Nonlinear Dynamics and Systems Theory, **12** (3) (2012) 289–302

Sum of Linear Ratios Multiobjective Programming Problem: A Fuzzy Goal Programming Approach

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Received: August 30, 2011; Revised: July 11, 2012

Abstract: Sum of ratios optimization is an interesting field of research. This paper presents a solution method for sum of linear ratios multiobjective programming (SOLR – MOP) problem using the fuzzy goal programming technique. Each membership function of fuzzy objectives is approximated into linear function by using first order Taylor's theorem about the vertex of the feasible region where the objective function has maximum value. Then the resulted approximated linearized membership functions may be used for the formulation of fuzzy goal programming. So the problem is solved using fuzzy goal programming technique. The efficiency of the method is measured by numerical examples.

Keywords: *multiobjective programming; fractional programming; fuzzy multiobjective fractional programming; sum of ratio fractional program; fuzzy goal programming.*

Mathematics Subject Classification (2010): 90C29, 90C32.

1 Introduction

Ratio criteria are used to measure the efficiency of a system in any different fields of engineering and management sciences. The ratio optimization problem is called the fractional programming. These may be applied to different disciplines such as financial sector, inventory management, production planning, banking sector and others. Basically it is used for modeling real life problems with one or more objectives such as debt/equity, profit/cost, inventory/sales, actual cost/standard cost, output/employees, nurses/patients ratios etc. with respect to some constraints.

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The ratio optimization problem with linear functions and linear constraints is called linear fractional programming (LFP) problem. If these problems have more than one objective then the problem is known as multiobjective linear fractional programming (MOLFP) problem.

If the ratio optimization problem has sum of linear ratios (ratios of affine functions), then the fractional programming problem (LFP) is known as sum of linear ratios programming (SOLR-P) problem.

A general sum of linear ratios programming (SOLR-P) problem is defined in the following way:

$$\begin{aligned}
&\text{Max} \;\; F(x) = \text{Max}\left\{\sum_{j}^{p} \frac{f_j(x)}{m_j(x)}\right\} = \text{Max}\sum_{j}^{p} \frac{c_{0j}^T x + \alpha_{0j}}{d_{0j}^T x + \beta_{0j}} \\
&\text{subject to} \\
&x \in S, \;\; x \geq 0, \qquad\qquad (1) \\
&\text{where} \;\; p \geq 2, x, c_{0j},\, d_{0j} \in R^n, \;\; \alpha_{0j}\,, \beta_{0j} \;\in R.
\end{aligned}$$

The feasible region S is a nonempty, compact, convex set in R^n. The function $f_j(x) = c_{0j}^T x + \alpha_{0j}$, and $m_j(x) = d_{0j}^T x + \beta_{0j}$ are positive for all $x \in S$. Note that under these assumptions, the global maximum for problem (1) is attained by at least one point in S.

If we take more than one objectives in problem (1), then the problem is known as sum of linear ratios multiobjective programming (SOLR-MOP) problem, mathematically it can be written as:

$$\begin{aligned}
&\text{Max}\, F(x) = [F_1(x),\, F_2(x),\, \ldots F_k(x)], \text{where} \\
&F_i(x) = \sum_{j}^{p} \frac{f_{ij}(x)}{m_{ij}(x)}, \qquad\qquad (2) \\
&x \in S, \;\; x \geq 0, \;\; p \geq 2, \;\; x, \;\; c_{ij}, \;\; d_{ij} \in R^n, \;\; \alpha_{ij}\,, \beta_{ij} \;\in R.
\end{aligned}$$

and $f_{ij}(x) = c_{ij}^T x + \alpha_{ij}$, $m_{ij}(x) = d_{ij}^T x + \beta_{ij}$ are positive for all $x \in S$, where $S = \{x \,:\, Ax\,(\leq, =, \geq)\, b,\, x \geq 0,\, x \in R^n,\, b \in R^m,\, A \in R^{m\times n}\}$, $(i = 1, 2, \ldots, k,\; j = 1, 2, \ldots, p)\,\forall\, x \in S$. Here, S is assumed to be non-empty compact convex set in R^n and all $F_i(x)$ having continuous partial derivative in the feasible region S.

Sum of ratios fractional program was one of the least researched fractional program until about 1990. During last decade, interest in these programs has become especially strong. This is because, from a practical point of view, the sum of ratios fractional programs have numerous applications in the fields as discussed above but still multiobjective sum of ratios problem has least attention.

