Nonlinear Dynamics and Systems Theory, **12** (1) (2012) 105–109

Global Stability Given Local Stability Via Curvature of Some Nonautonomous Differential Equations

N. Nilakantan * and V. Raghavendra

Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur, India

Received: January 27, 2011; Revised: January 19, 2012

Abstract: In this article, the global stability, (given local stability) of a class of nonautonomous differential equations is obtained. The boundedness of the curvature of the trajectories on sets with certain properties is used to determine the stability.

Keywords: *stability; nonautonomous differential equation.*

Mathematics Subject Classification (2010): 34D23, 37C75.

1 Introduction

In this study, we revisit the asymptotic stability of ordinary differential equations via the curvature properties of the trajectories. We have studied the global stability (given the local stability) of the zero solution of a class of non-autonomous linear differential equations of the type

$$x^{'}(t) = A(t)x$$

under certain conditions on the matrix A. The usual conditions on $A + A^T$ appear to be restrictive, whereas the arguments via the curvature yield the global stability given the local stability. Apparently, the theory depends on the boundedness of the curvature and a property of the trajectory called *the negative property on compact sets*. The idea of the proof is borrowed from [1], although it deals with only the autonomous systems. In this paper, the stress is on the nonautonomous differential equations.

Section 2 deals with the necessary preliminaries. The main result is stated in Section 3. Examples have been given here for illustration.

* Corresponding author: mailto:nandini@iitk.ac.in

2 Preliminaries

Let the curve Γ be represented by the C^1 map $\phi : \mathbb{R} \to \mathbb{R}^n$. The curvature at each point $x = \phi(t)$, for some $t \in \mathbb{R}$ on a C^1 curve Γ, where $\phi^{'}(t) \neq 0$, is given by

$$\kappa(x) = \frac{\| \| \phi^{'}(t) \|^2 \phi^{''}(t) - \left\langle \phi^{'}(t), \phi^{''}(t) \right\rangle \phi^{'}(t) \|}{\| \phi^{'}(t) \|^4}.$$

Let $K(x)$ be the curvature at a point $x \in \Gamma$, where Γ is a C^1 curve given by $\phi : \mathbb{R} \to \mathbb{R}^n$. A consequence of the curvature being bounded everywhere along Γ is:

Proposition 2.1 *Let K be an upper bound for the curvature on a C^1 curve Γ. If there exist t_-, t_0 and $t_+ \in \mathbb{R}$ with $t_- < t_0 < t_+$ such that the tangent at $\phi(t_0)$ is orthogonal to the tangents at $\phi(t_-)$ and $\phi(t_+)$ and $\left\langle \phi^{'}(t), \phi^{'}(t_0) \right\rangle > 0$ for all $t \in (t_-, t_+)$, then $d(\phi(t_-), \phi(t_0)) \geq \frac{\sqrt{2}}{K}$ and $d(\phi(t_0), \phi(t_+)) \geq \frac{\sqrt{2}}{K}$. Here, $d(\phi(t_-), \phi(t_0))$ denotes the euclidean distance between $\phi(t_-)$ and $\phi(t_0)$.*

Proof Let T_t denote the unit tangent vector to the curve Γ at $\phi(t)$, and let $\theta(t)$ be the angle between T_{t_0} and T_t.

$\langle (\phi(t_+) - \phi(t_0)), T_{t_0} \rangle = \int\limits_{t_0}^{t_+} \left\langle \phi^{'}(t), T_{t_0} \right\rangle dt = \int\limits_{t_0}^{t_+} \| \phi^{'}(t) \| \cos\theta(t)\, dt.$

Let $s(t)$ denote the arc length along the curve Γ, at time t from the fixed point $\phi(t_0)$. From [2], we know that $\left| \frac{d\theta}{ds} \right| \leq K$. Hence,

$$\langle (\phi(t_+) - \phi(t_0)), T_0 \rangle = \int\limits_{s(t_0)}^{s(t_+)} \cos\theta(s)\, ds \geq \frac{1}{K} \int\limits_{\theta(t_0)}^{\theta(t_+)} \cos\theta\, d\theta.$$

Since T_{t_0} and T_{t_+} are orthogonal to each other, $\theta(t_0) = 0$ and $\theta(t_+) = \frac{\pi}{2}$.

So, $\langle \phi(t_+) - \phi(t_0)), T_{t_0} \rangle \geq \frac{1}{K}$. By a similar argument,

$\left\langle \phi(t_+) - \phi(t_0), T_{t_+} \right\rangle \geq \frac{1}{K}$ or $d(\phi(t_0), \phi(t_+)) \geq \frac{\sqrt{2}}{K}$. By a similar argument $d(\phi(t_0), \phi(t_-)) \geq \frac{\sqrt{2}}{K}$, thereby proving the proposition.

