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Existence and Uniqueness of Solutions of Strongly Damped Wave Equations with Integral Boundary Conditions

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Abstract: In this work, we consider a strongly damped wave equation with integral boundary conditions. We apply the method of semi-discretization in time, also known as the method of lines, to establish the existence and uniqueness of a weak solution.

Keywords: *method of lines; strongly damped wave equation equation; integral boundary conditions; weak solution.*

Mathematics Subject Classification (2000): 34K30, 34G20, 47H06.

1 Introduction

In this paper, we are concerned with the following strongly damped wave equation involving nonlocal boundary conditions

$$\frac{\partial^2 w}{\partial t^2}(x,t) - \frac{\partial^3 w}{\partial x^2 \partial t}(x,t) - \frac{\partial^2 w}{\partial x^2}(x,t) = g(x,t),\ (x,t) \in (0,1) \times [0,T], \tag{1}$$

with the initial conditions

$$w(x,0) = w_0(x), \quad \frac{\partial w}{\partial t}(x,0) = w_1(x), \quad x \in [0,1], \tag{2}$$

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and the integral boundary conditions

$$\int_0^1 w(x,t)dx = m(t), \quad \int_0^1 xw(x,t)dx = k(t), \quad (x,t) \in (0,1) \times [0,T], \tag{3}$$

where $0 < T < \infty$, the map g is defined from $(0,1) \times [0,T]$ into $\mathbb{R}$. Such type of the equations aries in the motion of mechanical systems. Our aim is to apply the method of semi-discretization in time, also known as the method of lines or Rothe's method, to establish the existence, uniqueness of a weak solution.

In [1] Bahuguna studied a strongly damped wave equation as an abstract differential equation in a Banach space and established the existence and uniqueness of a strong solution with the help of Rothe's method. Beilin [13] has considered the wave equation with an integral condition using the method of separation of variables and Fourier series. Pulkina [14] has dealt with a hyperbolic problem with two integral conditions and has established the existence and uniqueness of the generalized solutions using the fixed point arguments. Bouziani and Merazga [10] have considered the quasilinear wave equation with the two integral boundary conditions and proved the existence and uniqueness of a solution by Rothe's method. The initial work on the nonlocal boundary conditions (integral conditions) has been carried out by Cannon [12]. Subsequently, similar studies have been carried out by Kamynin [16], Ionkin [15] and others.

Recently, the study of an initial boundary value problem with the integral boundary conditions has received considerable attention of researchers. For relevant references with the consideration of the nonlocal boundary conditions we refer to the papers [2, 3, 5, 8, 9, 10, 11] and the references cited in these papers. In these papers authors have used the method of semi-discretization in time and have established the existence and uniqueness of a weak solution. Our analysis is motivated by the works of Bahuguna [1], Bahuguna and Dabas [2, 3, 5] and Bouziani and Merazga [9, 10]. For more references on Rothe's method we refer to the papers [4, 6, 7] and the references cited in these papers.

Using the transformation $u(x,t) = w(x,t) - r(x,t)$ we reduce the nonhomogeneous integral boundary conditions in the problem (1)–(3) into homogeneous boundary conditions. We look for $r(x,t) := \chi(t)x + \xi(t)$, where χ and ξ are to be chosen suitably, with

$$\int_0^1 r(x,t)dx = m(t) \quad \text{and} \quad \int_0^1 xr(x,t)dx = k(t). \tag{4}$$

From (4), we have

$$\frac{1}{2}\chi(t) + \xi(t) = m(t), \tag{5}$$

$$\frac{1}{3}\chi(t) + \frac{1}{2}\xi(t) = k(t). \tag{6}$$

Hence the linear system (5)-(6) is uniquely solvable and $\chi(t)$ and $\xi(t)$ are given by

$$\chi(t) = 12k(t) - 6m(t), \tag{7}$$

$$\xi(t) = 4m(t) - 6k(t). \tag{8}$$

By using this transformation problem (1)–(3) equivalently reduces to the problem

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^3 u}{\partial x^2 \partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t),\ (x,t) \in (0,1) \times [0,T], \tag{9}$$

$$u(x,0) = U_0(x), \quad \frac{\partial u}{\partial t}(x,0) = U_1(x), \quad x \in [0,1], \tag{10}$$

$$\int_0^1 u(x,t)dx = 0, \quad \int_0^1 xu(x,t)dx = 0, \quad (x,t) \in (0,1) \times [0,T], \tag{11}$$

where $f(x,t) = g(x,t) + \frac{\partial^2 r}{\partial t^2}$, $U_0(x) = w_0(x) - r(x,0)$ and $U_1(x) = w_1(x) - \frac{\partial r}{\partial t}(x,0)$. Hence the solution of the problem (1)–(3) will be directly obtained by $w(x,t) = u(x,t) + r(x,t)$.

In the next section we define some function spaces required to establish the existence and uniqueness of weak solution to (9)–(11). The definition of weak solution and assumptions are also stated in this section.

2 Preliminaries

The problem (9)–(11) may be treated as an abstract equation in the real Hilbert space $\mathbf{H} = L^2(0,1)$ of square-integrable functions defined from $(0,1)$ into $\mathbb{R}$ with the inner product and the norm respectively

$$(u,v) = \int_0^1 u(x)v(x)\,dx, \quad \|u\|^2 = \int_0^1 |u(x)|^2\,dx, \quad u,v \in \mathbf{H}.$$

For $k \in \mathbb{N}$, the Sobolev space $\mathbf{H}^k$ is the Hilbert space of all functions $u \in \mathbf{H}$ such that the distributional derivative $u^{(j)} \in \mathbf{H}$ with the inner product and the norm respectively

$$(u,v)_k = \sum_{j=0}^{k} (u^{(j)}, v^{(j)}), \quad \|u\|_k^2 = \sum_{j=0}^{k} \|u^{(j)}\|^2, \quad u,v \in \mathbf{H}^k.$$

We shall incorporate the integral condition (11) with the space itself under consideration by taking $\mathbf{V} \subset \mathbf{H}$ defined by

$$\mathbf{V} = \left\{ u \in \mathbf{H} : \int_0^1 u(x)\,dx = \int_0^1 xu(x)dx = 0 \right\}. \tag{12}$$

$\mathbf{V}$ is a closed subspace of $\mathbf{H}$ and hence is a Hilbert space itself with the inner product $(.,.)$. and the corresponding norm $\|.\|$.

For any Banach space X with the norm $\|.\|_X$ and an interval $I = [a,b]$, $-\infty < a < b < \infty$, we shall denote $C(I;X)$ the space of all continuous functions u from $[a,b]$ into X with the norm

$$\|u\|_{C(I;X)} = \max_{a \le t \le b} \|u(t)\|_X.$$

The space $L^2(I;X)$ consists of all square-Bochner integrable functions (equivalent classes) u such that with the norm

$$\|u\|_{L^2(I;X)}^2 = \int_a^b \|u(t)\|_X^2\,dt.$$

Similarly $L^\infty(I;X)$ is the Banach space of all essentially bounded functions from I into X with the norm

$$\|u\|_{L^\infty(I;X)} = \operatorname*{ess\,sup}_{t\in I} \|u(t)\|_X,$$

and the Banach space $Lip(I;X)$ is the space of all Lipschitz continuous functions from I into X with the norm

$$\|u\|_{Lip(I;X)} = \|u\|_{C(I;X)} + \sup_{t,s\in I;\, t\neq s} \frac{\|u(t)-u(s)\|}{|t-s|}.$$

In addition, to the spaces mentioned above, we need the space $B_2^1(0,1)$ introduced by Merazga and A. Bouziani [9] being the completion of the space $C_0(0,1)$ of all real continuous functions having compact supports in (0,1) with the inner product

$$(u,v)_{B_2^1} = \int_0^1 \left(\int_0^x u(\xi)d\xi\right)\left(\int_0^x u(\xi)d\xi\right)dx.$$

It is clear that $v \in B_2^1(0,1)$ if and only if $\int_0^x v(\xi)d\xi \in L^2(0,1)$ and the corresponding norm $\|u\|^2_{B_2^1} = \int_0^1 \left(\int_0^x u(\xi)d\xi\right)^2 dx$. It follows that the inequality $\|v\|^2_{B_2^1} \leq \frac{1}{2}\|v\|^2$ holds for every $v \in L^2(0,1)$, and the embedding $L^2(0,1) \to B_2^1(0,1)$ is continuous.