Various solution approaches have been proposed in the literature for sum of ratios fractional program. In [6], Cambini et. al. proposed a simplex type finite algorithm for the case $p = 2$ in problem (1) and find the global optimal solution. Later, Konno et. al.[13] proposed a finite parametric simplex type algorithm for the solution of linear sum of fractional programs. They give the minimization of the sum of two ratios.

In [4], Benson presented a branch- and - bound algorithm for globally solving the nonlinear sum of ratios problem. The algorithm has reduced the computational difficulty by conducting branch - and - bound search in R^p space rather than R^n space and the algorithm is applied in numerical examples for verification. Benson [3] proposed a branch - and - bound algorithm using the concave envelopes for the same problem.

In the algorithm, upper bounds are computed by maximizing concave envelopes of a sum of ratios function over intersection of the feasible region of the equivalent problem with rectangular sets systematically subdivided as branch and bound search procedure. The convergence of the algorithm is also presented and computational advantage is also highlighted. Other algorithms are also presented by Benson in [2, 23, 25].

In [8], Shen et. al. solved the sum of convex - convex ratios problem with non-convex feasible region. They used a branch bound scheme where the Lagrange duality theory is used to obtain lower bounds and the convergence of the algorithm is also proved. Shen and Wang [5] proposed also a branch bound algorithm for globally solving the sum of ratios with coefficients. They reduced the problem in equivalent sequence of linear programming problem by utilizing linearization technique.

In [10], Dür et. al. gave a branch bound solution algorithm for sum of ratios problem using rectangular partitions in Euclidean space of dimension p. For the bounding procedures, they used dual constructions and the calculation of efficient points of a corresponding multiobjective optimization problem.

Jaberipour and Khorram [11] proposed a harmony search algorithm for solving a sum - of - ratios problem. They also presented the numerical examples for demonstration, effectiveness and robustness of the proposed method and they claimed that all the solution obtained by their method are superior to those obtained by other methods.

In [16], Kuno developed a branch- and- bound algorithm for maximizing a sum of $p \geq 2$ linear ratios on a polytope. They embedded the problem in $2p$-dimensional space and constructed the bounding operations. The operations are carried out in p-dimensional space and rectangular branch bound method is used to find the solution. They also discussed the convergence criteria and also reviewed some computational results.

Konno and Yamashita [15] proposed a method to minimize the sums and products of linear fractional functions. They developed efficient deterministic algorithms for globally minimizing the sum and the product of several linear fractional functions over a polytope using outer approximation algorithm in given problem. They showed that the Charnes Cooper transformation plays an essential role in solving these problems. Also a simple bounding technique using linear multiplicative programming techniques has remarkable effects on structured problems.

In [14], Konno and Fukaisi presented a practical algorithm for solving low rank linear multiobjective programming problems and minimize the sum of product of two linear functions and also solved low rank linear fractional programming problems as minimization of sum of linear fractional functions over a polytope. Recently Gao et. al. [22] gave the extension of branch bound algorithm as maximization of sum of nonlinear ratios problem. They also presented the complexity of the problem and discussed some numerical experiments on the extended algorithm.

In [26], Gao and Shi presented a comprehensive review on branch - and bound algorithms for solving sum of ratios problem and they made a comparison between two branch-and bound approaches for solving the sum-of ratios problem. They also modify the algorithm for nonlinear sum-of ratios problem.

Multiobjective programming problems have been extensively studied for several decades and the research is based on the theoretical background. As a matter of fact many ideas and approaches have their foundation in the theory of fractional programming. Multiobjective linear fractional programming problems using fuzzy set theory has been studied in [19, 21, 27, 28]. Luhandjula [28] has given a solution method for MOLFP using linguistic approach. Dutta, Rao and Tewari [27] modified linguistic approach of

Luhandjula [28] to solve MOLFP using fuzzy set theoretic approach. Recently, Güzel and Sivri [20] have given Taylor series approach to solve MOLFP and in [16], they developed another approach. Toksari [21], developed an algorithm to solve FMOLFP by Taylor series approach and he linearized the membership functions instead of objection functions.

Fuzzy set theory becomes the efficient tool for solving various types of non-linear systems [30, 31, 32].

Our objective in this paper is to propose a simple method to the solution of sum of linear ratio multiobjective programming (SOLR-MOP)(2) problem using fuzzy goal programming approach. In this approach, each membership function associated with each objective of SOLR – MOP is approximated into linear function and then it is solved by fuzzy goal programming method. In the proposed article, we have attempted to handle multiobjective case for sum of linear ratios using fuzzy goal programming approach which is not attempted in the literature. The proposed algorithm is applied to three numerical examples.

