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Existence of a Regular Solution to Quasilinear Implicit Integrodifferential Equations in Banach Space

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Abstract: In the present work, we establish first the existence of a unique local mild solution using contraction mapping theorem and after that the existence of a local classical solution to a class of quasilinear implicit integrodifferential equations in a Banach space. Finally, we demonstrate one application of the results established.

Keywords: *quasilinear evolution equation; mild solution; classical solution; contraction mapping theorem; C_0-semigroups.*

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1 Introduction

Let X and Y be two real Banach spaces such that the embedding $Y \hookrightarrow X$ is dense and continuous. Consider the following quasilinear implicit integrodifferential equation in X

$$\frac{du(t)}{dt} + A(t, u(t))u(t) = f(t, u(t), G(u)(t)), \quad 0 < t \leq T, \quad u(0) = u_0, \tag{1}$$

where $0 < T < \infty$, $A(t, u)$ is a linear operator in X for each u in an open subset W of X, G is a nonlinear Volterra integral operator defined from $C(J, X)$ into $C(J, X)$ where $J = [0, T]$ and the nonlinear map f is defined from $J \times W \times W$ into X. We follow the approach of T. Kato [13, 16, 17] to establish the existence of a unique *classical solution* to (1) under the assumptions (H1)-(H8) to be stated in the next section.

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Crandall and Souganidis [4] have used a different method to prove the existence, uniqueness and continuous dependence of a continuously differentiable solution to the quasilinear evolution equation

$$\frac{du(t)}{dt} + A(u(t))u(t) = 0, \quad 0 < t \leq T, \quad u(0) = u_0,$$

under similar assumptions considered by T. Kato [16].

T. Kato [16] has proved two general theorems on the nonhomogeneous quasilinear evolution equation

$$\frac{du(t)}{dt} + A(t, u(t))u(t) = f(t, u(t)), \quad 0 < t \leq T, \quad u(0) = u_0, \tag{2}$$

one on the existence and uniqueness, and the other on the continuous dependence of a solution on the initial data. Also, he has shown that these theorems are applicable to the different kinds of quasilinear differential equations such as Korteweg-de Vries equation, Navier-Stokes equation and Euler equation, equations for compressible fluids, magnetohydrodynamics equations, coupled Maxwell and Dirac equations etc. The results in [16] are based on the theory of linear 'hyperbolic' equation which was developed in [14, 15].

Murphy [19] constructed a family of approximate solutions to the homogeneous quasilinear evolution equation

$$\frac{du(t)}{dt} + A(t, u(t))u(t) = 0, \quad 0 < t \leq T, \quad u(0) = u_0. \tag{3}$$

He showed that the approximate solution converges to a "limit solution" and this "limit solution" becomes a unique solution to (3) under certain additional assumptions. [12] has extended the result in [19] to the nonhomogeneous equation (2) under slightly more general conditions than those of [16].

In [2], Bahuguna had used the method of lines (also known as Rothe's method) and the techniques of Crandall and Souganidis [4] to prove the existence, uniqueness and continuous dependence of a *strong solution* to the quasilinear explicit integrodifferential equaton

$$\frac{du(t)}{dt} + A(u(t))u(t) = K(u)(t) + f(t), \quad 0 < t \leq T, \quad u(0) = u_0,$$

in a Banach space X whose dual X^* is assumed to be uniformly convex under the additional assumption of compactness on the embedding of Y in X and where K is the nonlinear Volterra operator. Using technique of [2], Bahuguna and Shukla [3] have established similar results for the quasilinear implicit integrodifferential equation

$$\frac{du(t)}{dt} + A(u(t))u(t) = f(t, u(t), G(u)(t)), \quad 0 < t \leq T, \quad u(0) = u_0,$$

in Banach spaces. Further, using same technique of papers [2] and [3], Dubey [5] has established the similar result for the equation (1).

For the application of analytic semigroups to related quasilinear evolution equations we refer to Amann [1], Lunardy [18] while for the applications of fixed point theorems the reader may refer to Kartsatos [9, 10], Kartsatos and Parrott [11] and references cited therein.

