Nonlinear Dynamics and Systems Theory, **11** (2) (2011) 183–198

Analysis of Periodic Nonautonomous Inhomogeneous Systems

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Received: July 30, 2010; Revised: March 31, 2011

Abstract: This paper addresses the analysis of a class of near-periodic systems in which the dynamics can be described by a set of nonlinear differential equations with no known equilibrium solution. Linear models are developed by performing a power series expansion about a time-periodic reference motion. The result is a nonautonomous, inhomogeneous system consisting of a set of parametrically excited linear differential equations with time-periodic forcing excitations. The method of linearization assures that the time period of the parametric and forcing excitations is the same.

Floquet theory is used to address the stability of the homogeneous parametrically excited system. However, the linear system is inhomogeneous due to the forcing excitation. A modification of Floquet theory allows the use of Floquet multipliers or characteristic exponents to analytically examine the transitory and steady-state behavior of the inhomogeneous system.

Keywords: *dynamical; Floquet theory; linear control.*

Mathematics Subject Classification (2000): 34A34, 93C05.

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1 Introduction

There are several methods available to determine the long-term behavior of a system that can be described by a set of nonlinear differential equations

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}). \tag{1}$$

The most obvious method to investigate the behavior of a nonlinear system is to numerically solve the differential equations subject to specified initial conditions. One example would be the ordinary differential equations solvers in MATLAB [19]. Modern computational capability makes the issue of computer processor and memory limitations virtually irrelevant. However, numerical simulations can not always provide the definitive label of *stability.* The solutions may be virtually identical for a system that is slightly asymptotically stable, neutrally stable, or slightly unstable unless the simulation is carried out for a very long time.

Stability theorems generally address stability about an equilibrium or about a known solution. Stability in the sense of Lyapunov [17, 18] requires that, for motion about an equilibrium, the system output be dependent upon the magnitude of the initial conditions. Similar theorems loosen the requirement for stability about an equilibrium and address stability about a known solution to the system [5].

The direct method of Lyapunov uses a Lyapunov function, $v(x)$ to directly assess the stability of the differential equations in question without having to determine a first variation [11, 15]. Furthermore, the converse of the theorem is also true. If the equilibrium is stable, then the function $v(x)$ exists. However, there is no "prescription" for determining an appropriate Lyapunov function. The function can be difficult to determine, particularly for a complex system.

The strength of the nonlinearity of a system determines whether it is periodic, quasi-periodic or chaotic. Poincaré introduced the concept of a *phase-space* where all possible motions of a system are represented by a family of trajectories [7]. The degree to which a system is chaotic is determined by the sensitivity of the trajectories to initial conditions or perturbations, where small changes can cause widely diverging outcomes. The sensitivity to initial conditions can be quantified by a Lyapunov exponent. In general, Lyapunov exponents can not be found analytically and require the use of numerical methods [8].

A Poincaré Section maps the intersection of a dynamical orbit in state-space with a one-dimension lower subspace (phase-space) that is transverse to the flow of the orbit. While a Poincaré map can aid in determining stability [4, 10, 16], it is essentially a schematic for presenting the results of a numerical simulation at discrete time periods. If the period of a solution is many times the fundamental sample period used for the map, it may require simulation for a long time before the repetition appears.

The existence of many linear analysis tools justifies the attempt to linearize the system of (1). In general, a power series expansion about an equilibrium results in a reduced equation known as the first variation or first approximation of (1) with respect to the equilibrium condition [11]. The result is a constant coefficient linear system given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}. \tag{2}$$

Classical linear analysis methods for determining the stability of this linear system are well known [2, 6].

If the known solution used for linearization is nonautonomous, the matrix $\mathbf{A}$ is time varying. In general, linear analysis methods can be applied to time-varying systems with some modification such as canonical transformations or substitution of dynamic eigenvalues [13, 14, 23]. The stability of homogeneous linear equations with a time-periodic $\mathbf{A}$ matrix can be assessed by applying methods such as Hill's method of infinite determinants [20, 22] or Floquet–Lyapunov theory [3, 11, 26]. Floquet theory can also be used to determine if an inhomogeneous system has periodic solutions [3]. For time-periodic systems with time-periodic forcing functions, the literature typically addresses this form of steady-state behavior only [12, 24]. If the homogeneous equation exhibits asymptotic stability then the forced oscillations tend toward a periodic steady-state [3, 11].

This paper presents an extension of Floquet theory to a system which has no equilibrium or known solution. Equations are linearized about a time-periodic motion which closely approximates the nonlinear behavior. The behavior is almost periodic (a dynamical system that appears to almost retrace an orbit through phase space [1]). The result is a time-periodic linear system driven by a time-periodic forcing excitation having the same time period T, as the coefficients of $\mathbf{A}(t)$. The extension applies to the general case and is not limited to asymptotic behavior. Based on Floquet multipliers, the stability of the inhomogeneous system can be analyzed and performance metrics analogous to classical control theory settling time can be determined. The theoretical development is validated using a spinning pendulum.