2 Sum of Linear Ratios Fuzzy multiobjective Programming Problem (SOLR-FMOP)

If an uncertain aspiration level is introduced to each of the objectives of SOLR-MOP, then these fuzzy objectives are called fuzzy goals. The sum of linear ratios fuzzy multiobjective programming (SOR-FMOP) problem can be defined as

$$\begin{aligned}&\text{Find } X(x_1,\, x_2, ...x_n) \quad \text{such that}\\ &F_i(x) \lessapprox g_i \ \text{ or } \ F_i(x) \ \gtrapprox \ g_i\,\forall \quad (i=1,\,2,\,\ldots,k,\ j=1,2,\ldots,p) \qquad (3)\\ &\text{subject to}\\ &x\in S=\{x\in R^n,\ Ax(\leq,=,\geq)b, x\geq 0\,\text{with}\ b\in R^m,\ A\in R^{m\times n}\},\\ &F_i(x)=\sum_j^p \frac{c_{ij}^T x+\alpha_{ij}}{d_{ij}^T x+\beta_{ij}},\end{aligned}$$

where g_i is the aspiration level of the i^{th} objective F_i and $\lessapprox, \gtrapprox$ indicate fuzziness of the aspiration level. The membership function $\mu_i(x)$ must be described for each fuzzy goal. A membership function can be explained as given below.

If $F_i(x) \lessapprox g_i,$ then

$$\mu_i(x)=\begin{cases}1, & if \quad F_i(x)\leq g_i,\\ \dfrac{\overline{t_i}-F_i(x)}{\overline{t_i}-g_i}, & if \quad g_i\leq F_i(x)\leq \overline{t_i},\\ 0 & if \quad F_i(x)\geq \overline{t_i}.\end{cases} \tag{4}$$

If $F_i(x) \geq g_i,$ then

$$\mu_i(x)=\begin{cases}1, & if \quad F_i(x)\gtrapprox g_i,\\ \dfrac{F_i(x)-\underline{t_i}}{g_i-\underline{t_i}}, & if \quad \underline{t_i}\leq F_i(x)\leq g_i,\\ 0 & if \quad F_i(x)\leq \underline{t_i},\end{cases} \tag{5}$$

and $\overline{t_i}$ and $\underline{t_i}$ are the upper tolerance limit and lower tolerance limit, respectively, for the i^{th} fuzzy goal. Then the problem (3) is called sum of linear ratios fuzzy multiobjective programming problem (SOLR-FMOP).

3 Goal Programming

The concept of goal programming (GP) was first introduced by Charnes and Cooper in 1961 [7] as a tool to resolve infeasible linear programming problems. Thereafter, significant methodological development of GP was made by Ignizio [18] and others. The overall purpose of GP is to minimize the deviations between the achievement of goals and their aspiration levels. A typical GP is expressed as follows

$$\begin{aligned} & \text{Minimize} \sum_{i=1}^{k} |F_i(x) - g_i| \\ & \text{subject to} \\ & x \in X = \{x \in R^n;\, Ax \le b,\, x \ge 0\}, \end{aligned} \tag{6}$$

where F_{ij} is the linear function of the i^{th} goal and g_i is the aspiration level of the i^{th} goal.
Let $F_i(x) - g_i = d_i^+ - d_i^-, \qquad d_i^-, d_i^+ \ge 0$. Problem (6) can be formulated as follows

$$\begin{aligned} & \text{Minimize} \sum_{i=1}^{k} (d_i^+ + d_i^-) \\ & \text{subject to} \\ & F_i(x) - d_i^+ + d_i^- - g_i = 0, \qquad i = 1, 2, \ldots k, \\ & d_i^+, d_i^- \ge 0,\ x \in X = \{x \in R^n;\, Ax \le b,\, x \ge 0\}, \end{aligned} \tag{7}$$

where $d_i^- \ge 0$, $d_i^+ \ge 0$ are, respectively under - and over - deviations of the i^th goal. Problem (7) has been applied to solve many real world problems.

3.1 Fuzzy goal programming

In fuzzy goal programming approaches, the highest degree of membership function is 1. So, for the defined membership function in (4) and (5), the flexible membership goals with aspiration levels 1 can be expressed as

$$\frac{F_i(x) - \underline{t_i}}{g_i - \underline{t_i}} + d_i^- - d_i^+ = 1 \qquad \text{or} \qquad \frac{\overline{t_i} - F_i(x)}{\overline{t_i} - g_i} + d_i^- - d_i^+ = 1, \tag{8}$$

where $d_i^- \ge 0$, $d_i^+ \ge 0$ with $d_i^+.d_i^- = 0$ are, respectively, under - and over -deviations from the aspiration levels.

