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Existence and Uniqueness of Solutions to Quasilinear Integro-differential Equations by the Method of Lines

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Abstract: In this work we consider a class of quasilinear integro-differential equations. We apply the method of lines to establish the wellposedness for a strong solution. The method of lines is a powerful tool for proving the existence and uniqueness of solutions to evolution equations. This method is oriented towards the numerical approximations.

Keywords: *method of lines; integro-differential equation; semigroups; contractions; strong solution.*

Mathematics Subject Classification (2000): 34K30, 34G20, 47H06.

1 Introduction

Let X and Y be two real reflexive Banach spaces such that Y is densely and compactly embedded in X. In the present analysis we are concerned with the following quasilinear integro-differential equation

$$\begin{cases} \frac{du}{dt}(t) + A(t, u(t))u(t) = \int_0^t k(t, s)A(s, u(s))u(s)ds + f(t, u_t), \ 0 < t \le T, \\ u_0 = \phi \in C([-T, 0], X), \end{cases} \tag{1}$$

where $A(t, u)$ is a linear operator in X, depending on t and u, defined on an open subset W of Y. We denote by $J = [0, T]$, k is a real valued function defined on $J \times J \to \mathbb{R}$ and f is defined from $J \times C([-T, 0], X)$ into Y. Here $C([a, b], Z)$, for $-\infty \le a \le b < \infty$, is the

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Banach space of all continuous functions from $[a,b]$ into Z endowed with the supremum norm

$$\|\chi\|_{C([a,b],Z)} := \sup_{a\leq s\leq b} \|\chi(s)\|_Z, \quad \chi \in C([a,b],Z).$$

For $u \in C([-T,t],X)$, we denote by $u_t \in C([-T,0],X)$ a history function defined by

$$u_t(\theta) = u(t+\theta), \ \ \theta \in [-T,0].$$

By a strong solution to (1) on $[0,T']$, $0 < T' \leq T$, we mean an absolutely continuous function u from $[-T,T']$ into X such that $u(t) \in W$ with $u_0 = \phi$ and satisfies (1) almost everywhere on $[0,T']$.

Kato [8] has proved the existence of a unique continuously differentiable solution to the quasilinear evolution equation in X

$$\frac{du}{dt} + A(u)u = f(u), \quad 0 < t \leq T, \quad u(0) = u_0, \tag{2}$$

under the assumptions that there exists an open subset W of Y such that for each $w \in W$ the operator $A(w)$ generates a C_0-semigroup in X, $A(\cdot)$ is locally Lipschitz continuous on W from X into X, f defined from W into Y, is bounded and globally Lipschitz continuous from Y into Y, and there exists an isometric isomorphism $S : Y \to X$ such that

$$SA(w)S^{-1} = A(w) + B(w), \tag{3}$$

where $B(w)$ is in the set $B(X)$ of all bounded linear operators from X into X.

Crandall and Souganidis [6] have established the existence of a unique continuously differentiable solution to the quasilinear evolution equation (2) with $f = 0$ under more general assumptions on $A(w)$. Kato [10] has proved the existence of a strong solution to the quasilinear evolution equation

$$\frac{du}{dt} + A(t,u)u = f(t,u), \quad 0 < t \leq T, \quad u(0) = u_0, \tag{4}$$

under similar conditions on $A(t,u)$ and $f(t,u)$ as considered by Crandall and Souganidis [6].

Recently Oka [11] has dealt with the abstract quasilinear Volterra integrodifferential equation

$$\begin{cases} \frac{du}{dt}(t) + A(t,u(t))u(t) = \int_0^t b(t-s)A(s,u(s))u(s)ds + f(t), \ t \in [0,T], \\ u(0) = \phi, \end{cases} \tag{5}$$

in a pair of Banach spaces $X \supset Y$, where $b : [0,T] \to \mathbb{R}$ is a scalar kernel and $A(t,w)$ is a linear operator in X, depending on t and w, defined on an open subset W of Y. Oka has proved the existence, uniqueness and continuous dependence on the data.

Our analysis is motivated by the work of Bahuguna [1]. In [1] the author considered the following quasilinear integrodifferential equation in a Banach space

$$\frac{du(t)}{dt} + A(u(t))u(t) = \int_0^t a(t-s)k(s,u(s))ds + f(t), \ 0 < t < T, \ u(0) = u_0, \tag{6}$$

by using the application of Rothe's method, the author has established the existence and uniqueness of a strong solution which depends continuously on the initial data.

We shall use Rothe's method to establish the existence and uniqueness results. Rothe's method, introduced by Rothe [15] in 1930, is a powerful tool for proving the existence and uniqueness of a solution to a linear, nonlinear parabolic or a hyperbolic problem of higher order. This method is oriented towards the numerical approximations. For instance, we refer to Rektorys [14] for a rich illustration of the method applied to various interesting physical problems. It has been further developed for nonlinear differential and Volterra integro-differential equations (VIDEs) see [1–4, 7, 14] and references cited in these papers.

