Nonlinear Dynamics and Systems Theory, **11** (3) (2011) 319–336

Periodic and Subharmonic Solutions for a Class of Noncoercive Superquadratic Hamiltonian Systems

M. Timoumi *

Dpt of Mathematics. Faculty of Sciences. 5000 Monastir. Tunisia

Received: September 15, 2010; Revised: July 18, 2011

Abstract: Some existence theorems are obtained for periodic and subharmonic solutions to noncoercive first order Hamiltonian systems and to similar second order Hamiltonian systems, when the Hamiltonian satisfies a superquadratic condition and need not satisfy the global Ambrosetti–Rabinowitz condition. For the resolution, we use minimax methods in critical point theory, especially a Local Linking Theorem and a Generalized Mountain Pass Theorem.

Keywords: *Hamiltonian systems; periodic solutions; subharmonics; critical points.*

Mathematics Subject Classification (2000): 34C25, 34A34, 37J45, 35Q40.

1 Introduction

Consider the nonautonomous first order Hamiltonian systems

$$J\dot{x} - u^*A(t)u(x) + u^*G'(t,u(x)) = 0, \tag{1.1}$$

where $u : \mathbb{R}^{2N} \longrightarrow \mathbb{R}^m$ $(1 \leq m \leq 2N)$ is a linear operator, A is a continuous $T-$periodic function $(T > 0)$ from $\mathbb{R}$ into the space of symmetric $(m\times m)$-matrices, $G : \mathbb{R}\times\mathbb{R}^m \longrightarrow \mathbb{R}$ is a continuous function, $T-$ periodic in the first variable, differentiable with respect to the second variable and its derivative $G'(t,x) = \frac{\partial G}{\partial x}(t,x)$ is continuous, and J is the standard symplectic matrix:

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

When $A(t) = 0$ for all $t \in \mathbb{R}$, $m = 2N$ and $u = id_{\mathbb{R}^{2N}}$, Rabinowitz has proved in [7] the existence of periodic solutions for (1.1) under some suitable conditions, in particular the following superquadratic condition:

* Corresponding author: mailto:m_timoumi@yahoo.com

© 2011 InforMath Publishing Group/1562-8353 (print)/1813-7385 (online)/http://e-ndst.kiev.ua

There exist two constants $\mu > 2$ and $r > 0$ such that for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^{2N}$, $|x| \geq r$

$$0 < \mu G(t,x) \leq G'(t,x)\,x, \tag{1.2}$$

where $x.y$ denotes the standard inner product of x, y in $\mathbb{R}^{2N}$ and $|.|$ denotes the corresponding norm. Since then, condition (1.2) has been used extensively in the literature, see [2–7,9,10]. If $m = 2N$, $u = id_{\mathbb{R}^{2N}}$ and G satisfies the superquadratic condition (1.2), the existence of nontrivial periodic solutions for the Hamiltonian systems (1.1), was studied by Li-Szulkin in [3] when A is a constant symmetric $(2N \times 2N)$ matrix, and by Li-Willem when $A(t)$ is a continuous periodic map from $\mathbb{R}$ into the space of symmetric $(2N \times 2N)$ matrices, not necessary constant. In [10], the author has studied the same problem as in [4] in the general case when u is not necessary the identity.

By remarking that the condition (1.2) does not cover some superquadratic nonlinearity like

$$G(t,y) = |y|^2\,[ln(1+|x|^p)]^q,\ p,q > 1, \tag{1.3}$$

the author has studied, recently in [11], the existence of nontrivial periodic solution for (1.1) when the function G satisfies some superquadratic conditions which cover the cases as in (1.3). In particular, the author has assumed that the function G satisfies the two following assumptions:

there exist constants $1 < \alpha < 2$ and $a > 0$ such that

$$|G'(t,y)| \leq a(|y|^\alpha + 1),\ \forall (t,y) \in \mathbb{R} \times \mathbb{R}^m; \tag{1.4}$$

there exist constants $\beta > \frac{1}{2-\alpha}$, $b > 0$ and $r > 0$ such that

$$G'(t,y).y - 2G(t,y) \geq b\,|y|^\beta,\ \forall t \in \mathbb{R},\ \forall\,|y| \geq r. \tag{1.5}$$

Consider the function G defined in $\mathbb{R} \times \mathbb{R}^m$ by

$$G(t,y) = \left|cos(\frac{2\pi}{T}t)\right|\,|y|^{\alpha+1} + |y|^2\,ln(1+|y|^2), \tag{1.6}$$

where $\frac{3}{2} < \alpha < 2$. A simple computation shows that G neither satisfies the condition (1.2), nor (1.5). In section 3, we will extend the ranges of α and β and obtain the existence of nontrivial $T-$ periodic solutions of (1.1) under some superquadratic conditions covering the cases as in (1.6). For the resolution, we shall use a Local Linking Theorem.

The existence of subharmonic solutions for (1.1), i.e. of distinct $kT-$periodic solutions of (1.1), has been investigated in [2,6,9] when $A(t) = 0$ for all $t \in \mathbb{R}$, $m = 2N$, $u = id_{\mathbb{R}^{2N}}$ and G satisfies the condition (1.2). In section 4, we are interested in the existence of infinitely many subharmonic solutions of the Hamiltonian systems (1.1) when $A(t) = 0$ for all $t \in \mathbb{R}$, u is not necessary the identity and the function G satisfies some superquadratic conditions covering the cases as in (1.6). The main obstacle in obtaining such solutions is the fact that any $T-$periodic solution is also $kT-$periodic. For the resolution, we shall use the minimax methods in critical point theory, specially, a Generalized Mountain Pass Theorem.

2 Preliminaries

We will recall here some basic results needed in the proof of our next results.

2.1 Linking theorem [4]

Let X be a real Banach space with a direct sum decomposition

$$X = X^1 \oplus X^2.$$

Consider two sequences of subspaces

$$X_0^1 \subset X_1^1 \subset \cdots \subset X^1,\ X_0^2 \subset X_1^2 \subset \cdots \subset X^2$$

such that

$$X^j = \overline{\cup_{n\in\mathbb{N}} X_n^j},\ j = 1, 2.$$

For every multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, we denote by X_α the space

$$X^1_{\alpha_1} \oplus X^2_{\alpha_2}.$$

Let us recall that

$$\alpha \leq \beta \Leftrightarrow \alpha_1 \leq \beta_1, \alpha_2 \leq \beta_2.$$

A sequence $(\alpha_n) \subset \mathbb{N}^2$ is admissible if, for every $\alpha \in \mathbb{N}^2$, there exists $m \in \mathbb{N}$ such that

$$n \geq m \Rightarrow \alpha_n \geq \alpha.$$

For every function $f : X \longrightarrow \mathbb{R}$, we denote by f_α the function f restricted to the space X_α.

Definition 2.1 Let $f \in C^1(X, \mathbb{R})$. The function f satisfies the $(PS)^*$ condition if every sequence (x_{α_n}) such that (α_n) is admissible and

$$x_{\alpha_n} \in X_{\alpha_n},\ \sup_{n\in\mathbb{N}} f(x_{\alpha_n}) < \infty,\ f'_{\alpha_n}(x_{\alpha_n}) \longrightarrow 0,$$

possesses a subsequence which converges to a critical point of f.