Dubey [6] has established the local existence and uniqueness of a classical solution of an abstract second order integrodifferential equation in a Banach space by using the theory of analytic semigroups and contraction mapping theorem . The continuation of classical solution, the maximal interval of the existence and the global existence of the classical solution have been also studied. Pandey, Ujlayan and Bahuguna [8]considered an abstract semilinear hyperbolic integrodifferential equation and used the theory of resolvent operators to establish the existence and uniqueness of a mild solution under local Lipschitz conditions on the nonlinear maps and an integrability condition on the kernel. Under some additional conditions on the nonlinear maps they also proved the existence of a classical solution.

The plan of the paper is as follows. In the second section, we collect some preliminaries, notations and some results which easily follow from the hypotheses. In the third section, first, we establish the existence of a unique local mild solution using contraction mapping theorem and also the existence of a local classical solution to (1). Finally, in the last section, we demonstrate one application of the results established in earlier sections.

2 Preliminaries

Let X and Y be as in the earlier section. The norm in any Banach space Z is denoted by $\|.\|_Z$. $\bar{B}_Z(r, z_0)$ is the closure of the open ball $B_Z(r, z_0) = \{z \in Z \mid \|z - z_0\|_Z < r\}$ with radius r and center at z_0 in the Banach space Z. The space of all bounded linear operators from a Banach space X to a Banach space Y is denoted by $B(X, Y)$ and $B(X, X)$ is written as $B(X)$. Let J denote the interval $[0, T]$. The space $C^m(J, Z)$ represents the space of all m-times continuously differentiable functions defined from J into Z, $m = 1, 2, ...$; endowed with the supremum norm

$$\|u\|_{C^m(J,Z)} = \sum_{1 \le i \le m} \sup_{t \in J} \|u^{(i)}(t)\|, \quad u \in C^m(J, Z),$$

where $u^{(i)}$ denotes the ith derivative of u with $u^{(0)} = u$. Let W be a subset of X. A family $\{A(t, w) : (t, w) \in J \times W\}$, of infinitesimal generators of C_0-semigroups $S_{t,w}(s), s \ge 0$ on X is called stable if there exists real numbers $M \ge 1$ and ω, known as *stability constants*, such that

$$\rho(A(t, w)) \supset (\omega, \infty) \quad \text{for} \quad (t, w) \in J \times W,$$

where $\rho(A(t, w))$ is the resolvent set of $A(t, w)$ and

$$\left\| \prod_{j=1}^{k} R(\lambda; A(t_j, w_j)) \right\|_{B(X)} \le M(\lambda - \omega)^{-k} \quad \text{for } \lambda > \omega$$

and every finite sequence

$$0 \le t_1 \le t_2 \le .... \le t_k \le T, \quad w_j \in W, \quad 1 \le j \le k.$$

For a linear operator S in X, by the part $\tilde{S}$ of S in a subspace Y of X, we mean a linear operator $\tilde{S}$ with domain $D(\tilde{S}) = \{x \in D(S) \cap Y \mid Sx \in Y\}$ and values $\tilde{S}x = Sx$ for $x \in D(\tilde{S})$.

Let $S_{t,w}(s)$, $s \ge 0$, be the C_0-semigroup generated by $A(t, w)$, $(t, w) \in J \times W$. A subset Y of X called $A(t, w)$-admissible if Y is an invariant subspace of operator $S_{t,w}(s)$, $s \ge 0$, and the restriction of $S_{t,w}(s)$ to Y is a C_0-semigroup in Y.