In conventional GP, the under- and over-deviational variables are included in the achievement function or minimized and that depends upon the type of the objective functions to be optimized.

In this approach, only the under - deviational variable d_i^- is required to achieve the aspired levels of the fuzzy goals. It may be noted that any over - deviation from fuzzy goal indicates the full achievement of the membership value. Recently, B. B. Pal. et.al [19] proposed an efficient goal programming (GP) method for solving fuzzy multiobjective linear fractional programming problems.

4 Mathematical Modeling of Problem

We consider the sum of linear ratios multiobjective programming (SOLR-MOP) problem of the form:

$$\begin{aligned}
&\text{Max}\, F(x) = \{\, F_1(x),\, F_2(x),\, \ldots, F_k(x)\,\} \\
&\quad F_i(x) = \sum_{j}^{p} \frac{c_{ij}^T x + \alpha_{ij}}{d_{ij}^T x + \beta_{ij}}, \\
&\text{where} \quad d_{ij}^T x + \beta_{ij} > 0,\, \forall \quad (i = 1,\, 2,\, \ldots, k,\ j = 1, 2, \ldots p) \\
&\text{subject to} \\
&x \in S = \{Ax \leq\ b,\, x \geq\ 0,\, x,\, c_{ij}^T,\, d_{ij}^T,\, \in R^n,\, b \in R^m, \\
&A = (m \times n)\ \text{matrix},\, \alpha_{ij},\, \beta_{ij},\, \in R\}.
\end{aligned} \tag{9}$$

Assume fuzzy aspiration level g_i and tolerance limit $(\overline{t_i},\, \underline{t_i})$ for each objective function $F_i(x)$. We construct the membership function for each objective function using Zimmermann max-min approach [29]. Then the problem (9) becomes

$$\begin{aligned}
&\text{Find} \quad X(x_1,\, x_2, \ldots, x_n) \\
&\text{so as to satisfy} \\
&F_i(x) \lessapprox g_i \\
&\text{or} \\
&F_i(x) \gtrapprox g_i \\
&\text{subject to} \qquad x \in S = \{x \in R^n,\ Ax \leq b, x \geq 0 \text{ with } b \in R^m,\ A \in R^{m \times n}\} \\
&\text{and} \qquad F_i(x) = \sum_{j}^{p} \frac{c_{ij}^T x + \alpha_{ij}}{d_{ij}^T x + \beta_{ij}}, \ \text{where} \quad d_{ij}^T x + \beta_{ij} > 0, \forall\, i \,\text{and}\, j,
\end{aligned} \tag{10}$$

where g_i is the aspiration level of the i^{th} objective function $F_i(x)$. The membership function $\mu_i(x)$, described for each fuzzy goal, is given by equation (4) and equation (5). Suppose that all $F_i(x)$ and all of their partial derivatives of order less than or equal to $n+1$ are continuous on the feasible region S. So the membership functions $\mu_i(x)$ of each $F_i(x)$ are having same property in the feasible region.

The proposed algorithm can be explained in three steps and linear approximation of membership functions is motivated by Toksari [21].

Step 1: Determine the vertex of the feasible region, $x_q^* = \{\, x_{q1}^*,\, x_{q2}^*,\, \ldots, x_{qn}^*\}$ for which the i^{th} membership function is maximized associated with the i^{th} objective $F_i(x), \forall\ i = 1,\, 2,\, \ldots, k$ and $j = 1, 2, \ldots, p$, where n is the number of variable and q is finite.

Step 2: Transform each fractional membership function into linear membership function by using first order Taylor's theorem

$$\mu_i(x) = \widetilde{\mu_i}(x) \cong \mu_i(x_q^*) + [(x_1 - x_{q1}^*)\frac{\partial \mu_i(x_q^*)}{\partial x_1} + (x_2 - x_{q2}^*)\frac{\partial \mu_i(x_q^*)}{\partial x_2} + \dots \quad (11)$$
$$+ (x_n - x_{qn}^*)\frac{\partial \mu_i(x_{q^*})}{\partial x_n}] + O(h^2),$$
$$\mu_i(x) = \widetilde{\mu_i}(x) \cong \mu_i(x_q^*) + \sum_{j=1}^{n}[(x_j - x_{qj}^*)\frac{\partial \mu_i(x_q^*)}{\partial x_j}] + O(h^2), \quad (12)$$

where, if $F_i(x) \lessapprox g_i$, then

$$\mu_i(x) = \begin{cases} 1, & if \quad F_i(x) \le g_i, \\ \dfrac{\overline{t_i} - F_i(x)}{\overline{t_i} - g_i}, & if \quad g_i \le F_i(x) \le \overline{t_i}, \\ 0 & if \quad F_i(x) \ge \overline{t_i}. \end{cases} \quad (13)$$

