Nonlinear Dynamics and Systems Theory, **11** (3) (2011) 253–274

Adaptive Regulation with Almost Disturbance Decoupling for Power Integrator Triangular Systems with Nonlinear Parametrization

Y. M. Fu*, M. Z. Hou, J. Hu and L. Niu

Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin 150001, P. R. China

Received: September 25, 2009; Revised: July 22, 2011

Abstract: The problem of almost disturbance decoupling for a class of nonlinear systems is considered. The controlled systems consist of a chain of power integrators perturbed by a lower-triangular vector field with nonlinear parametrization. By using the tool of adding a power integrator combined with the parameter separation technique, under a set of growth conditions a smooth adaptive controller is explicitly constructed to attenuate the influence of the disturbance on the output with an arbitrary degree of accuracy. The designed adaptive controller is in its minimum-order property, since the order of the dynamic compensator is equal to one. An illustrative example is given to verify the effectiveness of the proposed approach.

Keywords: *almost disturbance decoupling; smooth adaptive controller; adding a power integrator; nonlinear parametrization; parameter separation.*

Mathematics Subject Classification (2000): 93C10, 93C40.

1 Introduction

One of the main objectives in control theory is to suppress unknown disturbances. It will be ideal if the influence of disturbances on the output can be eliminated completely, or in other words, the disturbances are decoupled from the output. Unfortunately, in most practical situations it is impossible to achieve the exact disturbance decoupling. In this case, it is reasonable to aim at almost disturbance decoupling (ADD), which means that the influence of the disturbance on the output is attenuated to an arbitrary degree of accuracy via feedback control design. More precisely, the problem of ADD can be stated

* Corresponding author: mailto:fuyanming@hit.edu.cn

as: given a system and a prescribed positive scalar, find a feedback control law such that the resultant closed-loop system is stable and the gain between the exogenous input and the regulated output is less than or equal to the prescribed positive number. The start point of the problem of ADD on nonlinear systems is associated with the papers [1], [2] in the late 1980s. The performance of the ADD in [2] is characterized in terms of the L_∞-induced norm from the disturbance to the outputs, and the solution of the problem of ADD is explicitly constructed by applying singular perturbation methods. However, a drawback of the result in [2] is that internal stability, which is crucial for a meaningful application or a practical implementation, is not taken into account. Therefore, the internal stability of the closed-loop systems cannot be guaranteed even in the absence of the disturbance. This problem is solved later in [3]. By applying a recursive design technique, a global solution to the ADD problem with internal stability was presented for a chain of integrators perturbed by a lower triangular vector field. The result in [3] was later generalized to a larger class of nonlinear minimum phase systems in [4]. These two results in [3] and [4] were further extended to a class of nonminimum-phase nonlinear systems in [5]. The proposed approach in [5] required that the unstable part of the zero-dynamics was not affected by the disturbance. Such a restriction was relaxed in the results of [6] and [7]. The construction of control law in [7] is based on a recursive Lyapunov-based design approach. In the case of systems with vector relative degree $[1, 1, \cdots, 1]$, the ADD problem was tackled in [8] for a general minimum phase system subject to parameter uncertainty and with a controlled output that may be affected by the disturbance. In addition, the problem of ADD for general affine nonlinear systems was addressed in [9] for the case of state feedback and the solution was converted into the solution of the so-called Hamilton-Jacobi-Isaacs equation (HJIE). The global inverse L_2-gain design for a chain of integrators perturbed by lower triangular vector field was reported in [10]. For a class of multi-input multi-output nonlinear systems, the ADD problem was addressed in [11] for the systems with nested lower triangular structure, and the controller was explicitly constructed by applying the backstepping design technique.

If only the output information is available for the feedback design, only a few results are devoted to the ADD problem via output feedback in the existing literature. In [12], the problem of ADD via output feedback for general affine nonlinear systems was converted into the solution of Hamilton-Jacobi-Isaacs equation. In [13], a systematic design procedure to output feedback controller with the function of ADD was given for a class of systems with the nonlinear terms depending only on the output. For the nonlinear systems in the so-called output feedback form, in [14] the polynomial gain disturbance attenuation property was achieved via output feedback. For a class of nonlinear systems satisfying linear growth conditions, in [15] a linear dynamic output compensator attenuating the influence of the disturbance on the output was explicitly constructed by the feedback domination design. In the above-mentioned literature on the ADD problem, most of the considered nonlinear systems are feedback (partial) linearizable and/or linear in control input. Recently, the ADD problem was addressed in [16] for a class of inherently nonlinear systems. The class of the systems is in the form of a chain of power integrators perturbed by a lower-triangular vector field. Different from some previous results, the ADD problem is formulated in terms of $L_2 - L_{2p}$ gain for the inherently nonlinear systems, rather than the standard L_2-gain. The controller was explicitly constructed by applying the so-called technique of adding a power integrator developed in [17].

It is well-known that adaptive control is one of the effective ways to deal with control systems with parametric uncertainty [18]. When the ADD problem for nonlinear systems with unknown parameters is investigated, a natural idea is to design an adaptive control law to solve this problem. However, only a few results on adaptive regulation with almost adaptive decoupling for nonlinear systems are available in the existing literature. In [19], [20] and [21], adaptive controllers are designed to guarantee arbitrary disturbance attenuation on the output tracking error for smooth reference signals for uncertain systems with output depending nonlinearities. In [19] and [20], the disturbance enters linearly in the state space equation, while in [21] the disturbance enters nonlinearly. Very recently, in [22] the ADD problem was discussed for power integrator lower triangular nonlinear systems. The function of disturbance attenuation is characterized by $L_{2m} - L_{2mp}$ gain. The adaptive control law was explicitly constructed by employing the adaptive adding a power integrator technique proposed in [23]. However, the result in [22] is only applicable to the case where the unknown parameter enters linearly in the state space equation. In this paper we will deal with the almost disturbance decoupling problem for power integrator triangular systems with nonlinear parametrization. With the help of the parameter separation technique proposed in [24], a constructive solution that solves the ADD problem is derived by using the adaptive adding one power integrator. A key feature of our proposed adaptive controller with the function of disturbance attenuation is its minimum-order property, since the order of the dynamic compensator is equal to one.

It should be pointed out that some other problems have been investigated for nonlinear systems. In [25], by using a constructed Lyapunov function, the conditions of ultimate boundedness of solutions for a class of nonlinear systems were given. In [26], an original practical criterion of global stability analysis of nonlinear polynomial systems was proposed. In [27], as a generalization of Gronwalls inequality, generalized dynamic inequalities were introduced to the time scales scene. Then, linear systems with linear and nonlinear perturbations and their stability characteristics versus the unperturbed system were investigated.

For simplicity, throughout this paper we use $I[m, n]$ to denote the set $\{m, m+1, \cdots, n\}$ for two integers $m < n$. For a group of scalars x_i, $i \in I[1, j]$, we use $x_{[j]}$ to denote the vector $\begin{bmatrix} x_1 & x_2 & \cdots & x_j \end{bmatrix}^T$. $\|\cdot\|$ is used to denote the Euclidean norm of a vector.

2 Problem Formulation

We consider the following single-input single-output power integrator lower-triangular system with an unknown parameter vector θ:

$$\begin{cases} \dot{x}_i = x_{i+1}^{p_i} + f_i(x_{[i]}) + g_i(x_{[i]})w + \phi_i(x_{[i]}, \theta),\ i \in I[1, n-1], \\ \dot{x}_n = u^{p_n} + f_n(x_{[n]}) + g_n(x_{[n]})w + \phi_n(x_{[n]}, \theta), \\ y = h(x_1), \end{cases} \tag{1}$$

where $u \in \mathbb{R}$, $x = x_{[n]} \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $w \in \mathbb{R}^s$ are the control input, system state, system output and disturbance signal, respectively; p_i, $i \in I[1, n]$, are positive integers and $f_i(\cdot), g_i(\cdot)$, $i \in I[1, n]$, and $h(\cdot)$, are smooth functions with $f_i(0) = 0$, $i \in I[1, n]$, and $h(0) = 0$; $\phi_i(\cdot)$, $i \in I[1, n]$ are continuous functions with $\phi_i(0, \theta) = 0$.

The objective of this paper is to design, under appropriate conditions, a smooth adaptive controller such that the closed-loop system is globally stable in the sense of Lyapunov, and the influence of the disturbance $w(t)$ on the output $y(t)$ is not greater than the prescribed level. To be specific, the following problem called the adaptive regulation with almost disturbance decoupling will be dealt with in this paper. In [28], some new results regarding the boundedness, stability and attractivity were provided for a class of initial-boundary-value problems characterized by a quasi-linear third order equation which may contain time-dependent coefficients.

Adaptive Regulation with Almost Disturbance Decoupling (ARADD): Consider the power integrators with nonlinearly parameterized lower triangular structure (1). Given any real number $\gamma > 0$, find, if possible, a smooth adaptive controller

$$\begin{cases} \dot{\hat{\theta}} = \psi(x_{[n]}, \hat{\theta}),\ \ \psi(0,0) = 0, \\ u = u(x_{[n]}, \hat{\theta}),\ \ u(0,0) = 0, \end{cases} \tag{2}$$

such that the closed-loop system (1) – (2) satisfies the following:

1) when $w = 0$, the closed-loop system is globally stable in the sense of Lyapunov, and globally asymptotical regulation of the state is achieved, i.e., $\lim_{t\to\infty} x_{[n]}(t) = 0$.

2) for any disturbance $w \in L_2$, the response of the closed-loop system starting from the initial state $x(0) = 0$ is such that

$$\int_0^t |y(s)|^{2p_1}\, \mathrm{d}s \le \gamma^2 \int_0^t \|w(s)\|^2\, \mathrm{d}s, \text{ for any } t \ge 0.$$

In order to solve the ARADD problem, the following assumptions are needed.

Assumption A1: $p_1 \ge p_2 \ge \cdots \ge p_n$ are odd integers.

Assumption A2: For any $i \in I[1, n]$,

$$\left|f_i(x_{[i]})\right| \le \alpha_i(x_{[i]}) \sum_{j=1}^{i} |x_j|^{p_i}, \tag{3}$$

where $\alpha_i(\cdot)$ is a nonnegative smooth function.

Assumption A3: For any $i \in I[1, n]$, $\left\|g_i(x_{[i]})\right\| \le \varphi_i(x_{[i]})$, where $\varphi_i(\cdot)$ is known bounding function that is nonnegative and smooth.

