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Stability of Hybrid Mechanical Systems with Switching Linear Force Fields

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Abstract: Linear hybrid mechanical systems with switchings of force fields are studied. Some sufficient conditions are brought forward for the switched systems being asymptotically stable for any switched law. The results are obtained based on two approaches. The first one is called as the decomposition method, and the second one consists in an explicit construction of the common Lyapunov functions for the families of systems corresponding to the switched systems. Different cases of domination concerning one of the force field components (e.g., velocity, gyroscopic, dissipative, potential) are considered.

Keywords: *hybrid mechanical systems; switched systems; stability; decomposition; common Lyapunov functions.*

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1 Introduction

The stability analysis of hybrid systems which are described by differential equations with switching right-hand sides is one of the most important problems in modern automatic control theory [3–5, 9, 15]. In various cases, after the design of continuous controller has been finished, it is required to verify the stability of the closed system for any admissible switching law [7, 9, 17]. Such a situation naturally arises when the switching law is either unknown or is too complex to consider in the stability investigation.

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A well-known approach for the stability analysis is to construct a common Lyapunov function for the family of subsystems corresponding to the switched system. That is, the function is positive and monotonically decreases along the solutions to each subsystem from the family. However, the problem of the existence of a common Lyapunov function is not completely solved until to now even for families of linear stationary systems [7, 8]. Only in some special cases, e.g., for two-dimensional or three-dimensional linear systems [14, 15], necessary and sufficient conditions for the existence of common quadratic Lyapunov function are found. For the linear systems with higher dimension, the existence of common quadratic Lyapunov function is proved only under some additional conditions, for instance, under commutativity of systems matrices [13].

This paper deals with mechanical systems with switching force fields. The switchings can be caused by both intrinsic reasons, such as using computer or microprocessor in control loop, and external reasons, for instance, when movement of mechanical system occurs in environment with changeable resistance [4, 6, 10, 15]. Motion of mechanical systems is usually described by differential equations of the second order, that results in the occurrence of some special properties. In particular, in the presence of switchings in acting force field, conditions of commutativity will be obviously broken. Therefore, the corresponding results based on commutativity of systems matrices for the existence of common quadratic Lyapunov functions are nonapplicable to mechanical systems. This motives us to study extendedly the problem of the existence of common Lyapunov functions for mechanical systems.

In the paper, we present two approaches for constructing common Lyapunov functions for mechanical systems with switching force fields. The first one is to decompose the original system consisting of n differential equations of the second order into two first-order subsystems of the same dimension. The approach is also available for the systems without switchings in the force fields since it allows one to solve stability problem on the basis of analysis of matrices of twice smaller dimensions than that in the original system. In the presence of switchings, decomposition makes it possible to use the conditions of matrix coefficients commutativity, which guarantees the existence of a common quadratic Lyapunov function for the family of systems corresponding to the switched system. The second approach is to give out an explicit construction of the common Lyapunov functions for the mechanical systems. These Lyapunov functions are constructed on the base of elements possessing clear mechanical meaning. It should be noted that in certain situations both approaches stated above are practically close to each other and lead to similar stability conditions.

In sum, this paper provides some stability conditions on the basis of construction of the common Lyapunov functions with essential use of mechanical system specificity. This specificity impels us to investigate this special subclass of hybrid systems not following the ordinal line of thought. Thus, the results obtained possess certain theoretical features and are of undoubted practical interest.

2 Statement of the Problem

Consider a family of linear systems

$$A\ddot{q} + F_s\dot{q} + C_s q = 0, \quad s = 1, \dots, N, \tag{1}$$

where q and $\dot{q}$ are n-dimensional vectors of generalized coordinates and generalized velocities, respectively; A, F_s, C_s are constant matrices, and matrix A is nonsingular. A switching law is the piecewise constant function $\sigma : [0, +\infty) \to D = \{1, \dots, N\}$. Thus, the switched system generated by the family (1) and a switching signal σ is

$$A\ddot{q} + F_\sigma \dot{q} + C_\sigma q = 0. \tag{2}$$

In this paper, we assume that on every bounded time interval, the switching function has finite number of discontinuities, which are called switching instants of time, and takes a constant value on every interval between two consecutive switching instants. This kind of switching law is called admissible.

We will look for the conditions to guarantee the switched system (2) is asymptotically stable for any admissible switching law. As is well known, it is sufficient [7, 9] to construct a common Lyapunov function for the family (1) such that it satisfies the assumptions of Lyapunov asymptotic stability theorem.

