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Representation of the Solution for Linear System of Delay Equations with Distributed Parameters

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Abstract: The first boundary value problem for an autonomous system of linear delay partial differential equations of the second order has been solved. The solution is presented in an analytical form of formal series for the case, when matrices of coefficients are commutative and their eigenvalues are real and different. The obtained solution is studied on convergence and differentiability.

Keywords: *delay partial differential equation; first boundary value problem; time delay argument.*

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1 Introduction

Usually, when systems of differential equations are investigated, the main attention is paid to systems of ordinary differential equations (e.g., [1, 2]) or systems of partial differential equations [3]– [7]. Aside remains the analysis of systems of partial differential equations with delay. Their investigation is extremely rare [8]– [10].

Autonomous second-order systems of linear differential equations of with constant delay are considered in this paper:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = a_{11}\frac{\partial^2 u(x,t)}{\partial x^2} + a_{12}\frac{\partial^2 v(x,t)}{\partial x^2} + b_{11}u\left(x,t-\tau\right) + b_{12}v\left(x,t-\tau\right), \\ \frac{\partial v(x,t)}{\partial t} = a_{21}\frac{\partial^2 u(x,t)}{\partial x^2} + a_{22}\frac{\partial^2 v(x,t)}{\partial x^2} + b_{21}u\left(x,t-\tau\right) + b_{22}v\left(x,t-\tau\right). \end{cases} \tag{1}$$

We assume that matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

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are normal, i.e. $AA^* = A^*A$, $BB^* = B^*B$, where A^* is the conjugate transpose of A, B^* is the conjugate transpose of B; and they satisfy the commutativity condition, i.e.,

$$AB = BA.$$

Functions $u\,(x,t)$, $v\,(x,t)$ are defined in a semistrip $t \geq -\tau$, $0 \leq x \leq l$, where l is a positive constant, and the initial and boundary conditions are

$$\begin{aligned} u\,(0,t) = \mu_1\,(t)\,, u\,(l,t) = \mu_2\,(t)\,, v\,(0,t) = \theta_1\,(t)\,, v\,(l,t) = \theta_2\,(t)\,, t \geq -\tau, \\ u\,(x,t) = \varphi\,(x,t)\,, v\,(x,t) = \psi\,(x,t)\,, 0 \leq x \leq l, -\tau \leq t \leq 0. \end{aligned} \tag{2}$$

Compatibility conditions are fulfilled:

$$\mu_1\,(t) = \varphi\,(0,t)\,, \mu_2\,(t) = \varphi\,(l,t)\,, \theta_1\,(t) = \psi\,(0,t)\,, \theta_2\,(t) = \psi\,(l,t)\,, -\tau \leq t \leq 0.$$

A solution of the first boundary value problem has been obtained for the case, when eigenvalues of the matrices A and B are real and different.

2 Representation of Solution for Delay System

If the matrices A and B are normal and satisfy the commutativity condition, then, according to [11]– [13], there always exists a nonsingular matrix S, which simultaneously reduces matrices A and B to the Jordan forms Λ_1 and Λ_2:

$$\begin{aligned} & S^{-1}AS = \Lambda_1, \quad S^{-1}BS = \Lambda_2, \\ & S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}, \; S^{-1} = \frac{1}{\Delta} \begin{bmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{bmatrix}, \; \Delta = s_{11}s_{22} - s_{12}s_{22}. \end{aligned} \tag{3}$$

Therefore by a transformation

$$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = S \begin{pmatrix} \xi(x,t) \\ \eta(x,t) \end{pmatrix}$$

system (1) can be reduced to a form

$$\begin{pmatrix} \frac{\partial \xi(x,t)}{\partial t} \\ \frac{\partial \eta(x,t)}{\partial t} \end{pmatrix} = \Lambda_1 \begin{pmatrix} \frac{\partial^2 \xi(x,t)}{\partial x^2} \\ \frac{\partial^2 \eta(x,t)}{\partial x^2} \end{pmatrix} + \Lambda_2 \begin{pmatrix} \xi\,(x,t-\tau) \\ \eta\,(x,t-\tau) \end{pmatrix}, \tag{4}$$

where Λ_1 is the Jordan form of the matrix A and Λ_2 is the Jordan form of the matrix B. The initial and boundary conditions will be

$$\begin{aligned} \xi\,(0,t) = \overline{\mu}_1\,(t)\,, \xi\,(l,t) = \overline{\mu}_2\,(t)\,, \eta\,(0,t) = \overline{\theta}_1\,(t)\,, \eta\,(l,t) = \overline{\theta}_2\,(t)\,, t \geq -\tau, \\ \xi\,(x,t) = \overline{\varphi}\,(x,t)\,, \eta\,(x,t) = \overline{\psi}\,(x,t)\,, 0 \leq x \leq l, -\tau \leq t \leq 0, \end{aligned} \tag{5}$$

where

$$\begin{pmatrix} \overline{\mu}_1\,(t) \\ \overline{\theta}_1\,(t) \end{pmatrix} = S^{-1} \begin{pmatrix} \mu_1\,(t) \\ \theta_1\,(t) \end{pmatrix}, \begin{pmatrix} \overline{\mu}_2\,(t) \\ \overline{\theta}_2\,(t) \end{pmatrix} = S^{-1} \begin{pmatrix} \mu_2\,(t) \\ \theta_2\,(t) \end{pmatrix},$$

$$\begin{pmatrix} \overline{\varphi}\,(x,t) \\ \overline{\psi}\,(x,t) \end{pmatrix} = S^{-1} \begin{pmatrix} \varphi\,(x,t) \\ \psi\,(x,t) \end{pmatrix}.$$

We will consider the representation of solution of the first boundary value problem for the system (1), (2), when roots of the characteristic equations λ_1, λ_2, ς_1, ς_2 of the both matrices A and B are real and different, i.e. $\lambda_1 \neq \lambda_2$, $\varsigma_1 \neq \varsigma_2$. In this case, after the transformation the system (4) decouples into two independent equations:

$$\frac{\partial \xi(x,t)}{\partial t} = \lambda_1 \frac{\partial^2 \xi(x,t)}{\partial x^2} + \varsigma_1 \xi\left(x, t-\tau\right), \quad \frac{\partial \eta(x,t)}{\partial t} = \lambda_2 \frac{\partial^2 \eta(x,t)}{\partial x^2} + \varsigma_2 \eta\left(x, t-\tau\right). \tag{6}$$

We will consider the first equation of system (6)

$$\frac{\partial \xi(x,t)}{\partial t} = \lambda_1 \frac{\partial^2 \xi(x,t)}{\partial x^2} + \varsigma_1 \xi\left(x, t-\tau\right) \tag{7}$$

with initial and boundary conditions

$$\xi\left(0,t\right) = \overline{\mu}_1\left(t\right), \xi\left(l,t\right) = \overline{\mu}_2\left(t\right), t \geq -\tau, \xi\left(x,t\right) = \overline{\varphi}\left(x,t\right), 0 \leq x \leq l, -\tau \leq t \leq 0.$$

