Nonlinear Dynamics and Systems Theory, **11** (1) (2011) 93–97

An Oscillation Criteria for Second-order Linear Differential Equations

J. Tyagi *

Department of Mathematics, Indian Institute of Technology Gandhinagar
Vishwakarma Government Engineering College Complex,
Chandkheda, Visat–Gandhinagar Highway, Ahmedabad, Gujarat, India – 382424

Received: April 15, 2010; Revised: January 24, 2011

Abstract: We establish an oscillation criteria for a class of second-order linear differential equations

$$(p(t)x'(t))' + q(t)x(t) = 0,\ t \in [0, \infty),$$

via Levin's comparison theorem. We employ an interval oscillation technique for oscillation of the above equation. This approach depends only on the behavior of q in certain interval. In this study, we allow the sign-changing nature of q. Using this approach, we also ascertain to answer the oscillatory behavior of a number of linear differential equations.

Keywords: *linear ordinary differential equations; oscillation.*

Mathematics Subject Classification (2000): 34Cxx, 34C10.

1 Introduction

We consider the second-order linear differential equations of the form

$$(p(t)x'(t))' + q(t)x(t) = 0, \tag{1}$$

where $p, q \in C([0,\infty), \mathbb{R})$, $p(t) > 0$ and $p\,x' \in C^1([0, \infty), \mathbb{R})$. When $p(t) \equiv 1$, (1) reduces to

$$x''(t) + q(t)x(t) = 0. \tag{2}$$

There is an extensive literature for the oscillation/non-oscillation of (1) and (2) (see [1–12]). Most of these results require the integral of the function q on the entire half interval

* Corresponding author: mailto:jtyagi1@gmail.com

$[0,\,\infty)$. Also, it is well-known that if $q(t)$ is of mean value zero and $q(t) \neq 0$, then (2) is oscillatory, (cf. [1]). We emphasize that the behavior of nonoscillatory solutions to certain second-order functional differential equations can be ascertained in terms of the oscillatory behavior of (2) (see [9]). Assuming the nonoscillation of (1), Tunc obtained some nonoscillation theorem for third-order nonlinear differential equations (see [7]). Let us recall the definition of interval oscillation.

If for each given solution of (1), we find a sequence of intervals $[\tau_n,\,\eta_n]$, $\tau_n \to \infty$, $\eta_n < \tau_{n+1}$ such that the given solution has at least one zero in $(\tau_n,\,\eta_n)$, for each $n \in \mathbb{N}$, then the solution is oscillatory.

By the above approach El–Sayed [2], gave some interval oscillation criteria for forced second-order linear differential equations. In the present study, the ideas of [2] are used to establish an interval oscillation criteria for (1). This approach depends only on the behavior of q in certain interval. Also, we do not restrict the sign of q. By this approach, we ascertain to answer the oscillatory behavior of a number of linear differential equations. Section 2 contains the preliminaries. Section 3 is devoted to the main result and its applications.

2 Preliminaries

We need the following lemmas for the proof of our main result. We consider

$$(p_1(t)x'(t))' + q(t)x(t) = 0, \tag{3}$$

$$(p_2(t)y'(t))' + r(t)y(t) = 0,\ \alpha \leq t \leq \beta, \tag{4}$$

where $p_1,\, p_2,\, q,\, r \in C([\alpha,\,\beta],\,\mathbb{R}),\ \ p_1(t) > 0,\, p_2(t) > 0$ and $p_1x',\, p_2x' \in C^1([\alpha,\,\beta],\,\mathbb{R})$.