Linear systems (1) can be represented in the form

$$\dot{x} = P_s x, \quad s = 1, \dots, N, \tag{3}$$

where $x = (q^T, \dot{q}^T)^T$,

$$P_s = \begin{pmatrix} 0 & I \\ -A^{-1}C_s & -A^{-1}F_s \end{pmatrix},$$

and I denotes the identity matrix. Thus, one can investigate the stability of (3) for the general switched linear systems, and some well known conditions of the existence of a common quadratic Lyapunov function [8, 9, 13] can be used.

However, we point out that such approach is not always effective, partly owing to the following difficulties:

1) The transformation of (1) to the form (3) lose the mechanical meaning of the conditions;

2) The dimension of (3) becomes higher;

3) Systems (1) possess a special structure, therefore known results obtained for the linear switched systems of general form may be nonapplicable for (3).

For instance, commutativity of matrices $P_1, \dots, P_N$ in the family (3) is a simple condition of the existence of common Lyapunov function [13]. But due to the special structure of matrices of $P_1, \dots, P_N$ in systems (3), the commutativity results in the equalities $F_s = F_r$, $C_s = C_r$, $s, r = 1, \dots, N$, which is a trivial case.

In the paper, we consider two approaches for the stability analysis of switched mechanical system (2). The first one is based on a decomposition procedure, and the second one consists in an explicit construction of the Lyapunov functions for the switched system.

3 Decomposition Approach for Stability Analysis of Linear Mechanical Systems

In the section, we consider the decomposition conditions for mechanical systems without switchings.

3.1 Systems with the domination of velocity forces

The systems with the domination of velocity forces are described by the following differential equations

$$A\ddot{q} + hF\dot{q} + Cq = 0. \tag{4}$$

This kind of equations are generally considered as the linearized ones of motions for gyroscopic systems [18], where q and $\dot{q}$ are n-dimensional vectors of generalized coordinates and generalized velocities respectively; A, F and C are constant matrices; h is a large positive parameter. We assume that all the matrices in (4) are nonsingular.

System (4) is linear and stationary one. Therefore, some well-known criteria, for instance, the Hurwitz criterion or equivalent ones [12], can be used to determine the stability conditions for this system. However, in the case of high dimension of (4) or under the uncertainties in the matrices A, F, C this approach may be inefficient or even nonapplicable in practice.

Another way to investigate the stability in such situations is to decompose the system into several simpler systems, to study each of them separately, and then to appropriately apply the obtained results to the original system [1, 16].

V. I. Zubov has proposed the following result, which allows one to decompose, for sufficiently large values of parameter h, the problem of stability analysis of system (4) consisting of n differential equations of the second order into two analogical problems for the first-order systems of the same dimension.

Theorem 3.1 [18] *Let the following isolated subsystems*

$$F\dot{y} + Cy = 0, \tag{5}$$

$$A\dot{z} + Fz = 0 \tag{6}$$

be asymptotically stable. Then there exists $h_0 > 0$ such that for any $h > h_0$ system (4) is also asymptotically stable.

In applications, it is important to get the estimation of the lower bound h_0 for admissible values of h. Theorem 3.1 in [18] was proved on the base of the first Lyapunov method and by means of the expansion of the roots of the characteristic equation for (4) in the series with respect to the negative powers of h. However, this process did not give constructive estimation of h_0 value.

In this paper, we suggest another approach to prove Theorem 3.1, which is based on using Lyapunov direct method. The new proof contains a constructive procedure for determining the set of admissible values of large parameter h.

Proof Making the substitution of variables

$$\dot{q} = z, \qquad A\dot{q} + hFq = hFy, \tag{7}$$

we transform (4) into

$$\begin{aligned} F\dot{y} &= -\tfrac{1}{h}Cy + \tfrac{1}{h^2}CF^{-1}Az, \\ A\dot{z} &= -hFz - Cy + \tfrac{1}{h}F^{-1}Az. \end{aligned} \tag{8}$$