A solution will be in the form

$$\xi\left(x,t\right) = \xi_0\left(x,t\right) + \xi_1\left(x,t\right) + \overline{\mu}_1\left(t\right) + \frac{x}{l}\left[\overline{\mu}_2\left(t\right) - \overline{\mu}_1\left(t\right)\right], \tag{8}$$

where

- $\xi_0\left(x,t\right)$ is a solution of homogeneous equation

$$\frac{\partial \xi(x,t)}{\partial t} = \lambda_1 \frac{\partial^2 \xi(x,t)}{\partial x^2} + \varsigma_1 \xi\left(x, t-\tau\right) \tag{9}$$

with zero boundary $\xi\left(0,t\right) = 0$, $\xi\left(l,t\right) = 0$ and nonzero initial conditions $\xi\left(x,t\right) = \Phi\left(x,t\right)$, $\Phi\left(x,t\right) = \bar{\varphi}\left(x,t\right) - \bar{\mu}_1\left(t\right) - \frac{x}{l}\left[\bar{\mu}_2\left(t\right) - \bar{\mu}_1\left(t\right)\right]$, $-\tau \leq t \leq 0$, $0 \leq x \leq l$.

- $\xi_1\left(x,t\right)$ is a solution of inhomogeneous equation

$$\frac{\partial \xi(x,t)}{\partial t} = \lambda_1 \frac{\partial^2 \xi(x,t)}{\partial x^2} + \varsigma_1 \xi\left(x, t-\tau\right) + F\left(x,t\right), \tag{10}$$

$$F\left(x,t\right) = \varsigma_1 \left\{\overline{\mu}_1\left(t-\tau\right) + \frac{x}{l}\left[\overline{\mu}_2\left(t-\tau\right) - \overline{\mu}_1\left(t-\tau\right)\right]\right\} - \dot{\overline{\mu}}_1\left(t\right) - \frac{x}{l}\left[\dot{\overline{\mu}}_2\left(t\right) - \dot{\overline{\mu}}_1\left(t\right)\right]$$

with zero boundary $\xi\left(0,t\right) = 0$, $\xi\left(l,t\right) = 0$, $t \geq -\tau$ and zero initial conditions $\xi\left(x,t\right) = 0$, $-\tau \leq t \leq 0$, $0 \leq x \leq l$.

2.1 Homogeneous equation

For finding the solution $\xi_0\left(x,t\right)$ we will use the method of separation of variables. According to this method, the solution will be in a form of product of two functions $\xi_0\left(x,t\right) = X\left(x\right)T\left(t\right)$. After substitution in the equation (7) we obtain

$$X\left(x\right)T'(t) = \lambda_1 X''\left(x\right)T\left(t\right) + \varsigma_1 X\left(x\right)T\left(t-\tau\right).$$

Separating variables, we have

$$\frac{T'\left(t\right) - \varsigma_1 T\left(t-\tau\right)}{\lambda_1 T\left(t\right)} = \frac{X''\left(x\right)}{X\left(x\right)} = -k^2,$$

where k is an arbitrary constant. We will divide the obtained expression into two equations

$$T'(t)+\lambda_1 k^2 T(t)-\varsigma_1 T(t-\tau)=0,\ \ X''(x)+k^2X(x)=0. \tag{11}$$

Solutions of the second equation from (11), which is not identically zero and satisfies zero boundary conditions $X(0)=0$, $X(l)=0$, are

$$X_n(x)=A_n\sin\frac{\pi n}{l}x,\ k_n^2=\left(\frac{\pi n}{l}\right)^2,\ n=1,2,...$$

where A_n are arbitrary constants.

Now we will consider the first of equations from (11)

$$T_n'(t)=-\lambda_1\left(\frac{\pi n}{l}\right)^2 T_n(t)+\varsigma_1 T_n(t-\tau),\ n=1,2,\dots. \tag{12}$$

To obtain initial conditions for each of the equations (12) we will expand the corresponding initial condition $\Phi(x,t)$ into series under solutions of the second equation

$$\Phi(x,t)=\sum_{n=1}^{\infty}\Phi_n(t)\sin\frac{\pi n}{l}x, \tag{13}$$

$$\Phi_n(t)=\frac{2}{l}\int_0^l\overline{\varphi}(s,t)\sin\frac{\pi n}{l}sds+\frac{2}{\pi n}\left[(-1)^n\bar{\mu}_2(t)-\bar{\mu}_1(t)\right],n=1,2,...$$

Preliminary we should consider some results on linear homogeneous equations with constant delay

$$\dot{x}(t)=bx(t-\tau) \tag{14}$$

with an initial condition $x(t)=\beta(t)$, $-\tau\le t\le 0$, $b\in\mathbb{R}$.

Definition 2.1 [14] A delay exponential function $\exp_\tau\{b,t\}$ is a function which can be written as

$$\exp_\tau\{b,t\}=\begin{cases}0, & \text{if}\quad -\infty<t<-\tau,\\ 1, & \text{if}\quad -\tau\le t<0,\\ 1+b\frac{t}{1!}, & \text{if}\quad 0\le t<\tau,\\ \dots & \\ 1+b\frac{t}{1!}+b^2\frac{(t-\tau)^2}{2!}+...+b^k\frac{[t-(k-1)\tau]^k}{k!}, & \text{if}\quad (k-1)\tau\le t<k\tau,\end{cases} \tag{15}$$

a k-degree polynomial on intervals $(k-1)\tau<t\le k\tau$ "merged" in points $t=k\tau$, $k=0,1,2,...$, $b=const$.

Lemma 2.1 *A rule of differentiation for the delay exponential function can be formulated in the following way:*

$$\frac{d}{dt}\exp_\tau\{b,t\}=b\exp_\tau\{b,t-\tau\}. \tag{16}$$

I.e., the delay exponential function is a solution of the equation (14) with unitary initial conditions $x(t)\equiv 1$, $-\tau\le t\le 0$.

