynamics* (5.7)–(5.8) *has uniformly sustainable development with respect to two measures if the matrix* $\widehat{\Theta}(Y, w)$ *is negative semi-definite, the measure* H *is uniformly continuous with respect to the measure* H_0 *and condition (3)(b) is satisfied.*

Proof We note that function $V(t, Y, w)$ determined by formula (5.10) is a scalar pseudo-quadratic form with respect to $w \in R^m$. Therefore, the property of definite sign of function (5.10) with respect to the measure H does not require the H-sign-definiteness of the elements $u_{ij}(t, x)$ of matrix $U(t, Y)$. First we shall prove assertion (a) of Theorem 5.1. Conditions (1), (2), and (3a) imply that the function $V(t, Y, w)$ is weakly

H_0-decreasing. Thus, for $t_0 \in R$, $(t_0 \in R_+)$ there exists a constant $\Delta_0 = \Delta_0(t_0) > 0$ such that for $H_0(t_0, x_0) < \Delta_0$ the inequality

$$V(t_0, Y_0, w) \leq b(t_0, H_0(t_0, Y_0)) \tag{5.12}$$

holds true.

Also, condition (3a) implies that there exists a $\Delta_1 \in (0, H)$ such that

$$a(H(t, x)) \leq V(t, x, w) \quad \text{for} \quad H(t, x) \leq \Delta_1. \tag{5.13}$$

The fact that the measure H is continuous with respect to the measure H_0 implies that there exist a function $\varphi \in CK$ and a constant $\Delta_2 = \Delta_2(t_0) > 0$ such that

$$H(t_0, Y_0) \leq \varphi(t_0, H_0(t_0, Y_0)) \quad \text{for} \quad H_0(t_0, Y_0) < \Delta_2, \tag{5.14}$$

where Δ_2 is taken so that

$$\varphi(t_0, \Delta_2) < \Delta_1. \tag{5.15}$$

Let $\varepsilon \in (0, \Delta_0)$ and $t_0 \in R$ $(t_0 \in \mathcal{T}_\tau)$ be given. Since the functions $a \in K$ and $b \in CK$, given ε and t_0, one can choose $\Delta_3 = \Delta_3(t_0, \varepsilon) > 0$ so that

$$b(t_0, \Delta_3) < a(\varepsilon). \tag{5.16}$$

We take $\delta(t_0) = \min(\Delta_1, \Delta_2, \Delta_3)$. Conditions (5.12)–(5.16) imply that for $H_0(t_0, Y_0) < \delta$ the inequalities

$$a(H(t_0, Y_0)) \leq V(t_0, Y_0, w) \leq b(t_0, H_0(t_0, Y_0)) < a(\varepsilon)$$

are fulfilled. From this we get

$$H(t_0, Y_0) < \varepsilon.$$

Let $Y(t; t_0, Y_0) = Y(t)$ be a solution of system (5.9) with the initial conditions for which $H_0(t_0, Y_0) < \delta$. We shall make sure that under conditions of Theorem 5.1 the estimate

$$H(t, Y(t)) < \varepsilon \quad \text{for all} \quad t \geq t_0$$

holds true. Assume that there exists a $t_1 \geq t_0$ such that

$$H(t_1, Y(t)) = \varepsilon \quad \text{and} \quad H(t, Y(t)) < \varepsilon, \quad t \in [t_0, t_1),$$

for solution $Y(t; t_0, Y_0)$ with the initial conditions $H_0(t_0, Y_0) < \delta$.

Condition (4) and the fact that the matrix $\widehat{\Theta}(Y, w)$ is negative semi-definite in the domain $\mathcal{S}$ imply that the roots $\lambda_i = \lambda_i(Y, w)$ of the equation

$$\det[\widehat{\Theta}(Y, w) - \lambda E] = 0$$

satisfy the condition $\lambda_i(Y, w) \leq 0$, $i = 1, 2, \dots, m$, in the domain S. Therefore,

$$D^+V(t, Y, w) \leq e^T \widehat{\Theta}(Y, w) e \leq 0$$

and for all $t \in [t_0, t_1]$ the sequence of inequalities

$$a(\varepsilon) = a(H(t_1, Y(t_1))) \leq V(t, Y, w) \leq V(t_0, Y_0, w) \leq b(t_0, H_0(t_0, Y_0)) < a(\varepsilon)$$

is satisfied.

The contradiction obtained disproves the assumption that $t_1 \in [t_0, +\infty)$. Thus, system (5.7)–(5.8) is (H_0, H)-stable.

Assertion (b) of Theorem 5.1 is proved in the same way. Besides, it is taken into account that condition (3)(b) is satisfied and the measure H is uniformly continuous with respect to the measure H_0, the value δ can be taken independent of $t_0 \in R$ $(t_0 \in R_+)$. Hence the uniform (H_0, H)-stability of system (5.7)–(5.8) follows.

Note that the construction of a suitable function (5.10) in terms of the matrix function $U(t, Y)$ is essentially simplified because the elements $u_{ij}(t, Y)$ can be associated with the world dynamics equations on a certain system level.

