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Periodic Solutions of Singular Integral Equations

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Abstract: We consider a scalar integral equation

$$x(t) = a(t) - \int_{-\infty}^{t} C(t,s)g(s,x(s))ds$$

in which $C(t,s)$ has a singularity at $t = s$. There are periodic assumptions on a, C, and g. First we prove a fixed point theorem of the Krasnoselskii–Schaefer type. We then construct a Liapunov functional which allows us to satisfy the conditions of the fixed point theorem and to prove that there is a periodic solution.

Keywords: *integral equations; fixed point theorems; periodic solutions; Liapunov functionals.*

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1 Introduction

We consider a scalar integral equation

$$x(t) = a(t) - \int_{-\infty}^{t} C(t,s)g(s,x(s))ds \tag{1}$$

for which there is a $T > 0$ so that

$$a(t+T) = a(t),\ g(t+T,x) = g(t,x),\ C(t+T,s+T) = C(t,s) \tag{2}$$

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for all $t \in \Re$ and $s < t$ with a and g continuous. We denote by $(\mathcal{P}_T, \|\cdot\|)$ the Banach space of continuous T-periodic functions.

If g is Lipschitz and if C is small enough then a contraction mapping will yield a periodic solution. If C is convex then Liapunov arguments will produce *a priori* bounds. Under compactness conditions, Schaefer's fixed point theorem will yield a periodic solution. A collection of such results are found in Burton [7]. A recent n-dimensional result is given in [17].

In this paper we ask that g satisfies

$$|g(t,x) - g(t,y)| \leq K|x - y| \tag{3}$$

for all $x, y \in \Re$ and some $K > 0$, while C satisfies a truncated convexity condition, but has a significant singularity at $t = s$. We derive a set of conditions measuring the magnitude of the singularity that will still permit proof of the existence of a periodic solution using a combination Krasnoselskii–Schaefer fixed point theorem which we will prove in Section 2.

2 A Fixed Point Theorem

In this section, we will prove a fixed point theorem of Krasnoselskii-Schaefer type in which the mapping function has the form $Px = Bx + Ax$ with A being compact and $(I - B)^{-1}$ continuous on an appropriate subset M of a Banach space S. The theorem resembles that of Burton–Kirk [6] without having a λ term in B. See [8, 10, 11, 13, 14, 15] for work on Krasnoselskii and Schaefer theorems and their extended forms.

Since P is the sum of two operators, it is in general a non-self map; that is, P may not necessarily map a closed convex subset M of S into itself. To prove the existence of a fixed point of P, we apply topological degree theory or transversality method by constructing a homotopy U_λ on M with $U_1 = P$. It is assumed that $U_\lambda(\phi) = U(\lambda, \phi)$ is a continuous mapping of $[0, 1] \times M$ into a compact subset of S. In many applications, U_0 is a constant map sending M to a point $p \in M/\partial M$. In this case, U_0 is an "essential" map. If $U_\lambda(\phi)$ is fixed point free on ∂M for all $\lambda \in (0, 1)$, then $U_1(\phi)$ is essential having a fixed point property in M (Granas and Dugundji [9, p.120-123]). This fact is often written in the form of Leray–Schauder principle or its nonlinear alternatives which states that either

(A_1) U_1 has a fixed point in M or

(A_2) there exists $x \in \partial M$ and $\lambda \in (0, 1)$ with $x = U_\lambda(x)$

(see [1, p. 48], [9, p. 123], [15, p. 28], [16]).

Theorem 2.1 *Let $(S, \|\cdot\|)$ be a Banach space, $A, B : S \to S$ such that A is continuous with A mapping bounded sets into compact sets, $(I - B)^{-1}$ exists and is continuous on $(I - B)S$ with $\lambda A(M) \subset (I - B)S$ for each closed convex subset $M \subset S$ and $\lambda \in [0, 1]$. Then either*

(i) $x = Bx + \lambda Ax$ has a solution in S for $\lambda = 1$, or

(ii) the set of all such solutions, $0 < \lambda < 1$, is unbounded.