Assumption A4: For any $i \in I[1, n]$, $\left|\phi_i(x_{[i]}, \theta)\right| \le \beta_i(x_{[i]}, \theta) \sum_{j=1}^{i} |x_j|^{p_i}$, where $\beta_i(\cdot)$ is a nonnegative continuous function.

Before ending this section, we provide some useful lemmas. The first lemma is a slight extension of the well-known Young's inequality, and will be repeatedly used in the design of the adaptive controller. The proof can be found in [17].

Lemma 2.1 *For any positive integers m, n, and any real-valued function $\gamma(x, y) > 0$, the following inequality holds:*

$$|x|^m |y|^n \le \frac{m}{m+n}\gamma(x, y) |x|^{m+n} + \frac{n}{m+n}\gamma^{-m/n}(x, y) |y|^{m+n}.$$

By applying the above lemma, one can easily obtain the following conclusion [16]. This result will also play a vital role in the adding a power integrator design.

Lemma 2.2 *Let x, y and z, be real variables. Assume that $g_1 : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Then, for any positive integers m, n and real number $N > 0$, there exists a nonnegative smooth function $h_1 : \mathbb{R}^3 \to \mathbb{R}$ such that the following relation holds:*

$$|x^m [(y + xg_1(x, z))^n - (xg_1(x, z))^n]| \le \frac{|x|^{m+n}}{N} + |y|^{m+n} h_1(x, y, z).$$

The following lemma provides the parameter separation principle. It is this principle that enables us to deal with nonlinear parameterization. A constructive proof of the result can be found in [24].

Lemma 2.3 *For any real-valued continuous function $f(x, y)$, where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, there are smooth scalar functions $a(x) \geq 1$ and $b(y) \geq 1$, such that $|f(x, y)| \leq a(x)b(y)$.*

3 Global Adaptive Regulation

In this section we solve the problem of adaptive regulation with almost disturbance decoupling for the power integrator lower triangular system (1). Using the adding a power integrator technique as the design tool, we will explicitly construct a one-dimensional adaptive controller that solves the problem of ARADD with the help of the parameter separation technique provided in Lemma 2.3. Now we are ready to present the main result.

Theorem 3.1 *Under the condition of Assumptions A1 – A4, the ARADD problem for system (1) is solvable by a one-dimensional smooth adaptive controller*

$$\begin{cases} \dot{\hat{\Theta}} = \psi(x_{[n]}, \hat{\Theta}), \ \hat{\Theta} \in \mathbb{R}, \ \psi(0, 0) = 0, \\ u = u(x_{[n]}, \hat{\Theta}), \ u(0, 0) = 0. \end{cases} \tag{4}$$

Proof The proof is based on a feedback domination design approach which combines the technique of adding one power integrator [17, 23] with the parameter separation method [24]. The conclusion is obtained by applying mathematical induction method. Firstly, we need some preliminaries with the help of the parameter separation technique given in Lemma 2.3.

By Lemma 2.3, there exist two smooth functions $c_i(\theta) \geq 1$ and $\gamma_i(x_{[i]}) \geq 1$ satisfying

$$\beta_i(x_{[i]}, \theta) \leq \gamma_i(x_{[i]})c_i(\theta).$$

Since θ is a constant vector, $c_i(\theta)$ is also a constant. Let $\Theta := \sum_{i=1}^{n} c_i(\theta)$ be a new unknown constant. Then Assumption A4 implies that there are smooth function $\gamma_i(x_{[i]}) \geq 1$ and an unknown constant $\Theta \geq 1$, such that

$$\left|\phi_i(x_{[i]}, \theta)\right| \leq \gamma_i(x_{[i]})\Theta \sum\nolimits_{j=1}^{i} |x_j|^{p_i}. \tag{5}$$

Now we proceed to construct the smooth adaptive controller to solve the ARADD problem for the system (1).

Step 1: Define $\tilde{\Theta} = \Theta - \hat{\Theta}$, where $\hat{\Theta}(t)$ is the estimate of Θ to be designed later. Consider the Lyapunov function

$$V_1(x_1, \hat{\Theta}) = \frac{1}{p_1 + 1} x_1^{p_1+1} + \frac{1}{2}\tilde{\Theta}^2.$$

By (3), (5) and Lemma 2.1, there exist a smooth function $\rho_0(x_1) \geq 0$, such that for any $\beta > 0$

$$\begin{aligned}
& \dot{V}_1(x_1, \hat{\Theta}) + y^{2p_1} - \beta \|w\|^2 \\
= \;& x_1^{p_1}(x_2^{p_1} + f_1(x_1) + g_1(x_1)w + \phi_1(x_1, \theta)) - \dot{\hat{\Theta}}\tilde{\Theta} + y^{2p_1} - \beta \|w\|^2 \\
\leq \;& x_1^{p_1} x_2^{p_1} + x_1^{2p_1}\alpha_1(x_1) + |x_1^{p_1}|\,\varphi_1(x_1)\|w\| + x_1^{2p_1}\gamma_1(x_1)(\tilde{\Theta} + \hat{\Theta}) - \dot{\hat{\Theta}}\tilde{\Theta} \\
& + x_1^{2p_1}\rho_0(x_1) - \beta \|w\|^2 \\
\leq \;& x_1^{p_1} x_2^{p_1} + x_1^{2p_1}\alpha_1(x_1) + \frac{x_1^{2p_1}\varphi_1^2(x_1)}{4\beta} + x_1^{2p_1}\gamma_1(x_1)\hat{\Theta} \\
& + x_1^{2p_1}\rho_0(x_1) + (\Psi_1(x_1, \hat{\Theta}) - \dot{\hat{\Theta}})\tilde{\Theta} \\
\leq \;& x_1^{p_1} x_2^{p_1} + x_1^{2p_1}\rho_1(x_1, \hat{\Theta}) + (\Psi_1(x_1, \hat{\Theta}) - \dot{\hat{\Theta}})\tilde{\Theta},
\end{aligned}$$

where

$$\rho_1(x_1, \hat{\Theta}) = \alpha_1(x_1) + \gamma_1(x_1)\sqrt{\hat{\Theta}^2 + 1} + \frac{\varphi_1^2(x_1)}{4\beta} + \rho_0(x_1) \geq 0$$

and

$$\Psi_1(x_1, \hat{\Theta}) = x_1^{2p_1}\gamma_1(x_1) \geq 0.$$

It is easy to check that that the virtual controller

$$x_2^*(x_1, \hat{\Theta}) = -x_1\left[n + \rho_1(x_1, \hat{\Theta})\right]^{1/p_1} \tag{6}$$

satisfies

$$\dot{V}_1(x_1, \hat{\Theta}) + y^{2p_1} - \beta \|w\|^2 \leq -n x_1^{2p_1} + x_1^{p_1}(x_2^{p_1} - x_2^{*p_1}) + (\Psi_1(x_1, \hat{\Theta}) - \dot{\hat{\Theta}})\left(\tilde{\Theta} + \eta_1\right) \tag{7}$$

with $\eta_1 = 0$. Moreover, the virtual control function $x_2^*(x_1, \hat{\Theta})$ is smooth due to the smooth nonnegativeness of functions $\alpha_1(x_1)$,$\gamma_1(x_1)$ and $\varphi_1(x_1)$. In addition

$$\left|\Psi_1(\xi_1, \hat{\Theta})\right| \leq |x_1|^{2p_1}\bar{\alpha}_1(\xi_1, \hat{\Theta}),\ \bar{\alpha}_1(\xi_1, \hat{\Theta}) = \gamma_1(\xi_1) \geq 0. \tag{8}$$

Step 2: Consider the (x_1, x_2)-subsystem of (1). For convenience, we let $x_1^* = 0$ in the sequential discussion. The change of coordinate

$$\xi_1 = x_1, \quad \xi_2 = x_2 - x_2^*(\xi_1, \hat{\Theta})$$

transforms the (x_1, x_2)-subsystem of (1) into

$$\begin{aligned}
\dot{\xi}_1 &= \delta_1(\xi_{[2]}, \hat{\Theta}) + \Phi_1(\xi_1, \hat{\Theta}, \theta) + G_1(\xi_1)w - \omega_1(\hat{\Theta})\dot{\hat{\Theta}}, \\
\dot{\xi}_2 &= x_3^{p_2} + \Delta_2(\xi_{[2]}, \hat{\Theta}) + \Phi_2(\xi_{[2]}, \hat{\Theta}, \theta) + G_2(\xi_{[2]})w - \omega_2(\xi_1, \hat{\Theta})\dot{\hat{\Theta}},
\end{aligned}$$

where

$$\begin{aligned}
&\delta_1(\xi_{[2]}, \hat{\Theta}) = (\xi_2 + x_2^*)^{p_1} + f_1(\xi_1),\\
&\Delta_2(\xi_{[2]}, \hat{\Theta}) = f_2(\xi_{[2]} + x_{[2]}^*) - \tfrac{\partial x_2^*}{\partial \xi_1}\delta_1(\xi_{[2]}, \hat{\Theta}),\\
&\Phi_1(\xi_1, \hat{\Theta}, \theta) = \phi_1(\xi_1, \theta),\\
&\Phi_2(\xi_{[2]}, \hat{\Theta}, \theta) = \phi_2(\xi_{[2]} + x_{[2]}^*, \theta) - \tfrac{\partial x_2^*}{\partial \xi_1}\Phi_1(\xi_1, \hat{\Theta}, \theta),\\
&G_1(\xi_1) = g_1(\xi_1),\\
&G_2(\xi_{[2]}) = g_2(\xi_{[2]} + x_{[2]}^*) - \tfrac{\partial x_2^*}{\partial \xi_1}G_1(\xi_1),\\
&\omega_1(\hat{\Theta}) = 0,\\
&\omega_2(\xi_1, \hat{\Theta}) = \tfrac{\partial x_2^*}{\partial \hat{\Theta}} - \tfrac{\partial x_2^*}{\partial \xi_1}\omega_1(\hat{\Theta}).
\end{aligned}$$

Under the condition of Assumption A4, it follows from the relation (5) that

$$\left|\Phi_1(\xi_1, \hat{\Theta}, \theta)\right| \le |x_1|^{p_1}\, \bar{\beta}_1(\xi_1, \hat{\Theta})\Theta,\ \ \bar{\beta}_1(\xi_1, \hat{\Theta}) = \gamma_1(\xi_1) \ge 0.$$