From the asymptotic stability of isolated subsystems (5) and (6), it follows [1] the existence of quadratic forms $V_1(y)$ and $V_2(z)$ such that the inequalities

$$a_{11}\|y\|^2 \le V_1 \le a_{12}\|y\|^2, \qquad a_{21}\|z\|^2 \le V_2 \le a_{22}\|z\|^2,$$

$$\left\|\frac{\partial V_1}{\partial y}\right\| \le a_{13}\|y\|, \qquad \left\|\frac{\partial V_2}{\partial z}\right\| \le a_{23}\|z\|, \qquad \dot{V}_1\big|_{(5)} \le -a_{14}\|y\|^2, \qquad \dot{V}_2\big|_{(6)} \le -a_{24}\|z\|^2$$

are valid for any $y, z \in R^n$, where a_{ij} are positive constants, $i = 1, 2$, $j = 1, 2, 3, 4$. Construct the function $V(y, z) = \varepsilon h^2 V_1(y) + V_2(z)$, where ε is a positive parameter. Differentiating $V(y, z)$ along the solutions to (8), we get that the inequality

$$\dot{V}\Big|_{(8)} \leq -a_{14}\varepsilon h\|y\|^2 - \left(ha_{24} - \frac{b_1}{h}\right)\|z\|^2 + (b_2\varepsilon + b_3)\|y\|\|z\|$$

holds for any $y, z \in R^n$, where $b_1 = a_{23}\|A^{-1}CF^{-1}A\|$, $b_2 = a_{13}\|F^{-1}CF^{-1}A\|$, $b_3 = a_{23}\|A^{-1}C\|$. Hence, if the condition

$$h > \sqrt{\frac{(b_2\varepsilon + b_3)^2}{4\varepsilon a_{14}a_{24}} + \frac{b_1}{a_{24}}} \tag{9}$$

is satisfied, function $\dot{V}\Big|_{(8)}$ is negative definite.

To complete the proof, it remains to find a $\varepsilon_0 > 0$ such that for $\varepsilon = \varepsilon_0$ (9) gives us the largest region of admissible values of h. It is easy to show that $\varepsilon_0 = b_3/b_2$, and estimation (9) becomes $h > \sqrt{b_2 b_3/(a_{14}a_{24}) + b_1/a_{24}}$. □

3.2 Systems with the domination of gyroscopic forces

Along with (4), the following equations

$$A\ddot{q} + (B + hG)\dot{q} + Cq = 0 \tag{10}$$

are also used as a linear approximation for the equations of motions of gyroscopic systems [11], where $q, \dot{q} \in R^n$; A, B, G, C are constant matrices; h is a large positive parameter. It is assumed [11] that A is symmetric and positive definite matrix of inertial characteristics; B is symmetric matrix of dissipative and accelerating forces; G is skew-symmetric and nonsingular matrix of gyroscopic forces. Thus, the dominating forces in (4) are the velocity ones, while in (10) they are the gyroscopic ones.

The conditions of decomposition for (10) have been established in [11]. As mentioned above, in [11] as well as in [18], for justifying the possibility of decomposition, the first Lyapunov method was used, and the constructive estimation for the admissible values of large parameter was not obtained.

Next we propose the same approach based on Lyapunov direct method as in the proof of Theorem 3.1 to study the stability analysis of system (10).

Consider the isolated subsystems

$$G\dot{y} + Cy = 0, \tag{11}$$

$$A\dot{z} + (B + hG)z = 0. \tag{12}$$

Theorem 3.2 *Let the matrix B be positive definite and subsystem* (11) *be asymptotically stable. Then there exists $h_0 > 0$ such that for any $h > h_0$ system* (10) *is also asymptotically stable.*

Proof By using the substitution of variables $\dot{q} = z$, $A\dot{q} + (B + hG)q = (B + hG)y$, we transform (10) into the following system

$$\begin{aligned} \dot{y} &= -\tfrac{1}{h}G^{-1}Cy + \tfrac{1}{h}(B + hG)^{-1}BG^{-1}Cy + (B + hG)^{-1}C(B + hG)^{-1}Az, \\ A\dot{z} &= -(B + hG)z - Cy + C(B + hG)^{-1}Az. \end{aligned} \tag{13}$$

From the asymptotic stability of (11), it follows that for this subsystem there exists a quadratic Lyapunov function $V_1(y)$ satisfying all the assumptions of the Lyapunov asymptotic stability theorem. If the matrix B is positive definite, then subsystem (12) is asymptotically stable for any $h > 0$, and let $V_2(z) = z^T Az$ be its Lyapunov function.