Proof Since $\lambda A(M) \subset (I-B)S$, we have $0 \in (I-B)S$. If $x^* = (I-B)^{-1}(0)$, then x^* is the unique fixed point of B. For each positive integer n, define a closed and bounded set

$$M_n = \{x \in S : \|x\| \leq n\}.$$

We choose n sufficiently large so that $x^* \in M_n/\partial M_n$. Now $(I-B)^{-1}$ exists and is continuous on $(I-B)S$. Since A is continuous with A mapping M_n into a compact set, so is $(I-B)^{-1}(\lambda A)$ for each $\lambda \in [0,1]$. Define $U : [0,1] \times M_n \to S$ by

$$U(\lambda, \phi) = (I-B)^{-1}(\lambda A\phi).$$

Then $U_\lambda(\phi) = U(\lambda, \phi)$ is a continuous mapping of $[0,1] \times M_n$ into a compact subset of S. Indeed, set $\Gamma = \{\lambda A\phi : \lambda \in [0,1],\ \phi \in M_n\}$ and let $\{(\lambda_k, \phi_k)\}$ be a sequence in $[0,1] \times M_n$. We may assume that $\lambda_k \to \lambda_0 \in [0,1]$ as $k \to \infty$. Since AM_n is contained in a compact subset of S, there exists a convergent subsequence $\{A\phi_{k_j}\}$ of $\{A\phi_k\}$. Now $\{\lambda_{k_j} A\phi_{k_j}\}$ converges in S. This implies that Γ is pre-compact, and so is $(I-B)^{-1}\Gamma$. Observe that for all $\phi \in M_n$,

$$U_0(\phi) = (I-B)^{-1}(0) = x^*$$

is a constant map. Moreover, $x^* \in M_n/\partial M_n$. By the statement of nonlinear alternatives (A_1) and (A_2) above, either U_1 has a fixed point in M_n or there exists $x_n \in \partial M_n$ such that $x_n = U_\lambda(x_n)$ for some $\lambda \in (0,1)$. This implies that either $x = Bx + Ax$ has a solution in M_n or there exists $x_n \in \partial M_n$ with $x_n = Bx_n + \lambda Ax_n$ for some $\lambda \in (0,1)$. In the later case, we have $\|x_n\| = n$. Thus, if (i) does not hold, then $\|x_n\| \to \infty$ as $n \to \infty$ and (ii) must hold. This completes the proof.

Remark 2.1 It is clear that if B is a contraction mapping with contraction constant $0 < \alpha < 1$, then $(I-B)^{-1}$ exists and is continuous on S. Many generalized or nonlinear contractions satisfy this condition (see [2, 3, 8, 11, 12, 13]).

3 Technical Conditions

We now introduce the conditions which will produce the *a priori* bound needed in the fixed point theorem, as well as the required compactness. The kernel, $C(t,s)$, can have a singularity at $t = s$, but we ask that there exists a fixed $\epsilon > 0$ so that

$$C(t,s) \geq 0,\ C_s(t,s) \geq 0,\ C_t(t,s) \leq 0,\ C_{st}(t,s) \leq 0 \tag{4}$$

provided that

$$-\infty < s \leq t - \epsilon,\ t < \infty. \tag{5}$$

Moreover, if $x \in \mathcal{P}_T$, then

$$\int_{-\infty}^{t-\epsilon} C(t,s)g(s,x(s))ds \quad \text{and} \quad \int_{t-\epsilon}^{t} C(t,s)g(s,x(s))ds \quad \text{are continuous.} \tag{6}$$

The ϵ will play a central role. First, assume that there is a $\eta < 1$ with

$$K\int_{t-\epsilon}^{t} |C(t,s)|ds \leq \eta,\ t \in \Re. \tag{7}$$

Next, there are positive constants α and β with $2\alpha+\beta<2$ so that both

$$\int_s^{s+\epsilon} [\epsilon C_s(u,u-\epsilon)+C(u,u-\epsilon)+|C(u,s)|]du<\alpha,\ s\in\Re \tag{8}$$

and

$$C(t,t-\epsilon)\epsilon+\int_{t-\epsilon}^{t}|C(t,s)|ds<\beta,\ t\in\Re. \tag{9}$$