6 Stability Analysis of Takagi–Sugeno Impulsive Systems

6.1 General results

Consider the impulsive fuzzy dynamic model of Takagi–Sugeno. Given the properly defined input variables and membership functions, the T-S fuzzy rules for a multivariable system considered herein are of the form:

$$R^i,\ i = \overline{1, r}\colon \text{ if } z_1(t) \text{ is } M_{i1} \text{ and} \dots \text{and } z_n(t) \text{ is } M_{in}, \text{ then}$$
$$\begin{cases} \dfrac{dx(t)}{dt} = A_i x(t), & t \neq \tau_k, \\ x(t^+) = B_i x(t), & t = \tau_k, \quad k = 1, 2, \dots (k \in \mathbb{N}), \\ x(t_0^+) = x_0, \end{cases} \tag{6.1}$$

where $x(t) = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ is the state vector, $z = (z_1, \dots, z_n)^T \in \mathbb{R}^n$ is the premise variable vector associated with the systems states and inputs, $x(t^+)$ is the right value of $x(t)$, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times n}$ are the system matrices, $M_{ij}(\cdot)$ are the membership functions of the fuzzy sets M_{ij} and r is the number of fuzzy rules. We suppose that B_i are non-singular matrices and $0 < \theta_1 \le \tau_{k+1} - \tau_k \le \theta_2 < \infty$.

We also suppose that at the moments of impulsive effects $\{\tau_k\}$ the solution $x(t)$ is left continuous, i.e., $x(\tau_k^-) = x(\tau_k)$.

The state equation can be defined as follows

$$\begin{cases} \dfrac{dx(t)}{dt} = \sum\limits_{i=1}^{r} \mu_i(z(t)) A_i x(t), & t \neq \tau_k, \\ x(t^+) = \sum\limits_{i=1}^{r} \mu_i(z(t)) B_i x(t), & t = \tau_k, \quad k \in \mathbb{N}, \\ x(t_0^+) = x_0, \end{cases} \tag{6.2}$$

where

$$\mu_i(z) = \frac{\omega_i(z)}{\sum_{i=1}^{r} \omega_i(z)} \quad \text{with} \quad \omega_i(z) = \prod_{j=1}^{n} M_{ij}(z_j).$$

Clearly $\sum\limits_{i=1}^{r} \mu_i(z) = 1$ and $\mu_i(z) \ge 0$, $i = \overline{1, r}$. Next, without loss of generality we take $z = x$.

The stability analysis in the sense of Lyapunov of zero solution $x = 0$ of system (6.2) is the aim of this section. Before the main results, the following assumption is made regarding the T-S fuzzy system (6.2).

Assumption 6.1 There exist $\gamma > 0$ and $\varepsilon > 0$ such that the functions $\mu_i(x)$ for system (6.2) satisfy the inequality $\|D_x^+\mu_i(x)\| \le \gamma\|x\|^{-1+\varepsilon}$, $i = \overline{1,r}$.

In this assumption $D_x^+\mu_i(x)$ denotes the upper Dini derivative of $\mu_i(x)$, i.e.

$$D_x^+\mu_i(x) = \limsup\{(\mu_i(x(t+\Delta)) - \mu_i(x(t)))/\Delta\colon\ \Delta \to 0\,\}.$$

Remark 6.1 It should be noted that Assumption 3 admits unique existence of solutions for system (6.2).

Let $\mathcal{E}$ denote the space of symmetric $n \times n$-matrices with scalar product $(X, Y) = \mathrm{tr}(XY)$ and corresponding norm $\|X\| = \sqrt{(X, X)}$, where $\mathrm{tr}(\cdot)$ denotes the trace of corresponding matrix. Let $K \subset \mathcal{E}$ be a cone of positive semi-definite symmetric matrices. Next we will define the following linear operators $\mathfrak{F}_i X = A_i^T X + XA_i$, $\mathfrak{B}_{ij} X = B_i^T XB_j$, for all $X \in \mathcal{E}$, $i, j = \overline{1,r}$.

Several theorems are first proved to demonstrate that if certain hypotheses are satisfied, the stability of the above nonlinear system can be obtained using the direct Lyapunov method. It is shown that stability conditions can be formulated in terms of Linear Matrix Inequalities.

Theorem 6.1 *Under Assumption 6.1 the equilibrium state $x = 0$ of fuzzy system (6.2) is asymptotically stable if for all $\theta \in [\theta_1, \theta_2]$ there exists a common symmetric positive definite matrix X such that*

$$\left(\frac{1}{2}(\mathfrak{B}_{ji} + \mathfrak{B}_{ij}) - I + \sum_{k=1}^{p-1} \frac{(-1)^{k+1}(\mathfrak{F}_i)^k\theta^k}{k!}\right)X < 0, \quad i, j = \overline{1,r}, \tag{6.3}$$

$$(-1)^p(\mathfrak{F}_i)^p X \ge 0. \tag{6.4}$$

Before we prove Theorem 6.1 we have the following remark.

