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Mean Square Stability of Itô–Volterra Dynamic Equation

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Abstract: This paper presents a sufficient condition for the mean square stability of the Itô–Volterra dynamic equation on isolated time scales.

Keywords: *Itô–Volterra dynamic equation; mean square stability; convolution; time scale; stochastic dynamic equations.*

Mathematics Subject Classification (2000): 37C75, 45D05, 60H20.

1 Introduction

Given a time scale $\mathbb{T}$, a collection of measurable real functions $X = \{X(t) : t \in \mathbb{T}\}$, defined on a measurable space $(\Omega, \mathcal{F})$, will be referred to as a stochastic process indexed by $\mathbb{T}$ [15, 19, 30]. We consider the Itô–Volterra dynamic equation of the form

$$\Delta X = (a * X)(t)\Delta t + (b * X)(t)\Delta V, \quad X(t_0) = X_0, \tag{1.1}$$

where $a, b : \mathbb{T} \to \mathbb{R}$, $a * X$ is the convolution of a and X defined in Definition 2.2, V is the solution of

$$\Delta V = \sqrt{\mu(t)}\Delta W, \quad V(t_0) = V_0, \tag{1.2}$$

and $X = \{X(t) : t \in \mathbb{T}\}$ is a stochastic process indexed by an isolated time scale $\mathbb{T}$, and $\mu(t) = \sigma(t) - t$ with $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$. In (1.2), W is one-dimensional Brownian motion indexed by a time scale $\mathbb{T}$ which is defined as an adapted stochastic process $W = \{W(t), \mathcal{F}(t) : t \in \mathbb{T}\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the following properties: (a) $W(t_0) = 0$ a.s.; (b) if $t_0 \le s < t$ and $s, t \in \mathbb{T}$, then the increment $\Delta W(t) = W(\sigma(t)) - W(t)$ is independent of $\mathcal{F}(s)$ and is normally distributed with mean zero and variance $\mu(t)$.

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Since $V^{\Delta}(t) = \Delta V(t)/\Delta t = \Delta W(t)/\sqrt{\mu(t)}$, we observe that $\{V^{\Delta}(t) : t \in \mathbb{T}\}$ are i.i.d. random variables which generate a natural filtration $\{\mathcal{F}(t) : t \in \mathbb{T}\}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[V^{\Delta}(t)] = 0$ and $\mathbb{E}\left[\left(V^{\Delta}(t)\right)^2\right] = 1$, where $\mathbb{E}$ is the expectation with respect to the probability measure $\mathbb{P}$. Throughout the paper we assume that $X(\tau)$ is independent of $V^{\Delta}(t)$ for $\tau \in [t_0, t)$.

For time scale calculus we refer to [14]; for integral equations of Volterra type we refer to [16, 23, 24]. Stability and convergence of solutions of Volterra equations, likewise, has been discussed in [2–6, 17, 18, 20–22, 25–29]. For improper integrals and multiple integration on time scales we refer to [1, 7, 8, 10, 11, 13], and for partial differentiation on time scales we refer to [9].

The organization of the paper is as follows. Section 2 presents core definitions and concept of convolution on a time scale. In Section 3, we derive new conditions that guarantee the mean square stability of (1.1) on an isolated time scale. Our attempt is to make the mathematical discussion that follows as self contained as is practical.

2 Convolution

Convolution on time scales was introduced by Bohner and Guseinov in [12]. In this section we present a brief survey. Let $\sup \mathbb{T} = \infty$ and fix $-\infty < t_0 \in \mathbb{T}$.

Definition 2.1 For $b : \mathbb{T} \to \mathbb{R}$, the *shift* (or delay) $\tilde{b}$ of b is the function $\tilde{b} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ given by

$$\begin{aligned} \tilde{b}^{\Delta_t}(t, \sigma(s)) &= -\tilde{b}^{\Delta_s}(t, s), \quad t, s \in \mathbb{T},\ t \geq s \geq t_0, \\ \tilde{b}(t, t_0) &= b(t), \quad t \in \mathbb{T},\ t \geq t_0, \end{aligned} \tag{2.1}$$

where Δ_t is the partial Δ-derivative with respect to t.