A consequence of Proposition 2.1 is

Definition 2.1 Let $\Omega \subset \mathbb{R}^n$ be a set such that the curvature at each point along any regular curve in Ω is bounded above by K. If for each point x_0 in Ω there do not exist points x_1 and x_2 on any regular curve ϕ through x_0, (where $\phi(t_0) = x_0$, $\phi(t_1) = x_1$ and $\phi(t_2) = x_2$) with $\left\langle \phi^{'}(t_0), \phi^{'}(t_1) \right\rangle = 0$, $\left\langle \phi^{'}(t_0), \phi^{'}(t_2) \right\rangle = 0$ and $d(x_0, x_1)$, $d(x_0, x_2) \geq \frac{2}{K}$, then Ω is said to have the *negative property.*

We now turn our attention to nonautonomous ordinary differential equations of the type

$$x^{'} = A(t)x, \tag{1}$$

where $A : [0, \infty) \to M_n(\mathbb{R})$ is a C^1 matrix, such that $\lim\limits_{t\to\infty} A(t)$ exists and $\lim\limits_{t\to\infty} A(t) \neq 0$.

A solution of equation (1) passing through x_0 at time t_0 is denoted by $x(t, t_0, x_0)$.

Henceforth, if Ω has the negative property, we assume that it is with respect to the solution curves of the concerned differential equation.

The following is a result on the ω limit points of trajectories in a set with the negative property.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^n$ have no equilibrium points of equation* (1) *and have the negative property. Let $\Gamma = \{\phi(t) : t \in [0, \infty)\}$ be any forward trajectory of equation* (1) *which is contained entirely in Ω. Then, the omega limit set of Γ consists entirely of equilibrium points which lie in the closure of Ω but not in Ω.*

Proof Let $\overline{x}$ be an omega limit point of Γ. Assume that $\overline{x}$ is not an equilibrium point. There exists a sequence $(t_n) \uparrow \infty$ such that $\phi(t_n) \to \overline{x}$. By using the fundamental theorem of calculus, we get

$$\lim_{n\to\infty} \int_{t_n}^{t_{n+1}} A(t)\phi(t)\, dt = 0. \tag{2}$$

There exists a $T \in \mathbb{R}$ such that for all $t > T$, $\langle A(t)\overline{x}, A(t)\overline{x}\rangle > 0$. Hence, there exists a $\delta > 0$, $t_0 \geq T$, such that $\langle A(s)y, A(r)z\rangle > 0$, for all $s, r > t_0$ and $y, z \in B(\overline{x}, \delta)$, *i.e.*, there exists a $\tau > 0$, such that $\forall\ |t - t_0| < \tau$, $s, r > t_0$ and $y, z \in B(\overline{x}, \delta)$

$$\langle A(s)x(t, t_0, y), A(r)z\rangle > 0. \tag{3}$$

Fix a $s > t_0$ and $y \in B(\overline{x}, \delta)$ and call the vector $A(s)y$ as v. From equation (2), we see that

$$\lim_{n\to\infty} \int_{t_n}^{t_{n+1}} \langle A(t)\phi(t), v\rangle \ dt = 0. \tag{4}$$

When n is sufficiently large, $\phi(t_n) \to \overline{x}$. We get from equation (3):

$$\int_{t_n}^{t_n+\tau} \langle A(t)\phi(t), v\rangle \ dt > 0 \tag{5}$$

For n sufficiently large, there exists an interval (t_n, t_{n+1}), where the integrand must change sign, *i.e.*, there exist t_{m_-}, t_{m_+} with $t_{m_-} < t_m < t_{m_+}$ such that $\left\langle A(t_m)\phi(t_m), A(t_{m_+})\phi(t_{m_+})\right\rangle = 0$ and $\left\langle A(t_{m_-})\phi(t_{m_-}), A(t_m)\phi(t_m)\right\rangle = 0$.

Let $x_0 = A(t_m)\phi(t_m)$, $x_1 = A(t_{m_-})\phi(t_{m_-})$ and $x_2 = A(t_{m_+})\phi(t_{m_+})$. Now, $\left\langle \phi^{'}(t_m), \phi^{'}(t_{m_+})\right\rangle = 0$ and $\left\langle \phi^{'}(t_m), \phi^{'}(t_{m_-})\right\rangle = 0$. From Proposition 2.1, we conclude that $d(x_-, x_0)$ and $d(x_0, x_+) \geq \frac{\sqrt{2}}{K}$, which implies that Ω does not have the negative property, a contradiction. Hence $\overline{x}$ has to be an equilibrium point. Since $\overline{x} \notin \Omega$, $\overline{x}$ has to belong to $\overline{\Omega}$.

We turn our attention to a few applications. Let us now consider nonautonomous linear differential equations

$$x^{'}(t) = A(t)x + g(x), \tag{6}$$

where $A : [0, \infty) \to M_n(\mathbb{R})$ is a C^1 matrix, such that $\lim\limits_{t\to\infty} A(t)$ exists with $\lim\limits_{t\to\infty} A(t) \neq 0$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function and satisfies the smallness conditions.