For more details of the above mentioned notions, one may refer to the chapters 5 and 6 in Pazy [7]. On the family of operators $\{A(t,w) : (t,w) \in J \times W\}$, we make the same assumptions $(\tilde{H}1)$-$(\tilde{H}4)$ considered in §6.6.4 in Pazy [7] for the homogeneous quasilinear evolution equation, as restated below.
(H1) The family $\{A(t,w) : \ (t,w) \in J \times W\}$ is stable.
(H2) Y is $A(t,w)$-admissible for $(t,w) \in J \times W$ and the family $\{\tilde{A}(t,w) : \ (t,w) \in J \times W\}$ of the parts of $A(t,w)$ in Y is stable in Y.
(H3) For $(t,w) \in J \times W$, $D(A(t,w)) \supset Y$, $A(t,w)$ is a bounded linear operator from Y to X, and the map $t \mapsto A(t,w)$ is continuous in $B(Y,X)$ with associated norm $\|.\|_{Y \to X}$ for every $w \in W$.
(H4) There is a positive constant L_A such that

$$\|A(t,w_1) - A(t,w_2)\|_{Y \to X} \leq L_A \|w_1 - w_2\|_X$$

for every $w_1, w_2 \in W$ and $0 \leq t \leq T$.

A two parameter family of bounded linear operators $U(t,s)$, $0 \leq s \leq t \leq T$, on X is called an *evolution system* if the following two conditions are satisfied:

(i) $U(s,s) = I$ and $U(t,r)U(r,s) = U(t,s)$ for $0 \leq s \leq r \leq t \leq T$.

(ii) The map $(t,s) \mapsto U(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

If $u \in C(J,X)$ has values in W and the family $\{A(t,w) : (t,w) \in J \times W\}$ of the operators satisfies the assumptions(H1)-(H4) then there exists a unique evolution system $U_u(t,s)$ in X satisfying

$$\text{(i)} \quad \|U_u(t,s)\|_X \leq Me^{\omega(t-s)} \tag{4}$$

for $0 \leq s \leq t \leq T$, where M and ω are the stability constants;

$$\text{(ii)} \quad \frac{\partial^+}{\partial t} U_u(t,s)w|_{t=s} = A(s,u(s))w \tag{5}$$

for $w \in Y$ and $0 \leq s \leq T$;

$$\text{(iii)} \quad \frac{\partial}{\partial s} U_u(t,s)w = -U_u(t,s)A(s,u(s))w \tag{6}$$

for $\omega \in Y$ and $0 \leq s \leq T$.

Further, there exists a positive constant C_0 such that for every $u, v \in C(J,X)$ with values in W and for every $y \in Y$, we have

$$\|U_u(t,s)y - U_v(t,s)y\|_X \leq C_0 \|y\|_Y \int_s^t \|u(\tau) - v(\tau)\|_X d\tau. \tag{7}$$

For details of the above mentioned results, one may refer to Theorem 6.4.3 and Lemma 6.4.4 in Pazy [7].

We further assume that
(H5) For every $u \in C(J,X)$ satisfying $u(t) \in W$ for $t \in J$, we have

$$U_u(t,s)Y \subset Y, \quad \text{for} \quad t,s \in J \quad \text{and} \quad s \leq t$$

and $U_u(t,s)$ is strongly continuous in Y for $s,t \in J$ and $s \leq t$.
(H6) Closed convex subsets of Y are also closed in X.
(H7) The nonlinear map $G : C(J,X) \to C(J,X)$ satisfy the following:

(a) For all $u, v \in \bar{B}_{C(J,X)}(\tilde{u_0}, r)$,

$$\|G(u) - G(v)\|_{C(J,X)} \leq \mu_G(r)\|u - v\|_{C(J,X)},$$

where $\mu_G(r)$ is a nonnegative nondecreasing function and $\tilde{u_0} \in C(J, X)$ is defined by $\tilde{u}_0 = u_0$ for all $t \in J$.

(b) The subspace $C(J, Y)$ of space $C(J, X)$ is an invariant subspace of the map G, i.e. the map $G : C(J, Y) \to C(J, Y)$ satisfies

$$\|G(u)(t)\|_Y \leq \lambda_G(r) \quad \text{for} \quad u \in \bar{B}_Y(u_0, r),$$

where $\lambda_G(r)$ is a nonnegative nondecreasing function. In particular, we may take operator G as a Volterra operator defined by

$$G(u)(t) = \int_0^t a(t - s)k(s, u(s))ds,$$

where a is a real valued continuous function defined on J and k is defined on $J \times Y$ into Y and $\|k(t, w)\|_Y \leq C_k$ for every $(t, w) \in J \times Y$. Clearly, the map G satisfies (b).