With this relation combined with (5), we have by applying Lemma 2.1

$$\begin{aligned}
\left|\Phi_2(\xi_{[2]}, \hat{\Theta}, \theta)\right| &\le \left|\phi_2(\xi_{[2]} + x_{[2]}^*, \theta)\right| + \left|\frac{\partial x_2^*}{\partial \xi_1}\Phi_1(\xi_1, \theta)\right|\\
&\le (|\xi_1|^{p_2} + |\xi_2 + x_2^*|^{p_2})\, \gamma_2(\xi_{[2]} + x_{[2]}^*)\Theta + \left|\frac{\partial x_2^*}{\partial \xi_1}\right| |x_1|^{p_1}\, \gamma_1(x_1)\Theta\\
&\le (|\xi_1|^{p_2} + |\xi_2|^{p_2})\tilde{\beta}_2(\xi_{[2]}, \hat{\Theta})\Theta + |\xi_1|^{p_1}\left|\frac{\partial x_2^*}{\partial \xi_1}\right|\gamma_1(\xi_1)\Theta
\end{aligned}$$

for a smooth function $\tilde{\beta}_2(\cdot) \ge 0$. This inequality implies that

$$\left|\Phi_2(\xi_{[2]}, \hat{\Theta}, \theta)\right| \le (|\xi_1|^{p_2} + |\xi_2|^{p_2})\bar{\beta}_2(\xi_{[2]}, \hat{\Theta})\Theta \tag{9}$$

for a smooth function $\bar{\beta}_2(\cdot) \ge 0$, because $p_1 \ge p_2$. By similar ways, it is easy to show that the following two relations hold

$$\begin{aligned}
\left|\delta_1(\xi_{[2]}, \hat{\Theta})\right| &\le (|\xi_1|^{p_1} + |\xi_2|^{p_1})\, \bar{\tau}_1(\xi_{[2]}, \hat{\Theta}),\\
\left|\Delta_2(\xi_{[2]}, \hat{\Theta})\right| &\le (|\xi_1|^{p_2} + (|\xi_2|^{p_2})\, \tilde{\tau}_2(\xi_{[2]}, \hat{\Theta}),
\end{aligned} \tag{10}$$

for two nonnegative smooth functions $\bar{\tau}_1(\xi_{[2]}, \hat{\Theta})$ and $\tilde{\tau}_2(\xi_{[2]}, \hat{\Theta})$. By Assumption A3, it is known that there exist smooth nonnegative functions $\tilde{\varphi}_1(\xi_1)$ and $\tilde{\varphi}_2(\xi_{[2]})$, satisfying

$$\begin{aligned}
\|G_1(\xi_1)\| &\le \varphi_1(\xi_1) = \tilde{\varphi}_1(\xi_1),\\
\left\|G_2(\xi_{[2]})\right\| &\le \left\|g_2(\xi_{[2]} + x_{[2]}^*) - \frac{\partial x_2^*}{\partial \xi_1}G_1(\xi_1)\right\| \le \tilde{\varphi}_2(\xi_{[2]}).
\end{aligned} \tag{11}$$

Again using Lemma 2.1, we have from (9) and (10) that

$$\begin{aligned}
&\left|\xi_2^{2p_1 - p_2}\Phi_2(\xi_{[2]}, \hat{\Theta}, \theta)\right| \le \left[\frac{\xi_1^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_1^2)} + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]}, \hat{\Theta})\right]\Theta\\
\le\ &\frac{1}{6}\xi_1^{2p_1} + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]}, \hat{\Theta})\sqrt{\hat{\Theta}^2+1} + \left[\frac{\xi_1^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_1^2)} + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]}, \hat{\Theta})\right]\tilde{\Theta},
\end{aligned} \tag{12}$$

$$\left|\xi_2^{2p_1-p_2}\Delta_2(\xi_{[2]},\hat{\Theta})\right| \le \frac{1}{6}\xi_1^{2p_1} + \xi_2^{2p_1}\check{\rho}_2(\xi_{[2]},\hat{\Theta}), \tag{13}$$

for some nonnegative smooth functions $\bar{\rho}_2(\xi_{[2]},\hat{\Theta})$ and $\check{\rho}_2(\xi_{[2]},\hat{\Theta})$. By applying Lemma 2.2, it is easy to show from (6) that

$$\left|\xi_1^{p_1}((\xi_2+x_2^*)^{p_1} - x_2^{*p_1})\right| \le \frac{\xi_1^{2p_1}}{6} + \xi_2^{2p_1}\tilde{\rho}_2(\xi_{[2]},\hat{\Theta}), \tag{14}$$

for some nonnegative smooth function $\tilde{\rho}_2(\xi_{[2]},\hat{\Theta})$. Now, consider the Lyapunov function

$$V_2(\xi_{[2]},\hat{\Theta}) = V_1(x_1,\hat{\Theta}) + \frac{\xi_2^{2p_1-p_2+1}}{2p_1-p_2+1}$$

which is positive definite and radially unbounded. With the relations (7), (12), (13) and (14), a straightforward computation gives

$$\begin{aligned}
& \dot{V}_2(\xi_{[2]},\tilde{\Theta}) + y^{2p_1} - 2\beta\left\|w\right\|^2 \\
\le\ & -n\xi_1^{2p_1} + \xi_1^{p_1}((\xi_2+x_2^*)^{p_1} - x_2^{*p_1}) + (\Psi_1(x_1,\hat{\Theta}) - \dot{\hat{\Theta}})\left(\tilde{\Theta}+\eta_1\right) \\
& +\xi_2^{2p_1-p_2}\left(x_3^{p_2} + \Delta_2(\xi_{[2]},\hat{\Theta}) + \Phi_2(\xi_{[2]},\hat{\Theta},\theta) + G_2(\xi_{[2]})w\right) \\
& -\xi_2^{2p_1-p_2}\omega_2(\xi_1,\hat{\Theta})\dot{\hat{\Theta}} - \beta\left\|w\right\|^2 \\
\le\ & -\left(n-\frac{1}{2}\right)\xi_1^{2p_1} + \xi_2^{2p_1}\tilde{\rho}_2(\xi_{[2]},\hat{\Theta}) + (\Psi_1(x_1,\hat{\Theta}) - \dot{\hat{\Theta}})\left(\tilde{\Theta}+\eta_1\right) + \xi_2^{2p_1-p_2}G_2(\xi_{[2]})w \\
& +\xi_2^{2p_1-p_2}x_3^{p_2} + \xi_2^{2p_1}\check{\rho}_2(\xi_{[2]},\hat{\Theta}) + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]},\hat{\Theta})\sqrt{\hat{\Theta}^2+1} - \beta\left\|w\right\|^2 \\
& +\left[\frac{\xi_1^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_1^2)} + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]},\hat{\Theta})\right]\tilde{\Theta} - \xi_2^{2p_1-p_2}\omega_2(\xi_1,\hat{\Theta})\dot{\hat{\Theta}} \\
=\ & -\left(n-\frac{1}{2}\right)\xi_1^{2p_1} + \xi_2^{2p_1-p_2}x_3^{p_2} + \xi_2^{2p_1}\left[\tilde{\rho}_2(\xi_{[2]},\hat{\Theta}) + \bar{\rho}_2(\xi_{[2]},\hat{\Theta})\sqrt{\hat{\Theta}^2+1} + \check{\rho}_2(\xi_{[2]},\hat{\Theta})\right] \\
& +\xi_2^{2p_1-p_2}G_2(\xi_{[2]})w - \beta\left\|w\right\|^2 + \left(\Psi_2(\xi_{[2]},\hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_2(\xi_{[2]},\hat{\Theta})\right) + \Pi_2(\xi_{[2]},\hat{\Theta}),
\end{aligned} \tag{15}$$

where

$$\Psi_2(\xi_{[2]},\hat{\Theta}) = \Psi_1(\xi_1) + \frac{\xi_1^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_1^2)} + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]},\hat{\Theta}),$$

$$\eta_2(\xi_{[2]},\hat{\Theta}) = \eta_1 + \xi_2^{2p_1-p_2}\omega_2(\xi_1,\hat{\Theta}),$$

$$\Pi_2(\xi_{[2]},\hat{\Theta}) = -\Psi_2(\xi_{[2]},\hat{\Theta})\xi_2^{2p_1-p_2}\omega_2(\xi_1,\hat{\Theta}) - \left[\frac{\xi_1^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_1^2)} + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]},\hat{\Theta})\right]\eta_1. \tag{16}$$

By applying Lemma 2.1 again, it is easy to derive from the relation (8) that

$$\left|\Psi_2(\xi_{[2]},\hat{\Theta})\right| \le (\xi_1^{2p_1} + \xi_2^{2p_1})\bar{\alpha}_2(\xi_{[2]},\hat{\Theta}) \tag{17}$$

for a smooth function $\bar{\alpha}_2(\xi_{[2]}, \hat{\Theta}) \geq 0$. By the completion of square, it is easily derived from (11) that

$$\left\| \xi_2^{2p_1 - p_2} G_2(\xi_{[2]}) w \right\| \leq \left| \xi_2^{2p_1 - p_2} \right| \tilde{\varphi}_2(\xi_{[2]}) \|w\| \leq \xi_2^{2p_1} \frac{\xi_2^{2p_1 - 2p_2} \tilde{\varphi}_2^2(\xi_{[2]})}{4\beta} + \beta \|w\|^2 . \tag{18}$$