In the present study we extend the application of the method of lines to a class of nonlinear VIDEs. In earlier works on the application of the method of lines to integro-differential equations, only bounded perturbations to the heat equation in the integrands have been dealt with. In the problem considered in our paper we have a differential operator appearing in the integrand and hence we have the case of unbounded perturbation.

2 Preliminaries

Let X and Y be as in the first section. Let Z be either X or Y. We use $\| \|_Z$ to denote the norm of Z and by $B(X,Y)$ the set of all bounded linear maps on X to Y, with associated norm $\| \|_{B(X,Y)}$. We write $B(X)$ for $B(X,X)$ and corresponding norm by $\| \|_{B(X)}$. The domain of the operator T is denoted by $D(T)$. We denote by $\mathrm{C}(J_0, Z)$ and $\mathrm{Lip}(J_0, Z)$ the sets of all continuous and Lipschtz continuous functions from a subinterval J_0 of J into Z, respectively. Let $B_r(z_0, r)$ be the Z-ball of radius r at $z_0 \in Z$, i.e. the set $\{z \in Z \mid \ \|z - z_0\|_Z \le r\}$.

For a real number $\beta, N(Z, \beta)$ represents the set of all densely defined linear operators L in Z such that if $\lambda > 0$ and $\lambda\beta < 1$, then $(I + \lambda L)$ is one to one with a bounded inverse defined everywhere on Z and

$$\|(I + \lambda L)^{-1}\|_{B(Z)} \le (1 + \lambda\beta)^{-1},$$

where I is the identity operator on Z. The Hille-Yosida theorem states that $L \in N(Z, \beta)$ if and only if $-L$ is the infinitesimal generator of a strongly continuous semigroup $e^{-tL}, \ t \ge 0$ on Z satisfying $\|e^{-tL}\|_{B(Z)} \le e^{\beta t}, \ t \ge 0$.

A linear operator L on $D(L) \subseteq Z$ into Z is said to be accretive in Z if for every $u \in D(L)$

$$\langle Lu, u^* \rangle \ge 0 \quad \text{for some} \quad u^* \in F(u),$$

where $F : Z \to 2^{Z^*}$, Z^* is the dual of Z

$$F(z) = \{z^* \in Z \mid \langle z, z^* \rangle = \|z\|^2 = \|z^*\|^2\},$$

and $\langle z, f \rangle$ is the value of $f \in Z^*$ at $z \in Z$. If $L \in N(Z, \beta)$ then $(L + \beta I)$ is m-accretive in Z, i.e. $(L + \beta I)$ accretive and the range $R(L + \lambda I) = Z$ for some $\lambda > \beta$. (see corollary 1.3.8 and the remarks preceding it in Pazy [13], p.12). If Z^* is uniformly convex then F is single-valued and uniformly continuous on bounded subsets of Z.

In most of this paper X and Y will be related via a linear isometric isomorphism $S : Y \to X$. We assume, in addition, that the embedding of Y in X is compact and the dual of X^* is uniformly convex. Further, we make the following hypotheses.

(A1) There exists an open subset W of Y and $u_0 \in W$. Furthermore, there exists $\beta \ge 0$ such that $A : [0, T] \times W \to N(X, \beta)$.

(A2) $Y \subseteq D(A(t,w))$, for each $(t,w) \in [0,T] \times W$, which implies that $A(t,w) \in B(Y,X)$ by the closed graph theorem. For each $w \in W$, $t \to A(t,w)$ is continuous in $B(Y,X)$-norm, and for each $t \in [0,T]$, $t \to A(t,w)$ is Lipschitz continuous in the sense that

$$\|(A(t_1,w_1) - A(t_2,w_2))v\|_{B(Y,X)} \leq \mu_A[|t_1 - t_2| + \|w_1 - w_2\|_X]\|v\|_Y,$$

where μ_A is a constant and there exists a constant γ_A such that

$$\|A(t,w)v\|_{B(Y,X)} \leq \gamma_A \|v\|_Y,$$

for all $v \in Y$ and $(t,w) \in [0,T] \times W$.

(A3) There is a family $\{S\}$ of isometric isomorphism Y onto X such that

$$SA(t,w)S^{-1} = A(t,w) + P(t,w),$$

where $P : [0,T] \times W \to B(X)$, $\|P(t,w)\|_{B(X)} \leq \gamma_P$ for $(t,w) \in [0,T] \times W$, with $\gamma_P > 0$, is a constant and

$$\|P(t,w_1) - P(t,w_2)\|_{B(X)} \leq \mu_P \|w_1 - w_2\|_Y, \quad \forall\, w_1, w_2 \in W,$$

where μ_P is a positive constant.

(A4) The function $k : J \times J \to \mathbb{R}$ and $f : J \times C([-T,0],X) \to Y$ satisfy the Lipschitz conditions

$$\begin{aligned} |k(t_2,s) - k(t_1,s)| &\leq L_k|t_2 - t_1|, \\ \|f(t,u) - f(s,v)\|_X &\leq L_f[|t-s| + \|u - v\|_{C([-T,0],X)}], \end{aligned}$$

where L_k and L_f are Lipschitz constant.