Remark 6.2 It should be noted that

(1) $(\mathfrak{F}_i)^p X = \mathfrak{F}_{i_1}\mathfrak{F}_{i_2}\dots\mathfrak{F}_{i_p}X$, where $i_1 = i$, $i_2 = j$, $i_1, \dots, i_p = \overline{1,r}$;

(2) for $i_1, \dots, i_p = \overline{1,r}$

$$\left(\sum_{i=1}^{r}\mu_i(x)\mathfrak{F}_i\right)^p X = \sum_{i_p=1}^{r}\cdots\sum_{i_1=1}^{r}\mu_{i_p}(x)\dots\mu_{i_1}(x)\mathfrak{F}_{i_1}\mathfrak{F}_{i_2}\dots\mathfrak{F}_{i_p}X.$$

Proof Choose the Lyapunov function namely from class V_0, $V(t,x) = x^T P(t,x)x$, where

$$P(t,x) = \begin{cases} e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-\tau_k)}X - \int\limits_{\tau_k}^{t} e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-s)}ds Q, & \text{for} \quad t \in (\tau_k, \tau_{k+1}], \\ X, & \text{for} \quad t = \tau_{k+1}^+. \end{cases}$$

Q and X are symmetric positive definite $n \times n$-matrices. Later we shall show that $P(t,x) \overset{K}{>} 0$ in some neighborhood of the origin. First let us consider the derivative of

$V(t,x)$ with respect to time. If $t \neq \tau_k$, then we have

$$D_t^+ V(t,x)\big|_{(6.2)} = x^T \sum_{i=1}^{r} \mu_i(x)(A_i^T P(t,x) + P(t,x)A_i)x + x^T D_t^+ P(t,x)x$$
$$= x^T \sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i P(t,x)x + x^T D_t^+ P(t,x)x,$$

where

$$D_t^+ P(t,x)\big|_{(6.2)} = e^{-\sum\limits_{i=1}^{r} \mu_i(x)\mathfrak{F}_i(t-\tau_k)} \Big(-\sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i(t-\tau_k) - \sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i \Big) X$$
$$- \int_{\tau_k}^{t} e^{-\sum\limits_{i=1}^{r} \mu_i(x)\mathfrak{F}_i(t-s)} \Big(-\sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i - \sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i(t-s) \Big) dsQ - Q$$
$$= -\sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i \Big(e^{-\sum\limits_{i=1}^{r} \mu_i(x)\mathfrak{F}_i(t-\tau_k)} X - \int_{\tau_k}^{t} e^{-\sum\limits_{i=1}^{r} \mu_i(x)\mathfrak{F}_i(t-s)} ds\, Q \Big)$$
$$- e^{-\sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i(t-\tau_k)} \times \sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i X(t-\tau_k)$$
$$+ \int_{\tau_k}^{t} e^{-\sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i(t-s)} \sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i(t-s) ds\, Q - Q$$
$$= -\sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i P(t) - e^{-\sum\limits_{i=1}^{r} \mu_i(x)\mathfrak{F}_i(t-\tau_k)} \sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i X(t-\tau_k)$$
$$+ \int_{\tau_k}^{t} e^{-\sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i(t-s)} \sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i(t-s) ds\, Q - Q.$$

Hence, for the derivative $D_t^+ V(t,x)\big|_{(6.2)}$, we have the estimates:

$$D_t^+ V(t,x)\big|_{(6.2)} = x^T \sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i P(t,x)x - x^T \sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i P(t,x)x$$
$$- x^T Q x - x^T \Big[e^{-\sum\limits_{i=1}^{r} \mu_i(x)\mathfrak{F}_i(t-\tau_k)} \sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i X(t-\tau_k) \Big] x$$
$$+ x^T \Big[\int_{\tau_k}^{t} e^{-\sum\limits_{i=1}^{r} \mu_i(x)\mathfrak{F}_i(t-s)} \sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i(t-s) ds\, Q \Big] x$$
$$\leq -\lambda_{\min}(Q)\|x\|^2$$
$$+ \theta_2 e^{\sum\limits_{i=1}^{r} \mu_i(x)\|\mathfrak{F}_i\|\theta_2} \sum_{i=1}^{r} \|D_x^+ \mu_i(x)\| \, \|\mathfrak{F}_i\| \, \|X\| \left\| \frac{dx}{dt} \right\| \|x\|^2$$
$$+ \theta_2^2 e^{\sum\limits_{i=1}^{r} \mu_i(x)\|\mathfrak{F}_i\|\theta_2} \sum_{i=1}^{r} \|D_x^+ \mu_i(x)\| \, \|\mathfrak{F}_i\| \, \|Q\| \left\| \frac{dx}{dt} \right\| \|x\|^2,$$

where $\lambda_{\min}(\cdot) > 0$ is the minimal eigenvalue of corresponding matrix. Denote by $a = \max\limits_{i=\overline{1,r}} \|A_i\|$. Then since $\|\mathfrak{F}_i X\| \le \|A_i^T X + XA_i\| \le 2\|A_i\|\,\|X\|$ we get $\|\mathfrak{F}_i\| \le 2\|A_i\| \le 2a$, $i = \overline{1,r}$. It is also clear that

$$\Big\|\frac{dx}{dt}\Big\| \le \sum_{i=1}^{r} \mu_i(x)\|A_i\|\,\|x\| \le a\|x\|.$$

Hence the following inequality is fulfilled

$$\begin{aligned} D_t^+ V(t,x)\big|_{(6.2)} &\le -\lambda_{\min}(Q)\|x\|^2 + 2a^2\theta_2 e^{2a\theta_2} \sum_{i=1}^{r} \|D_x^+ \mu_i(x)\|\,\|X\|\,\|x\|^3 \\ &+ 2a^2\theta_2^2 e^{2a\theta_2} \sum_{i=1}^{r} \|D_x^+ \mu_i(x)\|\,\|Q\|\,\|x\|^3 \\ &\le \Big(-\lambda_{\min}(Q) + 2a^2 r\theta_2\gamma e^{2a\theta_2}\big(\|X\| + \theta_2\|Q\|\big)\|x\|^{\varepsilon}\Big)\|x\|^2. \end{aligned}$$