In view of the relation (17), by using Lemma 2.1 it is easily obtained from (16) that the following relation holds

$$\begin{aligned} & \left| \Pi_2(\xi_{[2]}, \hat{\Theta}) \right| \\ \leq\ & (\xi_1^{2p_1} + \xi_2^{2p_1}) \bar{\alpha}_2(\xi_{[2]}, \hat{\Theta}) \left| \xi_2^{2p_1 - p_2} \omega_2(\xi_1, \hat{\Theta}) \right| + \frac{1}{6} \xi_1^{2p_1} + \xi_2^{2p_1} \bar{\rho}_2(\xi_{[2]}, \hat{\Theta}) \sqrt{\eta_1^2 + 1} \\ \leq\ & \frac{\xi_1^{2p_1}}{2} + \xi_2^{2p_1} \hat{\rho}_2(\xi_{[2]}, \hat{\Theta}). \end{aligned} \tag{19}$$

for a nonnegative smooth functions $\hat{\rho}_2(\cdot)$. With the relations (14) – (19) in mind, it follows from (15) that

$$\begin{aligned} \dot{V}_2(\xi_{[2]}, \hat{\Theta}) \leq\ & -(n-1) x_1^{2p_1} + \left(\Psi_2(\xi_{[2]}, \hat{\Theta}) - \dot{\hat{\Theta}} \right) \left(\tilde{\Theta} + \eta_2(\xi_{[2]}, \hat{\Theta}) \right) \\ & + \xi_2^{2p_1} \rho_2(\xi_{[2]}, \hat{\Theta}) + \xi_2^{2p_1 - p_2} \left(x_3^{p_2} - x_3^{*p_2} \right) + \xi_2^{2p_1 - p_2} x_3^{*p_2}, \end{aligned}$$

where

$$\rho_2(\xi_{[2]}, \hat{\Theta}) = \tilde{\rho}_2(\xi_{[2]}, \hat{\Theta}) + \breve{\rho}_2(\xi_{[2]}, \hat{\Theta}) + \bar{\rho}_2(\xi_{[2]}, \hat{\Theta}) \sqrt{\hat{\Theta}^2 + 1} + \hat{\rho}_2(\xi_{[2]}, \hat{\Theta}) + \frac{\xi_2^{2p_1 - 2p_2} \tilde{\varphi}_2^2(\xi_{[2]})}{4\beta} \geq 0.$$

Choose

$$x_3^* = -\xi_2 \left[n - 1 + \rho_2(\xi_{[2]}, \hat{\Theta}) \right]^{1/p_2} .$$

This smooth virtual controller will satisfy

$$\begin{aligned} \dot{V}_2(\xi_{[2]}, \hat{\Theta}) + y^{2p_1} - 2\beta \|w\|^2 \leq\ & -(n-1)(\xi_1^{2p_1} + \xi_2^{2p_1}) + \xi_2^{2p_1 - p_2} \left(x_3^{p_2} - x_3^{*p_2} \right) \\ & + \left(\Psi_2(\xi_{[2]}, \hat{\Theta}) - \dot{\hat{\Theta}} \right) \left(\tilde{\Theta} + \eta_2(\xi_{[2]}, \hat{\Theta}) \right). \end{aligned}$$

Inductive Step: Suppose for the system (1) with dimension k, there is a global change of coordinates $\xi_i = x_i - x_i^*(\xi_{[i-1]}, \hat{\Theta})$, $i \in I[1, k]$, transforming (1) into the system

$$\begin{aligned} \dot{\xi}_1 &= \delta_1(\xi_{[2]}, \hat{\Theta}) + \Phi_1(\xi_1, \hat{\Theta}, \theta) + G_1(\xi_1) w - \omega_1(\hat{\Theta}) \dot{\hat{\Theta}}, \\ & \cdots \\ \dot{\xi}_{k-1} &= \delta_{k-1}(\xi_{[k]}, \hat{\Theta}) + \Phi_{k-1}(\xi_{[k-1]}, \hat{\Theta}, \theta) + G_{k-1}(\xi_{[k-1]}) w - \omega_{k-1}(\xi_{[k-2]}, \hat{\Theta}) \dot{\hat{\Theta}}, \\ \dot{\xi}_k &= x_{k+1}^{p_k} + \Delta_k(\xi_{[k]}, \hat{\Theta}) + \Phi_k(\xi_{[k]}, \hat{\Theta}, \theta) + G_k(\xi_{[k]}) w - \omega_k(\xi_{[k-1]}, \hat{\Theta}) \dot{\hat{\Theta}}, \end{aligned} \tag{20}$$

where

$$x_i^* = -\xi_{i-1} \left[n - i + 2 + \rho_{i-1}(\xi_{[i-1]}, \hat{\Theta}) \right]^{1/p_{i-1}} , \ i \in I[2, k], \tag{21}$$

$$\left|\Phi_i(\xi_{[i]},\hat{\Theta},\theta)\right| \leq \bar{\beta}_i(\xi_{[i]},\hat{\Theta})\Theta \sum_{j=1}^{i} |\xi_j|^{p_i}\ ,\ i \in I[1,k], \tag{22}$$

$$\left|\delta_i(\xi_{[i+1]},\hat{\Theta},\theta)\right| \leq \bar{\tau}_i(\xi_{[i+1]},\hat{\Theta}) \sum_{j=1}^{i} |\xi_j|^{p_i}\ ,\ i \in I[1,k-1], \tag{23}$$

$$\left|\Delta_k(\xi_{[k]},\hat{\Theta},\theta)\right| \leq \tilde{\tau}_k(\xi_{[k]},\hat{\Theta}) \sum_{j=1}^{k} |\xi_j|^{p_k}\ , \tag{24}$$

$$\left\|G_i(\xi_{[i]})\right\| \leq \tilde{\varphi}_i(\xi_{[i]}), \tag{25}$$

for some nonnegative smooth functions $\bar{\beta}_i(\cdot)$, $\tilde{\varphi}_i(\cdot)$, $i \in I[1,k]$, $\tilde{\tau}_k(\cdot)$, and $\rho_i(\cdot)$, $\bar{\tau}_i(\cdot)$, $i \in I[1,k-1]$. Moreover, there is a virtual controller

$$x_{k+1}^*(x_{[k]},\hat{\Theta}) = -\xi_k \left[n-k+1+\rho_k(\xi_{[k]},\hat{\Theta})\right]^{1/p_k},\ \rho_k(\xi_{[k]},\hat{\Theta}) \geq 0, \tag{26}$$

such that the closed-loop system (20) – (26) satisfies

$$\begin{aligned} \dot{V}_k(\xi_{[k]},\hat{\Theta}) + y^{2p_1} - k\beta\left\|w\right\|^2 \quad \leq \quad & -(n-k+1)\sum_{i=1}^{k} \xi_i^{2p_1} + \xi_k^{2p_1-p_k}\left(x_{k+1}^{p_k} - x_{k+1}^{*p_k}\right) \\ & + \left(\Psi_k(\xi_{[k]},\hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_k(\xi_{[k]},\hat{\Theta})\right), \end{aligned} \tag{27}$$

where

$$V_k(\xi_{[k]},\hat{\Theta}) = \frac{1}{2}\tilde{\Theta}^2 + \sum_{i=1}^{k} \frac{\xi_i^{2p_1-p_i+1}}{2p_1-p_i+1},$$

is a positive definite and proper Lyapunov function. Moreover,

$$\left|\Psi_k(\xi_{[k]},\hat{\Theta})\right| \leq \bar{\alpha}_k(\xi_{[k]},\hat{\Theta}) \sum_{i=1}^{k} \xi_i^{2p_1}. \tag{28}$$

Then, in the case when the dimension of system (1) is equal to $k+1$, introduce the transformation $\xi_{k+1} = x_{k+1} - x_{k+1}^*(\xi_{[k]},\hat{\Theta})$. This, together with (26), leads to the augmented system

$$\begin{aligned} \dot{\xi}_1 &= \delta_1(\xi_{[2]},\hat{\Theta}) + \Phi_1(\xi_1,\hat{\Theta},\theta) + G_1(\xi_1) - \omega_1(\hat{\Theta})\dot{\hat{\Theta}}, \\ & \cdots \\ \dot{\xi}_k &= \delta_k(\xi_{[k+1]},\hat{\Theta}) + \Phi_k(\xi_{[k]},\hat{\Theta},\theta) + G_k(\xi_{[k]}) - \omega_k(\xi_{[k-1]},\hat{\Theta})\dot{\hat{\Theta}}, \\ \dot{\xi}_{k+1} &= x_{k+2}^{p_{k+1}} + \Delta_{k+1}(\xi_{[k+1]},\hat{\Theta}) + \Phi_{k+1}(\xi_{[k+1]},\hat{\Theta},\theta), \\ & \quad + G_{k+1}(\xi_{[k+1]})w - \omega_{k+1}(\xi_{[k]},\hat{\Theta})\dot{\hat{\Theta}}, \end{aligned} \tag{29}$$

where

$$\omega_{k+1}(\xi_{[k]},\hat{\Theta}) = -\sum_{i=1}^{k} \frac{\partial x_{k+1}^*}{\partial \xi_i}\omega_i(\xi_{[i-1]},\hat{\Theta}) + \frac{\partial x_{k+1}^*}{\partial \hat{\Theta}},$$

$$\delta_k(\xi_{[k+1]}, \hat{\Theta}) = \Delta_k(\xi_{[k+1]}, \hat{\Theta}) + (\xi_{k+1} + x^*_{k+1})^{p_k},$$

$$\Delta_{k+1}(\xi_{[k+1]}, \hat{\Theta}) = f_{k+1}(\xi_{[k+1]} + x^*_{[k+1]}) - \sum_{i=1}^{k} \frac{\partial x^*_{k+1}}{\partial \xi_i} \delta_i(\xi_{[i+1]}, \hat{\Theta}),$$

$$\Phi_{k+1}(\xi_{[k+1]}, \hat{\Theta}, \theta) = \phi_{k+1}(\xi_{[k+1]} + x^*_{[k+1]}, \theta) - \sum_{i=1}^{k} \frac{\partial x^*_{k+1}}{\partial \xi_i} \Phi_i(\xi_{[i]}, \hat{\Theta}, \theta),$$

$$G_{k+1}(\xi_{[k+1]}) = g_{k+1}(\xi_{[k]}) - \sum_{i=1}^{k} \frac{\partial x^*_{k+1}}{\partial \xi_i} G_i(\xi_{[i]}).$$

Under the condition of Assumption 4, the relation (5) holds. Combining this relation with the inductive assumption (22) and (21), and applying Lemma 2.1, we have

$$\begin{aligned} \left|\Phi_{k+1}(\xi_{[k+1]}, \hat{\Theta}, \theta)\right| &\leq \left|\phi_{k+1}(\xi_{[k+1]} + x^*_{[k+1]}, \theta)\right| + \sum_{i=1}^{k} \left|\frac{\partial x^*_{k+1}}{\partial \xi_i}\right| \Phi_i(\xi_{[i]}, \hat{\Theta}, \theta) \\ &\leq \gamma_{k+1}(\xi_{[k+1]} + x^*_{[k+1]})\Theta \sum_{i=1}^{k+1} |\xi_i + x^*_i|^{p_{k+1}} \\ &\quad + \sum_{i=1}^{k} \left[\left|\frac{\partial x^*_{k+1}}{\partial \xi_i}\right| \bar{\beta}_i(\xi_{[i]}, \hat{\Theta})\Theta \sum_{j=1}^{i} |\xi_j|^{p_i} \right]. \end{aligned}$$