Example 2.1 For the forward difference operator, the problem (2.1) takes the form

$$\begin{aligned} \mu(s)\Delta_t \tilde{b}(t, \sigma(s)) &= -\mu(t)\Delta_s \tilde{b}(t, s), \quad t, s \in \mathbb{T},\ t \geq s \geq t_0, \\ \tilde{b}(t, t_0) &= b(t), \quad t \in \mathbb{T},\ t \geq t_0. \end{aligned} \tag{2.2}$$

Example 2.2 For $\mathbb{T} = \mathbb{R}$, the problem (2.1) takes the form

$$\frac{\partial \tilde{b}(t, s)}{\partial t} = -\frac{\partial \tilde{b}(t, s)}{\partial s}, \quad \tilde{b}(t, t_0) = b(t), \tag{2.3}$$

and its unique solution is $\tilde{b}(t, s) = b(t - s + t_0)$.

Example 2.3 For $\mathbb{T} = \mathbb{Z}$, the problem (2.1) takes the form

$$\tilde{b}(t + 1, s + 1) - \tilde{b}(t, s + 1) = -\tilde{b}(t, s + 1) + \tilde{b}(t, s), \quad \tilde{b}(t, t_0) = b(t), \tag{2.4}$$

and its unique solution is again $\tilde{b}(t, s) = b(t - s + t_0)$.

Lemma 2.1 *If $\tilde{b}$ is the shift of b, then $\tilde{b}(t, t) = b(t_0)$ for all $t \in \mathbb{T}$.*

Definition 2.2 The convolution of two functions $b, r : \mathbb{T} \to \mathbb{R}$, $b * r$ is defined as

$$(b * r)(t) = \int_{t_0}^{t} \tilde{b}(t, \sigma(s)) r(s) \Delta s, \quad t \in \mathbb{T}, \tag{2.5}$$

where $\tilde{b}$ is given by (2.1).

Example 2.4 For $\mathbb{T} = \mathbb{N}_0$ and $n \in \mathbb{N}_0$, (2.5) reduces to

$$(b * r)(n) = \sum_{i=0}^{n-1} b(n - i - 1) r(i). \tag{2.6}$$

Theorem 2.1 *The shift of a convolution is given by the formula*

$$(\widetilde{b * r})(t, s) = \int_{s}^{t} \tilde{b}(t, \sigma(l)) \tilde{r}(l, s) \Delta l. \tag{2.7}$$

Example 2.5 For $\mathbb{T} = \mathbb{N}_0$ and $m, n \in \mathbb{N}_0$, (2.7) reduces to

$$(\widetilde{b * r})(n, m) = \sum_{i=m}^{n-1} b(n - i - 1) r(i - m). \tag{2.8}$$

Theorem 2.2 *The convolution is associative, that is,*

$$(a * f) * r = a * (f * r). \tag{2.9}$$

Proof We use Theorem 2.1. Then

$$\begin{aligned} ((a * f) * r)(t) &= \int_{t_0}^{t} (\widetilde{a * f})(t, \sigma(s)) r(s) \Delta s \\ &= \int_{t_0}^{t} \int_{\sigma(s)}^{t} \tilde{a}(t, \sigma(u)) \tilde{f}(u, \sigma(s)) r(s) \Delta u \Delta s \\ &= \int_{t_0}^{t} \int_{t_0}^{u} \tilde{a}(t, \sigma(u)) \tilde{f}(u, \sigma(s)) r(s) \Delta s \Delta u \\ &= \int_{t_0}^{t} \tilde{a}(t, \sigma(u)) (f * r)(u) \Delta u \\ &= (a * (f * r))(t), \end{aligned}$$

where on the second equality we have used (2.7). Hence, the associative property holds.

Theorem 2.3 *If r is delta differentiable, then*

$$(r * f)^{\Delta} = r^{\Delta} * f + r(t_0) f \tag{2.10}$$

and if f is delta differentiable, then

$$(r * f)^{\Delta} = r * f^{\Delta} + r f(t_0). \tag{2.11}$$

Proof First note that

$$(r*f)^{\Delta}(t) = \int_{t_0}^{t} r^{\Delta_t}(t,\sigma(t))f(s)\Delta s + \tilde{r}(\sigma(t),\sigma(t))f(t).$$