Given a function $h : (0,1)\times[a,b] \to \mathbb{R}$ such that for each $t \in [a,b]$, $h(.,t) : [a,b] \to \mathbf{H}$, we may identify it with the function $h : [a,b] \to \mathbf{H}$ given by $h(t)(x) = h(x,t)$. We assume the following conditions:

(A1) The function $f : [0,T]\times H \to H$ satisfies the Lipschitz condition, i.e., there exists a positive constant L_f such that

$$\|f(t)-f(s)\|_{B_2^1} \leq L_f|t-s| \quad \text{for} \quad t,s\in[0,T] \quad \text{and} \quad u,v\in H.$$

(A2) $U_0(x), U_1(x) \in H^2(0,1)$ and $U_0(x), U_1(x) \in V$, i.e.

$$\int_0^1 U_0(x)dx = \int_0^1 xU_0(x)dx = 0, \text{ and } \int_0^1 U_1(x)dx = \int_0^1 xU_1(x)dx = 0.$$

Definition 2.1 By a weak solution of the problem (9)–(11) we mean a function $u : [0,T] \to \mathbf{H}$ such that

1. $u \in Lip([0,T],V)$,
2. u has $a.e.$ in $[0,T]$ a strong derivative $\frac{du}{dt} \in L^\infty([0,T];V)\cap Lip([0,T],B_2^1(0,1))$, and $\frac{d^2u}{dt^2} \in L^\infty([0,T],B_2^1(0,1))$,
3. u satisfying the initial boundary conditions (2) and the integral conditions (11),
4. also the following integral identity is satisfied

$$\left(\frac{d^2u(t)}{dt^2},v\right)_{B_2^1} + \left(\frac{du(t)}{dt},v\right) + (u(t),v) = (f(t),v)_{B_2^1}. \tag{13}$$

for all $v \in L^2([0,T],V)$ and a.e. $t \in [0,T]$.

We have the need of the following lemma due to Sloan and Thomme [17] for latter use.

Lemma 2.1 *Let $\{w_l\}$ be a sequence of nonnegative real numbers satisfying*

$$w_l \leq \alpha_l + \sum_{i=0}^{l-1} \beta_i w_i, \quad l > 0,$$

where $\{\alpha_l\}$ is a nondecreasing sequence of nonnegative real numbers and $\beta_l \geq 0$. Then

$$w_l \leq \alpha_l \exp\{\sum_{i=0}^{l-1} \beta_i\}, \quad l > 0.$$

3 Discretization Scheme and Priori Estimates

In this section we discretized the problem (9)–(11) and established the estimates. We shall prove Theorem 5.1 given in the last section with the help of Lemma 3.2 and 4.2 stated and proved in subsequent sections. For a positive integer n, we consider the discretization

$$[t_{j-1}^n, t_j^n], \quad t_j^n = jh_n, \quad h_n = \frac{T}{n}, \quad j = 0, 1, 2, \ldots, n;$$

of the interval $[0, T]$. We call u^n an approximate solution and set $u_0^n = U_0$,

$$u_{-1}^n(x) = U_0(x) - h_n U_1(x), \tag{14}$$

$$u_{-2}^n(x) = h_n^2 \left[f(0) + \frac{d^2 U_0}{dx^2} + \frac{d^2 U_1}{dx^2} \right] + U_0 - 2h_n U_1, \tag{15}$$

for all $n \in \mathbb{N}$. For $j = 1, 2, \ldots, n$, we define u_j^n the unique solutions of each of the equations

$$\delta^2 u_j^n - \frac{d^2 \delta u_j^n}{dx^2} - \frac{d^2 u_j^n}{dx^2} = f_j^n, \quad x \in (0, 1), \tag{16}$$

$$\int_0^1 u_j^n(x) dx = 0, \tag{17}$$

$$\int_0^1 x u_j^n(x) dx = 0, \tag{18}$$

where

$$\delta u_j^n = \frac{u_j^n - u_{j-1}^n}{h_n}, \quad \delta^2 u_j^n = \frac{\delta u_j^n - \delta u_{j-1}^n}{h_n}, \quad f_j^n = f(t_j^n). \tag{19}$$

The existence of unique $u_j^n \in \mathbf{H^2}$ satisfying $(16) - (18)$ is ensured similarly as established in [8] Lemma 3.1.

Lemma 3.1 *For each $n \in \mathbb{N}$ and each $j = 1, \ldots, n$, the problem (16)–(18) admits a unique solution $u_j \in H^2(0, 1)$.*

Proof For this purpose, we introduce the following functions

$$q_j^n = u_j^n + \delta u_j^n, \quad j = 1, \ldots, n. \tag{20}$$

If we solve this for u_j^n we have

$$u_j^n = \frac{h_n}{1+h_n} q_j^n + \frac{1}{1+h_n} u_{j-1}^n, \quad j = 1, 2, \dots, n.$$

And also

$$\delta u_j^n = \frac{1}{1+h_n}(q_j^n - u_{j-1}^n), \quad \delta^2 u_j^n = \frac{1}{1+h_n}(\delta q_j^n - \delta u_{j-1}^n), \ j = 1, \dots, n. \tag{21}$$

Then the problem (16)–(18) is equivalent to the following problem

$$-\tfrac{d^2 q_j^n}{dx^2} + \tfrac{1}{h_n(1+h_n)} q_j^n = f_j^n + \tfrac{1}{1+h_n}[\delta u_{j-1}^n + \tfrac{1}{h_n} q_{j-1}^n], \quad x \in (0,1), \tag{22}$$

$$\textstyle\int_0^1 q_j^n(x)dx = 0, \quad \int_0^1 x q_j^n(x)dx = 0. \tag{23}$$

For solving the system (22)–(23) we use an idea from [8]. Details are as follows. We first solve the equation (22) with classical Dirichlet boundary conditions

$$q_j^n(0) = \lambda, \quad \text{and} \quad q_j^n(1) = \mu, \tag{24}$$

where (λ, μ) is for the moment an arbitrary fixed ordered pair of real numbers. For $j = 1$, we have

$$F_1 = f_1 + \frac{1}{1+h_n}[\delta u_0^n + \frac{1}{h_n} q_0^n] \in \mathbf{H},$$

the Lax-Milgram Lemma guarantees the existence and uniqueness of a strong solution $q_1^n \in H^2(0,1)$ of the problem (22)–(24). Step by step each q_j is then uniquely determined in terms of $U_0, U_1, q_1^n, \dots, q_{j-1}^n$. Let us show that the parameters λ and μ can be chosen in a way such that the corresponding function $q_j^n(., \lambda, \mu)$ is also a solution of the problem (22)–(23) provided that n is large enough. The function $q_j^n(., \lambda, \mu)$ shall be a solution to problem (22)–(23) if and only if the pair (λ, μ) satisfies