For all $u, v \in B_X(u_0,R)$. Let $R > 0$ be such that $W_R = B_Y(u_0,R) \subseteq W$ and let

$$R_0 = \frac{R}{6}(1 + e^{2\theta T})^{-1}, \tag{7}$$

$$M_1 = Tk_T(\gamma_A + \gamma_P C_e)R + L_f[T + \|\tilde{u}_0 - \phi\|_{\mathcal{C}([-T,0],X)}] + \|f(0,\phi)\|_X, \tag{8}$$

$$M_2 = Tk_T(\gamma_A + \gamma_P C_e)R + L_f[T + \|\tilde{u}_{j-1} - \phi\|_{\mathcal{C}([-T,0],X)}] + \|f(0,\phi)\|_X, \tag{9}$$

where C_e is a positive embedding constant, $\theta = \beta + \|P\|_X$ and $k_T = \sup_{s,t \in J}|k(t,s)|$. Let $z_0 \in Y$ and $T_0, 0 < T_0 \leq T$ be such that for $i = 1, 2$

$$\|Su_0 - z_0\|_X \leq R_0, \tag{10}$$

$$T_0[\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M_i] \leq R_0. \tag{11}$$

We notice that (10) and (11) imply that

$$(1 + e^{2\theta T})[\|Su_0 - z_0\|_X + T_0\{\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M_i\}] \leq \frac{R}{3}. \tag{12}$$

We shall use later the following lemma due to Crandall and Souganidis [6].

Lemma 2.1 *Let $S : Y \to X$ be a linear isometric isomorphism, $Q \in N(X, \beta)$, $Y \subset D(Q)$, domain of Q, $P \in B(X)$, the space of all bounded linear operators on X and $SQ = QS + PS$. Set $\theta = \beta + \|P\|_{B(X)}$. Then for every $y \in X$ and $\lambda > 0$ such that $\lambda\theta < 1$, the problem*

$$x + \lambda Qx = y, \quad \tilde{x} + \lambda(Q\tilde{x} + P\tilde{x}) = y,$$

has a unique solution x and $\tilde{x}$ in X. Moreover

$$\|x\|_X \leq (1 - \lambda\theta)^{-1}\|y\|_X, \quad \|\tilde{x}\|_X \leq (1 - \lambda\theta)^{-1}\|y\|_X,$$

and if $y \in Y$, then $x \in Y$ and

$$\|x\|_Y \leq (1 - \lambda\theta)^{-1}\|y\|_Y.$$

We have the following main result.

Theorem 2.1 *Suppose that (A1)-(A4) hold. Then there exists a unique strong solution u to (1) such that $u \in Lip(J_0, X), J_0 = [0, T_0]$. Furthermore, if $v_0 \in B_Y(u_0, R_0)$ then there exists a strong solution v to (1) on $[0, T_0]$ with the initial point $v(0) = \psi$ such that*

$$\|u(t) - v(t)\|_X \leq C\|u_0 - v_0\|_X, \quad t \in [0, T_0], \tag{13}$$

where C is positive constant.

3 Construction of the Scheme and the Convergence

To apply Rothe's method, we use the following procedure. For any positive integer n we consider a partition t_j^n defined by $t_j^n = jh$; $h = \frac{T_0}{n}$, $j = 0, 1, 2, \ldots, n$. We set $u_0^n = \phi(0)$ for all $n \in N$. Let $w_0^n = Su_0^n$ for $n \geq N$ where N is a positive integer such that $\theta(\frac{T_0}{N}) < \frac{1}{2}$. We consider the following scheme

$$\delta u_j^n + A(t_{j-1}^n, u_{j-1}^n)u_j^n = h\sum_{i=0}^{j-1} k_{ji}^n \; A(t_i^n, u_i^n)u_i^n + f_j, \tag{14}$$

where

$$\delta u_j^n = \frac{u_j^n - u_{j-1}^n}{h}, \quad k_{ji}^n = k(t_j^n, t_i^n) \quad \text{and} \quad f_j^n = f(t_j^n, \tilde{u}_{j-1}^n), \quad 1 \leq i \leq j \leq n.$$

We define $\tilde{u}_0^n(t) = \phi(t)$ for $t \in [-T, 0]$, $\tilde{u}_0^n(t) = \phi(0)$ for $t \in [0, T_0]$ and for $2 \leq j \leq n$

$$\tilde{u}_{j-1}^n(\theta) = \begin{cases} \phi(t_j^n + \theta), & \theta \leq -t_j^n, \\ u_{i-1}^n + (t - t_{j-1})\delta u_i^n, & \theta \in [-t_{j+1-i}^n, -t_{j-i}^n],\ 1 \leq i \leq j. \end{cases} \tag{15}$$

For notational convenience, we occasionally suppress the superscript n, throughout, C will represent a generic constant independent of j, h and n. Our first result is concerned with the solvability of (14) in W_R.

