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Stability Analysis of Phase Synchronization in Coupled Chaotic Systems Presented by Fractional Differential Equations

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Abstract: In this paper, we have considered phase synchronizations in coupled chaotic systems presented by fractional differential equations. This synchronization occurs when some eigenvalues of the matrix found in the linear approximation of difference evolutional equation between coupled chaotic systems have zero real parts. Here, we have used nonlinear feedback function for synchronization. We have also demonstrated some numerical examples to show the accuracy of our analytical stability in some coupled chaotic fractional differential equations.

Keywords: *chaos; synchronization; fractional differential equations.*

Mathematics Subject Classification (2000): 34H10, 34D06, 34A08.

1 Introduction

As Pecora and Carroll have shown [1] in coupled chaotic systems, a complete synchronization occurs if the difference between various states of synchronized systems converges to zero. They have also shown that, synchronization stability depends upon the signs of the conditional Lyapunov exponents. That is, if all of the Lyapunov exponents of the response system under the action of the driver are negative, then there is a complete and stable synchronization between the drive and response systems. Stability of the synchronization can also be verified using the Jacobian matrix in the linearized system, where the linearized system represents the state difference between the drive and response chaotic systems [2]. Following this stability analysis and despite the theory of stability analysis in dynamical system, if this Jacobian matrix is of full rank and all of

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its eigenvalues are negative, then the system will converge to zero and yield complete synchronization. However, phase synchronization occurs when this Jacobian matrix has some zero eigenvalues. In this case, the difference between various states of synchronized systems may not necessarily converge to the zero, but will stay less than or equal to a constant.

Recently, fractional differential equations (FDEs) have been utilized to study dynamical systems in general, chaos, and synchronization in particular [3]–[7]. It is well-known that FDEs are useful because of their non-local nature, whereas for integer order (classical) differential equations that this property is the local one. Although the theory of fractional calculus is a 300-year-old topic which can trace back to Leibniz, Riemann, Liouville, Grnwald and Letnikov, the applications of fractional calculus to physics and engineering are just a recent focus of interest [8, 9]. Many systems are known to display fractional order dynamics, such as viscoelastic system [10], colored noise, dielectric polarization [11], electrode-electrolyte polarization [12] and electromagnetic wave [13], the control of fractional order dynamic systems [14] and so on. The main goal of this paper is to discuss the stability analysis of phase synchronization in coupled chaotic systems presented by FDEs. To do this, after some primarily definitions in the next section we implement the nonlinear coupling feedback function method for some coupled chaotic FDEs to discuse synchronization and phase synchronization in section 3. We also present two criteria for phase synchronization in both coupled chaotic ODE and FDE systems. Then in section 4, we illustrate the numerical results of two coupled chaotic systems in the form of FDEs in which the phase synchronizations and their convergences exist.

2 Preliminaries

In this section, we present some basic definitions and properties [8, 15].

2.1 Fractional Calculus

Definition 2.1 A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p(> \mu)$ such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0, \infty)$.

Definition 2.2 Let $f \in C_\mu$ and $\mu \geq 1$, then the (left-sided) Riemann–Liouville integral of order $\alpha, \alpha > 0$, is given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s.$$

Definition 2.3 The (left-sided) Caputo fractional derivative of $f, f \in C_{-1}^m$ with order $\alpha > 0$ and $m \in \mathbb{N} \cup 0$, is defined as

$$\frac{d^\alpha f(t)}{dt^\alpha} = D_*^\alpha f(t) = \begin{cases} [I^{m-\alpha} \frac{d^m}{dt^m} f(t)], & m-1 < \alpha \leq m, \quad m \in \mathbb{N}, \\ \frac{d^m}{dt^m} f(t), & \alpha = m. \end{cases}$$

2.2 Numerical method for solving FDEs

Recently, the approximate numerical techniques for FDEs have been developed in literature, which are numerically stable and can be applied to both linear and nonlinear FDEs. Diethelm et al. [16] presented a PECE (predict, evaluate, correct, evaluate) type method for numerical solution of FDEs with Caputo derivatives, which is a generalization of the classical one-step Adams–Bashforth–Moulton algorithm for first order ordinary differential equations.