Therefore $D_t^+ V(t,x)\big|_{(6.2)} < 0$ for all x from the ball $\|x\| < R$, where

$$R = \left(\frac{\lambda_{\min}(Q)}{2a^2 r\theta_2\gamma e^{2a\theta_2}\big(\|X\| + \theta_2\|Q\|\big)}\right)^{1/\varepsilon}.$$

Consider the difference $\Delta V = V(t^+, x(t^+)) - V(t,x)$:

$$\begin{aligned} \Delta V\big|_{(6.2)} &= x^T(t^+)P(t^+)x(t^+) - x^T(t)P(t)x(t) = x^T(t^+)Xx(t^+) \\ &- x^T\left(e^{-\sum\limits_{i=1}^{r}\mu_i(x(\tau_k))\mathfrak{F}_i(\tau_k-\tau_{k-1})}X - \int\limits_{\tau_{k-1}}^{\tau_k} e^{-\sum\limits_{i=1}^{r}\mu_i(x(\tau_k))\mathfrak{F}_i(\tau_k-s)}ds\,Q\right)x \\ &\le x^T\sum_{j=1}^{r}\sum_{i=1}^{r}\mu_j(x)\mu_i(x)B_j^T X B_i x - x^T e^{-\sum\limits_{i=1}^{r}\mu_i(x(\tau_k))\mathfrak{F}_i(\tau_k-\tau_{k-1})}Xx \\ &+ x^T\int\limits_{0}^{\theta_2} e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i y}dy\,Qx, \end{aligned}$$

where $y = \tau_k - s$.

Next we shall prove the following inequality

$$e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(\tau_k-\tau_{k-1})}X \overset{K}{\ge} \left(I - \sum_{k=1}^{p-1}\frac{(-1)^{k+1}\Big(\sum_{i=1}^{r}\mu_i(x)\mathfrak{F}_i\Big)^k(\tau_k-\tau_{k-1})^k}{k!}\right)X. \qquad (6.5)$$

Let us choose an arbitrary element $\Phi \in K^* = K$ and consider an expansion in a Maclaurin series of the scalar function

$$\begin{aligned} \psi_\Phi(h) = \operatorname{tr}\Bigg(\Phi\Bigg(& e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(\tau_k-\tau_{k-1})h}X - X \\ &+ \sum_{k=1}^{p-1}\frac{(-1)^{k+1}\Big(\sum_{i=1}^{r}\mu_i(x)\mathfrak{F}_i\Big)^k(\tau_k-\tau_{k-1})^k h^k}{k!}X\Bigg)\Bigg), \end{aligned}$$

$h \geq 0$, restricting p-order terms

$$\psi_\Phi(h) = \psi_\Phi(0) + \psi'_\Phi(0)h + \dots + \frac{\psi_\Phi^{(p-1)}(0)h^{p-1}}{(p-1)!} + \frac{\psi_\Phi^{(p)}(\xi)h^p}{p!}, \quad \xi \in (0,h).$$

Let $h = 1$, then since $\psi_\Phi(0) = \psi'_\Phi(0) = \dots = \psi_\Phi^{(p-1)}(0) = 0$, we get $\psi_\Phi(1) = \dfrac{\psi_\Phi^{(p)}(\xi)}{p!}$, where

$$\psi_\Phi^{(p)}(\xi) = \operatorname{tr}\biggl(\Phi\biggl((-1)^p\biggl(\sum_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(\tau_k - \tau_{k-1})\biggr)^p e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(\tau_k-\tau_{k-1})\xi} X\biggr)\biggr).$$

Inequality (6.4) and positivity of operator $e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(\tau_k-\tau_{k-1})\xi}$ give estimate $\psi_\Phi^{(p)}(\xi) \geq 0$. Thus $\psi_\Phi(1) \geq 0$ for all $\Phi \in K^*$ and therefore inequality (6.5) is satisfied.

Consider the function

$$f_x(\theta_2) = x^T \int_0^{\theta_2} e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i y}\, dy\, Qx.$$

By Lagrange theorem we have

$$f_x(\theta_2) = f'_x(\zeta)\theta_2 = x^T\theta_2 e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i\zeta} Qx,$$

where $\zeta \in (0,\theta_2)$ and therefore

$$\|f_x(\theta_2)\| \leq \|x\|^2 e^{\sum\limits_{i=1}^{r}\mu_i(x)\|\mathfrak{F}_i\|\theta_2}\|Q\|\theta_2 \leq \theta_2 e^{2a\theta_2}\|Q\|\|x\|^2. \tag{6.6}$$