Construct the function $V(y,z) = \varepsilon h^2 V_1(y) + V_2(z)$, $\varepsilon = const > 0$. Let $\bar{h}$ be a positive number. Differentiating $V(y,z)$ along the solutions to (13), one gets that the inequality

$$\dot{V}\Big|_{(13)} \leq -\varepsilon(a_1 h - a_2)\|y\|^2 - \left(a_3 - \frac{a_4}{h}\right)\|z\|^2 + (a_5\varepsilon + a_6)\|y\|\|z\|$$

holds for $h \geq \bar{h}$ and for all $y, z \in R^n$, where a_i are positive constants, $i = 1, \dots, 6$. It should be noted that values of a_2, a_4 and a_5 depend on the chosen value of $\bar{h}$. Hence, if the conditions $h \geq \bar{h}$, $h > a_2/a_1$ and

$$(a_1 h - a_2)\left(a_3 - \frac{a_4}{h}\right) > \frac{(a_5\varepsilon + a_6)^2}{4\varepsilon} \tag{14}$$

are satisfied, function $\dot{V}\Big|_{(13)}$ is negative definite.

It is easy to verify that for $\varepsilon = a_6/a_5$ inequality (14) gives out the largest region of admissible values h: $(a_1 h - a_2)(a_3 - a_4/h) > a_5 a_6$. □

4 Decomposition of Switched Mechanical Systems

Now we turn to consider the linear mechanical system with switching positional forces

$$A\ddot{q} + hF\dot{q} + C_\sigma q = 0. \tag{15}$$

The corresponding family of systems are

$$A\ddot{q} + hF\dot{q} + C_s q = 0, \quad s = 1, \dots, N, \tag{16}$$

where $q, \dot{q} \in R^n$; A, F, C_s are constant nonsingular matrices; h is a large positive parameter.

The decomposition method stated above is still used to obtain the asymptotic stability conditions for (15). We point out that in this case the approach suggested in [18] for justifying the possibility of decomposition can not anymore be used for switched system (15) since the negativeness of real parts of all roots of characteristic equations for systems (16) does not provide asymptotic stability of (15) [9].

Now we show that the approach proposed in the proof of Theorem 3.1 allows us to obtain decomposition conditions for the systems with switching positional forces.

Consider the isolated subsystem

$$A\dot{z} + Fz = 0 \tag{17}$$

and the family of isolated subsystems

$$F\dot{y} + C_s y = 0, \quad s = 1, \dots, N. \tag{18}$$

Theorem 4.1 *Let the following conditions be fulfilled:*

(a) Subsystem (17) *is asymptotically stable;*

(b) Subsystems (18) *are asymptotically stable, and moreover the family* (18) *admits a common quadratic Lyapunov function satisfying the assumptions of the Lyapunov asymptotic stability theorem.*

Then, for sufficiently large values of h and for any switching law, system (15) *is asymptotically stable.*

Proof By using the substitution (7), we transform (16) into the systems

$$\begin{aligned} F\dot{y} &= -\tfrac{1}{h}C_s y + \tfrac{1}{h^2}C_s F^{-1}Az, \\ A\dot{z} &= -hFz - C_s y + \tfrac{1}{h}C_s F^{-1}Az, \quad s = 1,\dots,N. \end{aligned} \tag{19}$$

Let $V_1(y)$ be a common quadratic Lyapunov function of family (18), and $V_2(z)$ be a quadratic Lyapunov function of (17), respectively, both of which satisfy all the assumptions of the Lyapunov asymptotic stability theorem. Construct the function $V(y,z) = \varepsilon h^2 V_1(y) + V_2(z)$, where ε is a positive parameter. By the analogy with the proof of Theorem 3.1, it is easy to show that, for sufficiently small values of ε and for sufficiently large values of h, the derivative of $V(y,z)$ along the solution to each of the systems in (19) is negative definite. Thus, $V(y,z)$ is a common Lyapunov function for family (19). It implies that, for any switching law σ, the zero solution of the system

$$F\dot{y} = -\tfrac{1}{h}C_\sigma y + \tfrac{1}{h^2}C_\sigma F^{-1}Az,$$
$$A\dot{z} = -hFz - C_\sigma y + \tfrac{1}{h}C_\sigma F^{-1}Az$$

is asymptotically stable. Hence, the zero solution of (15) possesses the same property. □

Now we turn to consider the linear mechanical system with the dominating gyroscopic forces and with the switching positional forces

$$A\ddot{q} + (B + hG)\dot{q} + C_\sigma q = 0. \tag{20}$$

The corresponding family of systems has the form

$$A\ddot{q} + (B + hG)\dot{q} + C_s q = 0, \quad s = 1,\dots,N,$$

where $q, \dot{q} \in R^n$; A, B, G, C_s are constant matrices; h is a large positive parameter. We assume that A is symmetric and positive definite matrix, B is symmetric matrix, G is skew-symmetric and nonsingular matrix.