$$\int_0^1 q_j^n(x, \lambda, \mu)dx = 0, \tag{25}$$

$$\int_0^1 x q_j^n(x, \lambda, \mu)dx = 0. \tag{26}$$

Solving (25)–(26) will provide all the solutions to the problem (22)-(23). Let us write $q_j^n(., \lambda, \mu)$ as the sum of two functions

$$q_j^n(x, \lambda, \mu) = q_j^n(x, 0, 0) + \overline{q}_j^n(x, \lambda, \mu),$$

where $q_j^n(x, 0, 0)$ and $\overline{q}_j^n(x, \lambda, \mu)$ are solutions to the following problems, respectively:

$$\begin{cases} -\frac{d^2 q_j^n}{dx^2} + \frac{1}{h_n(1+h_n)} q_j^n = F_j, \\ q_j^n(0) = 0 = q_j^n(1), \end{cases} \quad \begin{cases} -\frac{d^2 \overline{q}_j^n}{dx^2} + \frac{1}{h_n(1+h_n)} \overline{q}_j^n = 0, \quad x \in (0,1), \\ \overline{q}_j^n(0) = \lambda, \ \overline{q}_j^n(1) = \mu. \end{cases} \tag{27}$$

The solution of the second problem is given by

$$\overline{q}_j^n(x) = a_1 e^{px} + a_2 e^{-px}, \quad \text{where} \quad p = \frac{1}{\sqrt{h_n(1+h_n)}}, \tag{28}$$

using the boundary conditions, we find the constants as

$$a_1 = \frac{\mu - \lambda e^{-p}}{e^p - e^{-p}}, \quad a_2 = \frac{\mu - \lambda e^{p}}{e^{-p} - e^p}.$$

Now putting the function $q_j(x, \lambda, \mu)$ in (25)–(26), we have

$$\lambda + \mu = \frac{p \sinh p}{\cosh p - 1} \int_0^1 q_j(x, 0, 0)dx, \tag{29}$$

$$(p - \sinh p)\lambda + (\sinh p - p\cosh p)\mu = p^2 \sinh p \int_0^1 q_j(x, 0, 0)dx. \tag{30}$$

Determinant of the coefficient of the above system is

$$D(h_n) = 2\sinh p - p\cosh p - p.$$

It can be shown that the real function $D(h_n)$ admits a unique real root for all $h_n > 0$, hence the system (16)-(18) which is equivalent to (22)-(23) is uniquely solvable. This completes the proof of the lemma. □

We first obtain the estimates for δu_j^n and difference quotients $\{\frac{\delta u_j^n - \delta u_{j-1}^n}{h_n}\}$ using $(A1)$ and $(A2)$ which in turn imply the uniform bounds of $\{u_j^n\}$. To derive the estimates first we reformulate the discretized problem in the variational form. Let v be any function from the space V and

$$\int_0^x (x - \xi)v(\xi)d\xi = \Im_x^2 v, \quad \forall x \in (0, 1), \tag{31}$$

where

$$\Im_x v = \int_0^x v(\xi)d\xi, \quad \text{and} \quad \Im_x^2 v = \Im_x(\Im_x v) = \int_0^x d\xi \int_0^\xi v(s)ds, \tag{32}$$

with $x = 1$ in (31), for any $v \in V$ we have $\Im_1^2 v = 0$. Now multiplying (16) by $\Im_x^2 v$, $j = 1, 2, \ldots, n$, and integrating over $(0, 1)$, we get

$$\int_0^1 \delta^2 u_j^n(x)\Im_x^2 v\, dx - \int_0^1 \frac{d^2\delta u_j^n}{dx^2}(x)\Im_x^2 v dx - \int_0^1 \frac{d^2 u_j^n}{dx^2}(x)\Im_x^2 v dx$$
$$= \int_0^1 f_j^n(x)\Im_x^2 v dx. \tag{33}$$

Now integrating by parts for each term in (33) we have

$$\begin{aligned}
\int_0^1 \delta^2 u_j^n(x)\Im_x^2 v\, dx &= \int_0^1 \frac{d}{dx}\left(\Im_x(\delta^2 u_j^n)\right)\Im_x^2 v dx \\
&= \Im_x(\delta^2 u_j^n)\Im_x^2 v|_{x=0}^{x=1} - \int_0^1 \Im_x(\delta^2 u_j^n)\Im_x v dx \\
&= -(\delta^2 u_j^n, v)_{B_2^1}.
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \frac{d^2\delta u_j^n}{dx^2}(x)\Im_x^2 v dx &= \frac{d\delta u_j^n}{dx}(x)\Im_x^2 v|_{x=0}^{x=1} - \int_0^1 \frac{d\delta u_j^n}{dx}(x)\Im_x v dx \\
&= -\int_0^1 \frac{d\delta u_j^n}{dx}(x)\Im_x v dx \\
&= -\delta u_j^n(x)\Im_x v|_{x=0}^{x=1} + \int_0^1 \delta u_j^n(x)v\, dx \\
&= (\delta u_j^n, v),
\end{aligned}$$

and

$$\int_0^1 \frac{d^2u_j^n}{dx^2}(x)\Im_x^2 v dx = (u_j, v), \quad \int_0^1 f_j^n \Im_x^2 v \, dx = -(f_j^n, v)_{B_2^1}.$$

Finally, we have the following variational identity

$$(\delta^2 u_j^n, v)_{B_2^1} + (\delta u_j^n, v) + (u_j^n, v) = (f_j^n, v)_{B_2^1}. \tag{34}$$

Throughout, C will represent a generic constant independent of j, h_n and n and CT, Ce^{CT} are again replaced by C.

Lemma 3.2 *Assume that the hypotheses (A1) and (A2) are satisfied. Then there exists a positive constant C, independent of j, h_n and n such that*

$$\|\delta u_j^n\| \leq C, \tag{35}$$

$$\|\delta^2 u_j^n\|_{B_2^1} \leq C, \tag{36}$$

$n \geq 1$ *and* $j = 1, \ldots, n$.

Proof For $2 \leq j \leq n$, putting $v = \delta^2 u_j^n$, in (34) we have

$$(\delta^2 u_j^n - \delta^2 u_{j-1}^n, \delta^2 u_j^n)_{B_2^1} + h_n(\delta^2 u_j^n, \delta^2 u_j^n) + (\delta u_j^n, \delta u_j^n - \delta u_{j-1}^n)$$
$$= (f_j^n - f_{j-1}^n, \delta^2 u_j^n)_{B_2^1}.$$

Using the identity

$$2(u, u-w) = \|u\|^2 - \|w\|^2 + \|u-w\|^2,$$

we obtain

$$\|\delta^2 u_j^n\|_{B_2^1}^2 - \|\delta^2 u_{j-1}^n\|_{B_2^1}^2 + \|\delta^2 u_j^n - \delta^2 u_{j-1}^n\|_{B_2^1}^2 + h_n\|\delta^2 u_j^n\|^2$$
$$+ \|\delta u_j^n\|^2 - \|\delta u_{j-1}^n\|^2 + \|\delta u_j^n - \delta u_{j-1}^n\|^2 = 2(f_j^n - f_{j-1}^n, \delta^2 u_j^n)_{B_2^1}. \tag{37}$$