In view of the fact that and $p_1 \geq p_2 \geq \cdots \geq p_{k+1}$, there exists a smooth function $\bar{\beta}_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \geq 0$, such that

$$\left|\Phi_{k+1}(\xi_{[k+1]}, \hat{\Theta}, \theta)\right| \leq \Theta \bar{\beta}_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \sum_{i=1}^{k+1} |\xi_i|^{p_{k+1}}. \tag{30}$$

Under the condition of Assumption A3, by applying the inductive assumption (25) and the smoothness of x^*_i, $i \in I[1, k]$, it is known that there is a smooth function $\tilde{\varphi}_{k+1}(\xi_{[k+1]})$ to satisfy

$$\left\|G_{k+1}(\xi_{[k+1]})\right\| \leq \tilde{\varphi}_{k+1}(\xi_{[k+1]}). \tag{31}$$

According to the inductive assumptions (24) and (21), we can obtain by using Lemma 2.1 again

$$\begin{aligned} \left|\delta_k(\xi_{[k+1]}, \hat{\Theta})\right| &\leq \left|\Delta_k(\xi_{[k]}, \hat{\Theta})\right| + \left|(\xi_{k+1} + x^*_{k+1})^{p_k}\right| \\ &\leq \tilde{\tau}_k(\xi_{[k]}, \hat{\Theta}) \sum_{i=1}^{k} |\xi_i|^{p_k} + \left|(\xi_{k+1} + x^*_{k+1})^{p_k}\right| \\ &\leq \bar{\tau}_k(\xi_{[k+1]}, \hat{\Theta}) \sum_{i=1}^{k+1} |\xi_i|^{p_k} \end{aligned} \tag{32}$$

for some smooth function $\bar{\tau}_k(\xi_{[k+1]}, \hat{\Theta}) \geq 0$. According to Assumption 2 and by the relations (32) and (23), it can be derived that

$$
\begin{aligned}
\left|\Delta_{k+1}(\xi_{[k+1]}, \hat{\Theta})\right| &\leq \left| f_{k+1}(\xi_{[k+1]} + x^*_{[k+1]}) - \sum_{i=1}^{k} \frac{\partial x^*_{k+1}}{\partial \xi_i} \delta_i(\xi_{[i+1]}, \hat{\Theta}) \right| \\
&\leq \alpha_{k+1}(\xi_{[k+1]} + x^*_{[k+1]}) \sum_{i=1}^{k+1} |\xi_i + x_i^*|^{p_{k+1}} \qquad (33) \\
&\quad + \sum_{i=1}^{k} \left[\left| \frac{\partial x^*_{k+1}}{\partial \xi_i} \right| \bar{\tau}_i(\xi_{[i+1]}, \hat{\Theta}) \sum_{j=1}^{i+1} |\xi_j|^{p_i} \right] \\
&\leq \tilde{\tau}_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \sum_{j=1}^{k+1} |\xi_j|^{p_{k+1}}
\end{aligned}
$$

for a smooth function $\tilde{\tau}_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \geq 0$. Again using Lemma 2.1, we have from (30)

$$
\begin{aligned}
& \left| \xi_{k+1}^{2p_1 - p_{k+1}} \Phi_{k+1}(\xi_{[k+1]}, \hat{\Theta}, \theta) \right| \\
\leq & \left[\frac{\sum_{i=1}^{k} \xi_i^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(x_{[k]}, \hat{\Theta}))} + \xi_{k+1}^{2p_1} \bar{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \right] \Theta \\
\leq & \frac{1}{6} \sum_{i=1}^{k} \xi_i^{2p_1} + \xi_{k+1}^{2p_1} \bar{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \sqrt{\hat{\Theta}^2 + 1} \qquad (34) \\
+ & \left[\frac{\sum_{i=1}^{k} \xi_i^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(x_{[k]}, \hat{\Theta}))} + \xi_{k+1}^{2p_1} \bar{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \right] \tilde{\Theta}, \qquad (35)
\end{aligned}
$$

$$
\left| \xi_{k+1}^{2p_1 - p_2} \Delta_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \right| \leq \frac{1}{6} \sum_{i=1}^{k} \xi_i^{2p_1} + \xi_{k+1}^{2p_1} \check{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta}), \qquad (36)
$$

for some nonnegative smooth functions $\bar{\rho}_{k+1}(\xi_{[2]}, \hat{\Theta})$ and $\check{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta})$. By applying Lemma 2.2, it is easy to show that

$$
\left| \xi_k^{2p_1 - p_k} \left(\left(\xi_{k+1} + x^*_{k+1} \right)^{p_k} - x^{*p_k}_{k+1} \right) \right| \leq \frac{1}{6} \xi_k^{2p_1} + \xi_{k+1}^{2p_1} \tilde{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta}), \qquad (37)
$$

for some nonnegative smooth function $\tilde{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta})$. Now consider the Lyapunov function

$$
V_{k+1}(\xi_{[k+1]}, \hat{\Theta}) = V_k(\xi_{[k]}, \hat{\Theta}) + \frac{\xi_{k+1}^{2p_1 - p_{k+1} + 1}}{2p_1 - p_{k+1} + 1}.
$$

With the relations (35), (36) and (37) and the inductive assumption (27), it is derived that the time derivative of V_{k+1} along the trajectories of system (29) satisfies

$$\begin{aligned}
& \dot{V}_{k+1} + y^{2p_1} - (k+1)\beta \|w\|^2 \\
\leq \;& -(n-k+1)\sum_{i=1}^{k} \xi_i^{2p_1} + \xi_k^{2p_1-p_k}\left(x_{k+1}^{p_k} - x_{k+1}^{*p_k}\right) \\
+ \;& \left(\Psi_k(\xi_{[k]},\hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_k(\xi_{[k]},\hat{\Theta})\right) \\
+ \;& \xi_{k+1}^{2p_1-p_{k+1}}\Big(x_{k+2}^{p_{k+1}} + \Delta_{k+1}(\xi_{[k+1]},\hat{\Theta}) + \Phi_{k+1}(\xi_{[k+1]},\hat{\Theta},\theta \\
+ \;& G_{k+1}(\xi_{[k+1]})w - \omega_{k+1}(\xi_{[k]},\hat{\Theta})\dot{\hat{\Theta}}\Big) - \beta\|w\|^2 \\
\leq \;& -(n-k+1)\sum_{i=1}^{k} \xi_i^{2p_1} + \frac{1}{2}\sum_{i=1}^{k}\xi_i^{2p_1} + \left(\Psi_k(\xi_{[k]},\hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_k(\xi_{[k]},\hat{\Theta})\right) \\
+ \;& \xi_{k+1}^{2p_1-p_{k+1}} x_{k+2}^{p_{k+1}} + \xi_{k+1}^{2p_1}\Big[\bar{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\sqrt{\hat{\Theta}^2+1} + \check{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta}) \\
+ \;& \tilde{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\Big] - \xi_{k+1}^{2p_1-p_{k+1}}\omega_{k+1}(\xi_{[k]},\hat{\Theta})\dot{\hat{\Theta}} \\
+ \;& \left[\frac{\sum_{i=1}^{k}\xi_i^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(\xi_{[k]},\hat{\Theta}))} + \xi_{k+1}^{2p_1}\bar{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\right]\tilde{\Theta} \\
+ \;& \xi_{k+1}^{2p_1-p_{k+1}} G_{k+1}(\xi_{[k+1]})w - \beta\|w\|^2
\end{aligned}$$

$$= \; -(n-k+\frac{1}{2})\sum_{i=1}^{k}\xi_i^{2p_1} + \xi_{k+1}^{2p_1}\Big[\bar{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\sqrt{\hat{\Theta}^2+1} + \check{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta}) \tag{38}$$

$$+ \; \tilde{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\Big] + \xi_2^{2p_1-p_2}G_{k+1}(\xi_{[k+1]})w - \beta\|w\|^2 + \xi_{k+1}^{2p_1-p_{k+1}}x_{k+2}^{p_{k+1}} \tag{39}$$

$$+\left(\Psi_{k+1}(\xi_{[k+1]},\hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_{k+1}(\xi_{[k+1]},\hat{\Theta})\right) + \Pi_{k+1}(\xi_{[k+1]},\hat{\Theta}),$$

where

$$\Psi_{k+1}(\xi_{[k+1]},\hat{\Theta}) = \Psi_k(\xi_{[k]},\hat{\Theta}) + \frac{\sum_{i=1}^{k}\xi_i^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(x_{[k]},\hat{\Theta}))} + \xi_{k+1}^{2p_1}\bar{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta}),$$

$$\eta_{k+1}(\xi_{[k+1]},\hat{\Theta}) = \eta_k(\xi_{[k]},\hat{\Theta}) + \xi_{k+1}^{2p_1-p_{k+1}}\omega_{k+1}(\xi_{[k]},\hat{\Theta})$$

$$\begin{aligned}
\Pi_{k+1}(\xi_{[k+1]},\hat{\Theta}) \;=\; & -\Psi_{k+1}(\xi_{[k+1]},\hat{\Theta})\xi_{k+1}^{2p_1-p_{k+1}}\omega_{k+1}(\xi_{[k]},\hat{\Theta}) \\
- \; & \left[\frac{\sum_{i=1}^{k}\xi_i^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(\xi_{[k+1]},\hat{\Theta}))} + \xi_{k+1}^{2p_1}\bar{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\right]\eta_k(\xi_{[k+1]},\hat{\Theta}).
\end{aligned} \tag{40}$$