Theorem 4.2 *Let the following conditions be fulfilled:*
(a) Matrix B is positive definite;
(b) Subsystems

$$G\dot{y} + C_s y = 0, \quad s = 1,\dots,N, \tag{21}$$

are asymptotically stable, and for family (21) *there exists a common quadratic Lyapunov function satisfying the assumptions of the Lyapunov asymptotic stability theorem.*
Then, for sufficiently large values of h and for any switching law, system (20) *is asymptotically stable.*

The proof of Theorem 4.2 is similar to that one of Theorem 3.2.

Remark 4.1 Just as in Section 3, the suggested approach permits one to develop a constructive procedure for the estimation of lower bounds of admissible values of parameter h in systems (15) and (20).

Remark 4.2 As mentioned in Section 2, for systems (15) and (20), the direct application of known results on the existence of a common Lyapunov functions may be ineffective or even impossible. Theorems 4.1 and 4.2 provide a possibility to reduce the problem of constructing a common Lyapunov function for system of dimension $2n$ with the special structure to the analogical problem for the subsystem of dimension n which, generally, does not possess the special structure.

For instance, in Section 2, it has been shown that well-known commutativity condition is nonapplicable to system (15). However, according to Theorem 4.1, under the sufficiently large values of parameter h, instead of (15), one can consider subsystem (17) and family of subsystems (18). In fact, for (18), the commutativity condition becomes $C_sF^{-1}C_r = C_rF^{-1}C_s$, $s, r = 1, \dots, N$.

5 Construction of the Common Lyapunov Functions

5.1 Domination of potential forces

Consider the linear switched mechanical system

$$A\ddot{q} + (B_\sigma + G_\sigma)\,\dot{q} + (hK + P_\sigma)\,q = 0. \tag{22}$$

The corresponding family of systems is described as follows

$$A\ddot{q} + (B_s + G_s)\,\dot{q} + (hK + P_s)\,q = 0, \qquad s = 1, \dots, N, \tag{23}$$

where A, B_s, G_s, K, P_s are constant matrices, h is positive parameter. We assume that matrices K and B_s are symmetric, while matrices G_s and P_s are skew-symmetric. Moreover, here and in what follows it is assumed that A is symmetric and positive definite matrix.

Theorem 5.1 *Let the following conditions be fulfilled:*
(a) Matrices $B_1, \dots, B_N$ are positive definite;
(b) Matrix K is positive definite;
(c) The value of parameter h is sufficiently large.
Then, for any switching law of all components of force field, with the exception of potential component, system (22) is asymptotically stable.

Proof Construct the common Lyapunov function for the family (23) in the form

$$V(q, \dot{q}) = \frac{1}{2}\dot{q}^T A\dot{q} + \frac{h}{2}q^T Kq + \varepsilon q^T A\dot{q}, \tag{24}$$

where $\varepsilon > 0$ is sufficiently small positive number.

For arbitrary symmetric matrix M, let $\lambda_{min}(M)$ and $\lambda_{max}(M)$ be the minimal and maximal eigenvalues of M, respectively. Introduce the following notations: $k_1 = \lambda_{min}(K)$, $k_2 = \lambda_{max}(K)$, $a_1 = \lambda_{min}(A)$, $a_2 = \lambda_{max}(A)$,

$$b_1 = \min_{s=1,\dots,N} \lambda_{min}(B_s), \quad b_2 = \max_{s=1,\dots,N} \lambda_{max}(B_s),$$

$$p = \max_{s=1,\dots,N} \sqrt{\lambda_{max}\left(P_s^T P_s\right)}, \qquad g = \max_{s=1,\dots,N} \sqrt{\lambda_{max}\left(G_s^T G_s\right)}.$$