We neglect the third, forth and the last terms on the left hand side of the equation (37) to get

$$\begin{aligned}\|\delta^2 u_j^n\|_{B_2^1}^2 + \|\delta u_j^n\|^2 &\leq \|\delta^2 u_{j-1}^n\|_{B_2^1}^2 + \|\delta u_{j-1}^n\|^2 + 2(f_j^n - f_{j-1}^n, \delta^2 u_j^n)_{B_2^1} \\ &\leq \|\delta^2 u_{j-1}^n\|_{B_2^1}^2 + \|\delta u_{j-1}^n\|^2 + 2\|f_j^n - f_{j-1}^n\|_{B_2^1}\|\delta^2 u_j^n\|_{B_2^1}.\end{aligned}$$

Repeating the above inequality, we obtain

$$\|\delta^2 u_j^n\|_{B_2^1}^2 + \|\delta u_j^n\|^2 \leq \|\delta^2 u_1^n\|_{B_2^1}^2 + \|\delta u_1^n\|^2 + 2\sum_{i=2}^{j-1} \|f_j^n - f_{j-1}^n\|_{B_2^1}\|\delta^2 u_j^n\|_{B_2^1}.$$

Using the Cauchy inequality $2ab \leq \frac{1}{\epsilon}a^2 + \epsilon b^2,\ a, b \in \mathbb{R},\ \epsilon > 0$, with $\epsilon = h_n$ and using assumption $(A1)$, we have the estimate

$$\begin{aligned}\|\delta^2 u_j^n\|_{B_2^1}^2 + \|\delta u_j^n\|^2 &\leq \|\delta^2 u_1^n\|_{B_2^1}^2 + \|\delta u_1^n\|^2 + \frac{1}{h_n}\sum_{i=2}^{j-1} \|f(t_j^n) - f(t_{j-1}^n)\|_{B_2^1}^2 \\ &\quad + h_n \sum_{i=2}^{j-1} \|\delta^2 u_j^n\|_{B_2^1}^2 \\ &\leq \|\delta^2 u_1^n\|_{B_2^1}^2 + \|\delta u_1^n\|^2 + CT + h_n \sum_{i=0}^{j-1} \|\delta^2 u_j^n\|_{B_2^1}^2.\end{aligned} \tag{38}$$

From Lemma 2.1 we get the estimate

$$\|\delta^2 u_j^n\|^2_{B_2^1} + \|\delta u_j^n\|^2 \leq \Big[\|\delta^2 u_1^n\|^2_{B_2^1} + \|\delta u_1^n\|^2 + CT\Big] \exp\{(j-1)h_n\}. \tag{39}$$

To estimate the right hand side in (39), we use the variational identity (34) for $j = 1$ and $v = \delta^2 u_1^n = \frac{\delta u_1^n - U_1}{h_n}$, to obtain

$$(\delta^2 u_1^n, \delta^2 u_1^n)_{B_2^1} + (\delta u_1^n, \delta^2 u_1^n) + \left(u_1, \frac{\delta u_1^n - U_1}{h_n}\right) = (f_1^n, \delta^2 u_1^n)_{B_2^1}. \tag{40}$$

Rearranging the terms, we get

$$\begin{aligned}\|\delta^2 u_1^n\|^2_{B_2^1} + h_n\|\delta^2 u_1^n\|^2 + (\delta u_1^n, \delta u_1^n - U_1) \;=\; & (f_1^n, \delta^2 u_1^n)_{B_2^1} - (\delta u_0, \delta^2 u_1^n) \\ & -(U_0, \delta^2 u_1^n).\end{aligned} \tag{41}$$

Again by using the equality $2(u, u-w) = \|u\|^2 - \|w\|^2 + \|u-w\|^2$, we have

$$\begin{aligned}&\|\delta^2 u_1^n\|^2_{B_2^1} + h_n\|\delta^2 u_1^n\|^2 + \tfrac{1}{2}\{\|\delta u_1^n\|^2 + \|\delta u_1^n - U_1\|^2 - \|U_1\|^2\} \\ &\quad = (f_1^n, \delta^2 u_1^n)_{B_2^1} - (U_1, \delta^2 u_1^n) - (U_0, \delta^2 u_1^n).\end{aligned} \tag{42}$$

The second term on the right hand side of (42) gives us

$$\begin{aligned}(U_1, \delta^2 u_1^n) &= \int_0^1 U_1(x) \frac{d}{dx}(\Im_x \delta^2 u_1^n) dx \\ &= U_1(x)\Im_x \delta^2 u_1|_{x=0}^{x=1} - \int_0^1 \frac{dU_1}{dx}(x)\Im_x \delta u_1^n dx \\ &= -\int_0^1 \frac{dU_1}{dx}(x)\Im_x \delta u_1^n dx.\end{aligned} \tag{43}$$

Using $\Im_x(\frac{d^2U_1}{dx^2}) = \frac{dU_1}{dx}(x) - \frac{dU_1}{dx}(0)$, for all $x \in (0,1)$, in equation (43) we obtain

$$(U_1, \delta^2 u_1^n) = -\left(\frac{d^2U_1}{dx^2}, \delta^2 u_1^n\right)_{B_2^1}.$$

Similarly, we may write

$$(U_0, \delta^2 u_1^n) = -\left(\frac{d^2U_0}{dx^2}, \delta^2 u_1^n\right)_{B_2^1}.$$

From (42), we obtain

$$\begin{aligned}&2\|\delta^2 u_1^n\|^2_{B_2^1} + \|\delta u_1^n\|^2 \\ &\leq \|U_1\|^2 + 2\left[\left\|f_1^n + \frac{d^2U_1}{dx^2} + \frac{d^2U_0}{dx^2}\right\|_{B_2^1}\right]\|\delta^2 u_1^n\|_{B_2^1} \\ &\leq \|U_1\|^2 + \left\|f_1^n + \frac{d^2U_1}{dx^2} + \frac{d^2U_0}{dx^2}\right\|^2_{B_2^1} + \|\delta^2 u_1^n\|^2_{B_2^1}.\end{aligned} \tag{44}$$

Thus, from (44), we have

$$\|\delta^2 u_1^n\|_{B_2^1}^2 + \|\delta u_1^n\|^2 \quad \leq \quad \|U_1\|^2 + \left\| f_1^n + \frac{d^2U_1}{dx^2} + \frac{d^2U_0}{dx^2}\right\|_{B_2^1}^2 = C_1. \tag{45}$$

Finally we estimate (39) as

$$\|\delta^2 u_j^n\|_{B_2^1}^2 + \|\delta u_j^n\|^2 \leq C_1 \exp\{CT\}. \tag{46}$$

This completes the proof of the lemma. □

Remark 3.1 Estimates of Lemma 3.2 imply that for all n and $j = 1, 2, \dots, n$, $\|u_j^n\| \leq C$.

Definition 3.1 We define Rothe's sequence $\{U^n\}$ and $\{V^n\}$ of functions from $[0,T]$ into $\mathbf{H}^2 \cap \mathbf{V}$, given by

$$\begin{aligned} U^n(t) &= u_{j-1}^n + (t - t_{j-1}^n)\delta u_j^n, \quad t \in [t_{j-1}^n, t_j^n], \quad j = 1, 2, \dots, n, \\ V^n(t) &= \delta u_{j-1}^n + (t - t_{j-1}^n)\delta^2 u_j^n, \quad t \in [t_{j-1}^n, t_j^n], \quad j = 1, 2, \dots, n. \end{aligned}$$

Furthermore, we define another set of sequences $\{X^n\}$, $\{Y^n\}$ and $\{\tilde{Y}^n\}$ of step functions given by

$$\begin{aligned} X^n(t) &= U_0, \quad t \in (-h_n, 0], \quad X^n(t) = u_j^n, \quad t \in (t_{j-1}^n, t_j^n], \\ Y^n(t) &= U_1, \quad t \in (-h_n, 0], \quad Y^n(t) = \delta u_j^n, \quad t \in (t_{j-1}^n, t_j^n], \\ \tilde{Y}^n(t) &= \delta^2 u_1^n, \quad t = 0, \qquad \tilde{Y}^n(t) = \delta^2 u_j^n, \quad t \in (t_{j-1}^n, t_j^n]. \end{aligned}$$

for $j = 1, 2, \dots, n$.