If $F_i(x) \ge g_i$, then

$$\mu_i(x) = \begin{cases} 1, & if \quad F_i(x) \ge g_i, \\ \dfrac{F_i(x) - \underline{t_i}}{g_i - \underline{t_i}}, & if \quad \underline{t_i} \le F_i(x) \le g_i, \\ 0 & if \quad F_i(x) \le \underline{t_i}. \end{cases}$$

subject to

$x \in X = \{Ax \le b,\ x \ge 0,\ x,\ c_{ij}^T,\ d_{ij}^T, \in R^n,\ b \in R^m,$

$A = (a_{ij})_{m\times n},\ \alpha_{ij},\ \beta_{ij}, \in R\}.$

Now in (12), these are linearized approximated membership function of fuzzy objectives. Then the problem can be solved by assuming fuzzy goals.

Step 3: Find $x^* = \{x_1^*, x_2^*, \dots, x_n^*\}$ using fuzzy goal formulation. Apply fuzzy goal programming approach for the linearized membership functions $\widetilde{\mu_i}(x)$ in (12) of F_i. The flexible membership goals with aspiration levels 1 can be expressed as

$$\widetilde{\mu_i}(x) + d_i^- - d_i^+ = 1, \quad (14)$$

where $d_i^-, d_i^+ \ge 0$, with $d_i^+ d_i^+ = 0$ are respectively under- and over- deviations from the aspiration levels.

Now the fuzzy goal programming formulation can be obtained as

$$\begin{aligned} &\text{Minimize} \sum_{i=1}^{k} d_i^- \\ &\text{subject to} \\ &\widetilde{\mu_i}(x) - d_i^+ + d_i^- = 1, \qquad i = 1, 2, \dots k \\ &d_i^+,\ d_i^- \ge 0 \\ &x \in S = \{x \in R^n;\ Ax \le b,\ x \ge 0\} \text{ with } d_i^+ . d_i^+ = 0. \end{aligned} \quad (15)$$

In the problem (15), S is a non empty convex bounded set having feasible points . The LPP (15) can be solved easily, which gives the efficient solution of (SOLR-MOP)

(3). The values of membership functions at the optimal point gives the satisfaction level (degree) of objective function to the solution.

5 Numerical Examples

Example 1: Consider a SOLR-MOP with two objective functions:

$$
\begin{aligned}
&\text{Max}\left\{\frac{x_1+2x_2}{2x_1+x_2+5}+\frac{9x_1+2x_2}{7x_1+3x_2+1},\quad \frac{2x_1+3x_2+5}{x_1+1}+\frac{5x_1+4x_2}{x_1+x_2}\right\}\\
&\text{subject to}\\
&\qquad\qquad x_1-x_2\geq 2,\\
&\qquad\qquad 4x_1+5x_2\leq 25,\\
&\qquad\qquad x_1+9x_2\geq 9,\\
&\qquad\qquad x_1\geq 5,\\
&\qquad\qquad x_1,\,x_2\geq 0.
\end{aligned}
\tag{16}
$$

It is observed that $f_{ij}\geq 0,\, m_{ij}\geq 0, (i=1,2$ and $j=1,2)$ for each x in the feasible region.

If the fuzzy aspiration levels of the two objectives are 1.806, and 7.83, then find x in order to satisfy the following fuzzy goals:

$$
\left(\frac{x_1+2x_2}{2x_1+x_2+5}+\frac{9x_1+2x_2}{7x_1+3x_2+1}\right)\gtrapprox 1.806,\quad \left(\frac{2x_1+3x_2+5}{x_1+1}+\frac{5x_1+4x_2}{x_1+x_2}\right)\gtrapprox 7.83.
$$

The tolerance limits for the two fuzzy goals are (1.620, 7.05) respectively. The membership functions for the two fuzzy goals are

$$
\mu_1(x)=\begin{cases}1, & if\quad F_1(x)\geq g_i,\\ \dfrac{F_i(x)-\underline{t_i}}{g_i-\underline{t_i}}, & if\quad \underline{t_i}\leq F_i(x)\leq g_i,\\ 0, & if\quad F_i(x)\leq \underline{t_i}.\end{cases}
$$

i.e.