For all $q, \dot{q} \in R^n$ the estimations

$$\frac{a_1}{2}\|\dot{q}\|^2 - \varepsilon a_2\|q\|\|\dot{q}\| + \frac{h}{2}k_1\|q\|^2 \le V(q, \dot{q}) \le \frac{a_2}{2}\|\dot{q}\|^2 + \varepsilon a_2\|q\|\|\dot{q}\| + \frac{h}{2}k_2\|q\|^2 \tag{25}$$

are valid. Differentiating the Lyapunov function (24) along the solution to sth system in family (23), we get

$$\begin{aligned} \dot{V} &= -\dot{q}^T B_s\dot{q} + \varepsilon\dot{q}^T A\dot{q} - \varepsilon hq^T Kq - \dot{q}^T P_s q - \varepsilon q^T\left(B_s + G_s\right)\dot{q} \\ &\le (-b_1 + \varepsilon a_2)\|\dot{q}\|^2 - \varepsilon hk_1\|q\|^2 + (p + \varepsilon(b_2 + g))\|q\|\|\dot{q}\|. \end{aligned} \tag{26}$$

By using the estimations (25) and (26), it is easy to show that, for sufficiently large value h and sufficiently small value ε, function (24) is positive definite, while its derivative along the solutions to any system in (23) is negative definite.

For instance, if $\varepsilon = b_1/(2a_2)$, then we have the following condition for h:

$$h > \max \left\{ \frac{b_1^2}{4a_1 k_1}; \frac{(2pa_2 + b_1(b_2 + g))^2}{4a_2 b_1^2 k_1} \right\}.$$

For these values of parameters, there exist positive numbers $\beta_1, \beta_2, \beta_3$ such that for all $q, \dot{q} \in R^n$ the inequalities

$$\beta_1 \left(\|\dot{q}\|^2 + \|q\|^2 \right) \leq V(q, \dot{q}) \leq \beta_2 \left(\|\dot{q}\|^2 + \|q\|^2 \right), \qquad \dot{V}\Big|_{(22)} \leq -\beta_3 \left(\|\dot{q}\|^2 + \|q\|^2 \right)$$

hold. Hence, system (22) is asymptotically stable. □

5.2 Domination of dissipative forces

Now we consider the switched system

$$A\ddot{q} + (hB + G_\sigma)\,\dot{q} + (K_\sigma + P_\sigma)\,q = 0. \tag{27}$$

The corresponding family of systems has the form

$$A\ddot{q} + (hB + G_s)\,\dot{q} + (K_s + P_s)\,q = 0, \qquad s = 1, \ldots, N, \tag{28}$$

where A, B, G_s, K_s, P_s are constant matrices, h is positive parameter. We assume that matrices B and K_s are symmetric, while matrices G_s and P_s are skew-symmetric.

Theorem 5.2 *Let the following conditions be fulfilled:*
(a) Matrices $K_1, \ldots, K_N$ are positive definite;
(b) Matrix B is positive definite;
(c) The value of parameter h is sufficiently large.
Then, for any switching law of all components of force field, with the exception of dissipative component, system (27) *is asymptotically stable.*

Proof After constructing the common Lyapunov function for the family of systems (28) in the form

$$V(q, \dot{q}) = \frac{1}{2}\dot{q}^T A\dot{q} + \frac{h}{2}q^T Bq + q^T A\dot{q},$$

the subsequent proof is similar to that one of Theorem 5.1. □

5.3 System with small nonconservative forces

Next consider the switched system with the small parameter at the nonconservative forces

$$A\ddot{q} + (B_\sigma + G_\sigma)\,\dot{q} + (K + \varepsilon P_\sigma)\,q = 0, \tag{29}$$

and the corresponding family of systems

$$A\ddot{q} + (B_s + G_s)\,\dot{q} + (K + \varepsilon P_s)\,q = 0, \qquad s = 1, \ldots, N, \tag{30}$$

where A, B_s, G_s, K, P_s are constant matrices, ε is small positive parameter. Assume that matrices K and B_s are symmetric, while matrices G_s and P_s are skew-symmetric.

Theorem 5.3 *Let the following conditions be fulfilled:*
(a) Matrices $B_1, \ldots, B_N$ are positive definite;
(b) Matrix K is positive definite;
(c) The value of parameter ε is sufficiently small.
Then, for any switching law of all components of force field, with the exception of potential component, system (29) *is asymptotically stable.*

Proof Construct the Lyapunov function in the form

$$V(q, \dot{q}) = \frac{1}{2}\dot{q}^T A\dot{q} + \frac{1}{2}q^T Kq + \varepsilon q^T A\dot{q}. \tag{31}$$

It is easy to verify that, for sufficiently small values of ε, function (31) is positive definite, and its derivative along the solutions of each system from (30) is negative definite. □

5.4 Domination of gyroscopic forces

Next we consider the switched system in the form

$$A\ddot{q} + (B_\sigma + hG)\,\dot{q} + (K + P)\,q = 0. \tag{32}$$

The corresponding family of systems is

$$A\ddot{q} + (B_s + hG)\,\dot{q} + (K + P)\,q = 0, \qquad s = 1, \ldots, N, \tag{33}$$

where A, B_s, G, K, P are constant matrices, h is positive parameter. We assume that matrices B_s and K are symmetric, while matrices G and P are skew-symmetric, and moreover matrix G is nonsingular.