Definition 2.2 The function $f \in C^1(X, \mathbb{R})$ has a local linking at 0, with respect to (X^1, X^2) if, for some $r > 0$,

$$f(x) \geq 0,\ x \in X^1,\ ||x|| \leq r,$$

$$f(x) \leq 0,\ x \in X^2,\ ||x|| \leq r.$$

Remark 2.1 If f has a local linking at 0, then 0 is a critical point of f.

Theorem 2.1 *Suppose that $f \in C^1(X, \mathbb{R})$ satisfies the following assumptions*
a) f has a local linking at 0 and $X^1 \neq \{0\}$,
b) f satisfies the $(PS)^*$ condition,
c) f maps bounded sets into bounded sets,
d) for every $m \in \mathbb{N}$, $f(x) \longrightarrow -\infty$ as $\|x\| \longrightarrow +\infty$, $x \in X^1_m \oplus X^2$.
Then f has at least two critical points.

2.2 Generalized Mountain Pass Theorem

Let X be a real Banach space. We shall say that $f \in C^1(X, \mathbb{R})$ satisfies the Cerami-condition (C) if every sequence (x_n) in X satisfying

$$(f(x_n)) \text{ is bounded and } \|f'(x_n)\| (1 + \|x_n\|) \to 0 \text{ as } n \to \infty$$

possesses a convergent subsequence.

As shown in [1], a deformation lemma can be proved with the weaker condition (C) replacing the used (PS) condition, and it turns out that the Generalized Mountain Pass Theorem holds true under condition (C). We then have:

Theorem 2.2 *Let X be a real Hilbert space with inner product $< .,. >$. Suppose $X = X^1 \oplus X^2$ and $f \in C^1(X, \mathbb{R})$ satisfies the Cerami-condition (C) and the following conditions:*
a) $f(x) = \frac{1}{2} < P^+x - P^-x, x > +b(x)$, where $P^+ : E \longrightarrow E^+$ and $P^- : E \longrightarrow E^-$ are the orthogonal projections and b' is compact,
b) there exist constants $m,\ \rho > 0$, such that

$$f(x) \geq m,\ \forall x \in \partial B_\rho \cap X^1,$$

c) there exist $e \in \partial B_1 \cap X^1$ and constants $r_1,\ r_2 > 0$ such that

$$f(x) \leq 0,\ \forall x \in \partial Q,$$

where

$$Q = \{se/0 \leq s \leq r_1\} \oplus \left\{x \in X^2/\, \|x\| \leq r_2\right\}.$$

Then f possesses a critical value $c \geq m$ which can be characterized as

$$c = \inf_{h \in \Gamma} \max_{x \in Q} f(h(x)),$$

where

$$\Gamma = \left\{h \in C(\overline{Q}, E)/h = id \text{ on } \partial Q\right\}.$$

3 Existence of Periodic Solutions

Let $u : \mathbb{R}^{2N} \to \mathbb{R}^m$ $(1 \leq m \leq 2N)$ be a nontrivial linear operator with adjoint u^*, A be a continuous $T-$periodic function $(T > 0)$ from $\mathbb{R}$ into the space of symmetric $(m \times m)$-matrices and $G : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, $(t, y) \to G(t, y)$ be a continuous function, $T-$ periodic in the first variable, differentiable with respect to the second variable and its derivative $G'(t, y) = \frac{\partial G}{\partial y}(t, y)$ is continuous. Consider the noncoercive Hamiltonian systems

$$(HS) \qquad J\dot{x} - u^*A(t)u(x) + u^*G'(t, u(x)) = 0.$$

We are interested in the existence of nontrivial $T-$periodic solutions for (HS).
Consider the following assumptions

$$(G_0) \qquad JKer\, u \subset Ker\, u.$$

$$(G_1) \qquad G(t,y) = o(|y|^2) \ as \ |y| \to 0, \ uniformly \ in \ t \in \mathbb{R}.$$

$$(G_2) \qquad \lim_{|y|\to\infty} \frac{G(t,y)}{|y|^2} = +\infty, \ uniformly \ in \ t \in \mathbb{R}.$$

(G_3) There exist constants $\alpha > 1$ and $a > 0$ such that

$$|G'(t,y)| \le a(|y|^\alpha + 1), \ \forall \ t \in \mathbb{R}, \ \forall y \in \mathbb{R}^m.$$

(G_4) There exist constants $\beta > \alpha$, $b > 0$ and $r > 0$ such that

$$G'(t,y).y - 2G(t,y) \ge b\,|y|^\beta\,, \ \forall \ t \in \mathbb{R}, \ \forall\,|y| \ge r.$$

(G_5) There exists a constant $\delta > 0$ such that either

$$(i) \qquad G(t,y) \ge 0, \ \forall t \in \mathbb{R}, \ \forall\,|y| \le \delta,$$

or

$$G(t,y) \le 0, \ \forall t \in \mathbb{R}, \ \forall\,|y| \le \delta.$$

Our first main result in this section is the following:

Theorem 3.1 *Assume conditions* $(G_0) - (G_4)$ *hold. If* 0 *is an eigenvalue of* $J\frac{d}{dt} - u^*Au$*, assume also* (G_5)*. Then the system* (HS) *possesses at least one nontrivial* $T-$*periodic solution.*

Example 3.1 Let $p, q > 1$ be two real numbers. The function

$$G(t,y) = |y|^2\,[ln(1+|x|^p)]^q$$

satisfies $(G_1)-(G_5)$. The linear map $u : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ defined by $u(p,q) = p$ satisfies (G_0). Let $A(t) = Id_N$. Therefore for all $T > 0$, the corresponding Hamiltonian system (HS) possesses at least a nontrivial $T-$periodic solution.

Remark 3.1 Observe that if x is a periodic solution of (HS) then $y(t) = x(-t)$ is a periodic solution of

$$J\dot{y}(t) + u^*A(-t)u(y) - u^*G'(-t, u(y)) = 0.$$

Hence, it is easy to see that we obtain the same result of Theorem 3.1 if we replace assumptions (G_2) and (G_4) respectively by the following ones

$$\lim_{|y|\to\infty} \frac{G(t,y)}{|y|^2} = -\infty, \ uniformly \ in \ t \in \mathbb{R}.$$

There exist constants $\beta > \alpha$, $b > 0$ and $r > 0$ such that

$$G'(t,y).y - 2G(t,y) \le -b\,|y|^\beta\,, \ \forall \ t \in \mathbb{R}, \ \forall\,|y| \ge r.$$

Now consider the noncoercive second order Hamiltonian systems

$$(NS) \qquad \ddot{x} - u^* A(t)u(x) + u^* W'(t, u(x)) = 0$$

where $u : \mathbb{R}^N \longrightarrow \mathbb{R}^m$, $(1 \leq m \leq N)$ is a linear operator with adjoint u^*, $A(t)$ is a symmetric $m \times m$ matrix, continuous and $T-$periodic, $W : \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is a continuous function $T-$periodic in the first variable and continuously differentiable with respect to the second variable. Consider the following assumptions:

$$(W_1) \qquad W(t, y) = o(|y|^2) \; as \; |y| \to 0, \; uniformly \; in \; t \in \mathbb{R}.$$

$$(W_2) \qquad \lim_{|y| \to \infty} \frac{W(t, y)}{|y|^2} = +\infty, \; uniformly \;\; in \; t \in \mathbb{R}.$$