Remark 3.2 From Lemma 3.2 it follows that

1. The functions $\{U^n(t)\}$ and $\{V^n(t)\}$ are Lipschitz continuous on $[0,T]$ with uniform Lipschitz constant C, i.e.

$$\|U^n(t) - U^n(s)\| \leq C|t-s|, \quad \|V^n(t) - V^n(s)\|_{B_2^1} \leq C|t-s|.$$

2. The sequences $\{U^n(t)\}, \{X^n(t)\}$ are bounded in the space $L^2([0,T];V)$ and the sequences $\{V^n(t)\}, \{Y^n(t)\}$ are bounded in the space $L^2([0,T];B_2^1(0,1))$ uniformly for all $t \in [0,T]$ and $n \in \mathbb{N}$. Also we have

$$\left\|\frac{dU^n}{dt}(t)\right\| \leq C, \quad \left\|\frac{dV^n}{dt}(t)\right\|_{B_2^1} \leq C.$$

3. The sequence $X^n(t) - U^n(t)$, $U^n(t) - X^n(t-h_n)$ and $Y^n(t) - Y^n(t-h_n) \to 0$ in $L^2([0,T],V)$ as $n \to \infty$. Also the sequence $Y^n(t) - V^n(t) \to 0$ in $L^2([0,T], B_2^1(0,1))$ as $n \to \infty$. These results follow due to the following inequalities

$$\begin{aligned} &\left\|V^n(t) - \frac{dU^n}{dt}(t)\right\|_{B_2^1} \leq Ch_n, \\ &\|X^n(t) - U^n(t)\| \leq \frac{C}{n}, \quad \text{and} \quad \|U^n(t) - X^n(t-h_n)\| \leq \frac{C}{n}, \\ &\|Y^n(t) - V^n(t)\|_{B_2^1} \leq Ch_n, \text{ and } \|Y^n(t) - Y^n(t-h_n)\| \leq Ch_n. \end{aligned}$$

4 Convergence and Existence Result

In this section we establish the existence and uniqueness of a weak solution to (9)–(11).

Lemma 4.1 *There exist two functions* $u \in L^2([0,T];V) \cap L^\infty([0,T];V)$ *with* $u' \in L^2([0,T];B_2^1(0,1)) \cap L^\infty([0,T];B_2^1(0,1))$ *and* $w \in L^2([0,T];B_2^1(0,1)) \cap L^\infty([0,T];B_2^1(0,1))$ *with* $w' \in L^2([0,T];B_2^1(0,1))$ *such that*

$$U^{n_p} \rightharpoonup u \quad in \quad L^2([0,T];V), \tag{47}$$

$$V^{n_p} \rightharpoonup w, \quad in \quad L^2([0,T];B_2^1(0,1)). \tag{48}$$

Furthermore, we have that

$$\frac{dU^{n_p}}{dt} \rightharpoonup \frac{du}{dt} \quad in \quad L^2([0,T];B_2^1(0,1)), \tag{49}$$

$$\frac{du}{dt} = w \quad on \quad [0,T] \quad and \quad \frac{d^2u}{dt^2} = \frac{dw}{dt} \quad a.e. \quad on \quad [0,T], \tag{50}$$

where " $\rightharpoonup$ *" stands for the weak convergence.*

Proof Remark 3.2 we know that the sequences $\{X^n\}$ and $\{Y^n\}$ are bounded in $L^2([0,T];V)$, while the sequence $\{\tilde{Y}^n\}$ is bounded in $L^2([0,T];B_2^1(0,1))$. It follows that subsequences $\{X^{n_p}\}$, $\{Y^{n_p}\}$ and $\{\tilde{Y}^{n_p}\}$ can be found such that

$$\begin{array}{ccll} X^{n_p} & \rightharpoonup & u & \text{in} \quad L^2([0,T];V), \\ Y^{n_p} & \rightharpoonup & w & \text{in} \quad L^2([0,T];B_2^1(0,1)), \\ \tilde{Y}^{n_p} & \rightharpoonup & \tilde{w} & \text{in} \quad L^2([0,T];B_2^1(0,1)). \end{array}$$

Similarly as in the preceding chapters, one finds that

$$\begin{array}{ccll} U^{n_p} & \rightharpoonup & u & \text{in} \quad L^2([0,T];V), \\ \dfrac{dU^{n_p}}{dt} & \rightharpoonup & \dfrac{du}{dt} & \text{in} \quad L^2([0,T];V), \\ V^{n_p} & \rightharpoonup & w & \text{in} \quad L^2([0,T];B_2^1(0,1)). \end{array}$$

Now, we show that $w = \frac{du}{dt}$. For all $v \in L^2([0,T],B_2^1(0,1))$, we have

$$\begin{aligned} &\left(V^{n_p} - \frac{du}{dt}, v\right)_{L^2([0,T];B_2^1(0,1))} \\ &= \left(V^{n_p} - \frac{dU^{n_p}}{dt}, v\right)_{L^2([0,T];B_2^1(0,1))} + \left(\frac{dU^{n_p}}{dt} - \frac{du}{dt}, v\right)_{L^2([0,T];B_2^1(0,1))} \\ &\leq \left\| V^{n_p} - \frac{dU^{n_p}}{dt} \right\|_{L^2([0,T];B_2^1(0,1))} \|v\|_{L^2([0,T];B_2^1(0,1))} \\ &+ \left(\frac{dU^{n_p}}{dt} - \frac{du}{dt}, v\right)_{L^2([0,T];B_2^1(0,1))}. \end{aligned} \tag{51}$$

From Remark 3.2 and $\frac{dU^{n_p}}{dt} \rightharpoonup \frac{du}{dt}$ in $L^2([0,T];V)$, we have

$$\begin{aligned} \left(V^{n_p} - \frac{du}{dt}, v\right)_{L^2([0,T];B_2^1(0,1))} &\leq Ch_n \|v\|_{L^2([0,T];B_2^1(0,1))} \\ &+ \left(\frac{dU^{n_p}}{dt} - \frac{du}{dt}, v\right)_{L^2([0,T];B_2^1(0,1))}. \end{aligned} \tag{52}$$

Hence we conclude that as $p \to \infty$,

$$V^{n_p} \rightharpoonup \frac{du}{dt} \quad \text{in} \quad L^2([0,T]; B_2^1(0,1)).$$

Since $V^{n_p} \rightharpoonup w$ as $p \to \infty$, we have $w = \frac{du}{dt}$ and also $\tilde{w} = \frac{dw}{dt} = \frac{d^2u}{dt^2}$. This can also be achieved by another way by considering the following equalities

$$U^{n_p}(t) - U_0 = \int_0^t Y^{n_p}(s)ds \quad \text{in} \quad L^2([0,T]; V), \tag{53}$$

$$V^{n_p}(t) - U_1 = \int_0^t \tilde{Y}^{n_p}(s)ds \quad \text{in} \quad L^2([0,T]; B_2^1(0,1)). \tag{54}$$