By using Lemma 2.1, it is easily derived from (28) that the following relation holds

$$\left|\Psi_{k+1}(\xi_{[k+1]},\hat{\Theta})\right| \leq \bar{\alpha}_{k+1}(\xi_{[k+1]},\hat{\Theta})\sum_{i=1}^{k+1}|\xi_i|^{2p_1} \tag{41}$$

for a smooth function $\bar{\alpha}_{k+1}(\xi_{[k+1]},\hat{\Theta})\geq 0$. In view of relation (41), we have

$$\begin{aligned}&\Pi_{k+1}(\xi_{[k+1]},\hat{\Theta})\\ \leq\;& \bar{\alpha}_{k+1}(\cdot)\left|\xi_{k+1}^{2p_1-p_{k+1}}\omega_{k+1}(\cdot)\right|\sum_{i=1}^{k+1}|\xi_i|^{2p_1}+\frac{1}{6}\sum\nolimits_{i=1}^{k}\xi_i^{2p_1} \qquad (42)\\ +\;& \xi_{k+1}^{2p_1}\bar{\rho}_{k+1}(\cdot)\sqrt{\eta_k^2(\cdot)+1}\leq\frac{1}{2}\sum_{i=1}^{k}\xi_i^{2p_1}+\xi_{k+1}^{2p_1}\hat{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\end{aligned}$$

for a smooth function $\hat{\rho}_{k+1}(\cdot)$. On the other hand, it follows from (31) that

$$\begin{aligned}\left\|\xi_2^{2p_1-p_2}G_{k+1}(\xi_{[k+1]})w\right\| &\leq \left|\xi_2^{2p_1-p_2}\right|\tilde{\varphi}_{k+1}(\xi_{[k+1]})\left\|w\right\|\\ &\leq \xi_2^{2p_1}\frac{\xi_2^{2p_1-2p_2}\tilde{\varphi}_{k+1}^2(\xi_{[k+1]})}{4\beta}+\beta\left\|w\right\|^2. \qquad (43)\end{aligned}$$

By substituting (42) and (43) into (39), it is clear that the following virtual controller

$$x_{k+2}^*(\xi_{[k+1]},\hat{\Theta})=-\xi_{k+1}\left[n-k+\rho_{k+1}(\xi_{[k+1]},\hat{\Theta})\right]^{\frac{1}{p_{k+1}}}$$

with

$$\rho_{k+1}(\cdot)=\bar{\rho}_{k+1}(\cdot)\sqrt{\hat{\Theta}^2+1}+\check{\rho}_{k+1}(\cdot)+\tilde{\rho}_{k+1}(\cdot)+\hat{\rho}_{k+1}(\cdot)+\frac{\xi_2^{2p_1-2p_2}\tilde{\varphi}_{k+1}^2(\xi_{[k+1]})}{4\beta},$$

renders

$$\begin{aligned}\dot{V}_{k+1}(\xi_{[k+1]},\hat{\Theta}) \;\leq\;& -(n-k)\sum_{i=1}^{k+1}|\xi_i|^{2p_1}+\xi_{k+1}^{2p_1-p_{k+1}}\left[x_{k+2}^{p_{k+1}}-x_{k+2}^{*p_{k+1}}\right]\\ &+\left(\Psi_{k+1}(\xi_{[k+1]},\hat{\Theta})-\dot{\hat{\Theta}}\right)\left(\tilde{\Theta}+\eta_{k+1}(\xi_{[k+1]},\hat{\Theta})\right).\end{aligned}$$

The aforementioned inductive argument shows that (27) holds for $k=n$. In fact, in the n-th step, one can construct explicitly a global change of coordinates $(\xi_1,\xi_2,\cdots,\xi_n)$, a positive-definite and proper Lyapunov function $V_n(\xi_{[n]},\tilde{\Theta})$ and a smooth controller

$$u^*(\xi_{[n]},\hat{\Theta})=-\xi_n\left[1+\rho_n(\xi_{[n]},\hat{\Theta})\right]^{1/p_n}$$

for some smooth functions $\rho_n(\cdot)\geq 0$ and $\Psi_{k+1}(\xi_{[k+1]},\hat{\Theta})$, such that

$$\begin{aligned}\dot{V}_n(\xi_{[n]},\tilde{\Theta})+y^{2p_1}-n\beta\left\|w\right\|^2 \;\leq\;& -\sum_{i=1}^{n}\xi_i^{2p_1}+\xi_n^{2p_1-p_n}\left(u^{p_n}-u^{*p_n}\right)\\ &+\left(\Psi_n(\xi_{[n]},\hat{\Theta})-\dot{\hat{\Theta}}\right)\left(\tilde{\Theta}+\eta_n(\xi_{[n]},\hat{\Theta})\right).\end{aligned}$$

Therefore, the one-dimensional smooth adaptive controller

$$\left\{\begin{array}{l}\dot{\hat{\Theta}}=\Psi_n(\xi_{[n]},\hat{\Theta}),\\ u=u^*(\xi_{[n]},\hat{\Theta}),\end{array}\right. \qquad (44)$$

is such that

$$\dot{V}_n(\xi_{[n]}, \hat{\Theta}) + y^{2p_1} - n\beta \|w\|^2 \leq -\sum_{i=1}^{n} \xi_i^{2p_1}. \tag{45}$$

Set $\beta = \gamma^2/n$, we have

$$\dot{V}_n(\xi_{[n]}, \hat{\Theta}) + y^{2p_1} - \gamma^2 \|w\|^2 \leq -\sum_{i=1}^{n} \xi_i^{2p_1}. \tag{46}$$

When $w = 0$, it is derived that

$$\dot{V}_n(\xi_{[n]}, \hat{\Theta}) \leq -\sum_{i=1}^{n} \xi_i^{2p_1}. \tag{47}$$

According to the classical Lyapunov stability theory, it is known that the closed-loop system is global stable at the equilibrium $(\xi_{[n]}, \hat{\Theta}) = (0, 0)$. Since the Lyapunov function $V_n(\xi_{[n]}, \hat{\Theta})$ is positive definite and proper, it follows from (47) and La Salle's invariance principle that all the bounded trajectories of the closed-loop system approach the largest invariant set contained in $\left\{(\xi_{[n]}, \hat{\Theta}) : \dot{V}_n = 0\right\}$. Hence, $\lim_{t\to\infty} \xi_{[n]}(t) = 0$. This, combined with (21) with $k = n$, implies $\lim_{t\to\infty} x_{[n]}(t) = 0$. Moreover, note that $V_n(\cdot)$ is positive definite with $V_n(0) = 0$. It follows from (46) that

$$\int_0^t |y(s)|^{2p_1} \, \mathrm{d}s \leq \gamma^2 \int_0^t \|w\|^2 \, \mathrm{d}s, \; \forall t \geq 0, \text{ when } x(0) = 0.$$

This completes the proof of the theorem.

The proof of Theorem 3.1 is constructive, thus the design procedure of the adaptive controller solving the ARADD problem is actually given. When $w = 0$ and $f_i(x_i) = 0$, $i \in I[1, n]$, it is easy to check that Theorem 3.1 recovers the global stabilization results obtained in [24]. In addition, for the case of linearly parameterized systems we have the following corollary from Theorem 3.1.

Corollary 3.1 *Consider the power integrator triangular system (1) in which* $\phi_i(x_{[i]}, \theta) = \phi_i(x_{[i]})\theta$. *If Assumptions A1–A3 hold and*

$$\phi_i(x_{[i]}) \leq \gamma_i(x_{[i]}) \sum\nolimits_{j=1}^{i} |x_j|^{p_i}, \; i \in I[1, n],$$

then the ARADD problem is solvable by the one-dimensional smooth adaptive controller (4).

According to the result in [22], for linearly parameterized system (1) with s-dimensional unknown parameter θ, the designed adaptive controller is s-dimensional. However, the results presented in this paper indicate that the global adaptive regulation with almost disturbance decoupling for systems (1) is achievable by a smooth one-dimensional adaptive controller, no matter how big the number of unknown parameters is. This shows the minimum-order property of the proposed adaptive controller controller.

4 An Illustrative Example

Consider the following high-order planar nonlinear system

$$\begin{cases} \dot{x}_1 = x_2^3 + \frac{\theta x_1^3}{1+(\sigma x_2)^2} + w, \\ \dot{x}_2 = u^3, \\ y = x_1, \end{cases},$$

where θ and σ are the unknown parameters and w is the disturbance. For this system, one has

$$p_1 = p_2 = 3, \quad f_1 = f_2 = 0, \quad g_1 = 1, g_2 = 0, \quad \phi_1(x) = \frac{\theta x_1^3}{1 + (\sigma x_2)^2}.$$

By letting

$$\alpha_1 = \alpha_2 = 0, \quad \varphi_1 = 1, \varphi_2 = 0, \quad \beta_1 = |\theta|,$$

it is easy to check that Assumptions A1-A4 are satisfied since

$$|\phi_1(x)| = \left| \frac{\theta x_1^3}{1 + (\sigma x_2)^2} \right| \le |\theta| |x_1|^3.$$

In addition, it is easily obtained that $\Theta = |\theta|$ and $\gamma_1 = 1$.

Define $V_1 = \frac{1}{4}x_1^4 + \frac{1}{2}\tilde{\Theta}^2$. Then one has

$$\begin{aligned} \dot{V}_1 &= x_1^3 \dot{x}_1 + \tilde{\Theta}\dot{\tilde{\Theta}} \\ &= x_1^3 \left(x_2^3 + \frac{\theta x_1^3}{1 + (\sigma x_2)^2} + w \right) - \tilde{\Theta}\dot{\hat{\Theta}} \\ &\le x_1^3 x_2^3 + x_1^6 \left(\hat{\Theta} + \tilde{\Theta} \right) + |x_1|^3 |w| - \tilde{\Theta}\dot{\hat{\Theta}} \\ &\le x_1^3 x_2^3 + x_1^6 \left(\hat{\Theta} + \tilde{\Theta} \right) + \frac{x_1^6}{4\beta} + \beta w^2 - \tilde{\Theta}\dot{\hat{\Theta}} \\ &\le x_1^3 x_2^3 + x_1^6 \left(\sqrt{1 + \hat{\Theta}^2} + \frac{1}{4\beta} + 1 \right) - x_1^6 + \beta |w|^2 + \left(x_1^6 - \dot{\hat{\Theta}} \right) \tilde{\Theta}. \end{aligned}$$

Let $\rho_1 = \sqrt{1 + \hat{\Theta}^2} + \frac{1}{4\beta} + 1$, $\Psi_1 = x_1^6$, $\rho_0 = 1$. Then one can obtian

$$\dot{V}_1 + y^6 - \beta w^2 \le x_1^3 x_2^3 + x_1^6 \rho_1 + \left(\Psi_1 - \dot{\hat{\Theta}} \right) \tilde{\Theta}.$$

By choosing $x_2^* = -x_1 (2 + \rho_1)^{1/3}$, one can further obtain

$$\dot{V}_1 + y^6 - \beta w^2 \le -2x_1^6 + x_1^3 \left(x_2^3 - x_2^{*3} \right) + \left(\Psi_1 - \dot{\hat{\Theta}} \right) \tilde{\Theta}.$$