Inequalities (6.3), (6.5), (6.6) yield

$$\begin{aligned}\Delta V\big|_{(6.2)} &\leq -x^T\sum_{i_1=1}^{r}\dots\sum_{i_{p-1}=1}^{r}\mu_{i_1}(x)\dots\mu_{i_{p-1}}(x)Q_{i_1i_2\dots i_{p-1}}x + \theta_2 e^{2a\theta_2}\|Q\|\|x\|^2\\ &\leq -\sum_{i_{p-1}=1}^{r}\dots\sum_{i_1=1}^{r}\mu_{i_{p-1}}(x)\dots\mu_{i_1}(x)\lambda_{\min}(Q_{i_1i_2\dots i_{p-1}})\|x\|^2\\ &\quad+\theta_2 e^{2a\theta_2}\|Q\|\,\|x\|^2 \leq \bigl(-\lambda^* + \theta_2 e^{2a\theta_2}\|Q\|\,\bigr)\|x\|^2,\end{aligned}$$

where $Q_{i_1i_2\dots i_{p-1}}$ are positive definite matrices,

$$\lambda^* = \min\lambda_{\min}(Q_{i_1i_2\dots i_{p-1}}), \quad i_1,\,\dots,\,i_{p-1} = \overline{1,r}.$$

It is clear that $\Delta V\big|_{(6.2)} \leq 0$ if $\|Q\| \leq \dfrac{\lambda^*}{\theta_2}e^{-2a\theta_2}$ (we can choose, for example, $Q = \dfrac{\lambda^*}{2\sqrt{n}\theta_2}e^{-2a\theta_2}I$).

Next we shall show that $P(t,x) \overset{K}{>} 0$ for all $t \in \mathbb{R}$ i.e., $V(t,x)$ is a positive definite function. Since $V(t,x)$ is decreasing function, we have for $\|x\| < R$ and $t \in [\tau_k, \tau_{k+1})$, $k \in \mathbb{N}$

$$\begin{aligned} x^T P(t,x)x &\geq x^T(\tau_{k+1})P(\tau_{k+1}, x(\tau_{k+1}))x(\tau_{k+1}) \\ &\geq x^T(\tau_{k+1}^+)P(\tau_{k+1}^+, x(\tau_{k+1}^+))x(\tau_k^+) \geq \lambda_{\min}(X)\|x(\tau_{k+1}^+)\|^2 > 0. \end{aligned}$$

As a result, we have $V(t,x) > 0$, $D_t^+ V(t,x)\big|_{(6.2)} < 0$ and $\Delta V\big|_{(6.2)} \leq 0$ for all $\|x\| < R$.

Therefore the zero solution of impulsive Takagi–Sugeno fuzzy system (6.2) is asymptotically stable. This completes the proof of Theorem 6.1.

Let p be fixed then we shall name the LMIs (6.3)–(6.4) by p-order stability conditions of system (6.2).

Next we shall formulate 2-nd order stability conditions of system (6.2).

Corollary 6.1 *Under Assumption 6.1 the equilibrium state $x = 0$ of fuzzy system (6.2) is asymptotically stable if for all $\theta \in [\theta_1, \theta_2]$ there exists a common symmetric positive definite matrix X such that*

$$\begin{aligned} \frac{1}{2}(B_j^T X B_i + B_i^T X B_j) - X + (A_j^T X + X A_j)\theta < 0, \qquad & i,j = \overline{1,r}, \\ A_i^T A_j^T X + X A_j A_i + A_j^T X A_i + A_i^T X A_j \geq 0, \qquad & i,j = \overline{1,r}. \end{aligned}$$

Suppose that fuzzy system (6.2) is such that $A_1 = A_2 = \cdots = A_n = A$. Then we have the following 4-th order stability conditions.

Corollary 6.2 *Under Assumption 6.1 the equilibrium state $x = 0$ of fuzzy system (6.2) is asymptotically stable if for all $\theta \in [\theta_1, \theta_2]$ there exists a common symmetric positive definite matrix X such that*

$$\begin{aligned} &\frac{1}{2}\,(B_j^T X B_i + B_i^T X B_j) - X + (A^T X + XA)\theta \\ &\quad - \frac{1}{2}(\,(A^T)^2 X + 2A^T X A + X A^2\,)\theta^2 + \frac{1}{6}\theta^3\big((A^T)^3 X \\ &\quad + 3((A^T)^2 X A + A^T X A^2) + X A^3\big) < 0, \quad i,j = \overline{1,r}, \end{aligned} \tag{6.7}$$

$$(A^T)^4 X + 4((A^T)^3 X A + A^T X A^3) + 6(A^T)^2 X A^2 + X A^4 \geq 0. \tag{6.8}$$

Example 6.1 Let us consider the impulsive system (6.2) with the following system matrices

$$A_1 = A_2 = A = \begin{pmatrix} -2 & 0.5 \\ 0.4 & 0.1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1.1 & 0.1 \\ 0.2 & 0.2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1.2 & 0.15 \\ 0.1 & 0.3 \end{pmatrix}.$$

Let the period of control action $\theta_1 = \theta_2 = \theta = 0.12$ and suppose that Assumption 6.1 holds. Then it is easy to check that matrix $X = \begin{pmatrix} 0.0756 & 0.0102 \\ 0.0102 & 0.3261 \end{pmatrix}$ satisfies LMIs (6.7), (6.8). Therefore by Corollary 6.2 the zero solution $x = 0$ of the considered fuzzy system is asymptotically stable.

Remark 6.3 It is easy to verify that 2-nd order stability conditions are not available to discuss stability analysis of the above fuzzy system.

Remark 6.4 It should be noted that it is impossible to take stability analysis of fuzzy system from Example 6.1 via paper [29] because the discrete components (matrices B_1 and B_2) are unstable and stability conditions from the paper are neglected. Note that matrix A is also unstable. So, our stability conditions are available to investigate the impulsive T-S fuzzy system in which continuous and discrete components may be all unstable.