(W_3) There exist constants $\alpha > 1$ and $a > 0$ such that

$$|W'(t, y)| \leq a(|y|^\alpha + 1), \; \forall \; t \in \mathbb{R}, \; \forall y \in \mathbb{R}^m.$$

(W_4) There exist constants $\beta > \alpha$, $b > 0$ and $r > 0$ such that

$$W'(t, y).y - 2W(t, y) \geq b\,|y|^\beta, \; \forall t \in \mathbb{R}, \; \forall\, |y| \geq r.$$

(W_5) There exists a constant $\delta > 0$ such that either

$$(i) \qquad W(t, y) \geq 0, \; \forall t \in \mathbb{R}, \; \forall\, |y| \leq \delta,$$

or

$$(ii) \qquad W(t, y) \leq 0, \; \forall t \in \mathbb{R}, \; \forall\, |y| \leq \delta.$$

Our second main result in this section is the following:

Theorem 3.2 *Assume conditions* $(W_1) - (W_4)$ *hold. If* 0 *is an eigenvalue of* $J\frac{d^2}{dt^2} - u^* A u$, *assume also* (W_5). *Then the system* (NS) *possesses at least one non-trivial* $T-$*periodic solution.*

3.1 Proof of Theorem 3.1

Let $S^1 = \mathbb{R}/T\mathbb{Z}$ and $E = H^{\frac{1}{2}}(S^1, \mathbb{R}^{2N})$ be the Sobolev space of $T-$periodic functions with inner product $< .,. >_{H^{1/2}}$ and norm $\|.\|_{H^{\frac{1}{2}}}$ defined by

$$< x, y >_{H^{\frac{1}{2}}} = \hat{x}_0.\hat{y}_0 + \pi \sum_{k \in \mathbb{Z}} |k|\, \hat{x}_k.\hat{y}_k$$

and

$$\|x\|_{H^{\frac{1}{2}}} = \left(|\hat{x}_0|^2 + \pi \sum_{k \in \mathbb{Z}} |k|\, |\hat{x}_k|^2 \right)^{\frac{1}{2}}$$

for $x, y \in H^{\frac{1}{2}}(S^1, \mathbb{R}^{2N})$, where

$$x(t) \cong \sum_{k \in \mathbb{Z}} exp(J\frac{2k\pi t}{T})\hat{x}_k, \; \hat{x}_k \in \mathbb{R}^{2N},$$

and

$$y(t) \cong \sum_{k\in\mathbb{Z}} exp(J\frac{2k\pi t}{T})\hat{y}_k, \ \hat{y}_k \in \mathbb{R}^{2N}.$$

Consider the closed subspace of $H^{1/2}(S^1, \mathbb{R}^{2N})$

$$X = \left\{ x \in H^{\frac{1}{2}}(S^1, \mathbb{R}^{2N})/x(t) \in (Ker\, u)^{\perp} \ a.e. \right\}.$$

It is well known that the space X is compactly embedded in $L^s(S^1, \mathbb{R}^{2N})$ for every $s \in [1, \infty[$ (see [5]) and as a consequence there exists a constant $\gamma_s > 0$ such that

$$\|x\|_{L^s} \leq \gamma_s \|x\|_{H^{\frac{1}{2}}}, \ \forall x \in X. \tag{3.1}$$

Define on X the bilinear form

$$B(x, y) = -\frac{1}{2}\int_0^T [J\dot{x}.y - A(t)u(x).u(y)]dt.$$

Let X^+ (resp.X^-) be the positive (resp. negative) space corresponding to the spectral decomposition of B in X and $X^0 = \ker B$. Then $X = X^+ \oplus X^- \oplus X^0$. In fact it is not difficult to check that X^+, X^- and X^0 are mutually orthogonal in $L^2(S^1, \mathbb{R}^{2N})$. Denote Q the quadratic form associated to B:

$$Q(x) = -\frac{1}{2}\int_0^T [J\dot{x}.x - A(t)u(x).u(x)]dt.$$

We prove (see [11]) that there exists a constant $\nu > 0$ such that

$$Q(x) \geq \nu \|x\|^2, \ \forall x \in X^+, \tag{3.2}$$

$$Q(x) \leq -\nu \|x\|^2, \ \forall x \in X^-. \tag{3.3}$$

Now, since X^0 is of finite dimension, there exists a constant $a_1 > 0$ such that

$$\|x\|_{H^{\frac{1}{2}}} \leq a_1 \|x\|_{L^2}, \forall x \in X^0. \tag{3.4}$$

We deduce from (3.2), (3.3) and (3.4) that the following expression

$$\|x\|^2 = \left\|x^+ + x^- + x^0\right\|^2 = Q(x^+) - Q(x^-) + \left|x^0\right|^2_{L^2} \tag{3.5}$$

where $x^i \in X^i, \ i = +, -, 0$, is an equivalent norm on X, which will be considered in the following. Therefore we deduce from (3.1) that for all $s \in [1, \infty[$, there exists a constant $\mu_s > 0$ such that

$$\|x\|_{L^s} \leq \mu_s \|x\|, \forall x \in X. \tag{3.6}$$

If zero is not an eigenvalue of $J\frac{d}{dt} - u^*Au$, we take

$$X^1 = X^+, \ X^2 = X^-.$$

If zero is an eigenvalue of $J\frac{d}{dt} - u^*Au$, we take

$$X^1 = X^+ \oplus X^0, \ X^2 = X^-, \ if \ G(t, y) \leq 0 \ for \ |y| \leq \delta,$$

$$X^1 = X^+,\ X^2 = X^- \oplus X^0,\ if\ G(t,y) \geq 0\ for\ |y| \leq \delta.$$

In the following, we will assume that zero is an eigenvalue of $J\frac{d}{dt} - u^*Au$ and

$$G(t,y) \leq 0,\ for\ |y| \leq \delta. \tag{3.7}$$

The other cases are similar.

Define a functional f in X by

$$f(x) = -\frac{1}{2}\int_0^T [J\dot{x}.x - A(t)u(x).u(x)]dt - \int_0^T G(t,u(x))dt.$$

It is easy to see that there exist two constants $m, M > 0$ such that

$$m|x| \leq |u(x)| \leq M|x|,\ \forall x \in (Ker\, u)^{\perp}. \tag{3.8}$$

Combine this with $(G_3)(ii)$, there are two constants $c, d > 0$ such that

$$|u^*G'(t,u(x))| \leq c|x|^{\beta} + d,\ \forall t \in \mathbb{R}, \forall x \in (Ker\, u)^{\perp}. \tag{3.9}$$

Therefore, we conclude that $f \in C^1(X,\mathbb{R})$ and maps bounded sets into bounded sets. Now, let us choose Hilbertian basis $(e_n)_{n\in\geq 1}$ for X^1 and $(e_n)_{n\leq -1}$ for X^2. Define

$$X_n^1 = space(e_1, ..., e_n),\ n \geq 1$$

$$X_n^2 = space(e_{-1}, ..., e_{-n}),\ n \geq 1$$

$$X^j = \overline{\bigcup_{n\geq 1} X_n^j},\ j = 1, 2.$$

We will proceed by successive lemmas.