Define $\xi_2 = x_2 - x_2^*$. Then it is derived that

$$\dot{\xi}_2 = \dot{x}_2 - \dot{x}_2^* = u^3 - \frac{\partial x_2^*}{\partial x_1} \dot{x}_1 - \frac{\partial x_2^*}{\partial \hat{\Theta}} \dot{\hat{\Theta}} = u^3 + \Delta_2 + \Phi_2 + G_2 w - \omega_2 \dot{\hat{\Theta}},$$

where

$$\Delta_2 = -\frac{\partial x_2^*}{\partial x_1}\left(\xi_2 + x_2^*\right)^3, \ \Phi_2 = -\frac{\partial x_2^*}{\partial x_1}\frac{\theta x_1^3}{1+\left(\sigma x_2\right)^2}, \quad G_2 = -\frac{\partial x_2^*}{\partial x_1}, \ \omega_2 = -\frac{\partial x_2^*}{\partial \hat{\Theta}}.$$

By defining $V_2 = V_1 + \frac{1}{4}\xi_2^4$, one has

$$\begin{aligned}
&\dot{V}_2 + y^6 - 2\beta w^2 \\
&= \dot{V}_1 + y^6 - \beta w^2 - \beta w^2 + \xi_2^3\dot{\xi}_2 \\
&\leq -2x_1^6 + x_1^3\left(x_2^3 - x_2^{*3}\right) + \left(\Psi_1 - \dot{\hat{\Theta}}\right)\tilde{\Theta} - \beta w^2 \\
&+ \xi_2^3 u^3 + \xi_2^3\Delta_2 + \xi_2^3\Phi_2 + \xi_2^3 G_2 w - \xi_2^3\omega_2\dot{\hat{\Theta}} \\
&\leq -2x_1^6 + \left|x_1^3\left(x_2^3 - x_2^{*3}\right)\right| + \left(\Psi_1 - \dot{\hat{\Theta}}\right)\tilde{\Theta} - \beta w^2 \\
&+ \xi_2^3 u^3 + \left|\xi_2^3\Delta_2\right| + \left|\xi_2^3\Phi_2\right| + \left|\xi_2^3 G_2 w\right| + \left|\xi_2^3\omega_2\dot{\hat{\Theta}}\right|.
\end{aligned}$$

Simple computations yield

$$\begin{aligned}
|\Delta_2| &= \left|\frac{\partial x_2^*}{\partial x_1}\left(\xi_2 + x_2^*\right)^3\right| \\
&= \left|\frac{\partial x_2^*}{\partial x_1}\right|\left|\xi_2^3 + x_2^{*3} + 3\xi_2^2 x_2^* + 3\xi_2 x_2^{*2}\right| \\
&\leq \left|\frac{\partial x_2^*}{\partial x_1}\right|\left(|\xi_2|^3 + |x_2^*|^3 + 3|\xi_2|^2|x_2^*| + 3|\xi_2||x_2^*|^2\right) \\
&\leq \left|\frac{\partial x_2^*}{\partial x_1}\right|\left(4|\xi_2|^3 + 4|x_2^*|^3\right) \\
&\leq 4\left(2+\rho_1\right)\left|\frac{\partial x_2^*}{\partial x_1}\right|\left(|\xi_2|^3 + |x_1|^3\right),
\end{aligned}$$

$$|\Phi_2| = \left|\frac{\partial x_2^*}{\partial x_1}\right|\left|\frac{\theta x_1^3}{1+\left(\sigma x_2\right)^2}\right| \leq |x_1|^3\left|\frac{\partial x_2^*}{\partial x_1}\right|\Theta,$$

$$\begin{aligned}
\left|\Delta_2\xi_2^3\right| &\leq 4\left(2+\rho_1\right)\left|\frac{\partial x_2^*}{\partial x_1}\right|\left(|\xi_2|^3 + |x_1|^3\right)|\xi_2|^3 \\
&\leq 4\left(2+\rho_1\right)\left|\frac{\partial x_2^*}{\partial x_1}\right|\xi_2^6 + 4\left(2+\rho_1\right)\left|\frac{\partial x_2^*}{\partial x_1}\right||x_1|^3|\xi_2|^3 \\
&\leq \check{\rho}_2\xi_2^6 + \frac{1}{6}x_1^6,
\end{aligned}$$

with

$$\check{\rho}_2 = 4\left(2+\rho_1\right)\left|\frac{\partial x_2^*}{\partial x_1}\right| + 24\left(2+\rho_1\right)^2\left(\frac{\partial x_2^*}{\partial x_1}\right)^2,$$

$$
\begin{aligned}
\left|\Phi_2\xi_2^3\right| &\le |\xi_2|^3\,|x_1|^3\left|\frac{\partial x_2^*}{\partial x_1}\right|\Theta \\
&\le \left[\frac{x_1^6}{3\left(1+\hat{\Theta}^2\right)}+\frac{3}{4}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2\left(1+\hat{\Theta}^2\right)\xi_2^6\right]\Theta \\
&\le \left(\frac{x_1^6}{6}+\bar{\rho}_2\sqrt{1+\hat{\Theta}^2}\xi_2^6\right)+\left[\frac{x_1^6}{3\left(1+\hat{\Theta}^2\right)}+\bar{\rho}_2\xi_2^6\right]\tilde{\Theta},
\end{aligned}
$$

with

$$
\bar{\rho}_2=\frac{3}{4}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2\left(1+\hat{\Theta}^2\right),
$$

$$
\begin{aligned}
&\left|x_1^3\left(x_2^3-x_2^{*3}\right)\right| \\
&=\left|x_1^3\left[\left(\xi_2+x_2^*\right)^3-x_2^{*3}\right]\right| \\
&\le |x_1|^3\,|\xi_2|\left(\frac{5}{2}\xi_2^2+\frac{9}{2}x_2^{*2}\right) \\
&=\frac{5}{2}|x_1|^3|\xi_2|^3+\frac{9}{2}|x_1|^5|\xi_2|\left(2+\rho_1\right)^{2/3} \\
&=\frac{1}{6}x_1^6+\tilde{\rho}_2\xi_2^6,
\end{aligned}
$$

with

$$
\tilde{\rho}_2=\frac{75}{4}+\frac{15}{64}9^5\left(2+\rho_1\right)^4,
$$

$$
\left|\xi_2^3G_2w\right|=\left|\xi_2^3\frac{\partial x_2^*}{\partial x_1}\right||w|\le\frac{1}{4\beta}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2\xi_2^6+\beta w^2.
$$

As a result, one has

$$
\begin{aligned}
&\dot{V}_2+y^6-2\beta w^2 \\
&\le -\frac{3}{2}x_1^6+\left[\tilde{\rho}_2+\check{\rho}_2+\bar{\rho}_2\sqrt{1+\hat{\Theta}^2}+\frac{1}{4\beta}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2\right]\xi_2^6 \\
&\quad+\left|\xi_2^3\frac{\partial x_2^*}{\partial\hat{\Theta}}\dot{\hat{\Theta}}\right|+u^3\xi_2^3+\left(\Psi_2-\dot{\hat{\Theta}}\right)\tilde{\Theta},
\end{aligned}
$$

with

$$
\Psi_2=x_1^6+\frac{x_1^6}{3\left(1+\hat{\Theta}^2\right)}+\bar{\rho}_2\xi_2^6.
$$

Choose $\dot{\hat{\Theta}} = \Psi_2$. Then one has

$$\begin{aligned}
\left|\xi_2^3 \frac{\partial x_2^*}{\partial \hat{\Theta}} \dot{\hat{\Theta}}\right| &= \left|\xi_2^3\right| \left|\frac{\partial x_2^*}{\partial \hat{\Theta}}\right| \left(x_1^6 + \frac{x_1^6}{3\left(1+\hat{\Theta}^2\right)} + \bar{\rho}_2 \xi_2^6\right) \\
&\le \left(\frac{4}{3}\left|\frac{\partial x_2^*}{\partial \hat{\Theta}}\right| |x_1|^3\right) |x_1|^3 |\xi_2|^3 + \bar{\rho}_2 \left|\frac{\partial x_2^*}{\partial \hat{\Theta}}\right| |\xi_2|^3 |\xi_2|^6 \\
&\le \frac{1}{2}\left(\frac{4}{3}\left|\frac{\partial x_2^*}{\partial \hat{\Theta}}\right| |x_1|^3\right)^2 \xi_2^6 + \frac{1}{2}x_1^6 + \frac{1}{4}\bar{\rho}_2 \left[\left(\frac{\partial x_2^*}{\partial \hat{\Theta}}\right)^2 + 1\right] \left(\xi_2^6 + 1\right) \xi_2^6 \\
&\le \hat{\rho}_2 \xi_2^6 + \frac{1}{2} x_1^6
\end{aligned}$$

with

$$\hat{\rho}_2 = \frac{8}{9}\left(\frac{\partial x_2^*}{\partial \hat{\Theta}}\right)^2 x_1^6 + \frac{1}{4}\bar{\rho}_2 \left[\left(\frac{\partial x_2^*}{\partial \hat{\Theta}}\right)^2 + 1\right] \left(\xi_2^6 + 1\right).$$

Sequentially,

$$\dot{V}_2 + y^6 - 2\beta w^2 \le -x_1^6 + \left[\tilde{\rho}_2 + \check{\rho}_2 + \bar{\rho}_2 \sqrt{1+\hat{\Theta}^2} + \hat{\rho} + \frac{1}{4\beta}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2\right] \xi_2^6 + u^3 \xi_2^3.$$

Choose $u = -\xi_2 \left[1+\rho_2\right]^{1/3}$ with $\rho_2 = \tilde{\rho}_2 + \check{\rho}_2 + \bar{\rho}_2\sqrt{1+\hat{\Theta}^2} + \hat{\rho} + \frac{1}{4\beta}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2$. Then it is easily derived that $\dot{V}_2 + y^6 - 2\beta w^2 \le -x_1^6 - \xi_2^6$.