Lemma 2.1 *Let $p_2(t) \geq p_1(t) > 0,\ \ \forall\, t \in [\alpha,\,\beta]$. Let x and y be nontrivial solutions of (3) and (4), respectively such that $x(t)$ does not vanish on $[\alpha,\,\beta]$, $y(\alpha) \neq 0$ and the inequality*

$$\frac{-p_1(\alpha)x'(\alpha)}{x(\alpha)} + \int_\alpha^t q(s)ds > \left|\frac{-p_2(\alpha)y'(\alpha)}{y(\alpha)} + \int_\alpha^t r(s)ds\right|, \tag{5}$$

holds for all $t \in [\alpha,\,\beta]$. Then $y(t)$ does not vanish on $[\alpha,\,\beta]$ and

$$-\frac{p_1(t)x'(t)}{x(t)} > \left|\frac{p_2(t)y'(t)}{y(t)}\right|,\ \alpha \leq t \leq \beta.$$

Proof Since $x(t)$ does not vanish on $[\alpha,\,\beta]$, so $w(t) = -\frac{p_1(t)x'(t)}{x(t)}$ on $[\alpha,\,\beta]$ transforms (3) to

$$w'(t) = q(t) + \frac{(w(t))^2}{p_1(t)},$$

which is equivalent to the integral equation

$$w(t) = w(\alpha) + \int_\alpha^t q(s)ds + \int_\alpha^t \frac{(w(s))^2}{p_1(s)}ds.$$

Since $y(\alpha) \neq 0$, so with the substitution $z(t) = -\frac{p_2(t)y'(t)}{y(t)}$ on some interval $[\alpha,\,\gamma]$, $\alpha < \gamma \leq \beta$ and using the hypothesis that $p_2(t) \geq p_1(t) > 0$, the proof of Lemma 2.1 is similar to the proof of Theorem 1.35 [6]. We omit the proof for the sake of brevity.

Lemma 2.2 Let $p_2(t) \geq p_1(t) > 0, \ \forall t \in [\alpha, \beta]$. Let x and y be nontrivial solutions of (3) and (4), respectively such that $x(t)$ does not vanish on $[\alpha, \beta]$, $y(\beta) \neq 0$ and the inequality

$$\frac{p_1(\beta)x'(\beta)}{x(\beta)} + \int_t^\beta q(s)ds > \left| \frac{p_2(\beta)y'(\beta)}{y(\beta)} + \int_t^\beta r(s)ds \right|, \tag{6}$$

holds for all $t \in [\alpha, \beta]$. Then $y(t)$ does not vanish on $[\alpha, \beta]$ and

$$\frac{p_1(t)x'(t)}{x(t)} > \left| \frac{p_2(t)y'(t)}{y(t)} \right|, \ \alpha \leq t \leq \beta.$$

Proof The proof of this lemma is similar to the proof of Theorem 1.36 [6]. For convenience, we give a brief sketch. We define new functions x_1, y_1, q_1, r_1, p_1^* and p_2^* on $[\alpha, \beta]$ by

$$\begin{aligned} &x_1(t) = x(\alpha + \beta - t), \quad y_1(t) = y(\alpha + \beta - t). \\ &q_1(t) = q(\alpha + \beta - t), \quad r_1(t) = r(\alpha + \beta - t). \\ &p_1^*(t) = p_1(\alpha + \beta - t), \quad p_2^*(t) = p_2(\alpha + \beta - t). \end{aligned}$$

Then $x_1(t)$ does not vanish on $[\alpha, \beta]$, $y_1(\alpha) = y(\beta) \neq 0$ and

$$\begin{aligned} -\frac{p_1^*(\alpha)x_1'(\alpha)}{x_1(\alpha)} + \int_\alpha^{\alpha+\beta-t} q_1(s)ds &= \frac{p_1(\beta)x'(\beta)}{x(\beta)} + \int_t^\beta q(s)ds, \\ -\frac{p_2^*(\alpha)y_1'(\alpha)}{y_1(\alpha)} + \int_\alpha^{\alpha+\beta-t} r_1(s)ds &= \frac{p_2(\beta)y'(\beta)}{y(\beta)} + \int_t^\beta r(s)ds. \end{aligned}$$

It is easy to observe that inequality (6) is equivalent to inequality (5) of Lemma 2.1 and using the fact that $t \in [\alpha, \beta] \Leftrightarrow \alpha + \beta - t \in [\alpha, \beta]$, the required conclusion follows from Lemma 2.1.