Proof. Within an interval $(k-1)\tau < t \leq k\tau$ the delay exponential function is represented as follows

$$\exp_\tau\{b,t\} = 1 + b\frac{t}{1!} + b^2\frac{(t-\tau)^2}{2!} + b^3\frac{(t-2\tau)^3}{3!} + ... + b^k\frac{[t-(k-1)\tau]^k}{k!}.$$

Differentiating this function we will obtain

$$\frac{d}{dt}\exp_\tau\{b,t\} = b + b^2\frac{t-\tau}{1!} + b^3\frac{(t-2\tau)^2}{2!} + b^4\frac{(t-3\tau)^3}{3!} + ... + b^k\frac{[t-(k-1)\tau]^{k-1}}{(k-1)!} =$$

$$= b\left\{1 + b\frac{t-\tau}{1!} + b^2\frac{(t-2\tau)^2}{2!} + b^3\frac{(t-3\tau)^3}{3!} + ... + b^{k-1}\frac{[t-(k-1)\tau]^{k-1}}{(k-1)!}\right\} =$$

$$= b\exp_\tau\{b,t-\tau\},$$

Q.E.D. □

Theorem 2.1 *A solution of the equation (14), which satisfies the initial condition $x(t) = \beta(t)$, $-\tau \leq t \leq 0$, can be presented as follows*

$$x(t) = \exp_\tau\{b,t\}\beta(-\tau) + \int_{-\tau}^{0}\exp_\tau\{b,t-\tau-s\}\beta'(s)\,ds. \tag{17}$$

Proof. As the expression (17) is a linear functional of the delay exponential function $\exp_\tau\{b,t\}$ which, as it was shown in Lemma 2.1, is the solution of the equation (14), then the functional (17) is a solution of the homogeneous equation (14) for any function $\beta(t)$. We will show that initial conditions are satisfied, i.e. for $-\tau \leq t \leq 0$ the following identity is correct:

$$\beta(t) \equiv \exp_\tau\{b,t\}\beta(-\tau) + \int_{-\tau}^{0}\exp_\tau\{b,t-\tau-s\}\beta'(s)\,ds.$$

Then we will divide an integral from the expression (17) into two integrals:

$$x(t) = \exp_\tau\{b,t\}\beta(-\tau) + \int_{-\tau}^{t}\exp_\tau\{b,t-\tau-s\}\beta'(s)\,ds +$$

$$+ \int_{t}^{0}\exp_\tau\{b,t-\tau-s\}\beta'(s)\,ds.$$

Using the definition of the delay exponential function, we can obtain that

- $\exp_\tau\{b,t\} \equiv 1$ at $-\tau \leq t \leq 0$;
- $\exp_\tau\{b,t-\tau-s\} \equiv 1$ at $-\tau \leq s \leq t$;
- $\exp_\tau\{b,t-\tau-s\} \equiv 0$ at $t < s \leq 0$.

Therefore,

$$x(t) = \beta(-\tau) + \int_{-\tau}^{t}\beta'(s)\,ds = \beta(-\tau) + \beta(t) - \beta(-\tau) = \beta(t),$$

Q.E.D. □

Remark 2.1 Under the hypothesis of the theorem, continuous differentiability of the initial function $\beta(t)$ is required. Computing the integral in (17) by parts we obtain

$$x\left(t\right)=\exp_{\tau}\left\{b,t-\tau\right\}\beta\left(0\right)+b\int_{-\tau}^{0}\exp_{\tau}\left\{b,t-2\tau-s\right\}\beta\left(s\right)ds. \tag{18}$$

The equality (18) is an integral representation of the solution under the assumption of only continuity of the function $\beta(t)$.

Further we will consider the differential equation

$$\dot{x}\left(t\right)=ax\left(t\right)+bx\left(t-\tau\right) \tag{19}$$

with an initial condition $x\left(t\right)=\beta\left(t\right)$, $-\tau\leq t\leq 0$, $a,b\in\mathbb{R}$.

Theorem 2.2 *A solution of the equation (19), which satisfies initial condition* $x\left(t\right)=\beta\left(t\right)$, $-\tau\leq t\leq 0$, *can be presented as*

$$\begin{gathered}x\left(t\right)=\exp_{\tau}\left\{b_1,t\right\}e^{a(t+\tau)}\beta\left(-\tau\right)+\int_{-\tau}^{0}\exp_{\tau}\left\{b_1,t-\tau-s\right\}e^{a(t-s)}\left[\beta'\left(s\right)-a\beta\left(s\right)\right]ds,\\ b_1=be^{-a\tau}.\end{gathered} \tag{20}$$

Proof. We will make a substitution $x\left(t\right)=e^{at}y\left(t\right)$, where $y\left(t\right)$ is a new unknown function

$$ae^{at}y\left(t\right)+e^{at}\dot{y}\left(t\right)=ae^{at}y\left(t\right)+be^{a(t-\tau)}y\left(t-\tau\right),$$

$$\dot{y}\left(t\right)=b_1y\left(t-\tau\right),\quad b_1=be^{-a\tau}. \tag{21}$$

Correspondingly, the initial condition for the equation (21) is

$$y\left(t\right)=e^{-at}\beta\left(t\right).$$

As follows from (17) a solution of the corresponding Cauchy problem for the equation (21) will be

$$y\left(t\right)=\exp_{\tau}\left\{b_1,t\right\}e^{a\tau}\beta\left(-\tau\right)+\int_{-\tau}^{0}\exp_{\tau}\left\{b_1,t-\tau-s\right\}\left[e^{-as}\beta'\left(s\right)-ae^{-as}\beta\left(s\right)\right]ds.$$

Again, using a substitution $x\left(t\right)=e^{at}y\left(t\right)$, we obtain

$$x\left(t\right)=\exp_{\tau}\left\{b_1,t\right\}e^{a(t+\tau)}\beta\left(-\tau\right)+\int_{-\tau}^{0}\exp_{\tau}\left\{b_1,t-\tau-s\right\}e^{a(t-s)}\left[\beta'\left(s\right)-a\beta\left(s\right)\right]ds,$$

i.e. the statement of Theorem 2.2. $\square$

Using the results obtained above, we will solve each of the equations (12). According to the equality (20), solutions of (12) will be

$$\begin{aligned}T_n\left(t\right)=&\exp_{\tau}\left\{r_1,t\right\}e^{q_1(t+\tau)}\Phi_n\left(-\tau\right)+\\&+\int_{-\tau}^{0}\exp_{\tau}\left\{r_1,t-\tau-s\right\}e^{q_1(t-s)}\left[\Phi_n'\left(s\right)-q_1\Phi_n\left(s\right)\right]ds,\end{aligned}$$

$$r_1 = \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, q_1 = -\lambda_1 \left(\frac{\pi n}{l}\right)^2.$$