The above equalities can be ensured directly from the construction of U^n, V^n, Y^n and $\tilde{Y}^n$. It follows due to the above convergence result that

$$u(t) - U_0 = \int_0^t w(s)ds \quad \text{in} \quad L^2([0,T]; V), \tag{55}$$

$$w(t) - U_1 = \int_0^t \tilde{w}(s)ds \quad \text{in} \quad L^2([0,T]; B_2^1(0,1)), \tag{56}$$

which imply that $u \in C([0,T]; V)$ and strongly differentiable a.e. in $[0,T]$ with $w = \frac{du}{dt}$ and also $\tilde{w} = \frac{dw}{dt} = \frac{d^2u}{dt^2}$. Now we show that $u \in L^\infty([0,T]; V)$ and $u', w \in L^\infty([0,T]; B_2^1(0,1))$. The estimate $\|u_j^n\| \le C$, implies that the Rothe' sequence $\{U^n\}$ is bounded in $L^\infty([0,T]; V)$. Hence a subsequence $\{U^{n_k}\}$ of $\{U^n\}$ can be found converging weakly to a function $z \in L^\infty([0,T], V)$, which is easily shown to be equal to the function u. The second assertion $u', w \in L^\infty([0,T]; B_2^1(0,1))$ is obtained similarly. This completes the proof of the lemma. □

Thus, from Lemma 4.1 we conclude the following:

$$\begin{aligned}
&u \in AC([0,T]; V),\\
&u' \in L^2([0,T]; V) \cap AC([0,T]; B_2^1(0,1)),\\
&u'' \in L^2([0,T]; B_2^1(0,1)),\\
&u(0) = U_0 \quad \text{and} \quad u'(0) = U_1 \quad \text{in} \quad C([0,T); B_2^1(0,1)),
\end{aligned}$$

where $AC([0,T]; V)$ denotes a space of all absolutely continuous functions from $[0,T]$ into V.

For the notational convenience, let

$$f^n(0) = f_0, \quad f^n(t) = f(t_j^n), \quad t \in (t_{j-1}^n, t_j^n], \quad 1 \le j \le n.$$

Then (34) may be rewritten as

$$\left(\frac{dV^n}{dt}(t), v\right)_{B_2^1} + (Y^n(t), v) + (X^n(t - h_n), v) = (f^n(t), v)_{B_2^1}, \tag{57}$$

for all $v \in \mathbf{V}$ and a.e. $t \in (0,T]$.

Lemma 4.2 *There exist $u \in C([0,T]; V)$ and $w \in C([0,T]; B_2^1(0,1))$ such that*

$$\|U^n - u\|_{C([0,T];V)} \to 0 \quad \text{and} \quad \|V^n - w\|_{C([0,T];B_2^1(0,1))} \to 0, \tag{58}$$

as $n \to \infty$. Moreover u and w are Lipschitz continuous on $[0,T]$.

Proof For $m > n > n_0$, we consider the Rothe functions U^n and U^m corresponding to the step lengths $h_n = \frac{T}{n}$ and $h_m = \frac{T}{m}$. From (57) taking $v = Y^n(t) - Y^m(t)$, we have

$$\begin{aligned}&\left(\frac{d}{dt}(V^n(t) - V^m(t)), Y^n(t) - Y^m(t)\right)_{B_2^1}\\&+ (Y^n(t) - Y^m(t), Y^n(t) - Y^m(t))\\&+ (X^n(t - h_n) - X^m(t - h_m), Y^n(t) - Y^m(t))\\&= (f^n(t) - f^m(t), Y^n(t) - Y^m(t))_{B_2^1}.\end{aligned} \tag{59}$$

Now the first term of the left hand side in (59) can be written as

$$\begin{aligned}&\left(\frac{d}{dt}(V^n(t) - V^m(t)), Y^n(t) - Y^m(t)\right)_{B_2^1}\\&= \left(\frac{d}{dt}(V^n(t) - V^m(t)), Y^n(t) - V^n(t) + V^m(t) - Y^m(t)\right)_{B_2^1}\\&+ \left(\frac{d}{dt}(V^n(t) - V^m(t)), V^n(t) - V^m(t)\right)_{B_2^1}.\end{aligned} \tag{60}$$

Similarly we may write the third term of (59) as

$$\begin{aligned}&(X^n(t - h_n) - X^m(t - h_m), Y^n(t) - Y^m(t))\\&= (X^n(t - h_n) - U^n(t) + U^m(t) - X^m(t - h_m), Y^n(t) - Y^m(t))\\&+ (U^n(t) - U^m(t), Y^n(t) - Y^m(t)).\end{aligned} \tag{61}$$

Combining the equations (60)–(61) and using the fact that $Y^n(t) = \frac{dU^n}{dt}(t)$, equation (59) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|V^n(t) - V^m(t)\|^2_{B_2^1} + \frac{1}{2}\frac{d}{dt}\|U^n(t) - U^m(t)\|^2 + \|Y^n(t) - Y^m(t)\|^2\\&= \left(\frac{d}{dt}(V^n(t) - V^m(t)), V^n(t) - Y^n(t) + Y^m(t) - V^m(t)\right)_{B_2^1}\\&+ (X^n(t - h_n) - U^n(t) + U^m(t) - X^m(t - h_m), Y^m(t) - Y^n(t))\\&+ (f^n(t) - f^m(t), Y^n(t) - Y^m(t))_{B_2^1}.\end{aligned} \tag{62}$$

The first term on the right hand side of (62) is estimated as

$$\begin{aligned}&\left(\frac{d}{dt}(V^n(t) - V^m(t)), V^n(t) - Y^n(t) + Y^m(t) - V^m(t)\right)_{B_2^1}\\&\leq \left[\left\|\frac{dV^n(t)}{dt}\right\|_{B_2^1} + \left\|\frac{dV^m(t)}{dt}\right\|_{B_2^1}\right]\left[\|V^n(t) - Y^n(t)\|_{B_2^1} + \|Y^m(t) - V^m(t)\|_{B_2^1}\right]\\&\leq C(h_n + h_m).\end{aligned} \tag{63}$$

Similarly, we have

$$\begin{aligned}&(X^n(t - h_n) - U^n(t) + U^m(t) - X^m(t - h_m), Y^m(t) - Y^n(t))\\&\leq [\|X^n(t - h_n) - U^n(t)\| + \|U^m(t) - X^m(t - h_m)\|]\,[\|Y^m(t)\| + \|Y^n(t)\|]\\&\leq C(h_n + h_m).\end{aligned} \tag{64}$$

The last term in (62) is estimated as

$$\begin{aligned}
&(f^n(t)-f^m(t), Y^n(t)-Y^m(t))_{B_2^1}\\
&\leq \|f^n(t)-f^m(t)\|_{B_2^1}\|Y^n(t)-Y^m(t)\|_{B_2^1}\\
&\leq \frac{1}{2}\|f^n(t)-f^m(t)\|^2_{B_2^1}+\frac{1}{2}\|Y^n(t)-Y^m(t)\|^2_{B_2^1}\\
&\leq \epsilon_{nm}+\frac{1}{2}\|V^n(t)-V^m(t)\|^2_{B_2^1}, \qquad (65)
\end{aligned}$$

where

$$\epsilon_{nm}=C(h_n+h_m)+C(h_n+h_m)^2+C(h_n+h_m)\|V^n(t)-V^m(t)\|_{B_2^1},$$

is a sequence of real numbers tending to zero as $n,m\to\infty$. Now using (63)–(64) and (65), (62) becomes

$$\begin{aligned}
&\frac{d}{dt}\|V^n(t)-V^m(t)\|^2_{B_2^1}+\frac{d}{dt}\|U^n(t)-U^m(t)\|^2\\
&=\epsilon^1_{nm}+\|V^n(t)-V^m(t)\|^2_{B_2^1}+\|U^n(t)-U^m(t)\|^2, \qquad (66)
\end{aligned}$$

where ϵ^1_{nm} is another sequence of numbers tending to zero as $n,m\to\infty$. Integrating the last inequality over $(0,t)$ and using $U^n(0)=U^m(0)=U_0,\ V^n(0)=V^m(0)=U_1$, we have

$$\begin{aligned}
&\|V^n(t)-V^m(t)\|^2_{B_2^1}+\|U^n(t)-U^m(t)\|^2\\
&=\epsilon^1_{nm}T+\int_0^t\|V^n(s)-V^m(s)\|^2_{B_2^1}ds+\int_0^t\|U^n(s)-U^m(s)\|^2ds. \qquad (67)
\end{aligned}$$