Lemma 2.3 *Let y be a nontrivial solution of* (4) *satisfying the conditions $y(\alpha) = 0 = y(\beta) = y'(\gamma), \ \alpha < \gamma < \beta$. Let $p_2(t) \geq p_1(t) > 0, \ \forall t \in [\alpha, \beta]$. If the inequalities*

$$\begin{aligned} \int_t^\gamma q(s)ds &\geq \left| \int_t^\gamma r(s)ds \right|, \\ \int_\gamma^t q(s)ds &\geq \left| \int_\gamma^t r(s)ds \right| \end{aligned}$$

hold for all $t \in [\alpha, \gamma]$ and $[\gamma, \beta]$ respectively, then every solution of (3) *has at least one zero on $[\alpha, \beta]$.*

Proof The proof of this lemma is similar to the proof of Theorem 1.37 [6] with the account of Lemmas 2.1 and 2.2. We omit the details.

3 Main Result

In this section, we prove the main result on oscillation for second-order linear differential equations.

Theorem 3.1 *Let there exist a monotonic sequence* $\{\tau_n\} \subset \mathbb{R}^+$ *such that* $\tau_n \to \infty$, *as* $n \to \infty$ *and a sequence* $\{k_n\}$ *of positive numbers such that*

$$\int_t^{\tau_n+\frac{\pi}{2\sqrt{k_n}}} q(s)ds \geq k_n \left(\tau_n + \frac{\pi}{2\sqrt{k_n}} - t\right), \forall\, t \in \left[\tau_n, \tau_n + \frac{\pi}{2\sqrt{k_n}}\right], \quad (7)$$

$$\int_{\tau_n+\frac{\pi}{2\sqrt{k_n}}}^t q(s)ds \geq k_n \left(t - \tau_n - \frac{\pi}{2\sqrt{k_n}}\right), \forall\, t \in \left[\tau_n + \frac{\pi}{2\sqrt{k_n}}, \tau_n + \frac{\pi}{\sqrt{k_n}}\right], \quad (8)$$

$\forall\, n \in \mathbb{N}$. *Also, let* $0 < p(t) \leq 1,\ \forall\, t \in [\tau_n,\, \tau_n + \frac{\pi}{\sqrt{k_n}}]$. *Then* (1) *is oscillatory.*

Proof We prove this theorem by contradiction. Let x be a nontrivial solution of (1). Suppose x has finitely many zeros on $[0, \infty)$, so there exists a $\tau_0 > 0$ such that $x(t) \neq 0,\ \forall\, t \geq \tau_0$. We consider

$$y''(t) + k_n y(t) = 0,\ t \in [\tau_n,\, \tau_n + \frac{\pi}{\sqrt{k_n}}],\ \tau_n \geq \tau_0 \text{ for some } n \in \mathbb{N}. \quad (9)$$

(9) has a solution $y(t) = \sin\sqrt{k_n}(t - \tau_n)$ which has two consecutive zeros at $t = \tau_n$ and at $t = \tau_n + \frac{\pi}{\sqrt{k_n}}$. Also, $y'(t) = 0$ at $t = \tau_n + \frac{\pi}{2\sqrt{k_n}}$. From (7) and (8), it is easy to observe that the hypotheses of Lemma 2.3 are fulfilled. An application of Lemma 2.3 yields that x has at least one zero on $[\tau_n,\, \tau_n + \frac{\pi}{\sqrt{k_n}}]$, which leads to a contradiction. Hence the proof is complete.

Remark 3.1 We introduce Liouville's transformation $x(t) = \sqrt{t}\, y(s),\ s = \log t$, which converts (2) to

$$y''(s) + Q(s)y(s) = 0, \quad (10)$$

where $Q(s) = q(e^s)e^{2s} - \frac{1}{4}$. Let $q \in C([0, \infty], \mathbb{R})$ and satisfies (7) and (8) $\forall\, n \in \mathbb{N}$, then (10) is oscillatory.

Remark 3.2 Let $P \in C^2([0, \infty), (0, \infty))$. The substitution $x(t) = y(t)P^{\frac{1}{2}}(t)$ converts (2) to

$$(P(t)y'(t))' + Q(t)y(t) = 0, \quad (11)$$

where $Q(t) = \frac{P''(t)}{2} + P(t)q(t) - \frac{(P'(t))^2}{4P(t)}$. An oscillation criteria for (2) gives an oscillation criteria for (11) and conversely.