Thus, the solution $\xi_0(x,t)$ of the homogeneous equation (9), which satisfies zero boundary $\xi(0,t) = 0$, $\xi(l,t) = 0$ and nonzero initial conditions $\xi(x,t) = \Phi(x,t)$, $-\tau \leq t \leq 0$, $0 \leq x \leq l$, is

$$\xi_0(x,t) = \sum_{n=1}^{\infty} \left\{ \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t \right\} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (t+\tau)} \Phi_n(-\tau) + \right.$$

$$\left. + \int_{-\tau}^{0} \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t - \tau - s \right\} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (t-s)} \left[\Phi_n'(s) + \lambda_1 \left(\frac{\pi n}{l}\right)^2 \Phi_n(s) \right] ds \right\} \times$$

$$\times \sin \frac{\pi n}{l} x,$$

$$\Phi_n(t) = \frac{2}{l} \int_0^l \overline{\varphi}(s,t) \sin \frac{\pi n}{l} s ds + \frac{2}{\pi n} \left[(-1)^n \bar{\mu}_2(t) - \bar{\mu}_1(t) \right], n = 1, 2, \ldots$$

2.2 Inhomogeneous equation

Further we will consider the inhomogeneous equation (10)

$$\frac{\partial \xi(x,t)}{\partial t} = \lambda_1 \frac{\partial^2 \xi(x,t)}{\partial x^2} + \varsigma_1 \xi(x, t-\tau) + F(x,t),$$

$$F(x,t) = \varsigma_1 \left\{ \overline{\mu}_1(t-\tau) + \frac{x}{l} \left[\overline{\mu}_2(t-\tau) - \overline{\mu}_1(t-\tau) \right] \right\} - \dot{\overline{\mu}}_1(t) - \frac{x}{l} \left[\dot{\overline{\mu}}_2(t) - \dot{\overline{\mu}}_1(t) \right]$$

with zero boundary $\xi(0,t) = 0$, $\xi(l,t) = 0$, $t \geq -\tau$ and zero initial conditions $\xi(x,t) = 0$, $-\tau \leq t \leq 0$, $0 \leq x \leq l$. We will try to find a solution in the form of series expansion in terms of the functions from the previous problem, i.e. in the form

$$\xi_1(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{\pi n}{l} x.$$

After substituting the series in the equation (10) and having equated coefficients of the same terms, we obtain a system of the equations

$$T_n'(t) = -\lambda_1 \left(\frac{\pi n}{l}\right)^2 T_n(t) + \varsigma_1 T_n(t-\tau) + f_n(t), \quad n = 1, 2, \ldots, \tag{22}$$

where

$$f_n(t) = \frac{2}{l} \int_0^l F(s,t) \sin \frac{\pi n}{l} s ds =$$

$$= \frac{2}{\pi n} \left[\varsigma_1 \left((-1)^{n+1} \overline{\mu}_2(t-\tau) + \overline{\mu}_1(t-\tau) \right) - \left((-1)^{n+1} \dot{\overline{\mu}}_2(t) + \dot{\overline{\mu}}_1(t) \right) \right], n = 1, 2, \ldots.$$

Preliminary we will consider a linear inhomogeneous equation with a constant delay:

$$\dot{x}(t) = a x(t) + b x(t-\tau) + f(t). \tag{23}$$

We will solve the Cauchy problem for (22) with a zero initial condition $x(t) \equiv 0$, $-\tau \leq t \leq 0$, where $a, b \in \mathbb{R}$, $f : [0, \infty) \to \mathbb{R}$.

Theorem 2.3 *A solution of the inhomogeneous equation (23), which satisfies zero initial conditions* $x(t) \equiv 0$, $-\tau \le t \le 0$, *will be*

$$x(t) = \int_0^t \exp_\tau \{b_1, t-\tau-s\} e^{a(t-s)} f(s)\, ds, \; b_1 = be^{-a\tau}. \tag{24}$$

Proof. As in the previous case, we apply the substitution $x(t) = e^{at} y(t)$ and obtain a differential equation

$$ae^{at} y(t) + e^{at} \dot{y}(t) = ae^{at} y(t) + be^{a(t-\tau)} y(t-\tau) + f(t).$$

It will be adduced to

$$\dot{y}(t) = b_1 y(t-\tau) + e^{-at} f(t), \; b_1 = be^{-a\tau}. \tag{25}$$

We will show that the solution of the inhomogeneous equation (25), which satisfies zero initial condition, is

$$y(t) = \int_0^t \exp_\tau \{b_1, t-\tau-s\} e^{-as} f(s)\, ds. \tag{26}$$

Substituting (26) in the equation (25)

$$\exp_\tau \{b_1, t-\tau-s\} e^{-as} f(s)\big|_{s=t} + b_1 \int_0^t \exp_\tau \{b_1, t-2\tau-s\} e^{-as} f(s)\, ds =$$

$$= b_1 \int_0^{t-\tau} \exp_\tau \{b_1, t-2\tau-s\} e^{-as} f(s)\, ds + e^{-at} f(t),$$

considering that

$$\exp_\tau \{b_1, t-\tau-s\} e^{-as} f(s)\big|_{s=t} = \exp\{b_1, -\tau\} e^{-at} f(t) = e^{-at} f(t),$$

and dividing the second integral into two, we obtain

$$e^{-at} f(t) + b_1 \left(\int_0^{t-\tau} \exp_\tau \{b_1, t-2\tau-s\} e^{-as} f(s)\, ds \right) +$$

$$+ b_1 \left(\int_{t-\tau}^t \exp_\tau \{b_1, t-2\tau-s\} e^{-as} f(s)\, ds \right) =$$

$$= b_1 \left(\int_0^{t-\tau} \exp_\tau \{b_1, t-2\tau-s\} e^{-as} f(s)\, ds \right) + e^{-at} f(t).$$

Hence

$$e^{-at} f(t) + b_1 \left(\int_{t-\tau}^t \exp_\tau \{b_1, t-2\tau-s\} e^{-as} f(s)\, ds \right) = e^{-at} f(t).$$

After substitution

$$t-2\tau-s = \omega, s = t-\tau \Rightarrow \omega = -\tau, s = t \Rightarrow \omega = -2\tau$$

we obtain

$$be^{-a\tau}\left(\int_{-2\tau}^{-\tau}\exp_\tau\{b_1,\omega\}\,e^{a(\omega-t+2\tau)}f\,(t-2\tau-\omega)\,d\omega\right)=0$$

and identity

$$e^{-at}f\,(t)=e^{-at}f\,(t)\,,$$

which proves correctness of the equality (26). Hence

$$x\,(t)=e^{at}y\,(t)=\int_0^t\exp_\tau\{b_1,t-\tau-s\}\,e^{a(t-s)}f\,(s)\,ds,$$

Q.E.D. □

Corollary 2.1 *A solution of the inhomogeneous equation (23) with initial condition* $x\,(t)\equiv\beta\,(t)$, $-\tau\le t\le 0$ *is*

$$\begin{aligned}x\,(t)=&\exp_\tau\{b_1,t\}\,e^{a(t+\tau)}\beta\,(-\tau)+\int_{-\tau}^{0}\exp_\tau\{b_1,t-\tau-s\}\,e^{a(t-s)}\left[\beta'\,(s)-a\beta\,(s)\right]ds+\\&+\int_0^t\exp_\tau\{b_1,t-\tau-s\}\,e^{a(t-s)}f\,(s)\,ds,\;\; b_1=be^{-a\tau}.\end{aligned}\tag{27}$$