(H8) The nonlinear map $f : J \times W \times W \to X$ satisfies

(a) For $(t, u, v) \in J \times (W \cap Y) \times (W \cap Y)$ and $f(t, u, v) \in Y$, we have

$$\|f(t, u, v)\|_Y \leq \lambda_f(r)$$

for all $(t, u, v) \in J \times W \times W$ with $\|u\|_Y + \|v\|_Y \leq r$, where $\lambda_f(r)$ is a nonnegative nondecreasing function.

(b) In both $Z = X,\ Y$, the map f satisfies the Lipschitz like condition

$$\|f(t_1, u_1, v_1) - f(t_2, u_2, v_2)\|_Z \leq \mu_f(r)[|\phi(t_1) - \phi(t_2)| + \|u_1 - u_2\|_Z + \|v_1 - v_2\|_Z],$$

for all $t_1, t_2 \in [0, T]$ and all $u_i, v_i \in \bar{B}_Y(u_0, r)$, $i = 1, 2$, where ϕ is a real-valued continuous function of bounded variation on $[0, T]$ and $\mu_f(r)$ is a nonnegative nondecreasing function.

By a *mild solution* to (1) on $J = [0, T]$, we mean a function $u \in C(J, X)$ with values in W satisfying the integral equation

$$u(t) = U_u(t, 0)u_0 + \int_0^t U_u(t, s)f(s, u(s), G(u)(s))ds, \quad t \in J. \tag{8}$$

By the *Classical solution* u to (1) on $J = [0, T]$, we mean a function $u \in C(J, X)$ such that $u(t) \in Y \cap W$ for $t \in (0, T]$, $u \in C^1((0, T], X)$ and satisfies (1) in X. If there exists a T' with $0 < T' \leq T$ and a function $u \in C(J', X)$, where $J' = [0, T']$ such that u is a mild (classical) solution to (1) on J', then u is called a *local mild (classical)* solution to (1).

3 Main Result

In this section, we prove the following result.

Theorem 3.1 *Suppose that $u_0 \in Y$ and the family $\{A(t, w)\}$ of linear operators for $t \in J = [0, T]$ and $w \in W = \{y \in Y : \|y - u_0\|_Y \leq r\}$, for fixed $r > 0$, satisfy the assumptions (H1)-(H6) and $A(t, w)u_0 \in Y$ satisfies*

$$\|A(t, w)u_0\|_Y \leq C_A \tag{9}$$

for all $(t, w) \in J \times W$.

Further, suppose that the nonlinear maps G *and* f *satisfy (H7) and (H8), respectively. Then, there exists a unique local classical solution to (1).*

Proof First, we establish the existence of a unique local mild solution to (1). We note that from assumption (H5), it follows that

$$\|U_u(t,s)\|_{B(Y)} \leq C_1 \tag{10}$$

for $s \leq t$, $s, t \in J$ and every $u \in C(J, X)$ with values in W. We choose

$$T_0 = \min\left\{T, \frac{r}{2C_A C_1}, \frac{r}{2C_1 \lambda_f(R_1)}, \frac{1}{2P}\right\}, \tag{11}$$

where

$$P = C_0\|u_0\|_Y + Me^{\omega T}\mu_f(R_1)(1 + \mu_G(r)) + C_0\lambda_f(R_1)T$$

and

$$R_1 = r + \|u_0\|_Y + \lambda_G(r).$$

Let S be the subset of $C(J_0, X)$ defined by

$$S = \{u \in C(J_0, X) \mid u(0) = u_0, \text{ and } u(t) \in W \text{ for } t \in J_0\},$$

where $J_0 = [0, T_0]$. From (H6), it follows that S is a closed convex subset of $C(J_0, X)$. Next, we define a mapping $F : S \to S$ by

$$Fu(t) = U_u(t,0)u_0 + \int_0^t U_u(t,s)f(s, u(s), G(u)(s))ds \tag{12}$$

and check that F is well defined. Clearly, $Fu(0) = u_0$, $Fu \in C(J_0, X)$ and (H5) implies that $Fu(t) \in Y$ for $t \in J_0$. It remains to show that $\|Fu(t) - u_0\|_Y \leq r$ for $t \in J_0$. Now,

$$Fu(t) - u_0 = U_u(t,0)u_0 - u_0 + \int_0^t U_u(t,s)f(s, u(s), G(u)(s))ds. \tag{13}$$