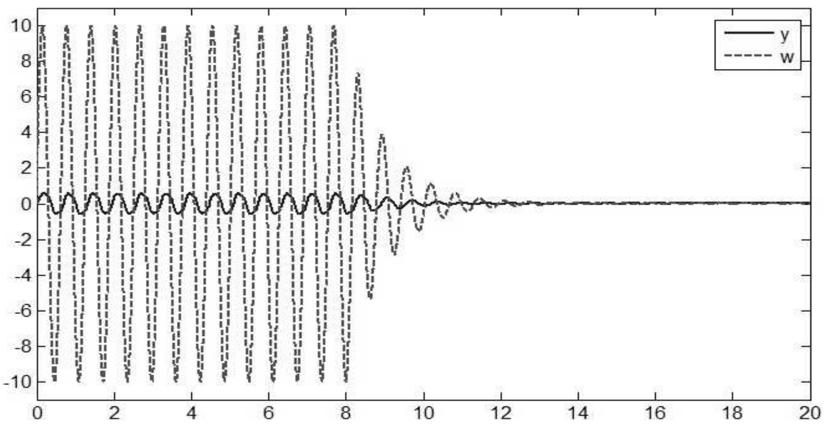

Figure 1: Disturbance signal and output response.

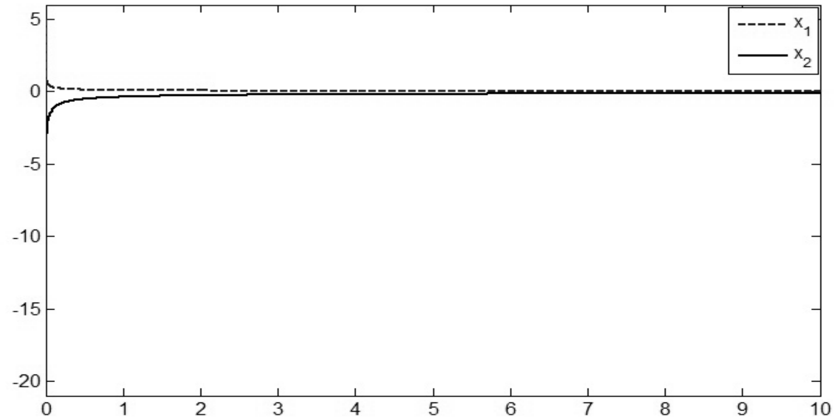

Figure 2: State response.

With the previous derivation, one can obtain the following control law

$$
\begin{aligned}
u &= -\xi_2 \left[1+\rho_2\right]^{1/3}, \\
\rho_2 &= \tilde{\rho}_2 + \check{\rho}_2 + \bar{\rho}_2 \sqrt{1+\hat{\Theta}^2} + \hat{\rho} + \frac{1}{4\beta}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2, \\
\hat{\rho}_2 &= \frac{8}{9}\left(\frac{\partial x_2^*}{\partial \hat{\Theta}}\right)^2 x_1^6 + \frac{1}{4}\bar{\rho}_2\left[\left(\frac{\partial x_2^*}{\partial \hat{\Theta}}\right)^2 + 1\right]\left(\xi_2^6+1\right), \\
\bar{\rho}_2 &= \frac{3}{4}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2\left(1+\hat{\Theta}^2\right), \\
\check{\rho}_2 &= 4\left(2+\rho_1\right)\left|\frac{\partial x_2^*}{\partial x_1}\right| + 24\left(2+\rho_1\right)^2\left(\frac{\partial x_2^*}{\partial x_1}\right)^2, \\
\tilde{\rho}_2 &= \frac{75}{4} + \frac{15}{64}9^5\left(2+\rho_1\right)^4,
\end{aligned}
$$

$$
\begin{aligned}
\rho_1 &= \sqrt{1+\hat{\Theta}^2} + \frac{1}{4\beta} + 1, \\
x_2^* &= -x_1\left(2+\rho_1\right)^{1/3}, \\
\frac{\partial x_2^*}{\partial x_1} &= -\left(2+\rho_1\right)^{1/3}, \\
\frac{\partial x_2^*}{\partial \hat{\Theta}} &= -\frac{x_1}{3}\left(2+\rho_1\right)^{-2/3}\left(1+\hat{\Theta}^2\right)^{-1/2}\hat{\Theta}, \\
\dot{\hat{\Theta}} &= x_1^6 + \frac{x_1^6}{3\left(1+\hat{\Theta}^2\right)} + \bar{\rho}_2\xi_2^6.
\end{aligned}
$$

Figures 1 and 2 give the simulation results of the resulstant closed-loop system under the obtained control law.

5 Conclusion

For the class of power integrator lower triangular systems with nonlinear parametrization, we formulated the problem of adaptive regulation with almost disturbance decoupling. Under a set of growth conditions, an explicit design of the adaptive smooth controller solving the ADD problem was provided. A significant feature of the obtained adaptive dynamical compensator is its minimum-order property. The results of this paper exploit a new application of the parameter separation technique proposed recently in [24].

Acknowledgment

This work is supported by the National Natural Science Foundation of China under Grant No. 60974044, and by the Basic Research Plan in Shenzhen City under Grant No. JC201005260145A.

References

[1] Saberi, A. and Sannuti, P. Global stabilization with almost disturbance decoupling of a class of uncertain nonlinear systems. *International Journal of Control* **47** (1988) 717–727.

[2] Marino, R., Respondek, W. and Van der Schaft, A. J. Almost disturbanc decoupling for single-input single-output nonlinear systems. *IEEE Transactions on Automatic Control* **34** (9) (1989) 1013–1017.

[3] Marino, R., Respondek, W. and Van der Schaft, A. J. Nonlinear H_∞ almost disturbanc decoupling. *Systems & Control Letters* **23** (1994) 159–168.

[4] Isidori, A. A note on almost disturbance decoupling for nonlinear minimum phase systems. *Systems & Control Letters* **27** (1996) 191–194.

[5] Isidori, A. Global almost disturbance decoupling with stability for non minimum-phase single-input single-output nonlinear systems. *Systems & Control Letters* **28** (1996) 115–122.

[6] Lin, Z. Almost disturbance decoupling with global asymptotic stability for nonlinear systems with disturbance-affected unstable zero dynamics. *Systems & Control Letters* **33** (1998) 163–169.

[7] Su, W., Xie, L. and de Souza, C. E. Global robust disturbance attenuation and almost disturbance decoupling for uncertain cascaded nonlinear systems. Automatica **35** (1999) 697–707.

[8] Xie, L. and Su, W. Robust H_∞ control for a class of cascaded nonlinear systems. *IEEE Transactions on Automatic Control*, **42** (10) (1997) 1465–1469.

[9] van der Schaft, A. J. L_2-gain analysis of nonlinear systems and nonlinear state feedback H_∞ control. *IEEE Transactions on Automatic Control* **37** (16) (1992) 770–784.

[10] Isidori, A. and Lin, W. Global L_2-gain design for a class of nonlinear systems. *Systems & Control Letters* **34** (1998) 295–302.

[11] Liu, X., Jutan, A. and Rohani, S. Almost disturbance decoupling of MIMO nonlinear systems and application to chemical process. *Automatica* **40** (2004) 465–471.

[12] Isidori, A. and Astolfi, A. Disturbance attenuation and H_∞-control via measurement feedback in nonlinear systems. *IEEE Transactions on Automatica Control* **37** (9) (1992) 1283–1293.

[13] Battilotti, S. Global output regulation and disturbance attenuation with global stability via measurement feedback for a class of nonlinear systems. *IEEE Transactions on Automatic Control* **41** (3) (1996) 315–327.

[14] Marino, R., Tomei, P. Nonlinear output feedback tracking with almost disturbance decoupling. *IEEE Transactions on Automatic Control* **44** (1) (1999) 18–28.

[15] Lin, W., Qian, C. and Huang, X. Disturbance attenuation of a class of non-linear system via output feedback. *International Journal of Robust and Nonlinear Control* **13** (2003) 1359–1369.

[16] Qian, C. and Lin, W. Almost disturbance decoupling for a class of high-order nonlinear systems. *IEEE Transactions on Automatic Control* **45** (6) (2000) 1208–1214.

[17] Lin, W. and Qian, C. Adding one power integrator: a tool for global stabilization of high-order lower-triangular systems. *Systems & Control Letters* **39** (2002) 339–351.

[18] Chang, Y. C. and Yen, H. M. Adaptive output feedback tracking control for a class of uncertain nonlinear systems using neural networks. *IEEE Transactions on Systems, Man, Cybernetics–Part B: Cybernetics***35** (6) (2005) 1311–1316.

[19] Ding, Z. Almost disturbance decoupling of uncertain nonlinear output feedback systems. *IEE Pro.-Control Theory Appl.* **146** (2) (1999) 220–226.

[20] Marino, R. and Tomei, P. Adaptive output feedback regulation with almost disturbance decoupling for nonlinearly parameterized systems. *International Journal of Robust and Nonlinear Control* **10** (2000) 655–669.

[21] Marino, R. and Tomei, P. Adaptive output feedback tracking with almost disturbance decoupling for a class of nonlinear systems. *Automatica* **36** (2000) 1871–1877.

[22] Bi, W. Robust adaptive disturbance attenuation for high-order uncertain nonlinear systems (In Chinese). *J. Sys. Sci. & Math. Scis.* **26** (2) (2006) 193–205.

[23] Lin, W. and Qian, C. Adaptive regulation of high-order lower-triangular systems: an adding a power integrator technique. *Systems & Control Letters* **39** (2002) 353–364.

[24] Lin, W. and Qian, C. Adaptive control of nonlinearly parameterized systems: the smooth feedback case. *IEEE Transactions on Automatic Control* **47** (8) (2002) 1249–1266.

[25] Aleksandrov, A. Yu. and Platonov, A. V. Conditions of ultimate boundedness of solutions for a class of nonlinear systems. *Nonlinear Dynamics and Systems Theory* **8** (2) (2008) 109–122.

[26] Mtar, R., Belhaouane, M. M., Ayadi, H. B. and Braiek, N. B. An LMI criterion for the global stability analysis of nonlinear polynomial systems. *Nonlinear Dynamics and Systems Theory* **9** (2) (2009) 171–183.

[27] DaCunha, J. J. Dynamic inequalities, bounds, and stability of systems with linear and nonlinear perturbations. *Nonlinear Dynamics and Systems Theory* **9** (3) (2009) 239–248.

[28] D'Anna, A. and Fiore, G. Stability properties for some non-autonomous dissipative phenomena proved by families of Liapunov Functionals. *Nonlinear Dynamics and Systems Theory* **9** (3) (2009) 249–262.