Proof. Proof is based on statements of the previous Theorems 2.2 and 2.3. □

Using the results obtained above, a solution of each of the equations (22)

$$T_n'\,(t)=-\lambda_1\left(\frac{\pi n}{l}\right)^2T_n\,(t)+\varsigma_1T_n\,(t-\tau)+f_n\,(t)\,,n=1,2,...$$

can be written as

$$T_n\,(t)=\int_0^t\exp_\tau\{r_1,t-\tau-s\}\,e^{q_1(t-s)}f_n\,(s)\,ds,\;r_1=\varsigma_1e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\;q_1=-\lambda_1\left(\frac{\pi n}{l}\right)^2.\tag{28}$$

Hence, a solution of the inhomogeneous equation (10) with zero boundary $\xi\,(0,t)=0$, $\xi\,(l,t)=0$, $t\ge-\tau$ and zero initial conditions $\xi\,(x,t)=0$, $-\tau\le t\le 0$, $0\le x\le l$, is

$$\xi_1\,(x,t)=\sum_{n=1}^{\infty}\left\{\int_0^t\exp_\tau\left\{\varsigma_1e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t-\tau-s\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)}f_n\,(s)\,ds\right\}\sin\frac{\pi n}{l}x,$$

$$f_n\,(t)=\frac{2}{\pi n}\left[\varsigma_1\left((-1)^{n+1}\overline{\mu}_2\,(t-\tau)+\overline{\mu}_1\,(t-\tau)\right)-\left((-1)^{n+1}\dot{\overline{\mu}}_2\,(t)+\dot{\overline{\mu}}_1\,(t)\right)\right],$$

$n=1,2,\ldots.$

2.3 General solution

Using all previous results, the solution of the first boundary value problem for the equation (7) can be written in the form of sum:

$$
\begin{aligned}
\xi(x,t) = \sum_{n=1}^{\infty} \Big\{ & \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t \right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 (t+\tau)} \Phi_n(-\tau) + \\
& + \int_{-\tau}^{0} \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t-\tau-s \right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 (t-s)} \left[\Phi_n'(s) + \lambda_1 \left(\frac{\pi n}{l}\right)^2 \Phi_n(s) \right] ds + \\
& + \int_{0}^{t} \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t-\tau-s \right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 (t-s)} f_n(s)\, ds \Big\} \sin \frac{\pi n}{l} x + \\
& + \overline{\mu}_1(t) + \frac{x}{l} \left[\overline{\mu}_2(t) - \overline{\mu}_1(t) \right],
\end{aligned}
\tag{29}
$$

$$
\Phi_n(t) = \frac{2}{l} \int_0^l \overline{\varphi}(s,t) \sin \frac{\pi n}{l} s ds + \frac{2}{\pi n} \left[(-1)^n \bar{\mu}_2(t) - \bar{\mu}_1(t) \right], n = 1, 2, ...,
$$

$$
f_n(t) = \frac{2}{\pi n} \left[\varsigma_1 \left((-1)^{n+1} \overline{\mu}_2(t-\tau) + \overline{\mu}_1(t-\tau) \right) - \left((-1)^{n+1} \dot{\overline{\mu}}_2(t) + \dot{\overline{\mu}}_1(t) \right) \right].
$$

Similarly, the second equation from (6) has a solution:

$$
\begin{aligned}
\eta(x,t) = \sum_{n=1}^{\infty} \Big\{ & \exp_\tau \left\{ \varsigma_2 e^{\lambda_2 \left(\frac{\pi n}{l}\right)^2 \tau}, t \right\} e^{-\lambda_2\left(\frac{\pi n}{l}\right)^2 (t+\tau)} \Psi_n(-\tau) + \\
& + \int_{-\tau}^{0} \exp_\tau \left\{ \varsigma_2 e^{\lambda_2 \left(\frac{\pi n}{l}\right)^2 \tau}, t-\tau-s \right\} e^{-\lambda_2\left(\frac{\pi n}{l}\right)^2 (t-s)} \left[\Psi_n'(s) + \lambda_2 \left(\frac{\pi n}{l}\right)^2 \Psi_n(s) \right] ds + \\
& + \int_{0}^{t} \exp_\tau \left\{ \varsigma_2 e^{\lambda_2 \left(\frac{\pi n}{l}\right)^2 \tau}, t-\tau-s \right\} e^{-\lambda_2\left(\frac{\pi n}{l}\right)^2 (t-s)} g_n(s)\, ds \Big\} \sin \frac{\pi n}{l} x + \\
& + \overline{\theta}_1(t) + \frac{x}{l} \left[\overline{\theta}_2(t) - \overline{\theta}_1(t) \right],
\end{aligned}
\tag{30}
$$

$$
\Psi_n(t) = \frac{2}{l} \int_0^l \overline{\psi}(s,t) \sin \frac{\pi n}{l} s ds + \frac{2}{\pi n} \left[(-1)^n \bar{\theta}_2(t) - \bar{\theta}_1(t) \right], n = 1, 2, ...
$$

$$
g_n(t) = \frac{2}{\pi n} \left[\varsigma_2 \left((-1)^{n+1} \overline{\theta}_2(t-\tau) + \overline{\theta}_1(t-\tau) \right) - \left((-1)^{n+1} \dot{\overline{\theta}}_2(t) + \dot{\overline{\theta}}_1(t) \right) \right].
$$

Then solutions of the boundary value problem of the initial system (1) with the conditions (2) finally are

$$
u(x,t) = s_{11} \xi(x,t) + s_{12} \eta(x,t), \ v(x,t) = s_{21} \xi(x,t) + s_{22} \eta(x,t), \tag{31}
$$

where the solutions $\xi(x,t)$, $\eta(x,t)$ of the reduced system (6) are defined in (29), (30), s_{ij}, $i,j = \overline{1,2}$ are the coefficients of the matrix of transformation S.

3 Existence Conditions for Solutions

The solution of the first boundary value problem of the equations (6) is presented in the form of formal series (29), (30). We will show that when certain conditions are satisfied the series converge and the representations are really the solutions of system of delay partial differential equations.

We will consider the first equations (7).