Let $p \geq 2$ and $G_{i_1 i_2 \cdots_{p-1}}$ be positive definite matrices. Consider the following matrix equations for $i_1, \dots, i_{p-1} = \overline{1, r}$

$$\left(\frac{1}{2}(\mathfrak{B}_{ji} + \mathfrak{B}_{ji}) - I + \sum_{k=1}^{p-1} \frac{(-1)^{k+1}(\mathfrak{F}_i)^k \theta^k}{k!}\right) X = -G_{i_1 i_2 \dots i_{p-1}}. \tag{6.9}$$

Similarly to Theorem 6.1 we have the following result.

Theorem 6.2 *Under Assumption 6.1 the equilibrium state $x = 0$ of fuzzy system (6.2) is asymptotically stable if for all $\theta \in [\theta_1, \theta_2]$ there exists a common symmetric positive definite solution X of (6.9) such that the following inequality is fulfilled*

$$e^{2a\theta} \frac{(2a\theta)^p}{p!} < \frac{\lambda^*}{\|X\|},$$

where $a = \max\limits_{i=\overline{1,r}} \|A_i\|$, $\lambda^* = \min \lambda_{\min}(G_{i_1 i_2 \dots i_{p-1}})$ for $i_1, \dots, i_{p-1} = \overline{1, r}$.

Next, we state the following assumption.

Assumption 6.2 *There exist $R_0 > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$ and $\varepsilon > 0$ such that the functions $\mu_i(x)$, $i = \overline{1, r}$, satisfy the inequality*

$$\|D_x^+ \mu_i(x)\| \leq \begin{cases} \gamma_1 \|x\|^{-1+\varepsilon}, & \text{for} \quad \|x\| \leq R_0, \\ \gamma_2 \|x\|^{-1-\varepsilon}, & \text{for} \quad \|x\| \geq R_0. \end{cases}$$

Taking into account Assumption 6.2 we can establish the following.

Theorem 6.3 *Let in Assumption 6.2 constants γ_1, γ_2, R_0 be such that*

$$\gamma_1 \gamma_2 < \frac{\lambda_{\min}^2(Q)}{4a^4 r^2 \theta_2^2 e^{4a\theta_2}(\,\|X\| + \theta_2 \|Q\|\,)^2}$$

and

$$\left(\frac{\lambda_{\min}(Q)}{2a^2 r \theta_2 \gamma_2 e^{2a\theta_2}\big(\,\|X\| + \theta_2\|Q\|\,\big)}\right)^{-1/\varepsilon} < R_0,$$

$$R_0 < \left(\frac{\lambda_{\min}(Q)}{2a^2 r \theta_2 \gamma_1 e^{2a\theta_2}\big(\,\|X\| + \theta_2\|Q\|\,\big)}\right)^{1/\varepsilon},$$

where $a = \max\limits_{i=\overline{1,r}} \|A_i\|$, Q is a symmetric positive definite $n \times n$- matrix and X is a common symmetric positive definite matrix such that conditions (6.3), (6.4) of Theorem 6.1 hold. Then the zero solution of impulsive fuzzy system (6.2) is globally asymptotically stable.

Proof Choose for a candidate the Lyapunov function from class V_0, $V(t,x) = x^T P(t,x)x$, where

$$P(t,x) = \begin{cases} e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-\tau_k)} X - \int\limits_{\tau_k}^{t} e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-s)} ds\, Q, & \text{for} \quad t \in (\tau_k, \tau_{k+1}], \\ X, & \text{for} \quad t = \tau_{k+1}^+, \end{cases}$$

where Q and X are symmetric positive definite $n \times n$-matrices. Let us consider the derivative of $V(t,x)$ with respect to time (notice that $V(t,x)$ is radially unbounded function). If $t \neq \tau_k$ then we have two cases:

(1) if $\|x\| \leq R_0$ then similar to the proof of Theorem 6.1 we get

$$D_t^+ V(t,x)\big|_{(6.2)} \leq \big(-\lambda_{\min}(Q) + 2a^2 r\theta_2 e^{2a\theta_2}\gamma_1\big(\|X\| + \theta_2\|Q\|\big)\|x\|^{\varepsilon}\big)\|x\|^2.$$

Clearly $D_t^+ V(t,x)\big|_{(6.2)} < 0$ by conditions of Theorem 6.3;

(2) if $\|x\| \geq R_0$ then by analogy we get

$$D_t^+ V(t,x)\big|_{(6.2)} \leq \big(-\lambda_{\min}(Q) + 2a^2 r\theta_2 e^{2a\theta_2}\gamma_2\big(\|X\| + \theta_2\|Q\|\big)\|x\|^{-\varepsilon}\big)\|x\|^2.$$

Clearly $D_t^+ V(t,x)\big|_{(6.2)} < 0$ by conditions of Theorem 6.3. Thus we have showed that $D_t^+ V(t,x)\big|_{(6.2)} < 0$ for all $x \in \mathbb{R}^n$.

Similar to the proof of Theorem 6.1 we can show (taking into account the conditions of Theorem 6.3) that $\Delta V\big|_{(6.2)} = V(t^+, x(t^+)) - V(t,x) \leq 0$ and $V(t,x) > 0$. Therefore the zero solution of impulsive Takagi–Sugeno fuzzy system (6.2) is globally asymptotically stable.