Integrating (6) in X from 0 to t, we get

$$U_u(t,0)u_0 - u_0 = \int_0^t U_u(t,\tau)A(\tau, u(\tau))u_0 d\tau. \tag{14}$$

Using (9) and (10) in (14), we obtain

$$\|U_u(t,0)u_0 - u_0\|_Y \leq C_1 C_A T_0 \leq \frac{r}{2}. \tag{15}$$

Further, using (10) and (H8), we get

$$\left\|\int_0^t U_u(t,s)f(s, u(s), G(u)(s))ds\right\|_Y \leq C_1\lambda_f(R_1)T_0 \leq \frac{r}{2}, \tag{16}$$

since $\|u(s)\|_Y + \|G(u)(s)\|_Y \leq R_1$. Using (15) and (16) in (13), we see that F is well defined. For $u, v \in S$, we have

$$\begin{aligned} Fu(t) - Fv(t) &= (U_u(t,0) - U_v(t,0))u_0 \\ &\quad + \int_0^t [U_u(t,s)f(s, u(s), G(u)(s)) - U_v(t,s)f(s, v(s), G(v)(s))]ds \\ &= T_1 + T_2, \end{aligned} \tag{17}$$

where T_1 and T_2 represent the first and second terms on the right hand side of (17). We use (7) to obtain $\|T_1\|_X \leq C_0\|u_0\|_Y T_0\|u-v\|_{C(J_0,X)}$. Further, from (H7), (H8) and (7) it follows that

$$\begin{aligned}\|T_2\|_X &\leq \left\|\int_0^t U_u(t,s)[f(s,u(s),G(u)(s)) - f(s,v(s),G(v)(s))]ds\right\|_X \\ &\quad + \left\|\int_0^t [U_u(t,s)-U_v(t,s)]f(s,v(s),G(v)(s))ds\right\|_X \\ &\leq [Me^{\omega T}\mu_f(R_1)(1+\mu_G(r)) + C_0\lambda_f(R_1)T]T_0\|u-v\|_{C(J_0,X)}.\end{aligned}$$

Also,

$$\begin{aligned}\|f(s,u(s),G(u)(s)) &- f(s,v,(s),G(v)(s)\|_X \\ &\leq \mu_f(R_1)[\|u(s)-v(s)\|_X + \|G(v)(s)-G(u)(s)\|_X] \\ &\leq \mu_f(R_1)[\|u-v\|_{C(J_0,X)} + \|G(u)-G(v)\|_{C(J_0,X)}] \\ &\leq \mu_f(R_1)(1+\mu_G(r))\|u-v\|_{C(J_0,X)}.\end{aligned}$$

Hence, from (17), we have

$$\|Fu-Fv\|_{C(J_0,X)} \leq PT_0\|u-v\|_{C(J_0,X)} \leq \frac{1}{2}\|u-v\|_{C(J_0,X)}.$$

This shows that, F is a contraction map from S to S. Since S is closed in X, by the contraction mapping theorem, F has a unique fixed point $u \in S$ which is the local mild solution to (1).

