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PERSONAGE IN SCIENCE

Professor Theodore A. Burton

J.H. Dshalalow [1*] and A.A. Martynyuk [2]

[1] *Department of Applied Mathematics Florida Institute of Technology, Melbourne, Fl, 32901, USA*
[2] *S.P. Timoshenko Institute of Mechanics, National Academy of Science of Ukraine, Nesterov Str. 3, Kiev, 03057, Ukraine*

1 Brief outline of T.A. Burton's life

T. A. Burton (denoted by T. A. throughout this paper) was born on September 7, 1935 on a farm in Kansas, the fifth child of seven in the family. It was a peak of the Great Depression and when the dust storms wrecked havoc on the Mid-Western United States. Entire buildings were buried in the dust. The economy was so poor that farmers turned their livestock loose and left the area. At the age of five, T. A. and his family moved to Idaho, then to California, and finally to the Cascade Mountains of the state of Washington where he completed an elementary and high school.

On the day he graduated from high school he was drafted into the army for two years, emerging with veteran's rights to a college education. In 1959 he graduated from the Washington State College with a Bachelor of Science with Honors. His record earned him a full fellowship for three years of study toward a Ph.D. in mathematics. On August 5, 1961 he married the love of his life, Fredda Jean Anderson.

His graduate work began in 1959 under the direction of the late Donald W. Bushaw. Bushaw was a student of Solomon Lefschetz and his dissertation concerned the first paper on optimal control. But Lefschetz was also deeply interested in differential equations of various Liénard types and Bushaw inherited that interest, assigning to T. A. a problem on global stability of a nonlinear oscillator. There was much literature on the problem and its generalizations. Lefschetz gathered a number of outstanding foreign and American researchers to study the problem, whose most important aspects were well-defined. The problem was old, going back to Lagrange, and it is taught to every student in a basic course on differential equations.

We consider a number of physical problems, such as a spring-mass-dashpot system, and use Newton's second law of motion under numerous assumptions to obtain the equation of motion

$$x'' + ax' + bx = 0$$

* Corresponding author: mailto:edshalalow@cfl.rr.com

with a and b being positive constants. It is obvious that all solutions and their derivatives tend to zero as t tends to infinity. Taking into account an ample amount of significant real-world problems it stands to reason that a and b must be replaced with general functions to obtain a solution which incorporates a more realistic behavior with many uncertainties. The problem proposed to T. A. in his MA thesis in mathematics was to replace the above linear problem with

$$x'' + f(x)x' + g(x) = 0,$$

where $f(x)$ is a positive function and $g(x)$ is a continuous odd function. His thesis ultimately led to the first necessary and sufficient condition for global asymptotic stability. The result also held when $f(x)$ was replaced with $f(x)h(x')$ where h was also positive. These results have been generalized ever since, but the original ones are still widely quoted.

[1] Burton, T. A. The generalized Liénard equation. *J. SIAM on Control* **5** (1965) 223–230.

[2] Burton, T. A. On the equation $x'' + f(x)h(x')x' + g(x) = e(t)$. *Annali di Math. Pura Appl.* **LXXXV** (1970) 277–286.

The problem had also brought to its attention a large group of Chinese mathematicians, including Qichang Huang who later became a university president where he used his position to promote research and collaboration with foreign mathematicians around the world. In 1965, just before [1] appeared, Mao Zedong started the Cultural Revolution which sent scholars to farms, steel mills, and other places of hard labor. Huang was sent to a rural village, given a cow, and sent to the mountains using the cow to drag firewood down to the village. Nevertheless, he continued to work on this same problem during the night times and when the Cultural Revolution ended in the early 1980's he returned to his university and got across Burton's paper [1] which he presented in a conference to his fellows mathematicians who were working on the same problem. As a result, Huang was paid a two-years visit to work with T. A. It was the beginning of a cultural and scientific exchange which is still active.

The three year fellowship for graduate study given to T. A. came to an end one year before he was supposed to graduate. To support his fourth year of the graduate study, T.A. took part in a national competition to earn a full scholarship. This was a successful endeavor and T.A. was awarded for his final year followed by his PhD thesis defense in 1964. In the same year he accepted an academic position at the University of Alberta, Edmonton, Canada.