The work here is motivated by and is an extension of [4]. Relations (7)–(9) specify the strength of the singularity. For a "mild" singularity such as $C(t,s)=[t-s]^{-p}$, $0<p<1$, then (4), (5), (7)–(9) are satisfied for any $K>0$ when it is allowed that ϵ can be taken sufficiently small. But (6) would fail. The following function satisfies (4)-(9) with $0<\epsilon\le 1$ and an appropriate constant $k>0$

$$C(t,s)=\frac{k}{(t-s)(1+|\ln(t-s)-\ln\epsilon|)^2}.$$

We now define for $0\le\lambda\le 1$ a companion equation to (1)

$$x(t)=\lambda\left[a(t)-\int_{-\infty}^{t-\epsilon}C(t,s)g(s,x(s))ds\right]-\int_{t-\epsilon}^{t}C(t,s)g(s,x(s))ds. \tag{1_λ}$$

The mappings $A,B:\mathcal{P}_T\to\mathcal{P}_T$ mentioned in the theorem are defined by $\phi\in\mathcal{P}_T$ which implies that

$$(A\phi)(t):=a(t)-\int_{-\infty}^{t-\epsilon}C(t,s)g(s,\phi(s))ds \tag{10}$$

and

$$(B\phi)(t):=-\int_{t-\epsilon}^{t}C(t,s)g(s,\phi(s))ds. \tag{11}$$

By (6), if $\phi\in\mathcal{P}_T$ then ϕ is continuous so these integrals are continuous functions. To see that $A\phi,B\phi\in\mathcal{P}_T$ we note that

$$\begin{aligned}(A\phi)(t+T)&=a(t+T)-\int_{-\infty}^{t+T-\epsilon}C(t+T,s)g(s,\phi(s))ds\\&=a(t)-\int_{-\infty}^{t-\epsilon}C(t+T,s+T)g(s+T,\phi(s+T))ds=(A\phi)(t)\end{aligned}$$

while

$$\begin{aligned}(B\phi)(t+T)&=-\int_{t+T-\epsilon}^{t+T}C(t+T,s)g(s,\phi(s))ds\\&=-\int_{t-\epsilon}^{t}C(t+T,s+T)g(s+T,\phi(s+T))ds=(B\phi)(t).\end{aligned}$$

Moreover, by (3) and (7), B is a contraction.

4 A Liapunov Functional

We begin with the assumption that there is an $L > 0$ with

$$xg(t,x) \geq 0 \text{ for } |x| \geq L \tag{12}$$

and that

$$\lim_{s \to -\infty} (t-s)C(t,s) = 0 \text{ for fixed } t. \tag{13}$$

Then define a Liapunov functional by

$$V(t,\epsilon) = \lambda \int_{-\infty}^{t-\epsilon} C_s(t,s)\left(\int_s^t g(v,x(v))dv\right)^2 ds. \tag{14}$$

This Liapunov functional in the continuous case with finite delay was recently discussed in [5].

Lemma 4.1 *If $x \in \mathcal{P}_T$ solves (1_λ) then $V'(t,\epsilon)$ satisfies*

$$\begin{aligned} V'(t,\epsilon) \leq\ & \lambda C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^t g(v,x(v))dv\right)^2 \\ & + 2g(t,x)\left[\lambda C(t,t-\epsilon)\int_{t-\epsilon}^t g(v,x(v))dv - \int_{t-\epsilon}^t C(t,s)g(s,x(s))ds\right] \\ & + 2g(t,x)[\lambda a(t) - x(t)]. \end{aligned} \tag{15}$$

Proof Taking into account that $C_{st} \leq 0$ we have

$$\begin{aligned} V'(t,\epsilon) \leq\ & \lambda C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^t g(v,x(v))dv\right)^2 \\ & + 2\lambda g(t,x) \int_{-\infty}^{t-\epsilon} C_s(t,s) \int_s^t g(v,x(v))dvds. \end{aligned}$$