Application of Gronwall's inequality implies that

$$\|V^n(t)-V^m(t)\|^2_{B_2^1}+\|U^n(t)-U^m(t)\|^2\leq(\epsilon^1_{nm}T)\exp\{T\}. \qquad (68)$$

Taking the supremum over $t\in[0,T]$ we conclude that there exist functions $u\in C([0,T];V)$ and $w\in C([0,T];B_2^1(0,1))$ such that $U^n\to u$ and $V^n\to w$ as $n\to\infty$. By Remark 3.2 it follows that u, and w are Lipschitz continuous functions. This completes the proof of the lemma. □

5 Main Result

In this section we conclude our main result. We summarize the result so far obtained by previous Lemmas 4.1 and 4.2 in Remark 5.1 below.

Remark 5.1 By Remark 3.2 and Lemma 4.2, we conclude the following:

1. $u\in L^2([0,T];V)\cap Lip([0,T];V)$;
2. u is strongly differentiable $a.e.$ in $[0,T]$ and $\frac{du}{dt}\in L^\infty([0,T];V)$;
3. $X^n(t)\rightharpoonup u(t)$ in V for all $t\in[0,T]$; and $\frac{dU^n}{dt}\rightharpoonup\frac{du}{dt}$ in $L^2([0,T];V)$;
4. $w\in Lip([0,T];B_2^1(0,1))$; w is strongly differentiable $a.e$ in $[0,T]$ and $\frac{dw}{dt}\in L^\infty([0,T];B_2^1(0,1))$;

5. $Y^n(t) \rightharpoonup w(t)$ in V for all $t \in [0,T]$; and $\frac{dV^n}{dt} \rightharpoonup \frac{dw}{dt}$ in $L^2([0,T]; B_2^1(0,1))$.

Thus, by the definition of weak solution stated in Definition 2.1 the function $u(t)$ possesses several characteristic properties. Since $u \in L^2([0,T_0]; V)$ we have for almost all $t \in [0,T],\ u \in V$. Hence the integral boundary conditions (11) are satisfied. The initial condition is fulfilled in the sense of the equations (55) and (56). Now the question is in what sense the given differential equation equation (9) is satisfied. The answer to this question lies in the proof of the main theorem of this article.

Theorem 5.1 *Suppose that the conditions* $(A1)$ *and* $(A2)$ *are satisfied. Then problem (9)–(11) has a unique weak solution on* $[0,T]$. *For the sets of data* (U_0^i, U_1^i, f^i), *the corresponding solutions* u^i, $i = 1, 2$, *satisfy the following estimate*

$$\left\| \frac{d}{dt}(u^1(t) - u^2(t)) \right\|_{B_2^1}^2 + \left\| u^1(t) - u^2(t) \right\|^2$$
$$\leq \left(\|U_1^1 - U_1^2\|_{B_2^1}^2 + \|U_0^1 - U_0^2\|^2 + \int_0^t \|f^1(s) - f^2(s)\|_{B_2^1}^2 ds \right) \exp\{t\} \tag{69}$$

which shows the continuous dependence of the solutions on the data.

Proof Now we prove the existence on $[0,T]$. Integrating the identity (57) over $(0,t) \subset [0,T]$ and invoking the fact that $V^n(0) = U_1$, we have

$$(V^n(t) - U_1, v)_{B_2^1} + \int_0^t (Y^n(s), v)\, ds + \int_0^t (X^n(s), v)\, ds$$
$$= \int_0^t (f^n(s), v)_{B_2^1}\, ds. \tag{70}$$

Since $V^n(t) \rightharpoonup \frac{du(t)}{dt}$ in $\mathbf{V}$ for all $t \in [0,T]$, we have

$$(V^n(t) - U_1, v)_{B_2^1} \to \left(\frac{du(t)}{dt} - U_1, v \right)_{B_2^1}, \quad \text{as} \quad n \to \infty. \tag{71}$$

The linear functionals $(Y^n(s), v)$ and $(X^n(s), v)$ are bounded on $\mathbf{V}$, hence by the bounded convergence theorem as $n \to \infty$,

$$\int_0^t (Y^n(s), v) ds \to \int_0^t \left(\frac{du(s)}{dt}, v \right) ds, \quad \forall\, t \in [0,T], \tag{72}$$

$$\int_0^t (X^n(s), v) ds \to \int_0^t (u(s), v) ds, \quad \forall\, t \in [0,T]. \tag{73}$$

Assumption (A1) implies that $\|f^n(s) - f(s)\|_{B_2^1} \leq \frac{C}{n}$ a.e. in $[0,T]$. Hence

$$\|f^n(s) - f(s)\|_{L^2([0,T]; B_2^1(0,1))} \to 0 \quad \text{as} \quad n \to \infty. \tag{74}$$

This implies that $f^n(s) \to f(s)$ in $L^2([0,T]; B_2^1(0,1))$ as $n \to \infty$. Now, by taking into account the convergence result (71)–(74) and passing to the limit as $n \to \infty$, in (70) we have

$$\left(\frac{du}{dt}(t) - U_1, v \right)_{B_2^1} + \int_0^t \left(\frac{du}{dt}(s), v \right) ds + \int_0^t (u(s), v) ds = \int_0^t (f(s), v)_{B_2^1} ds,$$

for all $v \in \mathbf{V}$ and $t \in [0,T]$. Differentiating the above identity we get the desired result,

$$\left(\frac{d^2u}{dt^2}(t), v\right)_{B_2^1} + \left(\frac{du}{dt}(t), v\right) + (u(t), v) = (f(t), v)_{B_2^1}.$$

Uniqueness: Let u_1 and u_2 be two such solutions of (9)-(11). Let we denote the difference of these two solutions by $u(t) = u_1(t) - u_2(t)$, Then from (13), by taking $v = \frac{du(t)}{dt}$, we have

$$\left(\frac{d^2u(t)}{dt^2}, \frac{du(t)}{dt}\right)_{B_2^1} + \left\|\frac{du(t)}{dt}\right\|^2 + \left(u(t), \frac{du(t)}{dt}\right) = 0. \tag{75}$$

Since

$$\left(\frac{d^2u(t)}{dt^2}, \frac{du(t)}{dt}\right)_{B_2^1} = \frac{1}{2}\frac{d}{dt}\left\|\frac{du(t)}{dt}\right\|^2_{B_2^1} \quad \text{and} \quad \left(u(t), \frac{du(t)}{dt}\right) = \frac{1}{2}\frac{d}{dt}\|u(t)\|^2.$$

Then, (75) is written as

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{du(t)}{dt}\right\|^2_{B_2^1} + \left\|\frac{du(t)}{dt}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 = 0. \tag{76}$$

Integrating over $(0,s)$ for $0 \le s \le t \le T$ and using the fact that $u(0) \equiv 0$ and $\frac{du(0)}{dt} = 0$, we get

$$\left\|\frac{du(t)}{dt}\right\|^2_{B_2^1} + \int_0^t \left\|\frac{du(s)}{ds}\right\|^2 ds + \|u(t)\|^2 = 0,$$

consequently

$$\left\|\frac{du(t)}{dt}\right\|^2_{B_2^1} + \|u(t)\|^2 \le 0.$$

Application of the Gronwall's inequality implies that $u \equiv 0$ on $[0,T]$.