$$
\mu_1(x)=\begin{cases}1, & if\qquad F_1(x)\geq 1.806,\\ \dfrac{\left(\dfrac{x_1+2x_2}{2x_1+x_2+5}+\dfrac{9x_1+2x_2}{7x_1+3x_2+1}\right)-1.620}{0.19}, & if\; 1.620\leq F_1(x)\leq 1.806,\\ 0, & if\qquad F_1(x)\leq 1.620.\end{cases}
\tag{17}
$$

$$
\mu_2(x)=\begin{cases}1, & if\quad F_2(x)\geq 7.83,\\ \dfrac{\left(\frac{2x_1+3x_2+5}{x_1+1}+\dfrac{5x_1+4x_2}{7x_1+3x_2+1}\right)-7.05}{0.78}, & if\; 7.05\leq F_2(x)\leq 7.83,\\ 0, & if\quad F_2(x)\leq 7.05.\end{cases}
\tag{18}
$$

Expand the membership functions $\mu_1(x)$ about point (5, 0.44) and $\mu_2(x)$ about point (5, 1)

$$\begin{aligned}
&\mu_1(x) \cong \widetilde{\mu_1}(x) = \mu_1(5,\, 0.44) + (x_1 - 5)\frac{\partial \mu_1(5,\, 0.44)}{\partial x_1} + (x_2 - 0.44)\frac{\partial \mu_1(5,\, 0.44)}{\partial x_2}, \\
&\mu_1(x) \cong \widetilde{\mu_1}(x) = 0.14x_1 + 0.54x_2 + 0.06, \qquad (19)\\
&\mu_2(x) \cong \widetilde{\mu_2}(x) = \mu_2(5,\, 1) + (x_1 - 5)\frac{\partial \mu_2(5,\, 1)}{\partial x_1} + (x_2 - 1),\frac{\partial \mu_2(5,\, 1)}{\partial x_2}, \\
&\mu_2(x) \cong \widetilde{\mu_2}(x) = -0.18x_1 + 0.50x_2 + 1.4. \qquad (20)
\end{aligned}$$

Now apply the fuzzy goal programming technique:

$$\begin{aligned}
&\text{Minimize } (d_1^- + d_2^-) \\
&\text{ subject to} \\
&\widetilde{\mu_1}(x) - d_1^+ + d_1^- = 1, \qquad (21)\\
&\widetilde{\mu_2}(x) - d_2^+ + d_2^- = 1, \\
&d_1^-, d_1^+, d_2^-, d_2^+ \geq 0, \\
&x \in S = \{x \in R^n;\, Ax \leq b,\, x \geq 0\} \text{ with } d_1^+.d_1^+ = 0 \,\text{and}\, d_2^+.d_2^+ = 0.
\end{aligned}$$

Thus new LPP is obtained

$$\begin{aligned}
&\text{Minimize } (d_1^- + d_2^-) \\
&\text{ subject to} \\
&0.14x_1 + 0.54x_2 - d_1^+ + d_1^- = 0.94, \qquad (22)\\
&-0.18x_1 - 0.50x_2 - d_2^+ + d_2^- = -0.4, \\
&\quad x_1 - x_2 \geq 2, \\
&\quad 4x_1 + 5x_2 \leq 25, \\
&\quad x_1 + 9x_2 \geq 9, \\
&\quad x_1 \geq 5, \\
&\quad x_1,\, x_2 \geq 0, \quad \text{with} \quad d_1^+.d_1^+ = 0 \text{ and } d_2^+.d_2^+ = 0.
\end{aligned}$$

The optimal solution of the above problem is given by $x_1 = 5$, $x_2 = 1$, $d_1^- = 0$, $d_1^+ = 0.30$, $d_2^- = 0$, $d_2^+ = 0$ and the membership values are $\mu_1 = 0.12$, $\mu_2 = 1$. The optimal solution of the problem (22) is at the point $(5, 1)$ and minimum value is 0. The point (5, 1) is the efficient solution of the given original problem in the feasible region with optimal values of the functions $F_1 = 1.643, F_2 = 7.83$. The membership function values at (5, 1) indicate that goals F_1 and F_2 are satisfied 12% and 100% respectively, for the obtained solution.