If we integrate the last term by parts and use (13) in the lower limiting evaluation, keeping in mind that x is bounded, we obtain

$$\begin{aligned} V'(t,\epsilon) \leq\ & \lambda C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^t g(v,x(v))dv\right)^2 \\ & + 2\lambda g(t,x)\left[C(t,s)\int_s^t g(v,x(v))dv\Big|_{-\infty}^{t-\epsilon} + \int_{-\infty}^{t-\epsilon} C(t,s)g(s,x(s))ds\right] \\ =\ & \lambda C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^t g(v,x(v))dv\right)^2 \\ & + 2\lambda g(t,x)\left[C(t,t-\epsilon)\int_{t-\epsilon}^t g(v,x(v))dv\right] \\ & + 2g(t,x)\left[\lambda \int_{-\infty}^{t-\epsilon} C(t,s)g(s,x(s))ds + \int_{t-\epsilon}^t C(t,s)g(s,x(s))ds\right] \\ & - 2g(t,x)\int_{t-\epsilon}^t C(t,s)g(s,x(s))ds. \end{aligned}$$

Using (1_λ) in the next-to-last term yields (15).

We will integrate (15) to relate $g(t,x(t))$ to $a(t)$ and then use that relation in a lower bound on the Liapunov functional to obtain the *a priori* bound. We now obtain that lower bound.

Lemma 4.2 *For any $q>0$, if $x\in\mathcal{P}_T$ solves (1_λ), then*

$$
\begin{aligned}
(x(t)-\lambda a(t))^2 &\le 2(1+q^{-1})\int_{-\infty}^{t-\epsilon} C_s(t,s)ds\,V(t,\epsilon)\\
&\quad+2(1+q^{-1})\epsilon C^2(t,t-\epsilon)\int_{t-\epsilon}^{t} g^2(s,x(s))ds\\
&\quad+(1+q)\Big(\int_{t-\epsilon}^{t}|C(t,s)|ds\Big)^2\Big(K\|x\|+\sup_{0\le u\le T}|g(u,0)|\Big)^2. \qquad (16)
\end{aligned}
$$

Proof Let $q>0$ be fixed and define $H=(1+\lambda q)\Big(\int_{t-\epsilon}^{t}C(t,s)g(s,x(s))ds\Big)^2$ so that from (1_λ) we obtain

$$
\begin{aligned}
&(x(t)-\lambda a(t))^2=\Big(\lambda\int_{-\infty}^{t-\epsilon}C(t,s)g(s,x(s))ds+\int_{t-\epsilon}^{t}C(t,s)g(s,x(s))ds\Big)^2\\
&\le\lambda(1+q^{-1})\Big(\int_{-\infty}^{t-\epsilon}C(t,s)g(s,x(s))ds\Big)^2+H\\
&=\lambda(1+q^{-1})\Big(-C(t,s)\int_s^t g(u,x(u))du\Big|_{-\infty}^{t-\epsilon}\\
&\quad+\int_{-\infty}^{t-\epsilon}C_s(t,s)\int_s^t g(u,x(u))duds\Big)^2+H
\end{aligned}
$$

(using (13) and $x\in\mathcal{P}_T$)

$$
\begin{aligned}
&=\lambda(1+q^{-1})\Big(-C(t,t-\epsilon)\int_{t-\epsilon}^{t}g(u,x(u))du\\
&\quad+\int_{-\infty}^{t-\epsilon}C_s(t,s)\int_s^t g(u,x(u))duds\Big)^2+H\\
&\le 2\lambda(1+q^{-1})C^2(t,t-\epsilon)\Big(\int_{t-\epsilon}^{t}g(u,x(u))du\Big)^2\\
&\quad+2(1+q^{-1})\Big(\int_{-\infty}^{t-\epsilon}C_s(t,s)\int_s^t g(u,x(u))duds\Big)^2+H\\
&\le 2\lambda(1+q^{-1})C^2(t,t-\epsilon)\epsilon\int_{t-\epsilon}^{t}g^2(u,x(u))du+H\\
&\quad+2(1+q^{-1})\int_{-\infty}^{t-\epsilon}C_s(t,s)ds\int_{-\infty}^{t-\epsilon}C_s(t,s)\Big(\int_s^t g(u,x(u))du\Big)^2ds\\
&\le 2\lambda(1+q^{-1})C^2(t,t-\epsilon)\epsilon\int_{t-\epsilon}^{t}g^2(u,x(u))du\\
&\quad+2(1+q^{-1})\int_{-\infty}^{t-\epsilon}C_s(t,s)ds\,V(t,\epsilon)
\end{aligned}
$$