Lemma 3.1 *The functional f satisfies the $(PS)^*$ condition.*

Proof Consider a sequence (x_{α_n}) such that (α_n) is admissible and

$$x_{\alpha_n} \in X_{\alpha_n},\ c = \sup_{n\in\mathbb{N}} f(x_{\alpha_n}) < \infty,\ f'_{\alpha_n}(x_{\alpha_n}) \to 0\ as\ n \to \infty. \tag{3.10}$$

We claim that (x_{α_n}) is bounded. Suppose by contradiction that (x_{α_n}) is not bounded, then going, if necessary, to a subsequence, we can assume that $\|x_{\alpha_n}\| \to \infty\ as\ n \to \infty$. By (G_4) and (3.8) there exists a constant $c_1 > 0$ such that for all $t \in \mathbb{R}$ and for all $x \in (Ker\, u)^{\perp}$

$$G'(t,u(x)).u(x) - 2G(t,u(x)) \geq b\,|x|^{\beta} - c_1. \tag{3.11}$$

Therefore, by noting $x_{\alpha_n} = x_n$ and $f_{\alpha_n} = f_n$, we have

$$-f'_n(x_n).x_n + 2f(x_n) = \int_0^T [G'(t,u(x_n)).u(x_n) - 2G(t,u(x_n))]dt$$

$$\geq b\int_0^T |x_n|^{\beta} - c_1T.$$

Combining this with (3.6), we obtain

$$\frac{\int_0^T |x_n|^\beta \, dt}{\|x_n\|} \longrightarrow 0, \ as \ n \longrightarrow \infty. \tag{3.12}$$

Let $x_n = x_n^+ + x_n^- + x_n^0 \in X^+ \oplus X^- \oplus X^0$. By (G_3), Hölder's inequality, (3.6) and (3.8), we have

$$\begin{aligned} f_n'(x_n).x_n^+ &= \left\|x_n^+\right\|^2 - \int_0^T G'(t,u(x_n)).u(x_n^+)dt \\ &\geq \left\|x_n^+\right\|^2 - \int_0^T |G'(t,u(x_n))| \left|u(x_n^+)\right| dt \\ &\geq \left\|x_n^+\right\|^2 - a\int_0^T (|u(x_n)|^\alpha + 1) \left|u(x_n^+)\right| dt \\ &\geq \left\|x_n^+\right\|^2 - a\int_0^T (M^\alpha |x_n|^\alpha + 1)M \left|x_n^+\right| dt \\ &\geq \left\|x_n^+\right\|^2 - aM^{\alpha+1}[\int_0^T (|x_n|^\alpha)^{\frac{\beta}{\alpha}} dt]^{\frac{\alpha}{\beta}} [\int_0^T \left|x_n^+\right|^{\frac{\beta}{\beta-\alpha}} dt]^{\frac{\beta-\alpha}{\beta}} - aM \left\|x_n^+\right\|_{L^1} \\ &\geq \left\|x_n^+\right\|^2 - aM^{\alpha+1}\mu_{\beta\beta-\alpha} \left\|x_n^+\right\|_{L^\beta}^\alpha \left\|x_n^+\right\| - aM\mu_1 \left\|x_n^+\right\| \end{aligned}$$

for all integer $n \in \mathbb{N}$, which implies that

$$\left\|x_n^+\right\| \leq \|f_n'(x_n)\| + c_2 \|x_n\|_{L^\beta}^\alpha + c_3, \ \forall n \in \mathbb{N}, \tag{3.13}$$

where c_2, c_3 are two constants. Since $1 < \alpha < \beta$, we deduce from (3.12) and (3.13) that

$$\frac{\|x_n^+\|}{\|x_n\|} \longrightarrow 0, \ as \ n \longrightarrow \infty. \tag{3.14}$$

Similarly

$$\frac{\|x_n^-\|}{\|x_n\|} \longrightarrow 0, \ as \ n \longrightarrow \infty. \tag{3.15}$$

By (G_4) and (3.8), there exist two constants $c_4, \ c_5 > 0$ such that

$$G'(t,u(y)).u(y) - 2G(t,u(y)) \geq c_4 |y| - c_5, \ \forall (t,y) \in \mathbb{R} \times (Ker\, u)^\perp, \tag{3.16}$$

which implies

$$\begin{aligned} 2f(x_n) - f_n'(x_n).x_n &= \int_0^T [G'(t,u(x_n)).u(x_n) - 2G(t,u(x_n))]dt \\ &\geq \int_0^T [c_4 |x_n| - c_5]dt \\ &\geq \int_0^T [c_4 \left|x_n^0\right| - c_4 \left|x_n^+\right| - c_4 \left|x_n^-\right| - c_5]dt. \end{aligned} \tag{3.17}$$

Moreover, it follows from the equivalence of the norms on the finite dimensional subspace X^0 that there exists a positive constant d such that

$$\|x\| \leq d \|x\|_{L^1}, \ \forall x \in X^0. \tag{3.18}$$

Combining (3.6), (3.17) and (3.18) we obtain

$$2f(x_n) - f_n'(x_n).x_n \geq c_4\frac{1}{d}\left\|x_n^0\right\| - c_4\mu_1\left\|x_n^+\right\| - c_4\mu_1\left\|x_n^-\right\| - c_5T. \tag{3.19}$$

Therefore, by (3.14), (3.15) and (3.19), we have

$$\frac{\left\|x_n^0\right\|}{\|x_n\|} \longrightarrow 0 \; as \; n \longrightarrow \infty. \tag{3.20}$$

We deduce from (3.14), (3.15) and (3.20) that

$$1 = \frac{\|x_n\|}{\|x_n\|} \leq \frac{\left\|x_n^0\right\| + \|x_n^-\| + \|x_n^+\|}{\|x_n\|} \longrightarrow 0 \; as \; n \longrightarrow \infty, \tag{3.21}$$

which is a contradiction. So (x_n) must be bounded. Since the space X is closed in the reflexive space $H^{\frac{1}{2}}(S^1, \mathbb{R}^{2N})$, then X is also reflexive and the sequence (x_n) possesses a subsequence (x_{n_k}) weakly convergent to a point x. Note that

$$Q(x_{n_k}^+ - x^+) = (f_{n_k}'(x_{n_k}) - f'(x)).(x_{n_k}^+ - x^+) \tag{3.22}$$

$$+ \int_0^T [G'(t, u(x_{n_k})) - G'(t, u(x))].[u(x_{n_k}^+) - u(x^+)]dt$$

which implies that $x_{n_k}^+ \longrightarrow x^+$ in X. Similarly, $x_{n_k}^- \longrightarrow x$ in X. It follows that $x_{n_k} \longrightarrow x$ in X and $f'(x) = 0$. So f satisfies the $(PS)^*$ condition. The proof of Lemma 3.1 is complete.