Remark 3.3 Consider the equation

$$x''(t) + \frac{1}{t^2}x(t) = 0. \quad (12)$$

Let $\{\tau_n\} \subset \mathbb{R}^+$ be any monotonic, divergent sequence. We choose

$$k_n = \frac{1}{(\tau_n + \frac{\pi}{2\sqrt{k_n}})(\tau_n + \frac{\pi}{\sqrt{k_n}})},\ n \in \mathbb{N},$$

or after simplifying we have $k_n = \frac{8+5\pi^2+3\pi\sqrt{\pi^2+16}}{8\tau_n^2}$. With this choice of τ_n and k_n, it is easy to satisfy the hypotheses of Theorem 3.1. So, an application of Theorem 3.1 implies that (12) is oscillatory, while none of the known criteria (see [4, 5, 12]) can be applied to (12).

Example 3.1 Consider the differential equation

$$((1-\alpha\sin^2 t)x'(t))' + (1+2\cos t)x(t) = 0,\ 0 \le \alpha < 1. \tag{13}$$

(13) can be viewed as (1) with $p(t) = 1-\alpha\sin^2 t,\ q(t) = 1+2\cos t$. With the choice of $\tau_n = 2n\pi,\ k_n = \frac{1}{16}$, inequalities (7) and (8) are converted to

$$2\sin t + \frac{15t}{16} \le \frac{15}{16}(2n\pi + 2\pi),\ \forall\, t \in [2n\pi,\ (n+1)2\pi], \tag{14}$$

$$2\sin t + \frac{15t}{16} \ge \frac{15}{16}(2n\pi + 2\pi),\ \forall\, t \in [(n+1)2\pi,\ (n+2)2\pi]. \tag{15}$$

By simple calculus, it is easy to verify the inequalities (14) and (15). An application of Theorem 3.1 implies that (13) is oscillatory.

Remark 3.4 In (13), $q(t) = 1+2\cos t$, which mean value is non-zero and therefore the result given in [1] cannot apply to (13).

References

[1] Coppel, W. A. *Disconjugacy.* Lecture Notes in Math., Vol. 220, Springer-Varleg, Berlin, 1971.

[2] El-Sayed, M. A. An oscillation criterion for a forced second order linear differential equations. *Proc. Amer. Math. Soc.* **118** (1993) 813–817.

[3] Hartman, P. On nonoscillatory linear differential equations of second order. *Amer. J. Math.* **74** (1952) 389–400.

[4] Kamenev, I. V. An integral criterion for oscillation of linear differential equations of second order. *Math. Zametki* **23** (1978) 249–251.

[5] Leighton, W. The detection of oscillation of solutions of a second order linear differential equation. *Duke. Math. J.* **17** (1950) 57–62.

[6] Swanson, C. A. Comparison and oscillation theory of linear differential equations, Vol. 148, Academic Press, New York and London, 1968.

[7] Tunc, C. On the non-oscillation of solutions of some nonlinear differential equations of third order. *Nonlinear Dynamics and Systems Theory* **7** (4) 419–430.

[8] Tyagi, J. An oscillation theorem for a second order nonlinear differential equations with variable potential. *Electron. J. Diff. Equations* (19) (2009) 1–5.

[9] Tyagi, J. Oscillation of solutions and behavior of the nonoscillatory solutions of second-order nonlinear functional equations. *Nonlinear Dynamics and Systems Theory* **9** (3) (2009) 317–326.

[10] Tyagi, J. and Raghavendra, V. A note on a generalization of Sturm's comparison theorem. *Nonlinear Dynamics and Systems Theory.* **8** (2) (2008) 213–216.

[11] Tyagi, J. and Raghavendra, V. An oscillation criteria for second-order nonlinear differential equations with functional arguments. *Electron. J. Diff. Equations* (30) (2009) 1–7.

[12] Wintner, A. A criterion of oscillatory stability. *Quat. Appl. Math.* **7** (1949) 115–117.