$$+ (1+q)\left(\int_{t-\epsilon}^{t} |C(t,s)|ds\right)^2 \left(K\|x\| + \sup_{0\le u\le T} |g(u,0)\|\right)^2,$$

as required.

Lemma 4.3 *If*

$$|g(t,x)| \le |x| \ \text{ for } \ |x| \ge L, \tag{17}$$

where L is defined in (12)*, then for any* $\gamma > 0$ *there is an* $M > 0$ *such that for any solution of* (1_λ) *in* $\mathcal{P}_T$ *we have*

$$V'(t,\epsilon) \le Ma^2(t) + [\gamma+\beta-2]g^2(t,x(t)) + M + \int_{t-\epsilon}^{t} [|C(t,s)| + \epsilon C_s(t,t-\epsilon) + C(t,t-\epsilon)]g^2(s,x(s))ds. \tag{18}$$

Proof By Cauchy inequality, for any $\gamma > 0$, there is an $M > 0$ such that

$$2g(t,x)a(t) \le \gamma g^2(t,x) + Ma^2(t).$$

By (17), we may choose M so large that

$$-2g(t,x)x \le -2g^2(t,x) + M$$

for all $t \ge 0$ and $x \in \Re$. Now from (15) we have

$$\begin{aligned}
V'(t,\epsilon) &\le \gamma g^2(t,x) + Ma^2(t)\\
&- 2g^2(t,x) + M + C_s(t,t-\epsilon)\epsilon\int_{t-\epsilon}^{t} g^2(v,x(v))dv\\
&+ C(t,t-\epsilon)\int_{t-\epsilon}^{t} [g^2(t,x(t)) + g^2(v,x(v))]dv\\
&+ \int_{t-\epsilon}^{t} |C(t,s)|[g^2(t,x(t)) + g^2(s,x(s))]ds\\
&= Ma^2(t) + g^2(t,x)\left[\gamma - 2 + \epsilon C(t,t-\epsilon) + \int_{t-\epsilon}^{t} |C(t,s)|ds\right] + M\\
&+ \int_{t-\epsilon}^{t} [\epsilon C_s(t,t-\epsilon) + C(t,t-\epsilon) + |C(t,s)|]g^2(s,x(s))ds\\
&\text{by (9)}\\
&\le Ma^2(t) + g^2(t,x)[\gamma+\beta-2] + M\\
&+ \int_{t-\epsilon}^{t} [\epsilon C_s(t,t-\epsilon) + C(t,t-\epsilon) + |C(t,s)|]g^2(s,x(s))ds,
\end{aligned}$$

as required.

Lemma 4.4 *If* (17) *holds, if* $\epsilon \le T$*, and if* γ *is small enough then there is a* $\mu > 0$ *so that if* x *solves* (1_λ) *and* $x \in \mathcal{P}_T$ *then*

$$\int_0^T g^2(s,x(s))ds \le (M/\mu)\int_0^T a^2(s)ds + TM/\mu. \tag{19}$$

Proof We are going to integrate (18) from 0 to T and note that $0 = V(T,\epsilon) - V(0,\epsilon)$. First, we estimate the integral of the last term in (18) as follows. We have

$$\begin{aligned}&\int_0^T \int_{t-\epsilon}^t [|C(t,s)| + \epsilon C_s(t,t-\epsilon) + C(t,t-\epsilon)]g^2(s,x(s))dsdt \\ &\leq \int_{-\epsilon}^T \int_s^{s+\epsilon} [|C(t,s)| + \epsilon C_s(t,t-\epsilon) + C(t,t-\epsilon)]dtg^2(s,x(s))ds \\ &\leq \alpha \int_{-\epsilon}^T g^2(s,x(s))ds \leq 2\alpha \int_0^T g^2(s,x(s))ds.\end{aligned}$$