2 Nonautonomous Inhomogeneous Systems from a Nonequilibrium Reference

The autonomous nonlinear differential equations that describe the motion of interest are given by

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{f}(\mathbf{z}), \\ \mathbf{f}(\mathbf{z}) &\neq \mathbf{0}, \end{aligned} \tag{3}$$

which have no known solution or equilibrium to be used as a reference for analyzing stability. Additionally, the solution is unknown except through numerical integration. However, the motion is known to be almost periodic and, in some type of limit behavior, periodic.

Lacking a traditional equilibrium or solution, the nominal periodic motion of the system will be used as a reference condition $\mathbf{z}_R$, and a series expansion is performed

$$\begin{aligned} \dot{\mathbf{z}}_R + \delta\dot{\mathbf{z}} &= \mathbf{f}(\mathbf{z}_R) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{z}}\right]_R \delta\mathbf{z}, \\ \dot{\mathbf{z}}_R &\neq \mathbf{f}(\mathbf{z}_R). \end{aligned} \tag{4}$$

The reference condition is not a solution to (3) and cannot be eliminated from the equations, resulting in a linear system containing a forcing excitation of the form

$$\begin{aligned} \delta\dot{\mathbf{z}} &= \mathbf{A}(t)\delta\mathbf{z} + \mathbf{g}(t), \\ \mathbf{A}(t) &= \left[\frac{\partial \mathbf{f}}{\partial \mathbf{z}}\right]_R, \\ \mathbf{g}(t) &= \mathbf{f}(\mathbf{z}_R) - \dot{\mathbf{z}}_{\mathbf{R}}, \end{aligned} \tag{5}$$

with a time-varying $\mathbf{A}$ matrix and a time-varying forcing excitation $\mathbf{g}$. By selection of the reference condition as time-periodic, the linear system of (5) has the following properties

$$\begin{aligned} \mathbf{A}(t) &= \mathbf{A}(t+T), \\ \mathbf{g}(t) &= \mathbf{g}(t+T), \end{aligned} \tag{6}$$

where T is the period common to both the matrix $\mathbf{A}$ and the vector $\mathbf{g}$.

3 Floquet Theory

Linear homogeneous differential equations with time-periodic coefficients given by

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}, \tag{7}$$

where $\mathbf{A}$ is time-periodic

$$\mathbf{A}(t) = \mathbf{A}(t+T) \tag{8}$$

and T is the time period can be assessed by applying Floquet–Lyapunov theory as given in Theorem 3.1[20, 26].

Theorem 3.1 *(Floquet–Lyapunov theorem). Any fundamental matrix* $\mathbf{X}(t)$ *of equation (7) with* T*-periodic coefficients is expressible in the form*

$$\mathbf{X}(t) = \mathbf{F}(t)e^{\mathbf{K}t}, \tag{9}$$

where $\mathbf{F}(t)$ *is a nonsingular continuous* T*-periodic* $n \times n$ *matrix-function whose derivative is an integrable piecewise-continuous function, and* $\mathbf{K}$ *is some constant matrix.*

Given that $\mathbf{F}(t)$ is time-periodic, the stability of the trivial solution to (7) depends entirely upon the eigenvalues of the matrix $\mathbf{K}$. The eigenvalues of $\mathbf{K}$ are known as the Floquet characteristic exponents, ε, and can be found by first determining the eigenvalues of $\mathbf{X}(T)$, known as the Floquet multipliers, σ. The matrix $\mathbf{X}(T)$, called the monodromy matrix, is the fundamental set of solutions to (7) when $t = T$ and with initial conditions of $\mathbf{X}(0) = \mathbf{I}$. The monodromy matrix can be determined numerically or through other means such as a multiple parameter perturbation method [25]. The characteristic exponents are then determined by

$$\epsilon = \frac{\ln \sigma}{T}. \tag{10}$$

Table 1 summarizes properties of solutions corresponding to the properties of the characteristic exponents and multipliers.

4 Extended Floquet Theory

Floquet theory does not address stability of the inhomogeneous system described by (5) where the forcing excitation $\mathbf{g}(t)$ is present. However, the T-periodic nature of $\mathbf{g}(t)$ allows for an extension to the theory. The solution to the inhomogeneous system of (5) can be expressed in terms of $\mathbf{X}(T)$ as follows [26]

$$\mathbf{z}(t) = \mathbf{X}(t)\left[\mathbf{x}(0) + \int_0^t \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{11}$$

Property of Solutions	**Characteristic Exponents, ε**	**Multipliers, σ**
Lyapunov Stability	Real parts nonpositive: zero or pure imaginary ϵ (if present) are semisimple eigenvalues of $\mathbf{K}$	Inside or on unit circle. Latter case corresponds to semisimple eigenvalues of $\mathbf{K}$
Asymptotic Stability	Real parts negative	Inside unit circle
Instability	At least one characteristic exponent with positive real part or a pure imaginary (or zero) exponent that is not semisimple	At least one multipier either outside the unit circle or on the unit circle and not semisimple

Table 1: Properties of solutions of systems with periodic coefficients.

Given that, according to Floquet theory, the monodromy matrix satisfies the following identity at time $t = t + T$

$$\mathbf{X}(t + T) \equiv \mathbf{X}(t)\mathbf{X}(T), \tag{12}$$

the following theorem for the solution to (11) after n time periods $\mathbf{z}(nT)$ can be established.