Remark 6.5 In spite of advantages of LMI method, the existence of solution that satisfies the sufficient conditions is not guaranteed. This happens when the number of fuzzy rules is increased or too many system's matrices are imposed.

Remark 6.6 The result of this section can be utilized on chaotic, inverted pendulum, biological, electrical dynamical systems etc. Moreover in practice it is enough to verify(using, for example Matlab LMI toolbox) 2-nd order or 4-th order stability conditions.

6.2 Impulsive Fuzzy Control for Ecological Prey–Predator Community

It is well-known that control problem is an important task for mathematical theory of artificial ecosystems. Impulsive control of such systems is more favorable due to seasonal functioning of this type of systems. Some problem of impulsive control for homotypical model has been considered in the paper [15]. But for practice it is suitable to consider models with fuzzy impulsive control because it is almost impossible to accurately measure the biomass of one or another biological species but possible to roughly estimate those.

Consider a Lotka–Volterra type prey-predator model (with interspecific competition among preys) whose evolution is described by the following equations

$$\begin{aligned} \frac{dN_1}{dt} &= \alpha N_1 - \beta N_1 N_2 - \gamma N_1^2, \\ \frac{dN_2}{dt} &= -mN_2 + s\beta N_1 N_2, \end{aligned} \tag{6.10}$$

where $N_1(t)$ is the biomass of preys, $N_2(t)$ is the biomass of predators, α is the growth rate of the preys, m is the death rate of the predators, γ is the rate of the interspecific competition among preys, β is the per-head attack rate of the predators, and s is the efficiency of converting preys to predators.

Suppose that the ecosystem is controlled via regulation of the number of species at certain fixed moments of time(impulsive control) $\theta, 2\theta, \dots, k\theta, \dots$ and the regulation is reduced either to elimination or fulmination of the representatives of species. Taking into account these assumptions we have to add the regulator equations to the system of the evolution as

$$\begin{aligned}
&\Delta N_1 = u_1(N_1, N_2),\\
&\Delta N_2 = u_2(N_1, N_2), \quad t = k\theta, \quad k \in \mathbb{N},
\end{aligned}$$

where u_1, u_2 are feedback functions, θ is a period of control action.

Under these assumptions the equations of closed controlled ecosystem become

$$\begin{aligned}
&\frac{dN_1}{dt} = \alpha N_1 - \beta N_1 N_2 - \gamma N_1^2,\\
&\frac{dN_2}{dt} = -mN_2 - s\beta N_1 N_2, \quad t \neq k\theta,\\
&\Delta N_1 = u_1(N_1, N_2),\\
&\Delta N_2 = u_2(N_1, N_2), \quad t = k\theta, \quad k \in \mathbb{N}.
\end{aligned} \tag{6.11}$$

Besides the trivial equilibrium state, equation (6.10) has also the positive asymptotically stable states

$$N_1^* = \frac{m}{s\beta}, \quad N_2^* = \frac{s\alpha\beta - m\gamma}{s\beta^2}.$$

It is clear that if the number of preys is much greater than the equilibrium ones then some amount of preys is eliminated and vice versa. Analogous situation occurs with the predators. Thus, the impulsive fuzzy controls are designed regarding the rules:

if $N_i \ll N_i^*$, then $u_i(N_1, N_2) = \psi_i(N_i^* - N_i)$, $\quad \psi_i > 0$, $\quad i = 1, 2$;
if $N_i \gg N_i^*$, then $u_i(N_1, N_2) = \chi_i(N_i^* - N_i)$, $\quad \chi_i \in (0, 1)$, $i = 1, 2$,

where ψ_i are the fulmination rates, χ_i are the elimination rates.

The fuzzy relation $x \gg y$ ("x is much larger than y") can be formalized using the following membership function

$$\omega(x, y) = \begin{cases} \dfrac{1}{1 + 1/(x-y)^2}, & \text{if } x > y,\\ 0, & \text{if } x \le y. \end{cases}$$

Next, we define the variables of disturbance of motion $x_1(t) = N_1(t) - N_1^*$, $x_2(t) = N_2(t) - N_2^*$. Then the equations for system (6.11) become (using linearization):

$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{m\gamma}{s\beta}x_1 - \dfrac{m}{s},\\ \dfrac{dx_2}{dt} = -\dfrac{\alpha\beta s - m\gamma}{\beta}x_1, \quad t \neq k\theta,\\ \Delta x_1 = u_1,\\ \Delta x_2 = u_2, \quad t = k\theta. \end{cases} \tag{6.12}$$

The Takagi–Sugeno fuzzy model (6.1) of system (6.12) is specified by the following four rules:

R^1: if $N_1 \ll N_1^*$ and $N_2 \ll N_2^*$, then

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t), & t \neq k\theta, \\ x(t^+) = B_1 x, & t = k\theta, \\ x(t_0^+) = x_0. \end{cases}$$

R^2: if $N_1 \ll N_1^*$ and $N_2 \gg N_2^*$, then

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t), & t \neq k\theta, \\ x(t^+) = B_2 x, & t = k\theta, \\ x(t_0^+) = x_0. \end{cases}$$

R^3: if $N_1 \gg N_1^*$ and $N_2 \gg N_2^*$, then

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t), & t \neq k\theta, \\ x(t^+) = B_3 x, & t = k\theta, \\ x(t_0^+) = x_0. \end{cases}$$