Lemma 3.1 *For each $n \geq N$, there exists a unique $u_j, j = 1, 2, \ldots, n$, in W_R satisfying (14).*

Proof Lemma 2.1 implies that there exists a unique $u_1 \in Y$ such that

$$u_1 + hA(t_0, u_0)u_1 = u_0 + h^2 k_{10} A(t_0, u_0)u_0 + hf_1. \tag{16}$$

Applying S on both the sides in (16) using (A3) and letting $w_1 = Su_1$, we have

$$\begin{aligned}(w_1 - z_0) + hA(t_0, u_0)(w_1 - z_0) \quad &+ \quad hP(t_0, u_0)(w_1 - z_0) \\ &= \quad (w_0 - z_0) - hA(t_0, u_0)z_0 - hP(t_0, u_0)z_0 \\ &\quad\quad +h^2 k_{10}[A(t_0, u_0) + P(t_0, u_0)]w_0 + hSf_1.\end{aligned}$$

The estimates in Lemma 2.1 imply that

$$\|w_1 - z_0\|_X \leq (1 - h\theta)^{-1}[\|w_0 - z_0\|_X + h\{\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M_1\}].$$

Since $h\theta < \frac{1}{2}$, we have

$$\|w_1 - z_0\|_X \leq e^{2h\theta}[\|w_0 - z_0\|_X + h\{\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M_1\}].$$

Therefore,

$$\|w_1 - z_0\|_X \leq (1 + e^{2h\theta})[\|w_0 - z_0\|_X + h\{\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M_1\}] \leq R,$$

in view of the estimates (12). Hence, $u_1 \in W_R$. Now, suppose that $u_j \in W_R$ for $i = 1, 2, \ldots, j-1$. Again, Lemma 2.1 implies that for $2 \leq j \leq n$, there exists a unique $u_j \in Y$ such that

$$u_j + hA(t_{j-1}, u_{j-1})u_j = u_{j-1} + h^2 \sum_{i=0}^{j-1} k_{ji} A(t_i, u_i)u_i + hf_j. \tag{17}$$

Proceeding as before and letting $w_j = Su_j$, we get the estimate

$$\|w_j - z_0\|_X \leq e^{2h\theta}[\|w_{j-1} - z_0\|_X + h\{\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M_2\}].$$

Reiterating the above inequality, we get

$$\|w_j - z_0\|_X \leq e^{2jh\theta}[\|w_1 - z_0\|_X + jh\{\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M_2\}].$$

Hence

$$\|w_j - z_0\|_X \leq (1 + e^{2T\theta})[\|w_1 - z_0\|_X + T_0\{\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M_2\}] \leq R.$$

The above inequality and equations (16) and (17) imply that $u_j \in W_R$ satisfy (14) for $1 \leq j \leq n, n \geq N$. This completes the proof of the lemma. □

Lemma 3.2 *There exists a positive constant C, independent of j, h and n such that*

$$\|\delta u_j\|_X \leq C, \qquad j = 1, 2, \ldots, n;\ n \geq N.$$

Proof In (14) for $j = 1$, we get

$$\delta u_1 + hA(t_0, u_0)\delta u_1 = -A(t_0, u_0)u_0 + hk_{10}A(t_0, u_0)u_0 + f_1.$$

Using Lemma 2.1 we have

$$\|\delta u_1\|_X \le e^{2hT}[(1+hk_T)\gamma_A\|u_0\|_Y + f_T] := C_1,$$

where $f_T = L_f[T + \|\tilde{u}_0 - \phi\|_{\mathcal{C}([-T,0],X)}] + \|f(0,\phi)\|_X$. Now, from (14) for $2 \le j \le n$, we have

$$\begin{aligned}\delta u_j + hA(t_{j-1},u_{j-1})\delta u_j &= \delta u_{j-1} - [A(t_{j-1},u_{j-1}) - A(t_{j-2},u_{j-2})]u_{j-1}\\ &\quad + hk_{jj-1}A(t_{j-1},u_{j-1})u_{j-1}\\ &\quad + h\sum_{i=0}^{j-2}[k_{ji} - k_{j-1i}]A(t_i,u_i)u_i + f_j - f_{j-1}.\end{aligned}$$

Applying Lemma 2.1 and using (A2) and (A4) we get

$$\begin{aligned}\|\delta u_j\|_X &\le e^{2h\theta}\big[(1+\mu_A hR)\|\delta u_{j-1}\|_X + \mu_A hR + h\gamma_A R\{|k_{jj-1}|\\ &\quad + \sum_{i=0}^{j-2}|k_{ji} - k_{j-1j}|\} + \|f_j - f_{j-1}\|_Y\big]\\ &\le e^{2h\theta}\big[(1+\mu_A hR)\|\delta u_{j-1}\|_X + M_3 h + L_f h\|\delta\tilde{u}_{j-1}\|_{\mathcal{C}([-T,0],X)}\big],\end{aligned}$$

where $M_3 = \mu_A R + \gamma_A R(k_T + L_k T) + L_f T$. Denoting by $C_2 = \mu_A R + L_f$, we have

$$\max_{1\le i\le j}\|\delta u_i\|_X \quad\le\quad e^{2h\theta}\big[(1+C_2h)\max_{1\le i\le j-1}\|\delta u_i\|_X + M_3 h\big].$$