On the lines of the proof of Lemma 2.1, we have the following result.

Corollary 2.1 *Let $\Omega \subset \mathbb{R}^n$ have the negative property with respect to equation* (6). *Let $\Gamma = \{\phi(t) : t \in \mathbb{R}\}$ be any forward trajectory which is contained entirely in Ω. Then, the omega limit set of Γ consists entirely of equilibrium points which lie in the closure of Ω and not in Ω.*

Remark 2.1 It is interesting to note that, if Ω is compact and has the negative property, then every forward trajectory has to leave Ω.

From [3], we have the following result for stability.

Proposition 2.2 *Let us consider equation* (1) *where $A(t)$ is a continuous real valued $n \times n$ matrix on $[0, \infty)$. Let $M(t)$ be the maximum eigenvalue of $A(t) + A(t)^T$.*

If $\lim\limits_{t\to\infty} \int\limits_{t_0}^{t} M(s)\, ds = -\infty$, where t_0 is fixed, then every solution of equation (1) *tends to zero as $t \to \infty$.*

3 Stability

In this section, we study the global stability given the local stability of equations of the type (1) or (6).

Theorem 3.1 *Let $D \subset \mathbb{R}^n$ be eventually positively invariant under either of the equations* (1) *or* (6) *and $\Omega \subset D$ be compact, with no equilibrium points of equation* (1) *and have the negative property. If the solutions of $D \setminus \Omega$ approach the equilibrium solution asymptotically, then the solutions in Ω must also approach the equilibrium solution asymptotically.*

Proof Since, Ω is compact and has the negative property, from Lemma 2.1 and Corollary 2.1, we know that every forward trajectory in Ω must leave Ω. As $\Omega \subset D$, which is a positively invariant set in $\mathbb{R}^n$, the result follows.

We now have a result on global stability given the local stability.

Corollary 3.1 *Let D be positively invariant and contain a unique equilibrium $\overline{x}$ (with respect to either equation* (1) *or* (6) *and $\Omega \subset D$ be compact and have the negative property. If the solutions of $D \setminus \Omega$ approach $\overline{x}$ asymptotically, then $\overline{x}$ is globally stable in D.*

Remark 3.1 Consider the ordinary differential equation $x^{'} = Ax$, where A is a stable matrix. Let B be a closed ball of radius $\frac{1}{2}$ in $\mathbb{R}^n$ such for each $y \in B$, $\|y\| \geq 2$. We see that in B the curvature along any solution curve of $x^{'} = Ax$ is bounded above by 1. B does not contain the critical point and has the negative property. Therefore, by Theorem 3.1, the zero solution is globally stable.

Remark 3.2 The autonomous equation $y^{'} = y(y-1)$ has critical points at $y = 0$ and $y = 1$. $(-\infty, 1)$ is a positively invariant set under $y^{'} = y(y-1)$ and $(-2, -1)$ satisfies the negative property. By Theorem 3.1, we see that the zero solution is stable in $(-\infty, 1)$.

Example 3.1 $x^{'} = A(t)x$ where $t > 0$ and $A(t)$ is

$$\begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{1+t} \end{bmatrix}.$$

The only equilibrium point is $(0,0)$. The maximum eigen value, $\lambda(t)$ of $A + A^T$ is $\dfrac{-2}{1+t}$.

Since, $\lim\limits_{s\to\infty} \int\limits_0^s \lambda(t)\; dt = 0$, we see from Proposition 2.2 that we cannot conclude whether the zero solution is stable or not.

However, the set $\overline{B((1,0),\ \frac{1}{2})}$ has the negative property and is compact. Using Corollary 3.1, the zero solution is globally asymptotically stable on $\mathbb{R}^2$.

Example 3.2 Consider $x^{'} = A(t)x$ where $t > 0$ and $A(t)$ is

$$\begin{bmatrix} 0 & -\frac{3}{(1+t)^2} \\ 0 & -\frac{5}{(1+t)} \end{bmatrix}.$$

The only equilibrium point is $(0,0)$. The maximum eigenvalue $\lambda(t)$ of $A + A^T$ is $\dfrac{-5+\sqrt{13}}{2(1+t)}$. $\lim\limits_{s\to\infty} \int\limits_0^s \lambda(t)\; dt = 0$.

Since, the set $\overline{B((1,1),\ \frac{1}{2})}$ has the negative property and is compact, by Corollary 3.1, the zero solution is globally asymptotically stable on $\mathbb{R}^2$.

References

[1] McCluskey, C. Connell. On using curvature to demonstrate stability. *Differ. Equ. Nonlinear Mech*, 2008, Art. ID 745242, 7 pp.

[2] Pressley, A. *Elementary Differential Geometry.* Springer-Verlag, London, 2001.

[3] Raghavendra, V. and Deo, S. *Ordinary Differential Equations and Stability Theory.* Tata McGraw Hill, New Delhi, 1980.