Theorem 4.1 *The solution to (11) after n time periods, where n is an integer, is given by*

$$\mathbf{z}(nT) = \mathbf{X}(T)^n\mathbf{x}(0) + \left[\mathbf{X}(T)^n + ... + \mathbf{X}(T)^2 + \mathbf{X}(T)\right] \int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau. \tag{13}$$

Proof At $t = T$, the solution to (11) becomes

$$\mathbf{z}(T) = \mathbf{X}(T) \left[\mathbf{x}(0) + \int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{14}$$

Extending (14) to two time periods $2T$, yields

$$\mathbf{z}(2T) = \mathbf{X}(2T) \left[\mathbf{x}(0) + \int_0^{2T} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{15}$$

Expanding the term inside the integer and applying the identity of (12) yield

$$\mathbf{z}(2T) = \mathbf{X}(T)^2 \left[\mathbf{x}(0) + \int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau + \int_T^{2T} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{16}$$

Applying the variable change $U = \tau - T$, $dU = d\tau$ to equation (16) results in

$$\mathbf{z}(2T) = \mathbf{X}(T)^2 \left[\mathbf{x}(0) + \int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau + \int_0^T \mathbf{X}(U + T)^{-1}\mathbf{g}(U + T)dU\right]. \tag{17}$$

Once again applying the identity from (12) and substitution of the time-periodic properties of (6) we get

$$\mathbf{z}(2T) = \mathbf{X}(T)^2 \left[\mathbf{x}(0) + \int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau + \int_0^T \mathbf{X}(T)^{-1}\mathbf{X}(U)^{-1}\mathbf{g}(U)dU \right]. \quad (18)$$

Finally, (18) can be reduced to

$$\mathbf{z}(2T) = \mathbf{X}(T)^2\mathbf{x}(0) + \left[\mathbf{X}(T)^2 + \mathbf{X}(T)\right] \int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau. \quad (19)$$

Repeated application of the identity from (12) and substitution of (6) leads to a solution to (11) after n time periods nT

$$\mathbf{z}(nT) = \underbrace{\mathbf{X}(T)^n\mathbf{x}(0)}_{Homogeneous} + \underbrace{\underbrace{\left[\mathbf{X}(T)^n + ... + \mathbf{X}(T)^2 + \mathbf{X}(T)\right]}_{Summation} \underbrace{\int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau}_{Integral,\ \mathbf{\Lambda}}}_{Inhomogeneous}. \quad (20)$$

Note that the behavior of (20) as n approaches to infinity can be predicted strictly based on knowledge of the response during the first time period T. The steady-state behavior can be evaluated by examining each term in (20) as n approaches infinity. The homogeneous and inhomogeneous terms will be evaluated separately in the following subsections.

4.1 Homogeneous behavior

The behavior as n increases to infinity of the homogeneous term in (20) is dependent upon the Floquet multipliers of the monodromy matrix $\mathbf{X}(T)$. The behavior is given in Table 2 for various properties of the magnitude of the largest Floquet multiplier $\rho[\mathbf{X}(T)]$.

The Limits of Powers Theorem [21], given in Theorem 4.2, guarantees the existence of $\lim_{n\to\infty} \mathbf{X}(T)^n$ for Properties 1 and 2.

Theorem 4.2 *(Limits of Powers Theorem). For* $\mathbf{X} \in C^{k\times k}$, $\lim_{n\to\infty} \mathbf{X}^n$ *exists if and only if* $\rho[\mathbf{X}] < 1$ *or* $\rho[\mathbf{X}] = 1$, *where* 1 *is the only eigenvalue on the unit circle and is semisimple.*

When it exists $\lim_{n\to\infty} \mathbf{X}^n =$ *the projector onto* $N(\mathbf{I} - \mathbf{A})$ *along* $R(\mathbf{I} - \mathbf{A})$, *where* N *is the null space and* R *is the range space.*

Property 2 is of particular interest. According to Floquet theory, as summarized in Table 1, a system with the largest multiplier(s) identically equal to one (semisimple) exhibits stability in the sense of Lyapunov, not asymptotic stability as in Property 1. Therefore, the $\lim_{n\to\infty} \mathbf{X}(T)^n$ exists but is not necessarily zero. The concept of Cesaro summability [21], given in Theorem 4.3, yields additional information about the value of the limit for Property 2.

Property	$\rho[\mathbf{X}(T)]$ Mag. of largest Floquet Multiplier	$\mathbf{X}(T)^n$
1	$\rho[\mathbf{X}(T)] < 1$ Semisimple	Converges to 0
2	$\rho[\mathbf{X}(T)] = 1$ One is the only multiplier on the unit circle and is semisimple	Converges to $\mathbf{G}$
3	$\rho[\mathbf{X}(T)] = 1$ Multipliers, other than one, on the unit circle are semisimple	Nonconvergent Bounded
4	$\rho[\mathbf{X}(T)] = 1$ Multiple eigenvalues, not semisimple	Divergent
5	$\rho[\mathbf{X}(T)] > 1$ Multipliers outside the unit circle	Divergent

Table 2: Properties of the homogeneous term of $\mathbf{z}(nT)$.