With this information we now integrate (18) and obtain

$$\begin{aligned}0 = V(T,\epsilon) - V(0,\epsilon) &\leq M \int_0^T a^2(s)ds + TM \\ &+ \int_0^T [\gamma - 2 + \beta + 2\alpha]g^2(s,x(s))ds \\ &\leq M \int_0^T a^2(s)ds - \mu \int_0^T g^2(s,x(s))ds + TM\end{aligned}$$

since $\beta + 2\alpha < 2$ and γ can be made as small as we please.

Lemma 4.5 *Let the conditions of Lemma* 4.4 *hold and suppose there is a* $Q > 0$ *with*

$$\int_{-\infty}^{t-\epsilon} C_s(t,s))(t+T-s)^2 ds \leq Q. \tag{20}$$

Then there is a $Q^* > 0$ *with* $V(t,\epsilon) \leq Q^*$.

Proof We have

$$\begin{aligned}V(t,\epsilon) &= \int_{-\infty}^{t-\epsilon} C_s(t,s)\left(\int_s^t g(u,x(u))du\right)^2 ds \\ &\leq \int_{-\infty}^{t-\epsilon} C_s(t,s)(t-s)\int_s^t g^2(u,x(u))duds \\ &\leq \int_{-\infty}^{t-\epsilon} C_s(t,s)(t-s)\left[\int_s^{t+T} (M/\mu)a^2(u)du + (t-s+T)TM/\mu\right]ds \\ &\leq \int_{-\infty}^{t-\epsilon} C_s(t,s)(t+T-s)^2 ds[(M/\mu)\|a^2\| + TM/\mu]\end{aligned}$$

from which the result follows.

Lemma 4.6 *Let the conditions of Lemma* 4.5 *hold. Then there exists a constant* $J > 0$ *such that* $\|x\| < J$ *whenever* x *is* T*-periodic solution of* (1_λ) *for* $0 < \lambda \leq 1$.

Proof By (9) and (13), we have

$$\int_{-\infty}^{t-\epsilon} C_s(t,s)ds = C(t,t-\epsilon) \leq \beta/\epsilon.$$

If $x \in \mathcal{P}_T$ solves (1_λ), then (19) holds, and by Lemma 4.5, $V(t,\epsilon) \leq Q^*$. Now taking into account that (7) holds with $\eta < 1$, we obtain from (16) that

$$(x(t)-\lambda a(t))^2 \leq 2(1+q^{-1})(\beta/\epsilon)Q^*+2(1+q^{-1})(\beta^2/\epsilon)TM(\|a^2\|+1)/\mu+(1+q)(\eta\|x\|+\beta g^*)^2,$$

where $g^* = \|g(t,0)\|$. Since $\eta < 1$, we may choose $q > 0$ small enough so that $(1+q)\eta^2 < 1$, and hence, there exists $J > 0$ such that $\|x\| < J$. The proof is complete.

5 Continuity and Compactness

We select part of (10) and define the mapping $U : \mathcal{P}_T \to \mathcal{P}_T$ by $\phi \in \mathcal{P}_T$ which implies that

$$(U\phi)(t) = \int_{-\infty}^{t-\epsilon} C(t,s)g(s,\phi(s))ds. \tag{21}$$

Then U is well defined on P_T by (6). By a change of variable we have

$$(U\phi)(t) = \int_{-\infty}^{t} C(t,s-\epsilon)g(s-\epsilon,\phi(s-\epsilon))ds$$

with a fully convex kernel.