From here, since $\tilde{r}(\sigma(t),\sigma(t)) = r(t_0)$ by Lemma 2.1, and since

$$\widetilde{r^{\Delta}}(t,s) = \tilde{r}^{\Delta_t}(t,s),$$

the first equal sign of the statement follows. For the second equal sign, we use the definition of $\tilde{r}$ and integration by parts:

$$\begin{aligned}
(r*f)^{\Delta}(t) &= -\int_{t_0}^{t} \tilde{r}^{\Delta_s}(t,s)f(s)\Delta s + r(t_0)f(t) \\
&= -\int_{t_0}^{t} \left((\tilde{r}(t,\cdot)f)^{\Delta} - \tilde{r}(t,\sigma(s))f^{\Delta}(s)\right)\Delta s + r(t_0)f(t) \\
&= -\tilde{r}(t,t)f(t) + \tilde{r}(t,t_0)f(t_0) + \int_{t_0}^{t} \tilde{r}(t,\sigma(s))f^{\Delta}(s)\Delta s + r(t_0)f(t) \\
&= (r*f^{\Delta})(t) + r(t)f(t_0).
\end{aligned}$$

This completes the proof.

3 Mean-Square Stability

In this section we study the mean-square stability of (1.1).

Definition 3.1 A stochastic process indexed by a time scale $X = \{X(t) : t \in \mathbb{T}\}$ and defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is mean square stable if $\mathbb{E}\left[X^2\right] \in L^1_{\Delta}(\mathbb{T})$, i.e.,

$$\int_{\mathbb{T}} \mathbb{E}\left[X^2(\tau)\right]\Delta\tau < \infty,$$

where $\mathbb{E}$ is the expectation with respect to the probability measure $\mathbb{P}$.

In the definition above and henceforth, $L^p_{\Delta}(\mathbb{T})$ for $p > 0$ would represent the space of all functions $f : \mathbb{T} \to \mathbb{R}$, such that $\int_{\mathbb{T}} |f|^p(\tau)\Delta\tau < \infty$.

Theorem 3.1 *If $X(t)$ is represented as*

$$X(t) = r(t)X_0 + (r*f)(t), \tag{3.1}$$

where

$$r^{\Delta}(t) = (a*r)(t), \quad r(t_0) = 1, \tag{3.2}$$

and

$$f(t) = (b*X)(t)V^{\Delta}(t). \tag{3.3}$$

then X is a solution of the Itô–Volterra dynamic equation

$$\Delta X = (a*X)(t)\Delta t + (b*X)(t)\Delta V, \quad X(t_0) = X_0. \tag{3.4}$$

Proof From (3.1) we have

$$\begin{aligned}\Delta X(t) &= r^{\Delta}(t)X_0\Delta t + (r*f)^{\Delta}(t)\Delta t\\ &= (a*r)(t)X_0\Delta t + (r^{\Delta}*f)(t)\Delta t + f(t)\Delta t\\ &= (a*(rX_0))(t)\Delta t + (r^{\Delta}*f)(t)\Delta t + f(t)\Delta t\\ &= (a*(X-r*f))(t)\Delta t + (r^{\Delta}*f)(t)\Delta t + f(t)\Delta t\\ &= (a*X)(t)\Delta t - (a*(r*f))(t)\Delta t + ((a*r)*f)(t)\Delta t + f(t)\Delta t\\ &= (a*X)(t)\Delta t + f(t)\Delta t\\ &= (a*X)(t)\Delta t + (b*X)(t)\Delta V(t),\end{aligned}$$

where on the second equality we have used (2.10) and on the sixth equality we have used Theorem 2.2.

Lemma 3.1 *If f is given by* (3.3)*, then* $\mathbb{E}[f(t)] = 0$ *and*

$$\mathbb{E}[f(t)f(s)] = \begin{cases} \displaystyle\int_{t_0}^{t}\int_{t_0}^{t}\tilde{b}(t,\sigma(t_1))\tilde{b}(t,\sigma(t_2))\mathbb{E}\left[X(t_1)X(t_2)\right]\Delta t_1\Delta t_2 =: \phi(t) & \text{if } s=t\\ 0 & \text{if } s\neq t.\end{cases}$$