Theorem 4.3 *(Cesaro summability).*

For $\mathbf{X} \in C^{k\times k}$, $\mathbf{X}$ is Cesaro summable if and only if

$$\rho[\mathbf{X}] < 1 \text{ or } \rho[\mathbf{X}] = 1 \text{ with each eigenvalue on the unit circle being semisimple.}$$

When it exists the Cesaro limit

$$\lim_{n\to\infty} \frac{\mathbf{I}+\mathbf{X}+\ldots+\mathbf{X}^{n-1}}{n} = \mathbf{G}$$

is the projector onto $N(\mathbf{I}-\mathbf{A})$ along $R(\mathbf{I}-\mathbf{A})$, exactly the same as the ordinary limit described above in the Limits of Powers Theorem, had it existed.

$\mathbf{G} \neq 0$ if and only if 1 is an eigenvalue of $\mathbf{X}$, in which case $\mathbf{G}$ is the spectral projector associated with an eigenvalue of 1.

Note that the existence of the $\lim_{n\to\infty}\mathbf{X}^n$ implies that the Cesaro sum $\mathbf{G}$ exists and they have the same value. However, the existence of $\mathbf{G}$ does not imply the existence of $\lim_{n\to\infty}\mathbf{X}^n$. The Cesaro sum also exists when the largest Floquet multiplier magnitude is equal to one. In other words, the multiplier is not identically one, but has both real and imaginary parts with magnitude equal to one. This is the case for Property 3. The Cesaro sum $\mathbf{G}$ exists and $\mathbf{G} = 0$, but $\lim_{n\to\infty}\mathbf{X}(T)^n$ does not exist. The Cesaro sum is essentially the mean value of $\mathbf{X}(T)^n$ as n increases to infinity, indicting that $\mathbf{X}(T)^n$ oscillates with both positive and negative values around a mean of zero. Therefore, the homogeneous portion of the solution to $\mathbf{z}(nT)$ does not converge but remains bounded and oscillates indefinitely. As predicted by Table 1, Property 3 also exhibits stability in the sense of Lyapunov.

For Properties 4 and 5, the $\lim_{n\to\infty}\mathbf{X}(T)^n$ does not exist and also the Cesaro Sum does not exist. Therefore, the solution for $\mathbf{z}(nT)$ diverges. This result is in accordance with Floquet theory which predicts instability.

The results shown in Table 2 are completely consistent with the behavior predicted by Floquet theory in Table 1. This is not surprising as the homogeneous term of (20) is the solution to the time-periodic system in (7) at discrete multiples of the time period T.

4.2 Inhomogeneous behavior

The inhomogeneous term in (20) consists of the product of a summation and an integral term. The integral term, $\mathbf{\Lambda}$, is a definite integral over the time span of zero to T and is therefore a constant vector. The convergent or divergent behavior of the inhomogeneous term will be determined by the summation term and the Floquet multipliers of $\mathbf{X}(T)$. This result is given in Table 3.

For Floquet multipliers with magnitude less than one, as in Property 1, the convergence characteristics of the summation $[\mathbf{X}(T)^n + ... + \mathbf{X}(T)^2 + \mathbf{X}(T)]$ are given by the Neumann series [21], shown in Theorem 4.4.

Property	$\rho[\mathbf{X}(T)]$ Mag. of largest Floquet Multiplier	$[\mathbf{X}(T)^n + ... + \mathbf{X}(T)^2 + \mathbf{X}(T)]$
1	$\rho[\mathbf{X}(T)] < 1$ Semisimple	Converges to $[\mathbf{I} - \mathbf{X}(T)]^{-1}[\mathbf{X}(T)]$
2	$\rho[\mathbf{X}(T)] = 1$ One is the only multiplier on the unit circle and is semisimple	Unbounded
3	$\rho[\mathbf{X}(T)] = 1$ Multipliers, other than one, on the unit circle are semisimple	Nonconvergent Bounded
4	$\rho[\mathbf{X}(T)] = 1$ Multiple eigenvalues, not semisimple	Unbounded
5	$\rho[\mathbf{X}(T)] > 1$ Multipliers outside the unit circle	Unbounded

Table 3: Properties of the summation term of $\mathbf{z}(nT)$.

Theorem 4.4 *(Neumann series). For $\mathbf{X} \in C^{k\times k}$, the following statements are equivalent:*

the Neumann series $\mathbf{I} + \mathbf{X} + \mathbf{X}^2 + \dots$ converges;

$\rho[\mathbf{X}] < 1$*;*

$\lim_{n\to\infty} \mathbf{X}^n = 0$*;*

In which case $[\mathbf{I} - \mathbf{X}]^{-1}$ exists and $\sum_{n=0}^{\infty} \mathbf{X}^n = [\mathbf{I} - \mathbf{X}]^{-1}$.

Although $\lim_{n\to\infty} \mathbf{X}^n$ does exist for Property 2, according to Cesaro summability, the limit is a non-zero constant $\mathbf{G}$. In the limit, the summation term $[\mathbf{X}(T)^n + ... + \mathbf{X}(T)^2 + \mathbf{X}(T)]$ becomes a diverging algebraic series increasing by $\mathbf{G}$ with each additional term. Therefore, the summation term diverges.