Lemma 3.2 *The functional f satisfies the local linking condition at zero.*

Proof By assumption (G_3) and (3.8), there exists a constant $b_1 > 0$ such that

$$|G(t, u(x))| \leq b_1(|x|^{\alpha+1} + |x|), \; \forall t \in \mathbb{R}, \; \forall x \in (Ker\, u)^{\perp}. \tag{3.23}$$

Assumption (G_1) and (3.8) imply that for any $\epsilon > 0$, there exists a constant $R > 0$ such that

$$|G(t, u(x))| \leq \epsilon\,|x|^2, \; \forall t \in \mathbb{R}, \; \forall\,|x| \leq R. \tag{3.24}$$

Combining (3.23) with (3.24), we obtain

$$|G(t, u(x))| \leq (\epsilon\,|x|^2 + M_1\,|x|^{\alpha+1}), \; \forall t \in \mathbb{R}, \; \forall x \in (Keru)^{\perp} \tag{3.25}$$

where $M_1 = b_1(1 + R^{\alpha})$. Hence we obtain by (3.6)

$$\left|\int_0^T G(t, u(x))dt\right| \leq \epsilon\mu_2^2\,\|x\|^2 + M_1\mu_{\alpha+1}^{\alpha+1}\,\|x\|^{\alpha+1}. \tag{3.26}$$

So for all $x \in X^2 = X^-$

$$f(x) \leq -\,\|x\|^2 + \epsilon\mu_2^2\,\|x\|^2 + M_1\mu_{\alpha+1}^{\alpha+1}\,\|x\|^{\alpha+1}. \tag{3.27}$$

Since $\alpha > 1$ and ϵ is arbitrary, we deduce that there exists a constant $r > 0$ small enough such that

$$f(x) \leq 0, \; \forall x \in X^2, \; \|x\| \leq r. \tag{3.28}$$

Now, let $\eta > 0$ be such that

$$\forall\, x \in (Ker\, u)^{\perp},\ |x| \leq \eta \Rightarrow |u(x)| \leq \delta \tag{3.29},$$

where δ is introduced in (G_5). Since X^0 is a finite dimensional space, there exists a constant $\rho > 0$ such that

$$\|x\|_{\infty} \leq \rho \|x\|,\ \forall x \in X^0. \tag{3.30}$$

Let $x = x^0 + x^+ \in X^1 = X^0 \oplus X^+$ such that $\|x\| \leq \frac{\eta}{2\rho}$ and set

$$I = \left\{ t \in [0,T] / \left| x^+(t) \right| \leq \frac{\eta}{2} \right\}.$$

On I, we have by (3.30)

$$|x(t)| \leq \left|x^0(t)\right| + \left|x^+(t)\right| \leq \left\|x^0\right\|_{\infty} + \frac{\eta}{2} \leq \eta,$$

hence, by (3.7) and (3.29)

$$\int_0^T G(t, u(x))dt \leq 0. \tag{3.31}$$

On $[0,T] \mid I$, we have also by (3.30)

$$|x(t)| \leq \left|x^0(t)\right| + \left|x^+(t)\right| \leq \rho \left\|x^0\right\| + \left|x^+(t)\right| \leq \frac{\eta}{2} + \left|x^+(t)\right| \leq 2\left|x^+(t)\right|.$$

Hence, by (3.6) and (3.25), we obtain

$$\left| \int_{[0,T]\mid I} G(t, u(x))dt \right| \leq 4\epsilon\mu_2^2 \left\|x^+\right\|^2 + 2^{\alpha+1} M_1 \mu_{\alpha+1}^{\alpha+1} \left\|x^+\right\|^{\alpha+1}.$$

Therefore, we have

$$f(x) \geq \left\|x^+\right\|^2 - 4\epsilon\mu_2^2 \left\|x^+\right\|^2 - 2^{\alpha+1} M_1 \mu_{\alpha+1}^{\alpha+1} \left\|x^+\right\|^{\alpha+1} - \int_I G(t, u(x))dt. \tag{3.32}$$

Since $\alpha > 1$, we deduce from (3.31) and (3.32), by taking ϵ small enough, that there exists a constant $0 < r < \frac{\eta}{2\rho}$ such that

$$f(x) \geq 0,\ \forall\, x \in X^1,\ \|x\| \leq r. \tag{3.33}$$

Properties (3.28) and (3.33) show that f satisfies the local linking condition at zero which completes the proof of Lemma 3.2.

Lemma 3.3 *For each $m \in \mathbb{N}$, $f(x) \longrightarrow -\infty$ as $\|x\| \longrightarrow \infty$, $x \in X_m^1 \oplus X^2$.*

Proof For $x = x^+ + x^0 + x^- \in X_m^1 \oplus X^2$, we have

$$f(x) = \left\|x^+\right\|^2 - \left\|x^-\right\|^2 - \int_0^T G(t, u(x^+ + x^0 + x^-))dt. \tag{3.34}$$

Since X_m^1 is of finite dimension, there exists a positive constant γ_1 such that

$$\left\|x^+ + x^0\right\| \leq \gamma_1 \left\|x^+ + x^0\right\|_{L^2},\ \forall x = x^+ + x^0 \in X_m^1. \tag{3.35}$$

On the other hand, by assumption $(G_2)(i)$ and (3.8), there exists a constant $c_6 > 0$ such that

$$-G(t,u(x)) \leq -2\gamma_1 |x|^2 + c_6,\ \forall\ x \in (Ker\, u)^{\perp}. \tag{3.36}$$

Combining (3.34), (3.35) and (3.36), we obtain for $x = x^+ + x^0 + x^- \in X_m^1 \oplus X^2$

$$\begin{aligned} f(x) &\leq \left\|x^+\right\|^2 - \left\|x^-\right\|^2 - 2\gamma_1[\left\|x^+\right\|_{L^2}^2 + \left\|x^-\right\|_{L^2}^2 + \left\|x^0\right\|_{L^2}^2] + c_6 T \\ &\leq -\left\|x^-\right\|^2 - \left\|x^+\right\|^2 - 2\left\|x^0\right\|^2 + c_6 T \end{aligned}$$

which concludes the proof of Lemma 3.3.

We deduce from the previous lemmas that the functional f satisfies all the assumptions of the Local Linking Theorem and hence the functional f possesses at least two distinct critical points on X. Therefore the Hamiltonian system (HS) has at least one non trivial $T-$periodic solution.

3.2 Proof of Theorem 3.2

We consider only the case when 0 is an eigenvalue of $-\frac{d^2}{dt^2} + u^*Au$ and

$$W(t,y) \leq 0,\ \forall t \in \mathbb{R},\ \forall |y| \leq \delta. \tag{3.37}$$

The other cases are similar and simpler.

We shall apply the Local Linking Theorem to the functional

$$f(x) = \frac{1}{2}\int_0^T [|\dot{x}|^2 + A(t)u(x).u(x)]dt - \int_0^T W(t,u(x))dt$$

defined on the following closed subspace X of $H^1(S^1,\mathbb{R}^N)$

$$X = \{x \in H^1(S^1,\mathbb{R}^N)/x(t) \in (Ker\, u)^{\perp} a.e.\}$$

where $H^1(S^1,\mathbb{R}^N)$ is the space of $T-$ periodic absolutely continuous vector functions from S^1 into $\mathbb{R}^N$ whose first derivatives have square integrable norm. The inner product on $H^1(S^1,\mathbb{R}^N)$ is given by

$$< u,v >_{H^1} = \int_0^T [u(t).v(t) + \dot{u}(t).\dot{v}(t)]dt.$$

The functional f is continuously differentiable on X and maps bounded sets into bounded sets. Moreover the critical points of f correspond to the $T-$ periodic solutions of the system (NS) (see [9]).