Continuous dependence: let u^1 and u^2 be two weak solutions of the problem (9)–(11), corresponding to (U_0^1, U_1^1, f^1) and (U_0^2, U_1^2, f^2), respectively and the initial data satisfy the assumptions (A1) and (A2), from (13), putting $v = \frac{d}{dt}(u^1(t) - u^2(t))$, we have

$$\begin{aligned}&\left(\frac{d^2}{dt^2}(u^1(t) - u^2(t)), \frac{d}{dt}(u^1(t) - u^2(t))\right)_{B_2^1} + \left\|\frac{d}{dt}(u^1(t) - u^2(t)\right\|^2 \\ &+ \left(u^1(t) - u^2(t), \frac{d}{dt}(u^1(t) - u^2(t)\right) = \left(f^1(t) - f^2(t), \frac{d}{dt}(u^1(t) - u^2(t)\right)_{B_2^1}.\end{aligned}$$

Similarly, as in the uniqueness we may drop the middle term, we get

$$\begin{aligned}&\frac{d}{dt}\left\|\frac{d}{dt}(u^1(t) - u^2(t))\right\|^2_{B_2^1} + \frac{d}{dt}\|u^1(t) - u^2(t)\|^2 \\ &\le 2\|f^1(t) - f^2(t)\|_{B_2^1}\left\|\frac{d}{dt}(u^1(t) - u^2(t))\right\|_{B_2^1} \\ &\le \|f^1(t) - f^2(t)\|^2_{B_2^1} + \left\|\frac{d}{dt}(u^1(t) - u^2(t))\right\|^2_{B_2^1}.\end{aligned}$$

Integrating over $(0,s)$ for $0 \le s \le t \le T$ and using the fact that $u^i(0) = U_0^i$ and $du^i(0)/dt = U_1^i$, for $i = 1, 2$, we get

$$\begin{aligned}&\left\|\frac{d}{dt}(u^1(t) - u^2(t))\right\|^2_{B_2^1} + \|u^1(t) - u^2(t)\|^2\\ &\le \|U_1^1 - U_1^2\|^2_{B_2^1} + \|U_0^1 - U_0^2\|^2 + \int_0^t \|f^1(s) - f^2(s)\|^2_{B_2^1} ds\\ &+ \int_0^t \left\|\frac{d}{dt}(u^1(s) - u^2(s))\right\|^2_{B_2^1} ds + \int_0^t \|u^1(s) - u^2(s)\|^2 ds.\end{aligned}$$

Application of the Gronwall inequality leads to the estimate

$$\begin{aligned}&\left\|\frac{d}{dt}(u^1(t) - u^2(t))\right\|^2_{B_2^1} + \|u^1(t) - u^2(t)\|^2\\ &\le \{\|U_1^1 - U_1^2\|^2_{B_2^1} + \|U_0^1 - U_0^2\|^2 + \int_0^t \|f^1(s) - f^2(s)\|^2_{B_2^1} ds\} \exp\{t\}.\end{aligned}$$

This completes the proof of the theorem. □

6 Application

Example 6.1 In this example we consider the following problem

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^3 u}{\partial t \partial x^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = \sin x \cos t,\ (x,t) \in (0,\pi) \times [0,T], \quad (77)$$

$$u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = \sin x, \quad x \in (0,\pi), \quad (78)$$

$$\int_0^\pi u(x,t)dx = 2\sin t,\ \int_0^\pi x u(x,t)dx = \pi \sin t, \quad t \in [0,T]. \quad (79)$$

We notice that $u = \sin x \sin t$ is an exact solution of the above problem. The results of the earlier sections may be used to ensure the well-posedness of this model. We shall be dealing with the problem involving the Neumann condition together with nonlocal integral conditions of first kind in our subsequent study.

References

[1] Bahuguna, D. Rothe's method to strongly damped wave equations. *Acta Appl. Math.* **38** (1995) 185–196.

[2] Bahuguna, D. and Dabas, J. Existence and uniqueness of solution to a semilinear partial delay differential equation with an integral condition. *Nonlinear Dynamics and System Theory* **8** (1) (2008) 7–19.

[3] Dabas, J. and Bahuguna, D. An integro-differential equation with an integral boundary condition. *Mathematical and Computer Modelling* **50** (2009) 123–131.

[4] Bahuguna, D., Dabas, J. and Shukla, R.K. Method of lines to Hyperbolic integro-differential Equations. *Nonlinear Dynamics and System Theory* **8** (4) (2008) 317–328.

[5] Bahuguna, D., Abbas, S.and Dabas,J. Partial functional differential equation with an integral condition and applications to population dynamics. *Nonlinear Analysis, TMA* **69** (2008) 2623–2635.

[6] Bahuguna, D. and Raghavendra, V. Application of Rothe's method to nonlinear integro-differential equations in Hilbert spaces. *Nonlinear Analysis, TMA.* **23** (1) (1994) 75–81.

[7] Kacur, J. Method of Rothe in evolution equations. *Teubner-Texte zur Mathematik*, vol.80, BSBB. G. Teubner Verlagsgesellschaft, Leipzig, 1985.

[8] Merazga, N. and Bouziani, A. On a time-discretization method for a semilinear heat equation with purly integral conditions in a nonclassical function space. *Nonlinear Analysis, TMA* **66** (2007) 604–623.

[9] Merazga, D. and Bouziani, A. Rothe time-discretization method for a nonlocal problem arising in thermoelasticity. *J. Appl. Math. Stoch. Anal.* **1** (2005) 13–28.

[10] Bouziani, A. and Merazga, N. Rothe time-discretization method applied to a quasilinear wave equation subject to integral conditions. *Adv. Difference Equ.* **3** (2004) 211–235.

[11] Merazga, N. and Bouziani, A. Rothe method for a mixed problem with an integral condition for the two dimensional diffusion equation. *Abstract and applied analysis.* **16** (2003) 899–922.

[12] Cannon, J.R. The solution of the heat equation subject to the specification of energy. *Quart. Appl. Math.* **21** (1963) 155–160.

[13] Belin, S.A. Existence of solutions for one dimensional wave equations with nonlocal conditions. *Electronic Journal of Differential Equations* **76** (2001) 1–8.

[14] Pulkina, L.S. A non-local problem with integral conditions for hyperbolic equations. *Electronic Journal of Differential Equations* **45** (1999) 1–6.

[15] Ionkin, N.I. Solutions of boundary value problem in heat conduction theory with nonlocal boundary conditions. *Differents. Urav.* **13** (1977) 294–304.

[16] Kamynin, L.I. A boundary value problem in the theory of the heat conduction with non-classical boundary condition. *Z. Vychisl. Mat. Fiz.* **6** (1964) 1006–1024.

[17] Sloan, I.H. and Thomee, V. Time discretization of an integro-differential equation of parabolic type. *SIAM J. Numer. Anal.* **23** (1986) 1052–1061.