Reiterating the above inequality, we get

$$\max_{1\le i\le j}\|\delta u_i\|_X \le e^{2jh\theta}(1+C_2h)^j\big[\|\delta u_1\|_X + M_3 T\big],$$

hence

$$\|\delta u_j\|_X \le e^{2(\theta+C_2)T}[C_2 + M_3 T] := C.$$

This completes the proof of the lemma. □

Definition 3.1 We define the Rothe sequence $\{U^n\} \in C([-T,T],Y)$ given by

$$U^n(t) = \begin{cases}\phi(t), & t\in[-T,0],\\ u_{j-1} + \frac{u_j - u_{j-1}}{h}(t - t_{j-1}), & t\in[t_{j-1},t_j],\ j=1,2,\dots,n.\end{cases} \tag{18}$$

Further, we define a sequence of functions $\{X^n\}$ from $[-T,T]$ into Y given by

$$X^n(t) = \phi(t) \quad\text{for}\quad t\in(-T,0], \quad X^n(t) = u_j \quad\text{for}\quad t\in(t_{j-1},t_j]. \tag{19}$$

Remark 3.1 Each of the functions $\{X^n(t)\}$ lies in W_R for all $t \in (-h,T_0]$ and $\{U^n\}$ is Lipschitz continuous with uniform Lipschitz constant, i.e.,

$$\|U^n(t) - U^n(s)\|_X \le C|t-s|, \quad t,s\in J_0.$$

Furthermore, $\|U^n(t) - X^n(t)\|_X \le \frac{C}{n}$. Also, we define

$$\begin{aligned}K^n(t) &= h\sum_{i=0}^{j-1}k_{ji}A(t_i,u_i)u_i, \quad t\in(t_{j-1},t_j], &(20)\\ f^n(t) &= f(t_j,\tilde{u}^n_{j-1}), \quad t\in(t_{j-1},t_j]. &(21)\\ A^n(t,u) &= A(t_{j-1},u), \quad t\in(t_{j-1},t_j),\ j=1,2,\dots,n. &(22)\end{aligned}$$

Lemma 3.3 *Under the given assumptions we have*
(a) $\{K^n(t)\}$ *is uniformly bounded;*
(b) $\int_0^t A^n(s, X^n(s-h))X^n(s)ds = u_0 - U^n(t) + \int_0^t K^n(s)ds + \int_0^t f^n(s)ds;$
(c) $\frac{d^-}{dt}U^n(t) + A^n(t, X^n(t-h))X^n(t) = K^n(t) + f^n(t), \quad t \in (0, T_0],$
where $\frac{d^-}{dt}$ *is the left-derivative.*

Proof (a) This is a direct consequence of the assumptions (A2)-(A4).
(b) For $2 \le j \le n$ and $t \in (t_{j-1}, t_j]$, by Definition 3.1, we have

$$\begin{aligned}
&\int_0^t A^n(s, X^n(s-h))X^n(s)ds \\
&= \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} A^n(s, X^n(s-h))X^n(s)ds + \int_{t_{j-1}}^{t} A^n(s, X^n(s-h))X^n(s)ds \\
&= -\sum_{i=1}^{j-1}(u_i - u_{i-1}) - \frac{1}{h}(t - t_{j-1})(u_j - u_{j-1}) + \ h\sum_{i=1}^{j-1}\left[h\sum_{p=0}^{i-1} k_{ip}A(t_p, u_p)u_p\right] \\
&\ + (t - t_{j-1})[h\sum_{p=0}^{j-1} k_{ip}A(t_p, u_p)u_p] + \ h\sum_{i=0}^{j-1} f_i^n - (t - t_{j-1})f_j^n \\
&= u_0 - U^n(t) + \int_0^t K^n(s)ds + \int_0^t f^n(s)ds.
\end{aligned}$$

When $j = 1$, $t \in (0, t_1]$, we have

$$\begin{aligned}
\int_0^t A^n(s, X^n(s-h))X^n(s)ds &= tA(t_0, u_0)u_1 \\
&= -\frac{t}{h}(u_1 - u_0) + thk_{10}A(t_0, u_0)u_0 + tf_1^n \\
&= u_0 - U^n(t) + \int_0^t K^n(s)ds + \int_0^t f^n(s)ds.
\end{aligned}$$

(c) for $t \in (t_{j-1}, t_j]$,

$$A^n(t, X^n(t-h))X^n(t) = A(t_{j-1}, u_{j-1})u_j \quad \text{and} \quad \frac{d^-u^n}{dt}(t) = \frac{1}{h}(u_j - u_{j-1}).$$

Therefore,

$$\begin{aligned}
\frac{d^-u}{dt}(t) - A^n(t, X^n(t-h))X^n(t) &= \frac{1}{h}(u_j - u_{j-1}) - A(t_{j-1}, u_{j-1})u_j \\
&= h\sum_{i=0}^{j-1} k_{ji}A(t_i, u_i)u_i + f_j^n \\
&= K^n(t) + f^n(t).
\end{aligned}$$

This completes the proof of the lemma. □

In the next lemma we prove the local uniform convergence of the Rothe sequence.