Theorem 5.4 *Let the following conditions be fulfilled:*
(a) Matrices $B_1, \ldots, B_N$ are positive definite;
(b) The subsystem

$$\dot{y} = -G^{-1}(K + P)y \tag{34}$$

is asymptotically stable;
(c) The value of parameter h is sufficiently large.
Then, for any switching law of dissipative forces, system (32) *is asymptotically stable.*

Proof The asymptotic stability of subsystem (34) implies that for any given symmetric positive definite matrix D, there exists a symmetric positive definite matrix L such that

$$LG^{-1}(K + P) + (K + P)^T \left(G^{-1}\right)^T L = D.$$

Construct the Lyapunov function

$$V(q, \dot{q}) = \frac{1}{2}\dot{q}^T A\dot{q} + \frac{1}{2}q^T Lq - \frac{1}{h}q^T FA\dot{q}, \tag{35}$$

where $F = (K - P - L)G^{-1}$. Differentiating $V(q, \dot{q})$ along the solutions of sth system from the family (33), one gets

$$\dot{V} = -\dot{q}^T B_s\dot{q} - \frac{1}{2h}q^T Dq - \frac{1}{h}\left(\dot{q}^T FA\dot{q} - q^T FB_s\dot{q}\right).$$

Hence, for sufficiently large values of h, (35) is a common Lyapunov function for family (33). □

Remark 5.1 The approach for the Lyapunov function construction, which was used in the proof of Theorem 5.4, is a generalization of that one suggested in [2].

Remark 5.2 Theorem 5.4 is similar to Theorem 4.2. However, switching forces in system (20) are positional ones, while in system (32) they are the dissipative ones.

Remark 5.3 The Lyapunov functions considered in the present section are constructed on the base of elements possessing clear mechanical meaning (kinetic energy, potential, matrices of acting forces).

Remark 5.4 By the use of the Lyapunov functions constructed the estimations for the admissible values of parameters h and ε in the systems investigated can be obtained.

6 Conclusion

Theorems 3.1 and 3.2 about decomposition of linear mechanical systems with a large parameter are very significant for the justification of precession theory of gyroscopic devices. These theorems were proved primarily in [11, 18] on the base of the first Lyapunov method by means of the expansion of the roots of the characteristic equations for systems considered in the series with respect to negative powers of parameter. However, it is not convenient in applying such approach, since in [11, 18] no constructive estimations for the lower bounds of the admissible values of large parameter were given. Furthermore, for mechanical system with switching force fields, negativity of real parts of all characteristic equation roots doesn't guarantee the stability of equilibrium position. This paper presents new proofs of the above theorems, which are based on Lyapunov direct method. The Lyapunov functions found for an auxiliary isolated subsystems are used for constructing the common Lyapunov function, which guarantees the asymptotic stability of the equilibrium position for the mechanical system with switching force fields.

For mechanical systems with two degrees of freedom and with switching linear force fields, Theorems 4.1 and 4.2 permit one to use the necessary and sufficient conditions of the existence of a common quadratic Lyapunov function for family of switched two-dimensional systems [14]. Direct application of the criterion in [14] to the system with two degrees of freedom is impossible since, in this case, dimension of full system is equal to four. Moreover, decomposition allows one to use for switched linear isolated subsystems the commutativity conditions guaranteeing the existence of a common Lyapunov function for them [13]. We note that these conditions can not be used directly for full original system.

In the present paper, linear systems are studied. However, the theorems proved guarantee exponential stability of equilibrium positions. Hence, these theorems determine the asymptotic stability conditions for nonlinear systems by the linear approximation. By the way, the decomposition method can also be utilized for the mechanical systems with essentially nonlinear forces. We will deal with them in our future work. Moreover, the results obtained can be used for the design of stabilizing controls for mechanical systems.

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