Proof We first note that

$$\begin{aligned}\mathbb{E}[f(t)] &= \mathbb{E}\left[\int_{t_0}^{t}\tilde{b}(t,\sigma(\tau))X(\tau)V^{\Delta}(t)\Delta\tau\right]\\ &= \int_{t_0}^{t}\tilde{b}(t,\sigma(\tau))\mathbb{E}[X(\tau)V^{\Delta}(t)]\Delta\tau\\ &= \int_{t_0}^{t}\tilde{b}(t,\sigma(\tau))\mathbb{E}\left[X(\tau)\right]\mathbb{E}[V^{\Delta}(t)]\Delta\tau\\ &= 0,\end{aligned}$$

by the assumption that $X(\tau)$ is independent of $V^{\Delta}(t)$ for $\tau\in[t_0,t)$ and $\mathbb{E}\left[V^{\Delta}(t)\right]=0$. Next, we consider

$$\begin{aligned}&\mathbb{E}[f(t)f(s)]\\ &= \mathbb{E}\left[\int_{t_0}^{t}\tilde{b}(t,\sigma(t_1))X(t_1)V^{\Delta}(t)\Delta t_1\int_{t_0}^{s}\tilde{b}(s,\sigma(t_2))X(t_2)V^{\Delta}(s)\Delta t_2\right]\\ &= \mathbb{E}\left[\int_{t_0}^{t}\int_{t_0}^{s}\tilde{b}(t,\sigma(t_1))\tilde{b}(s,\sigma(t_2))X(t_1)X(t_2)V^{\Delta}(t)V^{\Delta}(s)\Delta t_1\Delta t_2\right]\\ &= \int_{t_0}^{t}\int_{t_0}^{s}\tilde{b}(t,\sigma(t_1))\tilde{b}(s,\sigma(t_2))\mathbb{E}\left[X(t_1)X(t_2)V^{\Delta}(t)V^{\Delta}(s)\right]\Delta t_1\Delta t_2\\ &= \int_{t_0}^{t}\int_{t_0}^{s}\tilde{b}(t,\sigma(t_1))\tilde{b}(s,\sigma(t_2))\mathbb{E}\left[X(t_1)X(t_2)\right]\mathbb{E}\left[V^{\Delta}(t)V^{\Delta}(s)\right]\Delta t_1\Delta t_2\\ &= \begin{cases}\displaystyle\int_{t_0}^{t}\int_{t_0}^{t}\tilde{b}(t,\sigma(t_1))\tilde{b}(t,\sigma(t_2))\mathbb{E}\left[X(t_1)X(t_2)\right]\Delta t_1\Delta t_2 & \text{if } s=t\\ 0 & \text{if } s\neq t,\end{cases}\end{aligned}$$

where on the fourth equation we have used the assumption that $X(\tau)$ is independent of $V^{\Delta}(t)$ for $\tau\in[t_0,t)$ and on fifth equation we have used $\mathbb{E}[V^{\Delta}(t)]=0$ and $\mathbb{E}[(V^{\Delta}(t))^2] = 1 > 0$.

Lemma 3.2 *If $X(t) = r(t)X_0 + (r * f)(t)$, then*

$$\mathbb{E}[X(l)X(m)] = r(l)r(m)X_0^2 + \int_{t_0}^{l\wedge m} \tilde{r}(l,\sigma(s))\tilde{r}(m,\sigma(s))\phi(s)\Delta s,$$

where ϕ is as in Lemma 3.1 and $l \wedge m = \min(l, m)$.

Proof From (3.1) we have,

$$\begin{aligned}
\mathbb{E}[X(l)X(m)] &= \mathbb{E}\left[\{r(l)X_0 + (r*f)(l)\}\{r(m)X_0 + (r*f)(m)\}\right] \\
&= r(l)r(m)X_0^2 \\
&\quad + \int_{t_0}^{l}\int_{t_0}^{m} \tilde{r}(l,\sigma(s_1))\tilde{r}(m,\sigma(s_2))\mathbb{E}\left[f(s_1)f(s_2)\right]\Delta s_1 \Delta s_2 \\
&= r(l)r(m)X_0^2 + \int_{t_0}^{l\wedge m} \tilde{r}(l,\sigma(s))\tilde{r}(m,\sigma(s))\mathbb{E}\left[f^2(s)\right]\Delta s \\
&= r(l)r(m)X_0^2 + \int_{t_0}^{l\wedge m} \tilde{r}(l,\sigma(s))\tilde{r}(m,\sigma(s))\phi(s)\Delta s,
\end{aligned}$$

where on the second equality we have used the fact that $\mathbb{E}\left[f(t)\right] = 0$ and on the third equality we have used Lemma 3.1.