Lemma 3.4 *There exists a subsequence $\{U^{n_k}\}$ of the sequence $\{U^n\}$ and a function u in Lip(J_0, X) such that*

$$U^{n_k} \to u \quad in \quad C(J_0, X),$$

with supremum norm as $k \to \infty$.

Proof Since $\{X^n(t)\}$ is uniformly bounded in Y, the compact imbedding of Y implies that there exists a subsequence $\{X^{n_k}\}$ of $\{X^n\}$ and a function $u : J_0 \to X$ such that $X^{n_k}(t) \to u(t)$ in X as $k \to \infty$. The reflexivity of Y implies that $u(t)$ is the weak limit of $X^{n_k}(t)$ in Y hence $u(t) \in Y$ in fact in W_R since $X^{n_k}(t) \in W_R$. Now, $X^{n_k}(t) - U^{n_k}(t) \to 0$ in X, $U^{n_k}(t) \to u(t)$ as $k \to \infty$. The uniform Lipschitz continuity of $\{U^{n_k}\}$ on J_0 implies that $\{U^{n_k}\}$ is an equicontinuous family in $C(J_0, X)$ and the strong convergence of $U^{n_k}(t)$ to $u(t)$ in X implies that $\{U^{n_k}(t)\}$ is relatively compact in X. We use the Ascoli-Arzela theorem to assert that $U^{n_k} \to u$ in $C(J_0, X)$ as $k \to \infty$. Since U^{n_k} are in Lip(J_0, X) with uniform Lipschitz constant, $u \in$Lip(J_0, X). This completes the proof of the lemma. □

Lemma 3.5 *Let $\psi : [0, T] \to X$ be given by $\psi(t) = A(t, u(t))u(t)$. Then ψ is Bochner integrable on $[0, T]$.*

Proof Proof of this lemma can be established in similar way as that of Lemma 4.6 in Kato [9]. □

Lemma 3.6 *Let $\{K^n(t)\}$ be the sequence of functions defined by (20) and*

$$K(\psi)(t) = \int_0^t k(t, s)\psi(s)ds.$$

We have $K^{n_k}(t) \to K(\psi)(t)$, uniformly on $[0, T_0]$ as $k \to \infty$.

Proof For notational conveneince, we shall use the index n in place of n_k for the subsequence n_k of n. We first show that $K^n(t) - K(\psi_n)(t) \to 0$ uniformly on $[0, T_0]$ as $n \to \infty$ where $\psi_n : [0, T_0] \to X$ is given by $\psi_n(t) = A(t, X^n(t))X^n(t)$. For $t \in (t_{j-1}, t_j]$, we have

$$\begin{aligned} K^n(t) - K(\psi_n)(t) &= h\sum_{i=0}^{j-1} k^n_{ji} A(t_i, u_i)u_i - \int_0^t k(t, s)A(s, X^n(s))X^n(s)\, ds \\ &= \sum_{i=1}^{j-1}\left[\int_{t_{i-1}}^{t_i} [k_{ji}A(t_i, u_i) - k(t, s)A(s, X^n(s))]\, ds\right] u_i \\ &+ \quad hk(t_j, t_0)A(t_0, u_0)u_0 - \left[\int_{t_{j-1}}^{t} k(t, s)A(s, u_j)ds\right] u_j. \end{aligned}$$

Since $\|A(t, u_j)u_j\|_X \leq \gamma_A R$, and $k : [0, T_0] \to \mathbb{R}$ being Lipschitz continuous imply that the last two terms on the right hand side tend to zero strongly and uniformly on $[0, T_0]$ as $n \to \infty$ we have

$$\|K^n(t) - K(\psi_n)(t)\|_X \quad \leq \quad \gamma_A R\left[\sum_{i=0}^{j-2}\int_{t_i}^{t_{i+1}} |k_{ji} - k(t, s)|ds\right].$$

Now, since k satisfies (A4), $k(t,s)$ is uniformly continuous in t as well as in s on $[0,T_0]$. Hence for each $\epsilon > 0$ we can choose n sufficiently large such that for $|t_1 - t_2| + |s_1 - s_2| < h = \frac{T}{n},\ t_i, s_i \in [0,T_0]$, $i = 1,2$, we have

$$|k(t_1,s_1) - k(t_2,s_2)| < \frac{\epsilon}{\gamma_A RT}.$$

Then for sufficiently large n, we have

$$\|K^n(t) - K(\psi_n)(t)\|_X \leq \frac{\epsilon}{\gamma_A RT}\gamma_A Rjh < \epsilon,$$

Which show that $K^n(t) - K(\psi_n)(t) \to 0$ as $n \to \infty$, uniformly on $[0,T_0]$. Now we show that $K(\psi_n)(t) \to K(\psi)(t)$ uniformly as $n \to \infty$. For any $v \in X$, We note that $\langle A(t,u(t))u(t), v\rangle$ is continuous hence we may write

$$\Big\langle K(\psi)(t), v\Big\rangle = \int_0^t k(t,s)\langle A(s,u(s))u(s), v\rangle\, ds.$$