Theorem 3.1 *Let the functions $\Phi_n(t)$, $-\tau \le t \le 0$ and $f_n(t)$, $t \ge 0$, defined in (13), (22), satisfy the conditions*

$$\begin{aligned} &\lim_{n\to+\infty} \max_{-\tau\le t\le T-\tau} |f_n(t)|\, e^{-\lambda_2\left(\frac{\pi n}{l}\right)^2 (T-(k-1)\tau)} = 0,\\ &\lim_{n\to+\infty} e^{-\lambda_2\left(\frac{\pi n}{l}\right)^2 (T-(k-1)\tau)} \max_{-\tau\le t\le 0} |\Phi_n(t)| = 0 \end{aligned} \tag{32}$$

on an interval $(k-1)\tau \le T < k\tau$. Then the expression (29) is a solution of the equation (7) for t: $0 \le t \le T$. And the function $\xi(x,t)$ has a continuous first-order derivative with respect to t and a second-order derivative with respect to x.

Proof. We will write the representation (29) as a sum of three terms:

$$\xi(x,t) = S_1(x,t) + S_2(x,t) + S_3(x,t) + \overline{\mu}_1(t) + \frac{x}{l}\left[\overline{\mu}_2(t) - \overline{\mu}_1(t)\right], \tag{33}$$

where

$$S_1(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin\frac{\pi n}{l}x,\ S_2(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\frac{\pi n}{l}x,\ S_3(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin\frac{\pi n}{l}x,$$

$$A_n(t) = \exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, t\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t+\tau)}\Phi_n(-\tau),$$

$$B_n(t) = \int_{-\tau}^{0} \exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, t-\tau-s\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)} \times$$

$$\times \left[\Phi_n'(s) + \lambda_1\left(\frac{\pi n}{l}\right)^2 \Phi_n(s)\right] ds,$$

$$C_n(t) = \int_0^t \exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, t-\tau-s\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)} f_n(s)\, ds.$$

1. Firstly we will consider coefficients $A_n(t)$, $n = 1, 2, ...$ of the first series $S_1(x,t)$. As follows from the definition of delay exponential function, formulated in (15), for any moment of time $T : (k-1)\tau \le T < k\tau$, $k = 0, 1, 2, ...$ the following equality holds

$$A_n(T) = \exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, T\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T+\tau)}\Phi_n(-\tau) = e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T+\tau)}\Phi_n(-\tau) \times$$

$$\times \left[1 + \varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{T}{1!} + \varsigma_1^2 e^{2\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-\tau]}{2!} + \varsigma_1^3 e^{3\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-2\tau]^3}{3!} + \right.$$

$$\left.+...+\varsigma_1^k e^{k\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\left[T-(k-1)\tau\right]^k}{k!}\right].$$

Hence

$$S_1(x,T)=\sum_{n=1}^{\infty}A_n(T)\sin\frac{\pi n}{l}x=$$

$$=\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T+\tau)}\Phi_n(-\tau)\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{T}{1!}+\varsigma_1^2 e^{2\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-\tau]}{2!}+\right.$$

$$\left.+\varsigma_1^3 e^{3\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-2\tau]^3}{3!}+\ldots+\varsigma_1^k e^{k\left(\frac{\pi n}{l}\right)^2\tau}\frac{\left[T-(k-1)\tau\right]^k}{k!}\right]\sin\frac{\pi n}{l}x=$$

$$=\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T+\tau)}\Phi_n(-\tau)\sin\frac{\pi n}{l}x+\varsigma_1\frac{T}{1!}\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 T}\Phi_n(-\tau)\sin\frac{\pi n}{l}x+$$

$$+\varsigma_1^2\frac{[T-\tau]}{2!}\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-\tau)}\Phi_n(-\tau)\sin\frac{\pi n}{l}x+\ldots+$$

$$+\varsigma_1^k\frac{\left[T-(k-1)\tau\right]^k}{k!}\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-(k-1)\tau)}\Phi_n(-\tau)\sin\frac{\pi n}{l}x.$$

And if coefficients $\Phi_n(-\tau)$ are such that the following condition is satisfied

$$\lim_{n\to\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-(k-1)\tau)}\left|\Phi_n(-\tau)\right|=0,$$

then the series $S_1(x,t)$ converges absolutely and uniformly.

2. We will consider coefficients $B_n(t)$, $n=1,2,3,...$ of the second series $S_2(x,t)$. We will divide the integral into two and calculate the second integral by parts:

$$B_n(t)=\lambda_1\left(\frac{\pi n}{l}\right)^2\int_{-\tau}^{0}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t-\tau-s\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)}\Phi_n(s)\,ds+$$

$$+\int_{-\tau}^{0}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t-\tau-s\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)}\Phi_n'(s)\,ds=$$

$$=\lambda_1\left(\frac{\pi n}{l}\right)^2\int_{-\tau}^{0}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t-\tau-s\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)}\Phi_n(s)\,ds+$$

$$+\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t-\tau\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 t}\Phi_n(0)-$$

$$-\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t+\tau)}\Phi_n(-\tau)+$$

$$+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\int_{-\tau}^{0}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t-2\tau-s\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)}\Phi_n(s)\,ds-$$

$$-\lambda_1\left(\frac{\pi n}{l}\right)^2\int_{-\tau}^{0}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t-\tau-s\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)}\Phi_n(s)\,ds=$$

$$=\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t-\tau\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 t}\Phi_n(0)-$$

$$-\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t+\tau)}\Phi_n\left(-\tau\right)+$$

$$+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\int_{-\tau}^{0}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t-2\tau-s\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)}\Phi_n\left(s\right)ds=$$

$$=B_{n1}\left(t\right)-B_{n2}\left(t\right)+B_{n3}\left(t\right).$$

Now we will consider the first series

$$B_{n1}\left(t\right)=\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t-\tau\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 t}\Phi_n\left(0\right).$$

By analogy with the previous case, for any moment of time $T:(k-2)\,\tau\le T<(k-1)\,\tau$ the following holds:

$$B_{n1}\left(T\right)=\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},T\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 t}\Phi_n\left(0\right)=e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 T}\Phi_n\left(0\right)\times$$

$$\times\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{T}{1!}+\varsigma_1^2 e^{2\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-\tau]}{2!}+\varsigma_1^3 e^{3\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-2\tau]^3}{3!}+\right.$$

$$\left.+...+\varsigma_1^{(k-1)}e^{(k-1)\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-(k-2)\,\tau]^{(k-1)}}{(k-1)!}\right].$$

Hence

$$\sum_{n=1}^{\infty}B_{n1}\left(T\right)\sin\frac{\pi n}{l}x=\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 T}\Phi_n\left(0\right)\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{T}{1!}+\right.$$

$$+\varsigma_1^2 e^{2\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-\tau]}{2!}+\varsigma_1^3 e^{3\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-2\tau]^3}{3!}+$$

$$\left.+...+\varsigma_1^{(k-1)}e^{(k-1)\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-(k-2)\,\tau]^{(k-1)}}{(k-1)!}\right]\sin\frac{\pi n}{l}x=$$

$$=\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 T}\Phi_n\left(0\right)\sin\frac{\pi n}{l}x+\varsigma_1\frac{T}{1!}\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-\tau)}\Phi_n\left(0\right)\sin\frac{\pi n}{l}x+$$

$$+\varsigma_1^2\frac{[T-\tau]}{2!}\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-2\tau)}\Phi_n\left(0\right)\sin\frac{\pi n}{l}x+\ldots+$$

$$+\varsigma_1^{(k-1)}\frac{[T-(k-2)\,\tau]^{(k-1)}}{(k-1)!}\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-(k-1)\tau)}\Phi_n\left(0\right)\sin\frac{\pi n}{l}x.$$

And if coefficients $\Phi_n\left(0\right)$ are such that the following condition is satisfied

$$\lim_{n\to\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-(k-1)\tau)}\left|\Phi_n\left(0\right)\right|=0,$$

then the series $\sum_{n=1}^{\infty}B_{n1}\left(T\right)\sin\frac{\pi n}{l}x$ converges absolutely and uniformly.