The fractional Predictor–Corrector (PC) algorithm is based on the analytical property that the following FDE

$$D^\alpha y(t) = f(t, y(t)), \quad 0 \le t \le T,$$
$$y^{(k)}(0) = y_0^{(k)}, k = 0, 1, ..., m-1 \ (m = \lceil \alpha \rceil)$$

is equivalent to the Volterra integral equation [16]

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds.$$

Now, set $h = T/N, t_n = nh, n = 0, 1, ..., N$. Let $y_h(t_n)$ be approximation to $y(t_n)$. Assume that we have already calculated approximations $y_h(t_j)$ and we want to obtain $y_h(t_{n+1})$ by means of the equation

$$y_h(t_{n+1}) = \sum_{k=0}^{m-1} c_k \frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, y_h^p(t_{n+1})) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j)),$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha & if \ \ j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j-1)^{\alpha+1}, & if \ \ 1 \le j \le n, \\ 1, & if \ \ j = n+1, \end{cases}$$

and

$$y_h^p(t_{n+1}) = \sum_{k=0}^{m-1} c_k \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j)),$$

in which $b_{j,n+1} = \frac{h^\alpha}{\alpha}((n+1-j)^\alpha - (n-j)^\alpha)$. Therefore, the estimation error of this approximation is $\max_{j=0,1,...,N} |y(t_j) - y_h(t_j)| = O(h^p)$, where $p = \min(2, 1+\alpha)$.

3 Phase Synchronization in Fractional Order Dynamical Systems

Here, we use the nonlinear coupling feedback function method introduced by Ali and Fang [25] to couple two chaotic FDEs. Using this method on the FDE $\mathbf{D}^\alpha \mathbf{x}(t) = \mathbf{F}(\mathbf{x}(t))$, we suppose the vector-valued function $\mathbf{F}(\mathbf{x}(t))$ is decomposed into linear, $\mathbf{L}(\mathbf{x}(t))$, and non-linear, $\mathbf{N}(\mathbf{x}(t))$, components. That is,

$$\mathbf{F}(\mathbf{x}(t)) = \mathbf{L}(\mathbf{x}(t)) - \mathbf{N}(\mathbf{x}(t)). \tag{1}$$

Now consider two chaotic FDEs systems whose associated vector functions are decomposed as in (1) and coupled by using the non-linear parts of their vector functions as follows:

$$\mathbf{D}^\alpha \mathbf{x_1}(t) = \mathbf{L}(\mathbf{x_1}(t)) - \mathbf{N}(\mathbf{x_1}(t)) + s[\mathbf{N}(\mathbf{x_1}(t)) - \mathbf{N}(\mathbf{x_2}(t))], \tag{2}$$
$$\mathbf{D}^\alpha \mathbf{x_2}(t) = \mathbf{L}(\mathbf{x_2}(t)) - \mathbf{N}(\mathbf{x_2}(t)) + s[\mathbf{N}(\mathbf{x_2}(t)) - \mathbf{N}(\mathbf{x_1}(t))]. \tag{3}$$

Here, systems (2) and (3) serve as drive and response systems, respectively, and s measures the strength of their coupling. In a manner analogous to integer order differential equations, the stability of the synchronization in this fractional situation can be studied by using the evolutional equation of the difference between systems (2) and (3). This equation is determined by the linear approximation

$$\mathbf{D}^{\alpha}\mathbf{e}(t) = \left[\mathbf{L} + (2s-1)\frac{\partial \mathbf{N}}{\partial \mathbf{x}}\right]\mathbf{e}(t), \tag{4}$$

where $\mathbf{e}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$. It is well-known from linear stability theory in dynamical systems that if $\alpha = 1$ and $s = 0.5$, then the stability type of the zero equilibrium in Eq. (4) reflects the stability type of the synchronization between the two chaotic systems and depends upon the signs of the real parts of the eigenvalues $\mathbf{L}$ [6]. However, in the case $0 < \alpha < 1$ and $s = 0.5$ we cannot use this stability criterion, instead we can use the following Matignon's theorem [18].

Theorem 3.1 *The linearized system of fractional differential equations,* $\mathbf{D}^{\alpha}\mathbf{x}(t) = \mathbf{L}(\mathbf{x}(t))$*, is asymptotically stable if and only if* $|\arg(spec(\mathbf{L}))| > \alpha\pi/2$*.*

We recall that in the case of phase synchronization the error $\mathbf{e}(t)$ converges to a constant or remains bounded by a constant. So, by just some modification on Theorem 1, we can analyse the convergence of phase synchronization.

Theorem 3.2 *Define* $\mathbf{E}(t) = \mathbf{e}(t) - \mathbf{c}$ *and let* $s = 0.5$*. Then the linear system of fractional differential equations* $\mathbf{D}^{\alpha}\mathbf{E}(t) = \mathbf{L}(\mathbf{E}(t))$ *is asymptotically stable if and only if* $|\arg(spec(\mathbf{L}))| > \alpha\pi/2$*. In this case, the vector* $\mathbf{e}(t)$ *converges to* $\mathbf{c}$ *at the rate* $t^{-\alpha}$*.*

Note that stability exists if and only if either asymptotic stability exists or those eigenvalues which satisfy $|\arg(spec(\mathbf{L}))| = \alpha\pi/2$ have geometric multiplicity one.