Now, we show that $u \in C(J_0,Y)$. For $s,t \in J_0$ with $s \leq t$, we have

$$\begin{aligned}u(t)-u(s) &= (U_u(t,0)-U_u(s,0))u_0 \\ &\quad + \int_0^s (U_u(t,\eta)-U_u(s,\eta))f(\eta,u(\eta),G(u)(\eta))d\eta \\ &\quad + \int_s^t U_u(t,\eta)f(\eta,u(\eta),G(u)(\eta))d\eta.\end{aligned}$$

Since $U_u(t,s)$ is strongly continuous in Y for $s,t \in J$ and $s \leq t$. So, for every $\epsilon > 0$, there exist δ_1, $\delta_2 > 0$ such that

$$t,s \in J_0 \quad \text{with} \quad |t-s| \leq \delta_1 \quad \Rightarrow \quad \|U_u(t,0)-U_u(s,0)\|_{B(Y)} \leq \frac{\epsilon}{3\|u_0\|_Y}$$

and

$$t,s \in J_0 \quad \text{with} \quad |t-s| \leq \delta_2 \quad \Rightarrow \quad \|U_u(t,\eta)-U_u(s,\eta)\|_{B(Y)} \leq \frac{\epsilon}{3\lambda_f(R_1)T_0}.$$

Let $\delta = \min\{\delta_1,\delta_2,\frac{\epsilon}{3C_1\lambda_f(R_1)}\}$. Then, for $s,t \in J_0$

$$|t-s| \leq \delta \Rightarrow \|u(t)-u(s)\|_Y \leq \epsilon.$$

Thus, $u \in C(J_0,Y)$.

Consider the following linear evolution equation

$$\frac{dv(t)}{dt} + B(t)v(t) = h(t), \quad 0 < t \leq T_0, \quad v(0) = u_0, \tag{18}$$

where $B(t) = A(t, u(t))$ and $h(t) = f(t, u(t), G(u)(t))$ for $t \in J_0$ and u being the unique fixed point of F in S. We note that $B(t)$ satisfies (H1)-(H3) of §5.5.3 in Pazy [7].

We have to prove that $h \in C(J_0, Y)$. For $s, t \in J_0$ (we assume without loss of generality that $s \leq t$), we have

$$\begin{aligned} \|h(t) - h(s)\|_Y &= \|f(t, u(t), G(u)(t)) - f(s, u(s), G(u)(s))\|_Y \\ &\leq \mu_f(R_1)[|\phi(t) - \phi(s)| + \|u(t) - u(s)\|_Y + \|G(u)(t) - G(u)(s)\|_Y]. \end{aligned}$$

As ϕ is a continuous function of bounded variation on J, $u \in C(J_0, Y)$ and $G(u) \in C(J_0, Y)$ for $u \in C(J_0, Y)$. So, for every $\epsilon > 0$, there exist $\delta_1 > 0$, $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$t, s \in J_0 \quad \text{with} \quad |t - s| \leq \delta_1 \quad \Rightarrow \quad |\phi(t) - \phi(s)| \leq \frac{\epsilon}{3\mu_f(R_1)},$$

$$t, s \in J_0 \quad \text{with} \quad |t - s| \leq \delta_2 \quad \Rightarrow \quad \|u(t) - u(s)\|_Y \leq \frac{\epsilon}{3\mu_f(R_1)}$$

and

$$t, s \in J_0 \quad \text{with} \quad |t - s| \leq \delta_3 \quad \Rightarrow \quad \|G(u)(t) - G(u)(s)\|_Y \leq \frac{\epsilon}{3\mu_f(R_1)}.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then, for $s, t \in J_0$, we have: $|t - s| \leq \delta \ \Rightarrow \ \|h(t) - h(s)\|_Y \leq \epsilon$. Thus, $h \in C(J_0, Y)$. Theorem 5.5.2 in Pazy [7] implies that there exists a unique function $v \in C(J_0, Y)$ such that $v \in C^1(J_0/\{0\}, X)$ satisfying (18) in X and v is given by

$$v(t) = U_u(t, 0)u_0 + \int_0^t U_u(t, s)f(s, u(s), G(u)(s))ds, \quad t \in J_0,$$

where $U_u(t, s)$, $0 \leq s \leq t \leq T_0$ is the evolution system generated by the family $\{A(t, u(t))\}$, $t \in J_0$, of linear operators in X. The uniqueness of v implies that $v \equiv u$ on J_0 and hence u is a unique local classical solution to (1). This completes the proof.