We will consider the second series

$$B_{n2}\left(t\right)=\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t+\tau)}\Phi_n\left(-\tau\right)=A_n\left(t\right).$$

For any moment of time $T:(k-1)\tau \le T < k\tau$ the series $\sum_{n=1}^{\infty} B_{n2}(T)\sin\frac{\pi n}{l}x$ converges absolutely and uniformly if, as follows from the previous case, coefficients $\Phi_n(-\tau)$ are such that the following condition is satisfied

$$\lim_{n\to\infty} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-(k-1)\tau)}\left|\Phi_n(-\tau)\right| = 0,$$

Finally, for coefficients

$$B_{n3}(t) = \varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\int_{-\tau}^{0}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, t-2\tau-s\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)}\Phi_n(s)\,ds$$

at the moment of time $T:(k-1)\tau \le T < k\tau$, we make a substitution $T-2\tau-s=\omega$ and obtain:

$$B_{n3}(T) = \varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\int_{T-2\tau}^{T-\tau}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, \omega\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega+2\tau)}\Phi_n(T-2\tau-\omega)\,d\omega.$$

Dividing the integral into two we have:

$$B_{n3}(T) = \varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\int_{T-2\tau}^{(k-2)\tau}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, \omega\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega+2\tau)}\times$$
$$\times\Phi_n(T-2\tau-\omega)\,d\omega+$$
$$+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\int_{(k-2)\tau}^{T-\tau}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, \omega\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega+2\tau)}\Phi_n(T-2\tau-\omega)\,d\omega.$$

Therefore, owing to the mean value theorem, there exist values $\omega_1 : T-2\tau \le \omega_1 \le (k-2)\tau$, $\omega_2 : (k-2)\tau \le \omega_2 \le T-\tau$ for which the following holds:

$$B_{n3}(T) = \varsigma_1(k\tau - T)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_1+\tau)}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, \omega_1\right\}\Phi_n(T-2\tau-\omega_1)+$$
$$+\varsigma_1(T-(k-1)\tau)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_2+\tau)}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, \omega_2\right\}\Phi_n(T-2\tau-\omega_2).$$

Hence

$$B_{n3}(T) = \varsigma_1(k\tau - T)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_1+\tau)}\Phi_n(T-2\tau-\omega_1)\times$$
$$\times\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\omega_1}{1!}+\varsigma_1^2 e^{2\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[\omega_1-\tau]}{2!}+\varsigma_1^3 e^{3\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[\omega_1-2\tau]^3}{3!}+\right.$$
$$\left.+\ldots+\varsigma_1^{k-2}e^{(k-2)\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[\omega_1-(k-3)\tau]^{k-2}}{(k-2)!}\right]+$$
$$+\varsigma_1(T-(k-1)\tau)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_2+\tau)}\Phi_n(T-2\tau-\omega_2)\times$$
$$\times\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\omega_2}{1!}+\varsigma_1^2 e^{2\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[\omega_2-\tau]}{2!}+\varsigma_1^3 e^{3\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[\omega_2-2\tau]^3}{3!}+\right.$$
$$\left.+\ldots+\varsigma_1^{k-1}e^{(k-1)\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[\omega_2-(k-2)\tau]^{k-1}}{(k-1)!}\right].$$

And

$$\sum_{n=1}^{\infty} B_{n3}(T)\sin\frac{\pi n}{l}x = \varsigma_1 \sum_{n=1}^{\infty}\Big\{(k\tau - T)\, e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_1+\tau)}\Phi_n(T-2\tau-\omega_1) +$$

$$+ (T-(k-1)\tau)\, e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_2+\tau)}\Phi_n(T-2\tau-\omega_2)\Big\}\sin\frac{\pi n}{l}x +$$

$$+\varsigma_1^2 \sum_{n=1}^{\infty}\left\{\frac{\omega_1}{1!}(k\tau - T)\, e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\omega_1}\Phi_n(T-2\tau-\omega_1) +\right.$$

$$\left.+\frac{\omega_2}{1!}(T-(k-1)\tau)\, e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\omega_2}\Phi_n(T-2\tau-\omega_2)\right\}\sin\frac{\pi n}{l}x +$$

$$+\varsigma_1^3 \sum_{n=1}^{\infty}\left\{\frac{[\omega_1-\tau]}{2!}(k\tau - T)\, e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_1-\tau)}\Phi_n(T-2\tau-\omega_1) +\right.$$

$$\left.+\frac{[\omega_2-\tau]}{2!}(T-(k-1)\tau)\, e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_2-\tau)}\Phi_n(T-2\tau-\omega_2)\right\}\sin\frac{\pi n}{l}x + \ldots +$$

$$+\varsigma_1^{k-1} \sum_{n=1}^{\infty}\left\{\frac{[\omega_1-(k-3)\tau]^{k-2}}{(k-2)!}(k\tau - T)\, e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_1-(k-3)\tau)}\Phi_n(T-2\tau-\omega_1) +\right.$$

$$\left.+\frac{[\omega_2-(k-3)\tau]^{k-2}}{(k-2)!}(T-(k-1)\tau)\, e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_2-(k-3)\tau)}\Phi_n(T-2\tau-\omega_2)\right\}\sin\frac{\pi n}{l}x +$$

$$+\varsigma_1^{k}\frac{[\omega_1-(k-2)\tau]^{k-1}}{(k-1)!}(T-(k-1)\tau)\times$$

$$\times\sum_{n=1}^{\infty} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_2-(k-2)\tau)}\Phi_n(T-2\tau-\omega_2)\sin\frac{\pi n}{l}x.$$

If coefficients $\Phi_n(t)$ are such that the following condition is satisfied

$$\lim_{n\to+\infty} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-(k-1)\tau)} \max_{-\tau\le t\le 0}|\Phi_n(t)| = 0,$$

the series $\sum_{n=1}^{\infty} B_{n3}(T)\sin\frac{\pi n}{l}x$ converges.