4 Numerical Results

To see our assertion in above analytical justification for the phase synchronization in FDEs, we first consider the diffusionless Lorenz chaotic system presented by FDEs

$$\begin{cases} D^{\alpha}x = -x - y, \\ D^{\alpha}y = -xz, \\ D^{\alpha}z = -xy + r. \end{cases} \tag{5}$$

This system is chaotic for $\alpha = 1$ and $r \in (0, 5)$ [19]. With the same value of r, system (5) remains chaotic for $0.88 < \alpha < 1$. Now using the nonlinear coupling feedback function method, drive and response systems can be presented by

$$\begin{cases} D^{\alpha}x_1 = -x_1 - y_1, \\ D^{\alpha}y_1 = -x_1z_1 + s(x_1z_1 - x_2z_2), \\ D^{\alpha}z_1 = x_1y_1 + r + s(x_2y_2 - x_1y_1), \\ D^{\alpha}x_2 = -x_2 - y_2, \\ D^{\alpha}y_2 = -x_2z_2 + s(x_2z_2 - x_1z_1), \\ D^{\alpha}z_2 = x_2y_2 + r + s(x_1y_1 - x_2y_2). \end{cases} \tag{6}$$

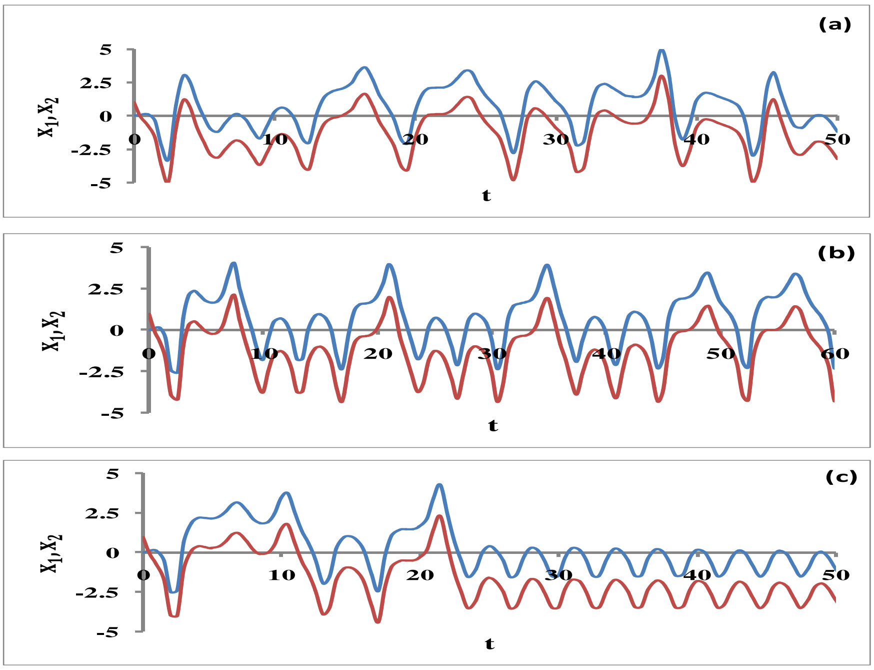

Figure 1: Phase synchronization in (x_1, x_2) plane for $\alpha = 0.95$ in (a), $\alpha = 0.91$ in (b) and $\alpha = 0.9$ in (c).

Here matrix **L** in error linear approximation (4) will be

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As we can see, the eigenvalues of matrix **L** are -1 and zero with multiplicity 2. So the condition for phase synchronization exists. In addition, it is easy to see that the condition in Theorem 2 is also satisfied for the convergence of this phase synchronization. Now using the PC method described in Section 2 to approximate the solutions of system (6), with $s = 0.5$, the results are illustrated in Figures 1 for different values of α. As we can see in Figure 1-c the phase synchronization exits, but the chaotic solution is merging to the limit cycle. This is because of the derivatives order $\alpha = 0.9$ which affects the system and changes its chaotic solution to the limit cycle.