Now, for any $v \in X$,

$$\begin{aligned}\langle K(\psi_n)(t), v\rangle &= \sum_{i=0}^{j-2}\int_{t_i}^{t_{i+1}} k(t,s)\langle A(s,u_{i+1})u_{i+1}, v\rangle ds\\ &\quad + \int_{t_{j-1}}^{t} k(t,s)\langle A(t,u_j)u_j, v\rangle ds.\end{aligned}$$

This implies that $\langle K(\psi_n)(t), v\rangle \to \langle K(\psi)(s), v\rangle$, as $n \to \infty$. This completes the proof of the lemma. □

3.1 Proof of Theorem 2.1.

Proof First we show that $A^m(t, X^m(t-h))X^m(t) \rightharpoonup A(t,u(t))u(t)$ in X as $m \to \infty$, where '$\rightharpoonup$' denotes the weak convergence in X,

$$\begin{aligned}&A(t_{j-1}, X^m(t-h))X^m(t) - A(t,u(t))u(t)\\ &= [A(t_{j-1}, X^m(t-h)) - A(t,u(t))]X^m(t) + A(t,u(t))[X^m(t) - u(t)].\end{aligned}$$

Since,

$$\|[A(t_{j-1}, X^m(t-h)) - A(t,u(t))]X^m(t)\|_X \leq \mu_A R[|t_{j-1} - t| + \|X^m(t-h) - u(t)\|_X],$$

as $m \to \infty$ the right hand side of the above equation tends to zero. Since $X^m(t) \to u(t)$ in X uniformly on J_0 and $A(t,u(t)) \in N(X,\beta), \beta I + A(t,u)$ is m-accretive in X. We use Lemma 2.5 due to Kato [9] and the fact that

$$\|A(t,u(t))[X^m(t-h) - u(t)]\|_X \leq 2\mu_A R,$$

to assert that $A(t,u(t))X^m(t) \rightharpoonup A(t,u(t))u(t)$ in X and, hence, $A^m(t, X^m(t-h))X^m(t) \rightharpoonup A(t,u(t))u(t)$ in X as $m \to \infty$. Now we show that $A(t,u(t))u(t)$ is weakly continuous on J_0, let $\{t_p\} \subset J_0$ be a sequence such that $t_p \to t$, as $p \to \infty$. Then $u(t_p) \to u(t)$ in X as $p \to \infty$ and we can follow the same arguments as above to prove that $A(t_p, u(t_p))u(t_p) \rightharpoonup A(t, u(t))u(t)$ in X as $p \to \infty$. Now from Lemma 3.3 for each $x^* \in X^*$ we have

$$\langle U^m(t), x^*\rangle = \langle u_0, x^*\rangle + \int_0^t \langle -A^m(s, X^m(s-h))X^m(s) + K^m(s) + f^m(s), x^*\rangle ds.$$

Letting $m \to \infty$ using bounded convergence theorem and Lemma 3.6 we get

$$\langle U(t), x^*\rangle = \langle u_0, x^*\rangle + \int_0^t \langle -A(s, u(s))u(s) + K(\psi)(s) + f(s, u_s), x^*\rangle ds.$$

Continuity of the integrand implies that $\langle u(t), x^*\rangle$ is continuously differentiable on J_0. The Bochner integrability of $A(t, u(t))u(t)$ implies that the strong derivative of $u(t)$ exists $a.e.$ on J_0 and

$$\frac{du}{dt} + A(t, u(t))u(t) = \int_0^t k(t, s)A(s, u(s))u(s)ds + f(t, u_t), \quad a.e \quad \text{on} \quad J_0.$$

Since $u(0) = u_0$, u is a strong solution to (1). Now for the uniqueness of the solution of (1). Let v be another strong solution to (1) on J_0. Let $U = u - v$, then for $a.e.$ $t \in J_0$

$$\begin{aligned}
&\left\langle \frac{dU}{dt}(t), F(U(t)) \right\rangle + \langle \beta I + A(t, u(t))U(t), F(U(t))\rangle \\
&= \beta\|U(t)\|_X^2 + \langle (A(t, u(t)) - A(t, v(t)))v(t), F(U(t))\rangle \\
&+ \left\langle \int_0^t k(t, s)[A(s, u(s)) - A(s, v(s))]u(s)ds, F(U(t)) \right\rangle \\
&+ \left\langle \int_0^t k(t, s)A(s, v(s))[u(s) - v(s)]ds, F(U(t)) \right\rangle \\
&+ \langle f(t, u_t) - f(t, v_t), F(U)\rangle.
\end{aligned}$$

Using m-accretivity of $\beta I + A(t, u(t))u(t)$ and Assumptions (A2) and (A4) we get

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|_X^2 \le C_T\|U\|_{C([0,t],X)}^2,$$

where $C_T = \beta + \mu_A R + k_T(\gamma_A C_e + \mu_A R) + L_f$. Integrating the above inequality on $(0, t)$ and taking the supremum we get

$$\frac{1}{2}\|U(t)\|_{C([0,t],X)}^2 \le C_T \int_0^t \|U\|_{C([0,s],X)}^2 ds.$$

Applying the Gronwall's inequality we get $U = 0$ on J_0.