From the convergence of series $\sum_{n=1}^{\infty} B_{n1}(T)\sin\frac{\pi n}{l}x$, $\sum_{n=1}^{\infty} B_{n2}(T)\sin\frac{\pi n}{l}x$, $\sum_{n=1}^{\infty} B_{n3}(T)\sin\frac{\pi n}{l}x$ follows the convergence of series $S_2(x,t)$.

3. Now we will consider coefficients $C_n(t)$, $n = 1, 2, ...$ of the third series $S_3(x,t)$. For the fixed moment of time T: $(k-1)\tau \le T < k\tau$ we will make a substitution and write:

$$C_n(T) = \int_0^T \exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, T-\tau-s\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-s)} f_n(s)\, ds =$$

$$= \int_{-\tau}^{T-\tau} \exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, \omega\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega+\tau)} f_n(T-\tau-\omega)\, d\omega =$$

$$= \int_{-\tau}^{0} \exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, \omega\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega+\tau)} f_n(T-\tau-\omega)\, d\omega +$$

$$+\int_0^{\tau}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\omega\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega+\tau)}f_n\left(T-\tau-\omega\right)d\omega+$$

$$+\int_{\tau}^{2\tau}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\omega\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega+\tau)}f_n\left(T-\tau-\omega\right)d\omega+$$

$$+\int_{(k-2)\tau}^{T-\tau}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\omega\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega+\tau)}f_n\left(T-\tau-\omega\right)d\omega.$$

As follows from the mean value theorem, for each of integrals there are time moments

$$-\tau\le\omega_1\le 0,\ 0\le\omega_2\le\tau,\ \ldots\ (k-2)\,\tau\le\omega_k\le T-\tau,$$

for which the following holds:

$$C_n\left(T\right)=\tau e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_1+\tau)}f_n\left(T-\tau-\omega_1\right)+$$

$$+\tau\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\omega_2}{1!}\right]e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_2+\tau)}f_n\left(T-\tau-\omega_2\right)+$$

$$+\tau\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\omega_3}{1!}+\varsigma_1^2 e^{2\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\left[\omega_3-\tau\right]}{2!}+\varsigma_1^3 e^{3\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\left[\omega_3-2\tau\right]^3}{3!}\right]\times$$

$$\times e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_3+\tau)}f_n\left(T-\tau-\omega_3\right)+...+\tau\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\omega_{k-1}}{1!}+...+\right.$$

$$\left.+\varsigma_1^{k-2}e^{(k-2)\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\left[\omega_{k-1}-(k-3)\,\tau\right]^{k-2}}{(k-2)!}\right]e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_{k-1}+\tau)}f_n\left(T-\tau-\omega_{k-1}\right)+$$

$$+\left[T-(k-1)\,\tau\right]\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\omega_k}{1!}+...+\right.$$

$$\left.+\varsigma_1^{k-1}e^{(k-1)\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\left[\omega_k-(k-2)\,\tau\right]^{k-1}}{(k-1)!}\right]e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_k+\tau)}f_n\left(T-\tau-\omega_k\right).$$

Hence, we obtain that

$$S_3\left(x,T\right)=\sum_{n=1}^{\infty}C_n\left(T\right)\sin\frac{\pi n}{l}x=\sum_{n=1}^{\infty}\left\{\tau\sum_{i=1}^{k-1}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_i+\tau)}f_n\left(T-\tau-\omega_i\right)+\right.$$

$$\left.+\left(T-(k-1)\,\tau\right)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_k+\tau)}f_n\left(T-\tau-\omega_k\right)\right\}\times$$

$$\times\sin\frac{\pi n}{l}x-\varsigma_1\sum_{n=1}^{\infty}\left\{\tau\sum_{i=2}^{k-1}\frac{\omega_i}{1!}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\omega_i}f_n\left(T-\tau-\omega_i\right)+\right.$$

$$\left.+\left(T-(k-1)\,\tau\right)\frac{\omega_k}{1!}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\omega_k}f_n\left(T-\tau-\omega_k\right)\right\}\sin\frac{\pi n}{l}x+$$

$$+\varsigma_1^2\sum_{n=1}^{\infty}\left\{\tau\sum_{i=3}^{k-1}\frac{\left[\omega_i-\tau\right]^2}{2!}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_i-\tau)}f_n\left(T-\tau-\omega_i\right)+\right.$$

$$\left.+\left(T-(k-1)\,\tau\right)\frac{\left[\omega_k-\tau\right]^2}{2!}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_k-\tau)}f_n\left(T-\tau-\omega_k\right)\right\}\sin\frac{\pi n}{l}x-$$

$$+...+\varsigma_1^{k-2}\sum_{n=1}^{\infty}\left\{\tau\frac{\left[\omega_{k-1}-(k-3)\tau\right]^{k-2}}{(k-2)!}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_i-(k-3)\tau)}f_n\left(T-\tau-\omega_{k-1}\right)+\right.$$

$$\left.+\left[T-(k-1)\tau\right]\frac{\left[\omega_k-(k-3)\tau\right]^{k-2}}{(k-2)!}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_k-(k-3)\tau)}f_n\left(T-\tau-\omega_k\right)\right\}\sin\frac{\pi n}{l}x+$$

$$+\varsigma_1^{k-1}\left[T-(k-1)\tau\right]\frac{\left[\omega_k-(k-2)\tau\right]^{k-1}}{(k-1)!}\times$$

$$\times\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_k-(k-2)\tau)}f_n\left(T-\tau-\omega_k\right)\sin\frac{\pi n}{l}x.$$

And, if coefficients $f_n(t)$ satisfy the following condition

$$\lim_{n\to+\infty}\max_{-\tau\le t\le T-\tau}\left|f_n(t)\right|e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-(k-1)\tau)}=0,$$

then the series $S_3(x,t)$ converges absolutely and uniformly.

Thus it was shown that for absolute and uniform convergence of the series $S_1(x,t)$, $S_2(x,t)$, $S_3(x,t)$ "fast reduction" on an index n of coefficients $\Phi_n(t)$, $-\tau\le t\le 0$ and $f_n(t)$, $0\le t\le T$ is required.

Convergence of derivatives ξ_t' and ξ_{xx}'' follows from the differentiability property of delay exponential function (Lemma 2.1). □

Proof of convergence of the series which represents the solution $\eta(x,t)$ is similar.

Corollary 3.1 *As the solutions $u(x,t)$, $v(x,t)$ are linear combinations of the functions $\xi(x,t)$, $\eta(x,t)$, they converge absolutely and uniformly, and their representations (31) are the solution of the boundary value problem of the initial system (1), (2).*

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