Let X^+ (resp. X^-) be the positive (resp. negative) space corresponding to the spectral decomposition of $-\frac{d^2}{dt^2}+u^*Au$ in X and $X^0 = Ker(-\frac{d^2}{dt^2}+u^*Au)$. Let $X^2 = X^-$ and $X^1 = X^0 \oplus X^+$ and choose a Hilbertian basis $(e_n)_{n\geq 0}$ for X^1. Define

$$X_n^1 = span(e_0, e_1, ..., e_n), n \in \mathbb{N},$$

$$X_n^2 = X^2, n \in \mathbb{N}.$$

It is well known that X^0, X^2 are of finite dimensional.

As in the proof of Theorem 3.1, we prove by using assumptions (W_3), (W_4) that f

satisfies the $(PS)^*$ condition and by using assumptions (W_1), (W_3) that f satisfies the local linking at zero. Assumption (W_2) implies that f satisfies assertion $d)$ of the Local Linking Theorem. Consequently the functional f satisfies all the Local Linking Theorem assumptions and then it has at least two critical points. Therefore the system (NS) possesses a nontrivial $T-$ periodic solution.

4 Subharmonic Solutions

Let u, u^* and G be defined as in Section 3, we are interested in the existence of infinitely many subharmonic solutions of the Hamiltonian systems

$$(HS) \qquad J\dot{x} + u^* G'(t, u(x)) = 0,$$

i.e. of distinct $kT-$ periodic solutions of (HS).
Let $\alpha > 1$ be as in (G_3) and consider the following assumptions:
(G'_4) There exist constants $\beta > \alpha - 1$, $b > 0$ and $r > 0$ such that

$$G'(t, y).y - 2G(t, y) \geq b\,|y|^{\beta}\,,\ \forall t \in \mathbb{R},\ \forall\ |y| \geq r.$$

$$(G'_5) \qquad G(t, y) \geq 0,\ \forall (t, y) \in \mathbb{R} \times \mathbb{R}^m.$$

Our main result in this section is

Theorem 4.1 *Assume $(G_0) - (G_3)$, (G'_4) and (G'_5) hold. Then the Hamiltonian system (HS) possesses infinitely many subharmonic solutions.*

Example 4.1 Let $\frac{3}{2} \leq \alpha < 2$ be a real number. The function

$$G(t, y) = \left| cos(\frac{2\pi}{T} t) \right| |y|^{\alpha+1} + |y|^2\, ln(1 + |y|^2)$$

satisfies $(G_1) - (G_3)$, (G'_4) and (G'_5). The linear map $u : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ defined by $u(p, q) = p$ satisfies (G_0). Therefore the corresponding Hamiltonian system (HS) possesses infinitely many subharmonic solutions.

Remark 4.1 We obtain the same result if we replace assumptions (G_2), (G'_4) and (G'_5) respectively by

$$\lim_{|y| \to \infty} \frac{G(t, y)}{|y|^2} = -\infty,\ uniformly\ in\ t \in \mathbb{R}.$$

There exist constants $\beta > \alpha - 1$, $b > 0$ and $r > 0$ such that

$$G'(t, y).y - 2G(t, y) \leq -b\,|y|^{\beta}\,,\ \forall t \in \mathbb{R},\ \forall\ |y| \geq r,$$

$$G(t, y) \leq 0,\ \forall (t, y) \in \mathbb{R} \times \mathbb{R}^m.$$

Proof of Theorem 4.1. Choose $k \in \mathbb{N}$. By making the change of variables $s = k^{-1}t$, (HS) transforms to

$$(HS_k) \qquad J\dot{y} + ku^* G'(ks, y(s)) = 0.$$

Hence, finding $kT-$periodic solutions of (HS) is equivalent to finding $T-$periodic solutions of (HS_k). Let X be the space introduced in section 3 and consider the functional f_k defined over X by

$$f_k(y) = -\frac{1}{2}\int_0^T J\dot{y}.yds - k\int_0^T G(ks, u(y(s)))ds.$$

The assumptions of Theorem 4.1 imply that f_k is continuously differentiable in X and critical points of f_k are $T-$periodic solutions of (HS_k). Let X^+, X^- and X^0 be respectively the positive, negative and null subspaces of X corresponding to the spectral decomposition of the quadratic form

$$Q(y) = -\frac{1}{2}\int_0^T J\dot{y}.yds.$$

Then $X = X^- \oplus X^0 \oplus X^+$ and as in Section 3, we consider the equivalent norm on X given by

$$\|y\|^2 = Q(y^+) - Q(y^-) + \left|y^0\right|^2,$$

where $y = y^- + y^0 + y^+ \in X = X^- \oplus X^0 \oplus X^+$. Then we have

$$f_k(y) = \left\|y^+\right\|^2 - \left\|y^-\right\|^2 - k\int_0^T G(ks, u(y))ds. \tag{4.1}$$

We will apply the Generalized Mountain Pass Theorem to the functional f_k over X with $X^1 = X^+$ and $X^2 = X^0 \oplus X^-$. We will proceed by successive lemmas.

Lemma 4.1 *The functional f_k satisfies the Cerami's condition (C).*

Proof Let (y_n) be a sequence such that $(f_k(y_n))$ is bounded from above and $\|f_k'(y_n)\|(1 + \|y_n\|) \longrightarrow 0$ as $n \longrightarrow \infty$. We claim that (y_n) is a bounded sequence in X. For otherwise, going if necessary to a subsequence, we can assume that $\|y_n\| \longrightarrow \infty$ *as* $n \longrightarrow \infty$.

By (G_4') and (3.8), there is a constant $c > 0$ such that

$$G'(ks, u(y)).u(y) - 2G(ks, u(y)) \geq b\,|y|^\beta - c,\ \forall (t, y) \in \mathbb{R} \times \mathbb{R}^m \tag{4.2}$$

which implies with that

$$2f_k(y_n) - f_k'(y_n).y_n = k\int_0^T [G'(ks, u(y_n)).u(y_n) - 2G(ks, u(y_n))]ds$$

$$\geq k[b\int_0^T |y_n|^\beta\, ds - cT].$$

Hence for a given $k \in \mathbb{N}$, we get

$$\int_0^T |y_n|^\beta\, ds \leq c_1 \tag{4.3}$$

for all integer n and some positive constant $c_1 > 0$.