For Property 3 with semisimple Floquet multipliers on the unit circle but not identically one, the Cesaro sum $\mathbf{G} = 0$. However, as mentioned in the previous section, the $\lim_{n\to\infty} \mathbf{X}(T)^n$ does not exist. $\mathbf{X}(T)^n$ oscillates with both positive and negative values around a mean of zero. Therefore, the summation term, oscillates around some constant value.

For Properties 4 and 5, the summation term is unbounded.

4.3 Stability of the inhomogeneous system

Table 4 shows how the addition of a forcing term affects the steady-state solution of the inhomogeneous system. With all Floquet multipliers of $\mathbf{X}(T)$ less than one, as for

Property	$\rho[\mathbf{X}(T)]$ Mag. of largest Floquet Multiplier	Floquet $\mathbf{x}(nT)$	Inhomogeneous $\mathbf{z}(nT)$
1	$\rho[\mathbf{X}(T)] < 1$ Semisimple	Asymptotic Stability	Bounded
2	$\rho[\mathbf{X}(T)] = 1$ One is the only multiplier on the unit circle and is semisimple	Lyapunov Stability	Unbounded
3	$\rho[\mathbf{X}(T)] = 1$ Multipliers, other than one, on the unit circle are semisimple	Lyapunov Stability	Bounded
4	$\rho[\mathbf{X}(T)] = 1$ Multiple eigenvalues not semisimple	Unstable	Unbounded
5	$\rho[\mathbf{X}(T)] > 1$ Multipliers outside the unit circle	Unstable	Unbounded

Table 4: Homogeneous vs inhomogeneous properties of $\mathbf{z}(nT)$.

Property 1, $\mathbf{z}(nT)$ converges to a nonzero value instead of to zero (asymptotic stability) for the homogeneous system. The solution converges to

$$\lim_{n\ to\infty} \mathbf{z}(nT) = [\mathbf{I} - \mathbf{X}(T)]^{-1}[\mathbf{X}(T)]\ \mathbf{\Lambda}. \tag{21}$$

For Property 2, with Floquet multipliers of $\mathbf{X}(T)$ identically equal to one (semisimple), $\mathbf{z}(nT)$ is driven from Lyapunov stable to unbounded with the addition of the forcing excitation. The summation in the homogeneous term is unbounded, causing the solution to diverge. The solution in the limit is given by

$$\lim_{n\ to\infty} \mathbf{z}(nT) = \mathbf{x}(0) + \left[\mathbf{X}(T)^n + ... + \mathbf{X}(T)^2 + \mathbf{X}(T)\right]\ \mathbf{\Lambda}. \tag{22}$$

For Property 3, if the largest Floquet multiplier of $\mathbf{X}(T)$ has magnitude equal to one, is semisimple, but is not identically one, then the Lyapunov stable homogeneous system remains bounded with the addition of the forcing term. Neither term in equation 20 converges to a limit, indicating oscillation within some finite bound. The basic behavior of the system has not changed with the addition of a forcing term.

For Properties 4 and 5, both the homogeneous and inhomogeneous terms of equation 20 diverge and $\mathbf{z}(nT)$ is unbounded. The basic behavior of the system has not changed with the addition of a forcing term.

To summarize, there are two instances where the addition of the forcing excitation changes the fundamental behavior of the system. First, for Property 1, $\mathbf{z}(nT)$ converges to a nonzero steady-state instead of to zero for the homogeneous system. Second, for Property 2, $\mathbf{z}(nT)$ is driven from Lyapunov stable to unbounded with the addition of the forcing excitation.

Lyapunov stability [17, 18] presupposes that motion is analyzed with respect to an equilibrium or rest condition. As explained earlier, the system of interest has no equilibrium, and the nonlinear differential equations are linearized about a time-varying reference condition. Therefore, when evaluating the steady-state behavior of $\mathbf{z}(nT)$ in Table 4, the results are with respect to the reference behavior $\mathbf{z}_R(t)$. In relevant literature, a forced time-periodic system with or without an equilibrium is termed stable or asymptotically stable according to the Floquet multipliers, and the steady-state behavior is time-periodic [12, 24]. However, Lyapunov stability requires that the solution can be made arbitrarily small by changing the value of the initial conditions. For Properties 1 and 3, the steady-state solution is not dependent only on the initial conditions. For this reason, Table 3 utilizes the terms *bounded* or *unbounded* to refer to $\mathbf{z}(nT)$ as opposed to *stable* or *unstable*.

4.4 Transient behavior

A linear homogeneous differential equation given by

$$\ddot{x} + B\dot{x} + Cx = 0, \tag{23}$$

where B and C are constants can be expressed as a set of first-order equations in the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{24}$$

where $\mathbf{A}$ is a constant coefficient matrix. In classical control theory, the transient behavior can be determined by the eigenvalues λ of the $\mathbf{A}$ matrix if the eigenvalues are a complex pair with real parts less than zero [2, 5, 6].