Continuous dependence. Let $v_0 \in B_Y(u_0, R_0)$. Then

$$\|Sv_0 - z_0\|_X \le \|Sv_0 - Su_0\|_X + \|Su_0 - z_0\|_X \le 2R_0.$$

Hence

$$(1 + e^{2\theta T})[\|Sv_0 - z_0\|_X + T_0\{\gamma_A\|z_0\|_Y + \gamma_A\|z_0\|_X + M\}] \le 3R_0 = \frac{R}{2}.$$

We can proceed as before to prove the existence of $v_j^n \in W_R$ satisfying scheme (14) with u_j^n and u_0 replaced by v_j^n and v_0 respectively. Convergence of v_j^n to $v(t)$ can be proved in a similar manner. Let $U = u - v$ then following the steps used to prove the uniqueness, we have for $a.e.\ t \in J_0$

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|_X^2 \le C_T\|U\|^2_{C([0,t],X)}.$$

Integrating the above inequality on $(0,t)$ and taking the supremum we get

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|^2_{C([0,t],X)} \le \frac{1}{2}T\|U(0)\|_X^2 + C_T\int_0^t \|U(t)\|^2_{C([0,s],X)}.$$

Applying the Gronwall's inequality we get

$$\|U(t)\|^2_{C([0,t],X)} \le C\|U(0)\|_X^2,$$

where C is a positive constant. This completes the proof of the theorem. □

4 Application

For illustration, we consider the existence and uniqueness of a solutions for the following model

$$\begin{cases} a_0(x,u)\frac{\partial u}{\partial t} + \sum_{j=1}^m a_j(t,x,u)\frac{\partial u}{\partial x_j} = \int_{-T}^0 g(t,u(t+\theta,x)d\theta, \\ + \sum_{j=1}^m \int_0^t k(t-s)a_j(s,x,u)\frac{\partial u}{\partial x_j}ds, \quad 0 < t \le T,\ x \in \mathbb{R}^m, \\ u(\theta,x) = \phi_0(\theta,x) \quad \text{for} \quad \theta \in [-T,0] \quad \text{and} \quad x \in \mathbb{R}^m, \end{cases} \tag{23}$$

where the unknown $u = (u_1, \dots, u_N)$ is an N-vector, a_0 and $a_j,\ j = 1, 2, \dots, m$, are $N \times N$ symmetric matrix-valued smooth functions on $\Omega \times \mathbb{R}^N$ and $[0,T] \times \Omega \times \mathbb{R}^N$, respectively, where $\Omega \subset \mathbb{R}^m$ is a bounded domain with sufficiently smooth bounday. We set

$$\begin{aligned} &Y = H^s(\Omega,\mathbb{R}^N), \quad Z = H^{s-1}(\Omega,\mathbb{R}^N), \quad X = H^0(\Omega,\mathbb{R}^N), \quad W = B_r(Y), \\ &S = (1-\Delta)^{s/2}, \quad s > m/2 + 1, \\ &A(t,w) = a_0(x,w)^{-1}\sum_{j=1}^m a_j(t,x,w)\frac{\partial}{\partial x_j}, \end{aligned}$$

and use the variable norm

$$\|v\|_w^2 = \int_\Omega a_0(x,w)v.vdx.$$

We suppose that for $j = 1, 2, \dots, m$, $a_j(t,x,u)$ are simultaneously diagonalizable by a common nonsingular C^1 matrix $q(t,x,w)$ and $a_0(x,w)$ is positive-definite. The function $g : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ is continuous and Lipschitzian with respect to the second argument, the function $\phi_0 : [-r,0] \times \Omega \to \mathbb{R}$ will be specified later.

Note that $A(t,w) \in G(X_w,1,\beta)$ with β depending on $\|w\|_Y$, and $G(X_w,1,\beta)$ denotes the set of all (negative) generators A of C_0-semigroups on X_w such that $\|e^{-tA}\| \le Me^{\beta t}$ for $t > 0$. Again verification of the conditions is straightforward, except that we have to prove that $-A(t,w)$ is the generator of C_0-semigroup (for details see [8]).

Let $f : [0,T] \times C([-T,0],X) \to Y$ be defined by

$$f(t,\chi)(x) = \int_{-T}^{0} g(t, \chi(\theta)(x) d\theta, \quad t \geq 0.$$

The initial data $\phi \in C([-T,0],X)$ is defined by

$$\phi(\theta)(x) = \phi_0(\theta, x) \quad \text{for} \quad \theta \in [-T, 0].$$

Then (23) takes the following abstract form

$$\begin{cases} \frac{d}{dt}u(t) + A(t,u(t))u(t) = \int_0^t k(t-s)A(s,u(s))u(s)ds + f(t,u_t), \quad 0 < t \leq T, \\ u_0 = \phi \in C([-T,0],X). \end{cases} \tag{24}$$

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