Now, let $y_n = y_n^- + y_n^0 + y_n^+ \in X^- \oplus X^0 \oplus X^+$ and set

$$p = \frac{2\beta+1}{2\alpha-1} > 1 \ and \ q = \frac{p}{p-1} = \frac{2\beta+1}{2\beta+1-\alpha}. \tag{4.4}$$

It follows from Hӧder's inequality, (3.1) and (4.3) that

$$\begin{aligned}
\int_0^T |y_n|^\alpha \left|y_n^+\right| ds &= \int_0^T |y_n|^{\frac{\beta}{p}} |y_n|^{\alpha-\frac{\beta}{p}} \left|y_n^+\right| ds \\
&\leq [\int_0^T (|y_n|^{\frac{\beta}{p}})^p ds]^{\frac{1}{p}} [\int_0^T (|y_n|^{\alpha-\frac{\beta}{p}} \left|y_n^+\right|)^q ds]^{\frac{1}{q}} \\
&\leq [\int_0^T (|y_n|^{\beta}) ds]^{\frac{1}{p}} [\int_0^T (|y_n|^{\alpha-\frac{\beta}{p}})^{2q} ds]^{\frac{1}{2q}} [\int_0^T \left|y_n^+\right|^{2q} ds]^{\frac{1}{2q}} \\
&\leq [\int_0^T |y_n|^{\beta} ds]^{\frac{1}{p}} \|y_n\|_{L^{\frac{\beta+\alpha}{\beta+1-\alpha}}}^{\frac{\beta+\alpha}{2\beta+1}} \left\|y_n^+\right\|_{L^{2q}} \\
&\leq c_1^{\frac{1}{p}} \gamma_{\frac{\beta+\alpha}{\beta+1-\alpha}}^{\frac{\beta+\alpha}{2\beta+1}} \gamma_{2q} \|y_n\|^{\frac{\beta+\alpha}{2\beta+1}} \left\|y_n^+\right\|
\end{aligned} \tag{4.5}$$

for all integer n. By (G_3), (3.1), (3.8), (4.3) and (4.5), we have

$$\begin{aligned}
f_k'(y_n).y_n^+ &= \left\|y_n^+\right\|^2 - k\int_0^T G'(ks, u(y_n)).u(y_n^+) ds \\
&\geq \left\|y_n^+\right\|^2 - k\int_0^T |G'(ks, u(y_n))| \left|u(y_n^+)\right| ds \\
&\geq \left\|y_n^+\right\|^2 - ka\int_0^T (|u(y_n))|^\alpha + 1) \left|u(y_n^+)\right| ds \\
&\geq \left\|y_n^+\right\|^2 - kaM^{\alpha+1}\int_0^T (|y_n|^\alpha \left|y_n^+\right| ds - kaM\int_0^T \left|y_n^+\right| ds \\
&\geq \left\|y_n^+\right\|^2 - kaM^{\alpha+1} c_1^{\frac{1}{p}} \gamma_{\frac{\beta+\alpha}{\beta+1-\alpha}}^{\frac{\beta+\alpha}{2\beta+1}} \gamma_{2q} (\|y_n\|^{\frac{\beta+\alpha}{2\beta+1}} \left\|y_n^+\right\| - kaM\gamma_1 \left\|y_n^+\right\|
\end{aligned}$$

for all integer n. Noting that $\frac{\beta+\alpha}{2\beta+1} < 1$, one sees

$$\frac{\|y_n^+\|}{\|y_n\|} \longrightarrow 0 \quad as \quad n \longrightarrow \infty. \tag{4.6}$$

Similarly for y_n^-, we have

$$\frac{\|y_n^-\|}{\|y_n\|} \longrightarrow 0 \quad as \quad n \longrightarrow \infty. \tag{4.7}$$

On the other hand, since X^0 is of finite dimension, there exists a constant $\gamma > 0$ such that

$$\|y\| \leq \gamma^2 \|y\|_{L^2}, \ \forall y \in X^0. \tag{4.8}$$

Therefore by Hӧlder's inequality, (3.1) and (4.8) we have

$$\frac{1}{\gamma^2} \left\|y_m^0\right\|^2 \leq \int_0^T \left|y_m^0\right|^2 ds \leq \int_0^T |y_n|^{\frac{\beta}{\beta+1}} |y_n|^{\frac{\beta+2}{\beta+1}} ds$$

$$\leq [\int_0^T |y_n|^\beta \, ds]^{\frac{1}{\beta+1}} [\int_0^T |y_n|^{\frac{\beta+2}{\beta}} \, ds]^{\frac{\beta}{\beta+1}}$$

$$\leq (c_1)^{\frac{1}{\beta+1}} (\gamma_{\frac{\beta+2}{\beta}})^{\frac{\beta+2}{\beta+1}} \|y_n\|^{\frac{\beta+2}{\beta+1}} . \tag{4.9}$$

Since $\frac{\beta+2}{\beta+1} < 2$, we deduce from (4.9)

$$\frac{\|y_n^0\|}{\|y_n\|} \longrightarrow 0 \quad as \quad n \longrightarrow \infty. \tag{4.10}$$

Hence by (4.6), (4.7) and (4.10) we have

$$1 = \frac{\|y_n\|}{\|y_n\|} \leq \frac{\|y_n^0\| + \|y_n^-\| + \|y_n^+\|}{\|y_n\|} \longrightarrow 0 \quad as \quad n \longrightarrow \infty, \tag{4.11}$$

which is a contradiction. Therefore (y_n) must be bounded. Then by a standard argument, (y_n) has a convergent subsequence, which shows that f_k satisfies the Cerami's condition.

Lemma 4.2 *There exist constants $m > 0$ and $\alpha > 0$ such that*

$$f_k(y) \geq m, \ \forall y \in \partial B_\rho \cap X^1. \tag{4.12}$$

Proof As in (3.26), for all $\epsilon > 0$, there exists a constant $M_1 > 0$ such that

$$\left| \int_0^T G(ks, u(y)) ds \right| \leq \epsilon \gamma_2^2 \|y\|^2 + M_1 \gamma_{\alpha+1}^{\alpha+1} \|y\|^{\alpha+1}, \forall y \in X. \tag{4.13}$$

Now for all $x \in X^1 = X^+$, we have by (4.13)

$$f_k(y) = \frac{1}{2} \|y\|^2 - k \int_0^T G(ks, u(y)) ds$$

$$\geq \frac{1}{2} \|y\|^2 - k\epsilon C^2 \|y\|^2 - kC^{\alpha+1} M_1 \|y\|^{\alpha+1},$$

where $C = sup(1, \gamma_2, \gamma_{\alpha+1})$. So letting $\epsilon = \frac{1}{4kC^2}$ and $\rho = \frac{1}{8}(kM_1 C^{\alpha+1})^{-\frac{1}{\alpha-1}}$, we have

$$f_k(y) \geq \frac{1}{4}\rho^2 - kM_1(C\rho)^{\alpha+1} = \frac{1}{8}\rho^2 = m > 0 \tag{4.15}$$

for $y \in X^1$ with $\|y\| = \rho$.