$$\lambda = \sigma \pm j\omega_d, \ \sigma < 0. \tag{25}$$

If so, the solution to (23) is expressed as

$$\mathbf{x}(t) = e^{\sigma t}(\mathbf{C_1} sin\omega_d t + \mathbf{C_2} cos\omega_d t) = \mathbf{C}e^{\sigma t}(sin\omega_d t + \boldsymbol{\phi}), \tag{26}$$

where $\mathbf{C}$, $\mathbf{C_1}$ and $\mathbf{C_2}$ are vector constants determined by the initial conditions $\mathbf{x}(0)$, $\sigma = -\zeta\omega_n$, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$. The parameter ζ is the damping ratio of the second-order system, ω_n is the natural frequency and $\boldsymbol{\phi}$ are phase angles. The exponential term $\mathbf{C}e^{\sigma t}$ defines a decaying *envelope* that determines the rate at which the sinusoidal oscillations decrease to zero with time. A transient characteristic is the time constant

$$T_c = \frac{1}{\sigma}, \tag{27}$$

which is the time at which the exponential decreases to 37 percent of the initial value. A related characteristic is the settling time

$$T_s = \frac{number\ of\ time\ constants}{\sigma} \tag{28}$$

which is the time at which the exponential decreases to a desired absolute percent of the initial value. For example, the settling time to within 2 percent is approximately 4 time constants. The time constant and settling time are characteristics that can be extended to the homogeneous portion of (20).

The assumption is made that the homogeneous portion of (20) is second-order with complex-conjugate Floquet characteristic exponents with negative real parts (Floquet multipliers will lie inside the unit circle). According to Table 2, $\mathbf{X}(T)^n$ will converge to zero. At each multiple of n, the matrix $\mathbf{A}(nT)$ has the same constant value. Therefore, the homogeneous solution, $\mathbf{x}(nT) = \mathbf{X}(T)^n\mathbf{x}(0)$ at each multiple of the time period is identical to the solution of a constant coefficient system and $\mathbf{x}(nT)$ will lie along a damped sinusoid given by

$$\tilde{\mathbf{x}}(t) = e^{\sigma t}(\mathbf{C_1} sin\omega_d t + \mathbf{C_2} cos\omega_d t) = \mathbf{C}e^{\sigma t}(sin\omega_d t + \boldsymbol{\phi}). \tag{29}$$

Therefore, the homogeneous solution, $\mathbf{x}(nT)$, will also converge within the exponential envelope $\mathbf{C}e^{\sigma t}$. The classical control theory concepts of time constant and settling time can be directly applied to the homogeneous portion of $\mathbf{z}(nT)$. The number of integer time periods to reach the required settling time is given by

$$n_s = \frac{number\ of\ time\ constants}{\sigma T} = \frac{T_s}{T}, \tag{30}$$

where n_s can be rounded to the next higher integer and guarantee that $\mathbf{x}(nT)$ is equal to (or less than) the required percent of its maximum value

$$\frac{x(nT)}{C} \leq e^{\sigma n_s T}. \tag{31}$$

The inhomogeneous portion of $\mathbf{z}(t)$ can be shown to be time-periodic. Consider the solution $\mathbf{z}(t+nT)$, where $0 < t < T$. The inhomogeneous part of the solution $\mathbf{z}_i(t+nT)$ is given by

$$\mathbf{z}_i(t+nT) = \mathbf{X}(t+nT)\int_0^{t+nT} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau \tag{32}$$

which can be written as

$$\mathbf{z}_i(t+nT) = \mathbf{X}(t)\mathbf{X}(nT)\left[\int_0^{nT} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau + \int_{nT}^{t+nT} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{33}$$

Using the definition of a Neumann series (Theorem 4.4) and a procedure similar to that of Theorem 4.1, (33) converges to

$$\mathbf{z}_i(t+nT) = \mathbf{X}(t)\left[[\mathbf{I} - \mathbf{X}(T)]^{-1}[\mathbf{X}(T)]\ \boldsymbol{\Lambda} + \int_0^{t} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{34}$$

Assuming that (34) is time-periodic, then the equation is also a steady-state solution given by

$$\mathbf{z}_{ss}(t) = \mathbf{X}(t)\left[[\mathbf{I} - \mathbf{X}(T)]^{-1}[\mathbf{X}(T)]\boldsymbol{\Lambda} + \int_0^{t} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{35}$$

If the assumption is true then

$$\mathbf{z}_{ss}(t) = \mathbf{z}_{ss}(t+T) = \mathbf{X}(t+T)\left[[\mathbf{I} - \mathbf{X}(T)]^{-1}[\mathbf{X}(T)]\ \boldsymbol{\Lambda} + \int_0^{t+T} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{36}$$

Expanding the integral term yields

$$\mathbf{z}_{ss}(t+T) =$$

$$\mathbf{X}(t)\mathbf{X}(T)\left[[\mathbf{I}-\mathbf{X}(T)]^{-1}[\mathbf{X}(T)]\ \boldsymbol{\Lambda} + \int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau + \int_T^{t+T} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right] \quad (37)$$

which, again using the procedure of Theorem 4.1, reduces back to

$$\mathbf{z}_{ss}(t) = \mathbf{z}_{ss}(t+T) = \mathbf{X}(t)\left[[\mathbf{I}-\mathbf{X}(T)]^{-1}[\mathbf{X}(T)]\ \boldsymbol{\Lambda} \int_0^t \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right] \quad (38)$$

proving the assumption that $\mathbf{z}_{ss}(t)$ is time-periodic is true.