R^4: if $N_1 \gg N_1^*$ and $N_2 \ll N_2^*$, then

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t), & t \neq k\theta, \\ x(t^+) = B_4 x, & t = k\theta, \\ x(t_0^+) = x_0. \end{cases}$$

It is obvious that Assumption 6.1 holds for membership function $\omega(x, y)$. Using Corollary 6.1 the stability analysis of nontrivial equilibrium position for ecosystem is reduced to checking the existence of symmetric positive definite matrix X such that the following LMIs hold true:

$$\begin{gathered} \frac{1}{2}(B_i^T X B_j + B_j^T X B_i) - X + (A^T X + XA)\theta < 0, \quad i, j = \overline{1,4}, \\ (A^T)^2 X + 2A^T X A + X A^2 \geq 0. \end{gathered} \tag{6.13}$$

Matrices A, B_1, B_2, B_3 and B_4 are as follows

$$A = \begin{pmatrix} -\dfrac{m\gamma}{s\beta} & -\dfrac{m}{s} \\ \dfrac{\alpha\beta s - m\gamma}{\beta} & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1 - \psi_1 & 0 \\ 0 & 1 - \psi_2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 - \psi_1 & 0 \\ 0 & 1 - \chi_2 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 1 - \chi_1 & 0 \\ 0 & 1 - \chi_2 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 - \chi_1 & 0 \\ 0 & 1 - \psi_2 \end{pmatrix}.$$

Next we consider the stability analysis of the obtained Takagi–Sugeno fuzzy model for ecosystem's evolution with the following parameters: $\alpha = 4$, $\gamma = 0.3$, $\beta = 0.5$, $m = 1.2$, $s = 0.4$, $\theta = 0.5$ and the parameters of impulsive control: $\psi_1 = 0.9$, $\psi_2 = 0.5$, $\chi_1 = 0.99$, $\chi_2 = 0.6$.

It is easy to check that matrix $X = \begin{pmatrix} 1.7427 & 1.8779 \\ 1.8779 & 8.2018 \end{pmatrix}$ satisfies inequalities (6.13). Therefore by Corollary 6.1 the equilibrium state of ecological system is asymptotically stable (see Figure 6).

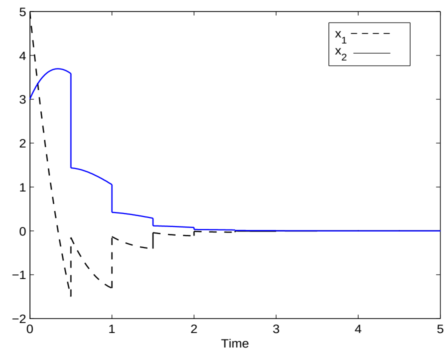

Figure 6. Evolution of $x_1(t)$ and $x_2(t)$ (stable result).

Let us change the parameters of impulsive control: $\psi_1 = 6$, $\psi_2 = 4$, $\chi_1 = 0.6$, $\chi_2 = 0.2$. In this case the solution of LMIs (6.13) is infeasible and computer simulation gives an unstable result (see Figure 7).

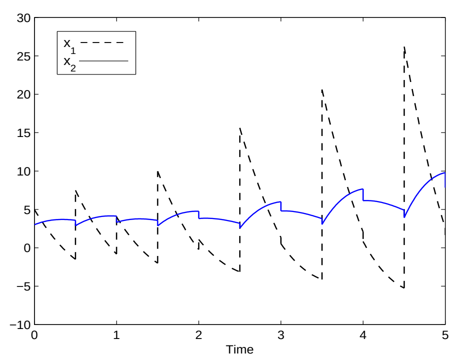

Figure 7. Evolution of $x_1(t)$ and $x_2(t)$ (unstable result).

Based on the well-known Lyapunov direct method, sufficient conditions have been derived to guarantee the asymptotic stability and globally asymptotic stability of the equilibrium point of impulsive T-S fuzzy systems. It is shown that these sufficient conditions are expressed easily as a set of LMIs. It is also concluded that the obtained stability conditions allow to investigate the impulsive T-S fuzzy system in which continuous and discrete components may be all unstable.

7 Conclusion

The results given in Section 2 are adopted from Martynyuk and Chernienko [21]. The model of robot interacting with dynamic environment is due to DeLuca and Manes [2]. It should be noted that the importance of studying the problem of stability of motion of a robot interacting with a dynamic environment was discussed in contemporary literature.

The contents of Section 3 are essentially new (see Martynyuk and Lukyanova [22]). For continuous neural networks see Hopfield [7], Wang and Michel [27], etc. and for discrete-time neural networks see Michel, Farrel and Sun [24], etc.

Section 4 is based on the results by Lila and Martynyuk [12, 13]. We note that the approach proposed for stability analysis of periodic solutions of system (4.1) can be extended for the cases where the presence of phase of coefficient of optic constraint between the lasers exists, neutral stability in linear approximation and in the study of dynamics of many-modulus systems.

In Section 5 the model (5.2) is taken from the monograph by Forrester [5]. The model (5.6) is a new. Theorem 5.1 is taken from Martynuuyuk [19]. Some other models of the world dynamics are in Egorov *et al.* [4], Levashov [9], etc.

Section 6 is adapted from Denysenko, Martynyuk and Slyn'ko [3].

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