4 Application

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the differential operator

$$A(t, x, u; D)w = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t, x, u(t, x))\frac{\partial w}{\partial x_j}\right) + c(t, x, u(t, x))w,$$

where $a_{ij}(t, x, u(t, x))$ and $c(t, x, u(t, x))$ are real valued functions defined on $I \times \overline{\Omega} \times \mathbb{R}$ and $I = [0, T]$, $0 < T < \infty$. We assume that $a_{ij} \in C[I \times \overline{\Omega} \times W, \mathbb{R}]$, where $W = C^{2l+1}(I \times \overline{\Omega}, \mathbb{R})$ with $1/2 < l < 1$, $a_{ij} = a_{ji}$, $(1 \leq i, j \leq n)$ and there exists some $\delta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(t, x, u(t, x))q_i q_j \geq \delta|q|^2, \quad q = (q_1, ..., q_n) \in \mathbb{R}$$

holds for each $(t, x, u(t, x)) \in I \times \overline{\Omega} \times \mathbb{R}$.

Consider the partial integrodifferential equation

$$\frac{\partial u(t,x)}{\partial t} + A(t,x,u;D)u(t,x) = f(t,x,u(t,x),K(u)(t,x)), \ \ (t,x) \in (0,T] \times \Omega \qquad (19)$$

with boundary condition

$$u(t,x) = 0 \quad \text{for} \quad (t,x) \in (0,T] \times \partial\Omega$$

and initial condition

$$u(0,x) = u_0(x) \quad \text{for} \quad x \in \Omega,$$

where

$$K(u)(t,x) = \int_0^t a(t-s)k(s,x,u(s,x),\nabla u(s,x))ds,$$

$$\nabla = (D_1, D_2, ..., D_n), \quad D_i = \frac{\partial}{\partial x_i},$$

the function a is a real valued continuous function of bounded variation in $\mathbb{R}$ and the function $f(t,x,u,v)$ is also a real valued continuous function defined on $I \times \overline{\Omega} \times W \times W$ and for every $t_0 > 0$, $r_0 > 0$ there exists $L_0 > 0$ such that if $\|u_1\| \leq r_0$, $\|u_2\| \leq r_0$, then

$$\|f(t,x,u_1,v_1) - f(s,x,u_2,v_2\| \leq L_0[|\psi(t) - \psi(s)| + \|u_1 - u_2\| + \|v_1 - v_2\|]$$

for $x \in \Omega$, $u_i, v_i \in W$, $i = 1,2$ and ψ is a real valued continuous function of bounded variation. $u : I \times \Omega \to \mathbb{R}$ is unknown function and u_0 is its initial value.

Further, we assume that $k : [0,\infty) \times \Omega \times W \times W \to \mathbb{R}$ is continuous and for every $t_0 > 0$, $r_0 > 0$ there exists $M_0 > 0$ such that if $\|u\| \leq r_0$, $\|v\| \leq r_0$, then

$$\|k(t,x,u,\xi) - k(t,x,v,\eta)\| \leq M_0[\|u - v\| + \|\xi - \eta\|]$$

for all $0 \leq t \leq t_0$, $x \in \Omega$ and $u, v, \xi, \eta \in W$.

Let $\frac{n}{2l-1} < p < \infty$ and $X = L^p(\Omega)$ with the usual norm

$$\|u\|_p = \left[\int_\Omega |u|^p dx\right]^{1/p},$$

then integrodifferential equation (19) can be reformulated as abstract integrodifferential equation (1) in Banach space X, where

$$A(t,u)w = A(t,x,u;D)w$$

with domain

$$D(A(t,u)) = \{w \in W_p^2(\Omega) : w(t,x) = 0, (t,x) \in (0,T] \times \partial\Omega\}$$

and

$$f(t,u,G(u))(x) = f(t,x,u(t,x),K(u)(t,x)).$$

We note that the assumptions (H1)-(H8) are satisfied thus we may apply the result of the earlier section to guarantee the existence of unique classical solution of (19).

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