The steady-state solution can be determined by integration over a single time period. Since the inhomogeneous portion of the solution to $\mathbf{z}(t)$ is time-periodic and, therefore, contains no "transient" terms, the entire solution converges to the steady-state with the settling time characteristics of the homogeneous system described above. As mentioned earlier, given that the initial conditions are the initial vector of the periodic solution, the entire solution is time-periodic [3, 11].

5 Spinning Pendulum Example

The system to be analyzed, as shown in Figure 1 is a mass attached to a fixed point by a rigid tether. The pendulum is spinning in a gravitational field. The nonlinear equations of motion are given by

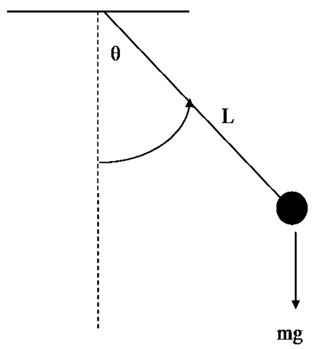

Figure 1: Pendulum spinning in a constant gravity field.

$$\ddot{\theta} = \frac{-g}{L}\sin\theta. \quad (39)$$

The reference condition chosen is the limiting behavior for $g << L$ which is a constant-rate spin

$$\begin{bmatrix} \theta \\ \dot{\theta}_0 \end{bmatrix}_R = \begin{bmatrix} \theta_0 \\ \theta_0 + \dot{\theta}_0 t \end{bmatrix}. \quad (40)$$

The linearized equations of motion are given by

$$\begin{bmatrix} \dot{\delta\theta} \\ \ddot{\delta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-g}{L}\cos\theta & 0 \end{bmatrix}_R \begin{bmatrix} \delta\theta \\ \dot{\delta\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-g}{L}\sin\theta \end{bmatrix}_R, \tag{41}$$

where g is the acceleration due to gravity and L is the length of the tether. The ratio g/L is set to 0.1 for this example. The reference condition is $\theta_0 = -100\ deg$ and $\dot{\theta}_0 = \pi/3\ rad/s$. The time period for both the parametric and the forced excitation is $T = 2\pi/\dot{\theta}_0 = 6\ s$.

The monodromy matrix $\mathbf{X}(T)$ is found by numerical simulation of the homogeneous portion of (41) with unity initial conditions for one time period. This results in the following Floquet multipliers

$$\boldsymbol{\sigma} = 0.918 \pm 0.397i, \quad |\boldsymbol{\sigma}| = 1. \tag{42}$$

The pendulum system has a pair of complex Floquet multipliers with magnitude equal to one. Therefore the system exhibits Property 3 from Tables 2, 3 and 4. The homogeneous portion of the system is Lyapunov stable (see Table 2) as shown in Figure 2 where $\delta\theta$ is plotted for 30 time periods. The inhomogeneous system response is nonconvergent but bounded (see Table 3) by some finite value as shown in Figure 3. Figures 2 and 3 show both the result of a numerical simulation for 30 time periods of (41) and also the response, $\mathbf{z}(nT)$, calculated from (20) at each time period.

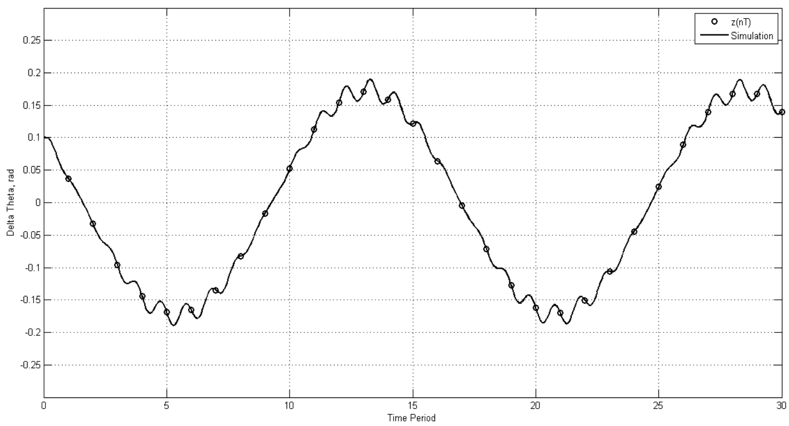

Figure 2: Homogeneous response to a small perturbation.

If a negative feedback controller with a proportional gain, K_p, and derivative gain, K_d, is applied to the pendulum system, the linear system of (41) becomes

$$\begin{bmatrix} \dot{\delta\theta} \\ \ddot{\delta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-g}{L}\cos\theta - K_p & -K_d \end{bmatrix}_R \begin{bmatrix} \delta\theta \\ \dot{\delta\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-g}{L}\sin\theta \end{bmatrix}_R. \tag{43}$$