Lemma 3.3 *The function ϕ defined in Lemma 3.1 is given by*

$$\phi(t) = (b*r)^2(t)X_0^2 + \int_{t_0}^{t} (\widetilde{b*r})^2(t,\sigma(s))\phi(s)\Delta s.$$

Proof Using Lemma 3.1, Lemma 3.2 and (2.5), we have

$$\begin{aligned}
\phi(t) &= \int_{t_0}^{t}\int_{t_0}^{t} \tilde{b}(t,\sigma(l))\tilde{b}(t,\sigma(m))\mathbb{E}\left[X(l)X(m)\right]\Delta l \Delta m \\
&= \int_{t_0}^{t}\int_{t_0}^{t} \tilde{b}(t,\sigma(l))\tilde{b}(t,\sigma(m))r(l)r(m)X_0^2 \Delta l \Delta m \\
&\quad + \int_{t_0}^{t}\int_{t_0}^{t} \tilde{b}(t,\sigma(l))\tilde{b}(t,\sigma(m)) \int_{t_0}^{l\wedge m} \tilde{r}(l,\sigma(s))\tilde{r}(m,\sigma(s))\phi(s)\Delta s \Delta l \Delta m \\
&= \int_{t_0}^{t}\int_{t_0}^{t} \tilde{b}(t,\sigma(l))\tilde{b}(t,\sigma(m))r(l)r(m)X_0^2 \Delta l \Delta m \\
&\quad + \int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{l\wedge m} \tilde{b}(t,\sigma(l))\tilde{b}(t,\sigma(m))\tilde{r}(l,\sigma(s))\tilde{r}(m,\sigma(s))\phi(s)\Delta s \Delta l \Delta m \\
&= \left(\int_{t_0}^{t} \tilde{b}(t,\sigma(l))r(l)\Delta l\right)^2 X_0^2 \\
&\quad + \int_{t_0}^{t}\int_{\sigma(s)}^{t}\int_{\sigma(s)}^{t} \tilde{b}(t,\sigma(l))\tilde{b}(t,\sigma(m))\tilde{r}(l,\sigma(s))\tilde{r}(m,\sigma(s))\phi(s)\Delta m \Delta l \Delta s \\
&= (b*r)^2(t)X_0^2 + \int_{t_0}^{t}\left(\int_{\sigma(s)}^{t} \tilde{b}(t,\sigma(l))\tilde{r}(l,\sigma(s))\Delta l\right)^2 \phi(s)\Delta s \\
&= (b*r)^2(t)X_0^2 + \int_{t_0}^{t} (\widetilde{b*r})^2(t,\sigma(s))\phi(s)\Delta s,
\end{aligned}$$

where on the last equality we have used Theorem 2.1.

Theorem 3.2 *If X is a solution of* (3.4)*, then*

$$\mathbb{E}\left[X^2(t)\right] = r^2(t)X_0^2 + \int_{t_0}^{t} \tilde{r}^2(t,\sigma(s))\phi(s)\Delta s.$$

Proof Squaring both sides of (3.1), we have

$$\begin{aligned} X^2(t) &= r^2(t)X_0^2 + 2r(t)X_0(r*f)(t) \\ &\quad + \int_{t_0}^{t} \tilde{r}(t,\sigma(s_1))f(s_1)\Delta s_1 \int_{t_0}^{t} \tilde{r}(t,\sigma(s_2))f(s_2)\Delta s_2 \\ &= r^2(t)X_0^2 + 2r(t)X_0\ (r*f)(t) \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{t} \tilde{r}(t,\sigma(s_1))\tilde{r}(t,\sigma(s_2))f(s_1)f(s_2)\Delta s_1\Delta s_2. \end{aligned}$$

Now taking the expectation on both sides of the above expression, we have

$$\begin{aligned} \mathbb{E}\left[X^2(t)\right] &= r^2(t)X_0^2 + 2r(t)X_0 \int_{t_0}^{t} \tilde{r}(t,\sigma(s))\mathbb{E}[f(s)]\Delta s \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{t} \tilde{r}(t,\sigma(s_1))\tilde{r}(t,\sigma(s_2))\mathbb{E}[f(s_1)f(s_2)]\Delta s_1\Delta s_2 \\ &= r^2(t)X_0^2 + \int_{t_0}^{t} \tilde{r}^2(t,\sigma(s))\phi(s)\Delta s, \end{aligned}$$

where on the second equality we have used Lemma 3.1.