Lemma 4.3 *There exist $e \in X^1$ and two constants $r_1, r_2 > 0$ such that*

$$f_k(y) \leq 0, \ \forall y \in \partial Q, \tag{4.16}$$

where

$$Q = \{se/0 \leq s \leq r_1\} \oplus \{y \in X^2 / \|y\| \leq r_2\}.$$

Proof Let $e \in X^1$ with $\|e\| = 1$. By (G_2) and (3.8), there exists a constant $M_2 > 0$ such that

$$G(ks, u(y)) \geq \gamma^2 |y|^2 - M_2, \ \forall t \in \mathbb{R}, \ \forall y \in (Ker\, u)^\perp, \tag{4.17}$$

where γ is the constant given by (4.8). It follows from (4.8) and (4.17) that for all $s > 0$ and $y \in X^2 = X^0 \oplus X^-$

$$\begin{aligned} f_k(se + y) &= \frac{1}{2}s^2 - \frac{1}{2}\left\|y^-\right\|^2 - k\int_0^T G(ks, u(se+y))ds \\ &\leq \frac{1}{2}(s^2 - \left\|y^-\right\|^2) - k\gamma^2 \|se + y\|_{L^2}^2 + kM_2T \\ &\leq \frac{1}{2}(s^2 - \left\|y^-\right\|^2) - k\gamma^2(s^2 \|e\|_{L^2}^2 + \|_-\|_{L^2}^2 + \|y_0\|_{L^2}^2) + kM_2T \\ &\leq \frac{1}{2}s^2 - ks^2 - \frac{1}{2}\left\|y^-\right\|^2 - \|y_0\|_{L^2}^2 + kM_2T. \end{aligned} \tag{4.18}$$

Let

$$r_1 = \frac{\sqrt{2kM_2T}}{2k-1}, \ r_2 = \sqrt{2kM_2T},$$

it is clear from (4.18) that

$$f_k(se + y) \leq 0 \ \textit{either} \ s \geq r_1 \ \textit{or} \ \ \|y\| \geq r_2. \tag{4.19}$$

Let

$$Q = \{se/0 \leq s \leq r_1\} \oplus \left\{y \in X^2/\, \|y\| \leq r_2\right\}. \tag{4.20}$$

Then we have $\partial Q = Q_1 \cup Q_2 \cup Q_3$, where

$$Q_1 = \left\{y \in X^0 \oplus X^-/\, \|y\| \leq r_2\right\}, \quad Q_2 = r_1 e \oplus \left\{y \in X^0 \oplus X^-/\, \|y\| \leq r_2\right\},$$

$$Q_3 = \{se/0 \leq s \leq r_1\} \oplus \left\{y \in X^0 \oplus X^-/\, \|y\| = r_2\right\}.$$

By (4.19), one has

$$f_k(y) \leq 0, \ \forall y \in Q_2 \cup Q_3.$$

It follows from $(G_5)(i)$ that $f_k(y) \leq 0$ for all $y \in X^0 \oplus X^-$, which implies that

$$f_k(y) \leq 0, \ \forall y \in Q_1.$$

Hence we obtain (4.16). The proof of Lemma 4.3 is complete.

By Lemma 4.1-3, we conclude that the functional f_k satisfies all the assumptions of the Generalized Mountain Pass Theorem. Therefore for a given $k \in \mathbb{N}$, there exists a critical point $y_k \in X$ of f_k such that $f_k(y_k) > 0$.
Finally, we claim that the system (HS) has infinitely many subharmonic solutions. Note that $y_1(ks)$ satisfies (HS_k), in fact

$$\frac{d}{ds}(y_1(ks)) = k\frac{dy_1}{ds}(ks) = kJu^*G'(ks, y_1(ks)).$$

If $y_k(s) = y_1(ks)$, it is easy to check that

$$c_k = f_k(y_k) = kf_1(y_1) = kc_1. \tag{4.21}$$

Since $c_1 = f_1(y_1) > 0$, one has that $c_k \longrightarrow +\infty$ as $k \longrightarrow \infty$. Noting that

$$c_k \leq \sup_{y\in Q} f_k(y) = \sup_{y\in Q}[\frac{1}{2}(s^2 - \left\|y^-\right\|^2) - k\int_0^T G(ks, u(y)ds] \leq \frac{1}{2}r_1^2 \leq M_2 T, \tag{4.22}$$

where Q is defined as in (4.20). Combining (4.21) with (4.22) yields a contradiction as $k \longrightarrow \infty$. Therefore the sequence (c_k) of critical values is bounded and there is a $k_1 \in \mathbb{N}$ such that $y_k(s) \neq y_1(ks)$ for all $k \geq k_1$.

Now, consider the $T-$periodic function $G_1(t,x) = k_1 G(k_1 t, x)$. By the same technicals as in the previous steps, we prove that the following Hamiltonian system

$$J\frac{dz}{ds} + jG_1^{'}(js, u(z)) = 0 \tag{3.23}$$

possesses a sequence of nonzero $T-$periodic solutions (z_j) such that there exists an integer k_2 satisfying $z_j(s) \neq z_1(js)$ for all $j \geq k_2$. Moreover, from the form of (3.23) and the corresponding variational problem we have

$$z_j(s) = y_{jk_1}(s) \; and \; y_{jk_1}(s) \neq y_1(jk_1 s) \; for \; all \; j \geq k_2.$$

By repeating this reasoning infinitely, we obtain a sequence $x_1(t) = y_1(t)$, $x_{k_1}(t) = y_{k_1}(\frac{t}{k_1})$, $x_{k_1k_2}(t) = y_{k_1k_2}(\frac{t}{k_1k_2})$, ... of distincts nonzero solutions of the system (HS) with x_l is $lT-$ periodic. The proof of Theorem 4.1 is complete.

References

[1] Bartolo, P., Benci, V and Fortunato, D. Abstract critical point theorems and applications to some nonlinear problems with strong resonance. *Nonlinear Analysis* **7** (9) (1983) 981–1012.

[2] Liu, C. G. Subharmonic solutions of Hamiltonian systems. *Nonlinear Analysis* **42** (2000) 185–198.

[3] Li, L. and Szulkin, A. Periodic solutions for a class of nonautonomous Hamiltonian vsystems. *J. Differential Equations* (1993) 226–238.

[4] Li, S. and Willem, M. Applications of local linking to critical point theory. *Seminaire Mathematiques U.C.L.* Rapport No. 203 (January 1992).

[5] Rabinowitz, P.H. Minimax Methods in critical point theory with applications to differential equations. *Amer. Math. Soc. Providence R. I* CBMS Reg. Conf. Ser. in Math., No 65 (1986).

[6] Rabinowitz, P.H. On subharmonic solutions of Hamiltonian systems. *Comm. Pure Appl. Math.* **XXXIII** (1980) 609–633.

[7] Rabinowitz, P.H. Periodic solutions of Hamiltonian systems. *Comm. Pure Appl. Math.* **31** (2) (1978) 157–184.

[8] Scarpello, G.M. and Ritelli, D. Nonlinear dynamics of a two-degrees of freedom Hamiltonian: bifurcations and integration. *Nonlinear Dynamics and Systems Theory* **8** (1) (2008) 97–108.

[9] Tianqing, A. Subharmonic solutions of Hamiltonian systems and the Maslov-type index theory. *J. Math. Anal. Appl.* **331** (1) (2007) 701–711.

[10] Timoumi, M. Closed orbits for a class of Hamiltonian systems. *Demonstratio Mathematica* **XXXI** (1) (1998) 1–10.

[11] Timoumi, M. Periodic solutions for noncoercive superquadratic Hamiltonian systems. *Demonstratio Mathematica* **XL** (2) (2007) 331–346.