As the next example, we introduce a new chaotic system in 4-dimensional space as follows.

$$\begin{cases} D^\alpha x = -ax - by + w, \\ D^\alpha y = -cy - axz, \\ D^\alpha z = -z + axy + d, \\ D^\alpha w = -fw - exz. \end{cases} \tag{7}$$

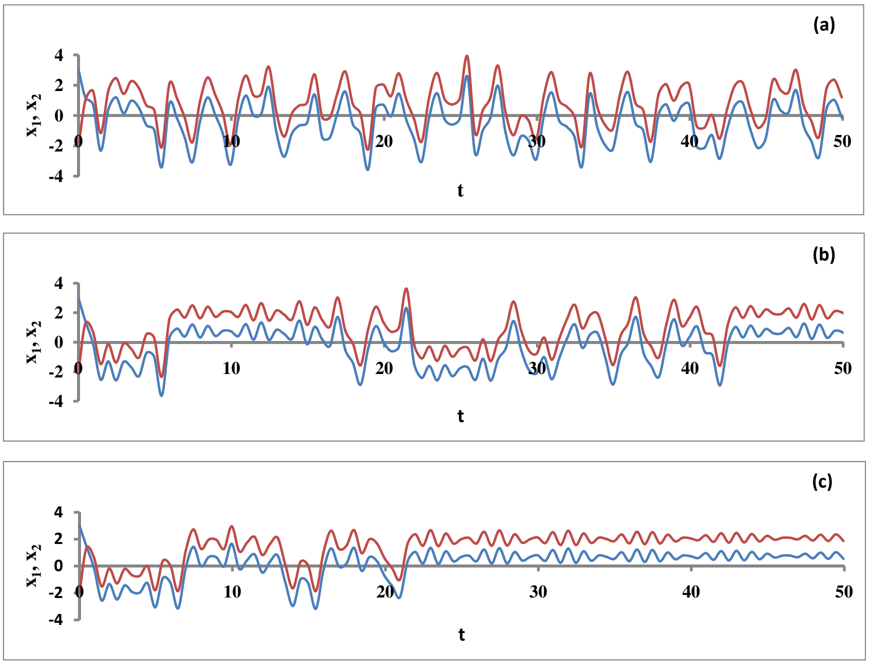

Figure 2: Phase synchronization in (x_1, x_2) plane for $\alpha = 0.95$ in (a), $\alpha = 0.9$ in (b) and $\alpha = 0.89$ in (c).

This system is chaotic for the parameters values $a = 3$, $b = 2$, $c = 1$, $d = 15$, $e = 0.2$ and $f = 1$. The system will remain chaotic for $0.92 \leq \alpha < 1$. Using nonlinear coupling feedback function method, system (7) is coupled as follows

$$\begin{cases} D^\alpha x_1 = -ax_1 - by_1 + w_1, \\ D^\alpha y_1 = -cy_1 - axz_1 + sa(x_1 z_1 - x_2 z_2), \\ D^\alpha z_1 = -z_1 + ax_1 y_1 + d + sa(x_2 y_2 - x_1 y_1), \\ D^\alpha w_1 = -fw_1 - ex_1 z_1 + se(x_1 z_1 - x_2 z_2), \\ D^\alpha x_2 = -ax_2 - by_2 + w_2, \\ D^\alpha y_2 = -cy_2 - axz_2 + sa(x_2 z_2 - x_1 z_1), \\ D^\alpha z_2 = -z_2 + ax_2 y_2 + d + sa(x_1 y_1 - x_2 y_2), \\ D^\alpha w_2 = -fw_2 - ex_2 z_2 + se(x_2 z_2 - x_1 z_1). \end{cases} \tag{8}$$

For this system matrix $\mathbf{L}$ in error linear approximation (4) will be

$$\begin{pmatrix} -a & -b & 0 & 1 \\ 0 & -c & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -f \end{pmatrix}$$

and its eigenvalues are $-a$, $-c$, -1 and $-f$. Some of the values for these parameters in which the phase synchronization happens are $a = 3$,$b = 2$, and $c = f = 0$. Obviously, the convergence criterion in Theorem 2 is satisfied here for system (8). Again, using the PC method to approximate the solutions of this system, with $s = 0.5$, the results are illustrated in Figures 2 for different values of α. Here, in Figure 2-c the phase synchronization exits, but the chaotic solution will change to the limit cycle. This change is again the affect of the derivatives order $\alpha = 0.89$, which turns the chaotic solution into the limit cycle.

5 Conclusions

As we discussed in this article, phase synchronization is a rare phenomenon, which occurs in some coupled chaotic systems. Direct stability criterion of the dynamical system cannot be applied for the convergence of phase synchronization. However, as we discussed in Theorem 1 and 2, these criteria can be adapted somehow in which we can apply for the convergence of phase synchronization in either ODE or FDE coupled chaotic systems. The illustrated diffusionless Lorenz system in Example 1 and the new 4-dimensional system in Example 2 showed our assertion for existence and stated convergence criterion.

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