Theorem 3.3 *Suppose X is a solution of* (3.4) *and r is a solution of* (3.2)*. Then*

$$r,\ \tilde{r}(\cdot,s),\ \text{and } b*r \in L^2_\Delta(\mathbb{T}) \tag{3.5}$$

and

$$\int_{\sigma(s)}^{\infty} (\widetilde{b*r})^2(t,\sigma(s))\Delta t \le k < 1 \quad \text{for all} \quad s\in\mathbb{T}, \tag{3.6}$$

imply that $\mathbb{E}\left[X^2\right] \in L^1_\Delta(\mathbb{T})$.

Proof From Lemma 3.3, we have

$$\begin{aligned} \int_{t_0}^{\infty} \phi(t)\Delta t &= X_0^2 \int_{t_0}^{\infty} (b*r)^2(t)\Delta t + \int_{t_0}^{\infty}\int_{t_0}^{t} (\widetilde{b*r})^2(t,\sigma(s))\phi(s)\Delta s\Delta t \\ &= X_0^2 \int_{t_0}^{\infty} (b*r)^2(t)\Delta t + \int_{t_0}^{\infty}\int_{\sigma(s)}^{\infty} (\widetilde{b*r})^2(t,\sigma(s))\phi(s)\Delta t\Delta s \\ &\le X_0^2 \int_{t_0}^{\infty} (b*r)^2(t)\Delta t + k\int_{t_0}^{\infty} \phi(s)\Delta s. \end{aligned}$$

Simplifying and using the fact that $b*r \in L^2_\Delta(\mathbb{T})$, we have

$$\int_{t_0}^{\infty} \phi(t)\Delta t \le \frac{X_0^2}{1-k}\int_{t_0}^{\infty} (b*r)^2(t)\Delta t < \infty, \tag{3.7}$$

which implies that $\phi \in L^1_\Delta(\mathbb{T})$. Then from Theorem 3.2, we have

$$\begin{aligned}\int_{t_0}^{\infty} \mathbb{E}\left[X^2(t)\right]\Delta t &= X_0^2 \int_{t_0}^{\infty} r^2(t)\Delta t + \int_{t_0}^{\infty}\int_{t_0}^{t} \tilde{r}^2(t,\sigma(s))\phi(s)\Delta s\Delta t \\ &\leq \alpha + \int_{t_0}^{\infty}\int_{\sigma(s)}^{\infty} \tilde{r}^2(t,\sigma(s))\phi(s)\Delta t\Delta s \\ &\leq \alpha + \beta\int_{t_0}^{\infty}\phi(s)\Delta s \\ &< \infty,\end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$ such that

$$X_0^2 \int_{t_0}^{\infty} r^2(t)\Delta t < \alpha$$

and

$$\int_{\sigma(s)}^{\infty} \tilde{r}^2(t,\sigma(s))\Delta t < \beta,$$

and this concludes the proof.

Example 3.1 For $\mathbb{T} = \mathbb{N}_0$, equation (3.5) reduces (with redundancy) to

$$\sum_{n=0}^{\infty} r^2(n) < \infty,$$

$$\sum_{n=m+1}^{\infty} r^2(n-m) < \infty, \quad \text{for all } m \in \mathbb{N}_0,$$

and

$$\sum_{n=1}^{\infty}\left(\sum_{i=0}^{n-1} b(n-i-1)r(i)\right)^2 < \infty.$$

Similarly, equation (3.6) reduces to

$$\sum_{n=m+1}^{\infty}\left(\sum_{i=m+1}^{n-1} b(n-i-1)r(i-m-1)\right)^2 \leq k < 1 \quad \text{for all } m \in \mathbb{N}_0.$$

Remark 3.1 For $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, and $\sup\mathbb{T} = \infty$, equations (3.5) and (3.6) reduce to

$$\int_0^{\infty} r^2(\tau)\mathrm{d}\tau < \infty,$$

$$\int_s^{\infty} r^2(\tau - s)\mathrm{d}\tau < \infty \quad \text{for all } s \in \mathbb{T},$$

$$\int_0^{\infty}\left(\int_0^{\tau} b(\tau-s)r(s)\mathrm{d}s\right)^2 d\tau < \infty,$$

and

$$\int_s^{\infty}\left(\int_s^{t} b(t-\tau)r(\tau-s)\mathrm{d}\tau\right)^2 \mathrm{d}t \leq k < 1 \quad \text{for all } s \in \mathbb{T}.$$

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