Example 2: Let us consider a SOLR - MOP with three objective functions

$$\text{Max} \quad \{F_1(x) = \frac{x_1 + 4x_2}{2x_1 + x_2 + 1} + \frac{9x_1 + 2x_2}{x_1 + 3x_2 + 1} + \frac{x_1 + 3x_2}{x_2 + 1}, \tag{23}$$

$$F_2 = \frac{3x_1 + 8x_2}{x_1 + x_2 + 3}, + \frac{x_1 + 2x_2}{2x_1 + 3x_2 + 2} + \frac{x_1 + 2x_2}{3x_1 + x_2 + 2}\}$$

subject to

$$\begin{aligned} &x_1 - x_2 \geq 5, \\ &4x_1 + 5x_2 \leq 25, \\ &x_1 \geq 5, \\ &x_1,\, x_2 \geq 0. \end{aligned} \tag{24}$$

If the fuzzy aspiration levels of two objectives are (9.08, 2.76) respectively, then find x in order to satisfy the following goals:

$$F_1(x) \gtrapprox 9.08, \quad F_2(x) \gtrapprox 2.76. \tag{25}$$

The tolerance limits for the three fuzzy goals are $(8.79, 2.51)$ respectively. The membership functions for the two fuzzy goals are given by

$$\mu_1(x) = \begin{cases} 1, & if \quad F_1(x) \quad \geq 9.08, \\ \dfrac{\frac{x_1+4x_2}{2x_1+x_2+1} + \dfrac{9x_1 + 2x_2}{x_1 + 3x_2 + 1} + \dfrac{x_1 + 3x_2}{x_2 + 1} - 8.79}{0.29}, & if\; 8.79 \leq F_1(x) \leq 9.08, \\ 0 & if \quad F_1(x) \leq 8.79. \end{cases} \tag{26}$$

$$\mu_2(x) = \begin{cases} 1, & if \quad F_2(x) \quad \geq 2.76, \\ \dfrac{\frac{3x_1+8x_2}{x_1+x_2+3}, + \dfrac{x_1 + 2x_2}{2x_1 + 3x_2 + 2} + \dfrac{x_1 + 2x_2}{3x_1 + x_2 + 2} - 2.51}{0.25}, & if 2.51 \leq F_2(x) \leq 2.76, \\ 0 & if \quad F_2(x) \leq 2.51. \end{cases} \tag{27}$$

Both membership functions are expanded by using first order Taylor's theorem about the point (6.25, 0) in the feasible region. The linearized forms of membership functions are obtained

$$\mu_1(x) \cong \widetilde{\mu_1}(x) = 0.68x_1 + 4.47x_2 - 3.25, \tag{28}$$

$$\mu_2(x) \cong \widetilde{\mu_2}(x) = 0.48x_1 + 3.09x_2 - 2. \tag{29}$$

Now apply the fuzzy goal programming technique and the new LPP is obtained

$$\begin{aligned} &\text{Minimize } (d_1^- + d_2^-) \\ &\text{ subject to} \\ &0.68x_1 + 4.47x_2 - d_1^+ + d_1^- = 4.25, \\ &0.48x_1 + 3.09x_2 - d_2^+ + d_2^- = 3, \\ &\quad x_1 - x_2 \geq 5, \\ &\quad 4x_1 + 5x_2 \leq 25, \\ &\quad x_1 \geq 5, \\ &\quad x_1,\, x_2 \geq 0. \quad \text{with} \quad d_1^+.d_1^+ = 0 \text{ and } d_2^+.d_2^+ = 0. \end{aligned} \tag{30}$$

The alternate optimal solution is obtained but the best minimum value is 0 at $x_1 = 5.56$, $x_2 = 0.56$, $d_1^- = 0$, $d_1^+ = 0.51$, $d_2^- = 0.01$, $d_2^+ = 0$ and the membership values are $\mu_1 = 0$, $\mu_2 = 1$. So the optimal solution of problem (31) is at $(5.56, 0.56)$. The point (5.56, 0.56) is the efficient solution of the given original problem in the feasible region with optimal values of the functions $F_1 = 7.93, F_2 = 3.12$. The membership function values at (5.56, 0.56) indicate that goals F_1 and F_2 are satisfied 0% and 100% respectively, for the obtained solution.

Example 3: Let us consider a SOLR - MOP with three objective functions

$$\begin{aligned} \text{Max} \quad \{F_1(x) &= \frac{x_1}{x_2+1} + \frac{x_2}{2x_1+3}, \\ F_2(x) &= \frac{x_2+4}{x_1+2x_2+1} + \frac{x_1+2}{3x_1+x_2+2}, \\ F_3(x) &= \frac{x_1+2x_2}{x_1+3x_2+2} + \frac{5x_1+x_2}{2x_1+5x_2+3}\} \\ \text{subject to} & \\ & x_1 \le 6, \\ & x_2 \le 6, \\ & 2x_1 + x_2 \le 9, \\ & -2x_1 + x_2 \le 5, \\ & x_1 - x_2 \le 5, \\ & x_1,\, x_2 \ge 0. \end{aligned} \tag{31}$$