With gains of $K_p = K_d = 0.06$ and an initial condition of $\delta\theta = 0.3\ rad$, the Floquet multipliers become

$$\sigma = 0.034 \pm 0.835i, \quad |\sigma| = 0.835, \tag{44}$$

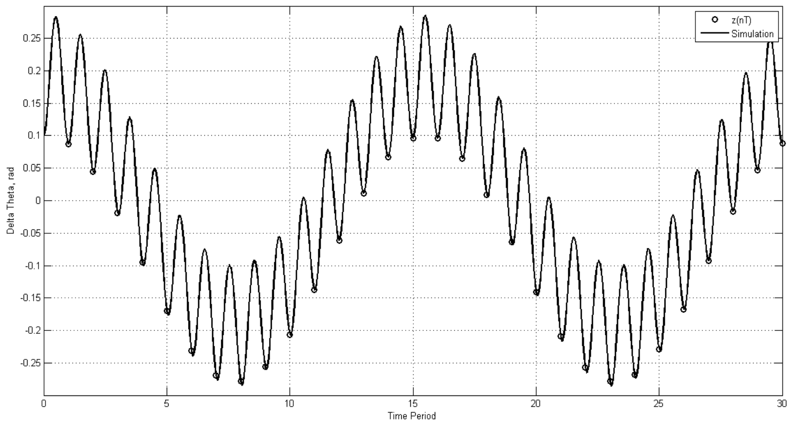

Figure 3: Inhomogeneous response to a small perturbation.

which are now a complex pair that lie inside the unit circle. The homogeneous system with negative feedback has become asymptotically stable as shown in Figure 4. As shown in Figure 5, the inhomogeneous system response is bounded and asymptotically approaches a time-periodic steady-state response.

Using (10) the corresponding Floquet characteristic exponents for (44) can be calculated

$$\epsilon = -0.030 \pm 0.255i \tag{45}$$

resulting in a time constant, $T_c = 33.3\ s = 5.6\ \textit{time periods}$ and a settling time to 2 percent, $T_s = 133.3\ s = 22.2\ \textit{time periods}$. The number of integer time periods for $\mathbf{z}(t)$ to settle to 2 percent is therefore $n_s = 23$. The homogeneous response in Figure 4 confirms that these results show good agreement with the simulated output. The solution for $\mathbf{z}(t)$ in Figure 5 shows the same settling time to the periodic steady-state.

6 Conclusions

When a near-periodic system is linearized about a time-periodic reference motion, the result is a linear parametrically excited system with a periodic forcing function. The solution to the system has been derived at each integer time period which requires knowledge of the system for the first time period only. The behavior of the homogeneous and inhomogeneous portions of the response can be predicted by using the Floquet characteristic exponents or multipliers. By adding a forcing excitation, the general behavior predicted by Floquet theory for the homogeneous system changed only for the case of semisimple multipliers that are identically equal to one. The presence of the forcing excitation caused the solution to diverge.

The classical control theory concept of settling time has been extended to the forced parametrically excited system. The homogeneous solution at each linear time period is the solution to a constant coefficient system which converges to zero at an exponential rate which can be determined from the Floquet characteristic exponents. It has been

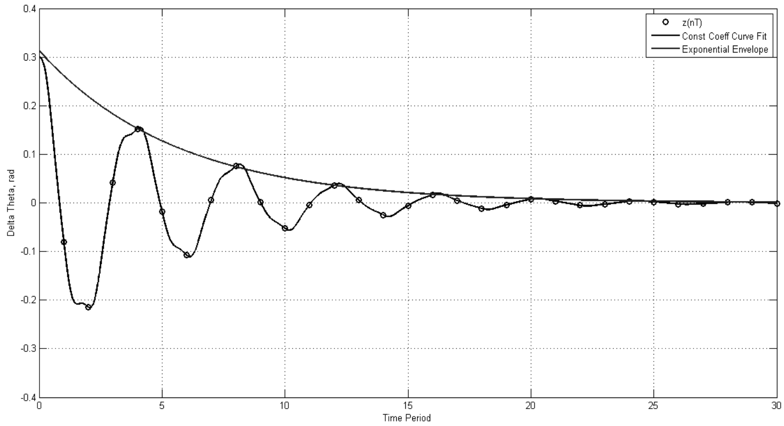

Figure 4: Homogeneous response to a small perturbation (controlled system).

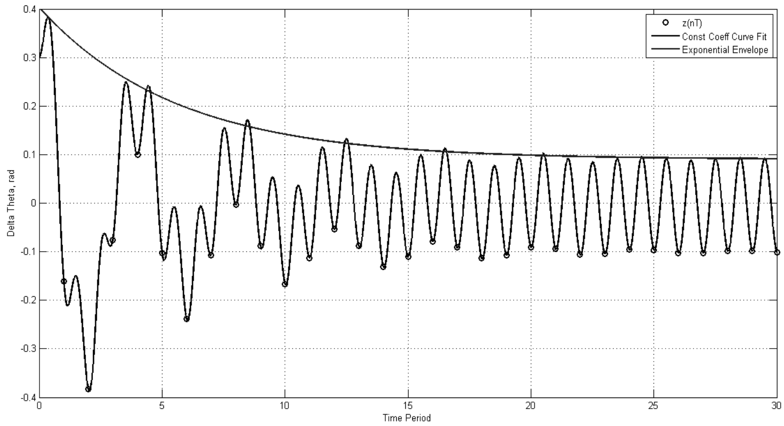

Figure 5: Inhomogeneous response to a small perturbation (controlled system).

shown that the entire inhomogeneous solution converges to a steady-state at the same exponential rate.

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