Lemma 5.1 *Suppose that $\int_{-\infty}^{t-\epsilon}[|C(t,s)|+|C_t(t,s)|]ds$ is bounded for all $t \in \Re$. Then U is continuous on P_T and for each $J > 0$, $\Gamma = \{U(\phi) : \phi \in \mathcal{P}_T|, \|\phi\| \leq J\}$ is uniformly bounded and equicontinuous.*

Proof First, there is a J^* such that $\phi \in \Gamma$ implies that $|g(t,\phi(t))| \leq J^*$ and there is a C^* with

$$\int_{-\infty}^{t-\epsilon}[|C(t,s)|+|C_t(t,s)|]ds \leq C^*, \quad t \in \Re. \tag{22}$$

It is clear that $U\phi \in P_T$ by (6) and the argument following (10). We now show that U is continuous on P_T. If $\tilde{\phi}$, $\phi \in P_T$, then

$$\begin{aligned} |U(\phi)(t) - U(\tilde{\phi})(t)| &= \left|\int_{-\infty}^{t-\epsilon} C(t,s)g(s,\phi(s))ds - \int_{-\infty}^{t-\epsilon} C(t,s)g(s,\tilde{\phi}(s))ds\right| \\ &= \left|\int_{-\infty}^{t-s} C(t,s)\left[g(s,\phi(s)) - g(s,\tilde{\phi}(s))\right]ds\right|. \end{aligned} \tag{23}$$

Since g is uniformly continuous on $[0,T] \times \{x \in R : |x| \leq \|\tilde{\phi}\|+1\}$, for any $\epsilon > 0$, there exists $0 < \delta < 1$ such that $\|\phi - \tilde{\phi}\| < \delta$ implies $|g(s,\phi(s)) - g(s,\tilde{\phi}(s))| < \varepsilon$ for all $s \in [0,T]$. It follows from (23) that $\|U(\phi) - U(\tilde{\phi})\| \leq \epsilon C^*$. Thus, F is continuous on P_T.

Next, for an arbitrary $\phi \in \Gamma$ we have

$$\frac{d}{dt}(U\phi)(t) = C(t,t-\epsilon)g(t-\epsilon,\phi(t-\epsilon)) + \int_{-\infty}^{t-\epsilon} C_t(t,s)g(s,x(s))ds.$$

and this derivative is bounded by

$$C(t,t-\epsilon)J^* + J^*\int_{-\infty}^{t-\epsilon}|C_t(t,s)|ds \leq J^* \sup_{0\leq t\leq T}\|C(t,t-\epsilon)\| + J^*C^*.$$

This implies that Γ is equicontinuous. The uniform boundedness of Γ follows from the inequality

$$|U(\phi)(t)| \le \int_{-\infty}^{t-\epsilon} |C(t,s)||g(s,\phi(s))|ds \le J^* C^*.$$

6 Periodic Solutions

We will show the existence of T-periodic solutions of (1) by applying Theorem 2.1. By (10) and (11), we see that $x \in P_T$ is a solution of (1_λ) if and only if it is a fixed point of $B + \lambda A$.

Theorem 6.1 *If* (2)-(9), (12), (13), (17), (20), *and* (22) *hold with* $\epsilon \le T$, *then* (1) *has a* T*-periodic solution.*

Proof Let the mappings A and B be defined in (10) and (11) with $S = P_T$. Then B is a contraction mapping with contraction constant η, and hence, $(I - B)^{-1}$ exists and is continuous on $(I - B)S = S$. By Lemma 5.1 and the Ascoli–Arzela theorem, we see that A is continuous and maps bounded sets into compact sets. It is also clear that $\lambda A(M) \subset (I - B)S$ for each closed convex subset $M \subset S$ and $\lambda \in [0, 1]$. Now by Lemma 4.6, the set of solutions to $x = Bx + \lambda Ax$ is bounded. Therefore, the alternative (i) of Theorem 2.1 must hold; that is, $B + A$ has a fixed point in P_T which is a T-periodic solution of (1).

Remark 6.1 Observe that the continuity of $C(t, s)$ with respect to s for $t-\epsilon < s < t$ is not required for fixed t. One may readily verify that the function $C(t, s)$ defined by $C(t, s) = k(t - s)^{-p}$ for $t - s \ge \epsilon$ and $C(t, s) = (t - s)^{-q}$ for $0 < t - s < \epsilon$ with $p > 2, 0 < q < 1, 0 < \epsilon \le 1, k > 0$ satisfy all conditions of Theorem 6.1 for an appropriately chosen constant k.

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