If the fuzzy aspiration levels of the three objectives are (4.5, 5, 2.57) respectively, then

$$F_1(x) \gtrapprox 4.5, \quad F_2(x) \gtrapprox 5, \quad F_3(x) \gtrapprox 2.57. \tag{32}$$

The tolerance limits for the two fuzzy goals are 0, 0.86, 0 respectively. The membership functions for the three fuzzy goals are

$$\mu_1(x) = \begin{cases} 1, & if \quad F_1(x) \ge 4.5, \\ \dfrac{\dfrac{x_1}{x_2+1} + \dfrac{x_2}{2x_1+3} - 0}{4.5}, & if \quad 0 \le F_1(x) \le 4.5, \\ 0, & if \quad F_1(x) \le 0. \end{cases} \tag{33}$$

$$\mu_2(x) = \begin{cases} 1, & if \quad F_2(x) \ge 5, \\ \dfrac{\dfrac{x_2+4}{x_1+2x_2+1} + \dfrac{x_1+2}{3x_1+x_2+2} - 0.86}{4.14}, & if \quad 0.86 \le F_2(x) \le 5, \\ 0 & if \quad F_2(x) \le 0.86. \end{cases} \tag{34}$$

$$\mu_3(x) = \begin{cases} 1, & if \quad F_3(x) \ge 2.57, \\ \dfrac{\dfrac{x_1+2x_2}{x_1+3x_2+2} + \dfrac{5x_1+x_2}{2x_1+5x_2+3} - 0}{2.57}, & if \quad 0 \le F_3(x) \le 2.57, \\ 0, & if \quad F_3(x) \le 0. \end{cases} \tag{35}$$

By expanding the first order Taylor's theorem for membership functions μ_1, μ_2 and μ_3 about points (4.5, 0), (0, 0) and (4.5, 0) respectively in the feasible region:

$$\mu_1(x) \cong \widetilde{\mu_1}(x) = 0.22x_1 - 4.22x_2 + 0.01, \tag{36}$$
$$\mu_2(x) \cong \widetilde{\mu_2}(x) = 1.33x_1 - 1.69x_2 + 1, \tag{37}$$
$$\mu_3(x) \cong \widetilde{\mu_3}(x) = 0.059x_1 - 0.71x_2 + 0.74. \tag{38}$$

Apply fuzzy goal programming technique, the new LPP is obtained

$$\begin{aligned}
&\text{Minimize } (d_1^- + d_2^- + d_3^-)\\
&\text{ subject to}\\
&0.22x_1 - 4.22x_2 - d_1^+ + d_1^- = 0.99,\\
&1.33x_1 - 1.69x_2 - d_2^+ + d_2^- = 0,\\
&0.059x_1 - 0.71x_2 - d_3^+ + d_3^- = 0.26,\\
&\quad x_1 \le 6,\\
&\quad x_2 \le 6,\\
&\quad 2x_1 + x_2 \le 9,\\
&\quad -2x_1 + x_2 \le 5,\\
&\quad x_1 - x_2 \le 5,\\
&\quad x_1,\, x_2 \ge 0, \quad \text{with} \quad d_1^+.d_1^+ = 0,\, d_2^+.d_2^+ = 0 \text{ and } d_3^-.d_3^+ = 0.
\end{aligned} \tag{39}$$

Optimal solution of the problem (40) is at the point $x_1 = 4.5$, $x_2 = 0$, $d_1^- = 0$, $d_2^- = 0$, $d_3^- = 0, d_1^+ = 0$, $d_2^+ = 5.99\, d_3^+ = 0.01$ and the minimum value is 0. The efficient solution of the given problem is $x_1 = 4.5, x_2 = 0$, $F_1 = 4.5$, $F_2 = 0.86$, $F_3 = 2.57$ and the membership values are $\mu_1 = 1,\ \mu_2 = 0$, $\mu_3 = 1$. The membership function values at (4.5, 0) indicate that goals F_1, F_2 and F_3 are satisfied 100%, 0% and 100% respectively, for the obtained solution.

6 Conclusion

In this paper, a new algorithm has been proposed to optimize sum of linear ratios multiobjective programming (SOLR-MOP)problem using fuzzy set theory and goal programming method. Most of the reported work is based on the single objective optimization. So, the proposed algorithm is a simple procedure to optimize sum of linear ratios in multiobjective case. This reduces computational complexity as compared to the previous reported work.

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