Due to the Russian satellite, Sputnik, American universities were awakening and gearing up for graduate study in all areas of science. Under the leadership of Delyte Morris, a little known university (Southern Illinois University at Carbondale) in Illinois received massive funding for graduate programs. T. A. joined the influx of young professors taking positions there. The opportunities were great and he attracted 13 doctoral students, all of whom succeeded as university professors. To this day he asserts that his main achievement was to guide his doctoral students. All of them were established in areas of mathematics with a future.

Much of his work was international. He was twice a Fulbright senior scholar to Eastern Europe at the University of Szeged, Hungary and at the Technical University of Budapest, once as senior lecturer and once as senior researcher. He had brief research appointments at the University of Florence and the University of Madrid. He was a plenary lecturer at conferences in Europe and Asia almost every year since 1984.

T. A. retired from teaching in 1998, moved to the state of Washington, and spent the last 14 years with the Northwest Research Institute (732 Caroline St., Port Angeles, WA, taburton at olypen.com) conducting research, writing, and lecturing at conferences. He returned to teaching for one semester at the University of Memphis in 2009, offering a graduate course on his book on Liapunov functionals for integral equations. In the summer of 2012 he delivered a plenary lecture at the Conference on Differential and Difference Equations and Applications 2012 in Terchova, Slovakia, a keynote talk at the 10^{th} International Conference on Fixed Point Theory and its Applications in Cluj-Napoca, Romania, and a plenary talk at the 8^{th} International Conference on Differential Equations and Dynamical Systems at the University of Waterloo, Ontario, Canada.

2 Basic Trends of His Scientific Work

Periodicity and Oscillation

In 1944, N. Levinson studied the equation $x'' + f(x, x')x' + g(x) = e(t)$ under conditions similar to those in [1] and with $e(t)$ periodic. Patterned after the behavior of the constant coefficient analog, he focused on the behavior of solutions of the system form when all solutions entered and remained in a bounded region. It was behavior later named uniform ultimate boundedness. He used a translation argument and Brouwer's fixed point theorem to get a periodic solution. Conditions showing this boundedness were widely sought. Exactly 20 years later, the year T. A. received the Ph.D., G. Sansone and R. Conti published an English version of a 533 page monograph studying such problems, their applications, generalizations, and history. In collaboration with C. G. Townsend, T. A. advanced the necessary and sufficient conditions for global stability of the unforced equation [1] to boundedness and periodicity of solutions of the forced equation under similar conditions. That work appeared as

[3] Burton, T. A. and Townsend, C. G. On the generalized Liénard equation with forcing function. *J. Differential Equations* **4** (1968) 620–633.

[4] Burton, T. A. and Townsend, C. G. Stability regions of the forced Liénard equation. *J. London Math. Soc.* (2) **3** (1971) 393–402.

It is to be remembered that all of this was in an effort to show that the behavior of solutions of the nonlinear equation was very similar to the behavior of the solutions of the linear constant coefficient equation. The results established clear boundaries for which nonlinear problems would have solutions like the linear problems. The problems are still vigorously studied under increasingly more general assumptions.

This is not to be confused with the problem encountered when the damping, $f(x, y)$, changes sign, a problem which one of T. A.'s doctoral students, John Graef, solved in a similar way in his doctoral dissertation obtaining necessary and sufficient conditions.

The study of periodicity led to the study of oscillation theory, as well as an introduction to functional differential equations. There was then a line of joint papers with R. Grimmer a typical of which was Burton, T. and Grimmer [5]:

[5] Burton, T. and Grimmer, R. Oscillation, continuation, and uniqueness of solutions of retarded differential equations. *Trans. Amer. Math. Soc.* **179** (1973) 193–209. (Correction **187** (1974) 429).

Uniform Asymptotic Stability

At the time T. A. began his work, one of the challenging questions on stability theory by Liapunov's direct method from 1950 to 1992 was to prove or give a counterexample to the following conjecture. Briefly, we have a functional differential equation with bounded delay denoted by $x' = f(t, x_t)$, $x_t(s) = x(t+s)$ for $-r \le s \le 0$. We denote by C_H the H-ball in the function space of all continuous functions on $[-r, 0]$ into $\Re^n$. The supremum norm is denoted by $\|\cdot\|$ and continuous increasing functions are denoted by W_i where $W_i(0) = 0$.

Conjecture 2.1 *If there are a continuous and locally Lipschitz functional* $V : [0, \infty) \times C_H \to [0, \infty)$ *and functions* W_i *with (i)* $W_1(|\phi(0)|) \le V(t, \phi) \le W_2(\|\phi\|)$ *and (ii)* $V'(t, \phi) \le -W_3(|\phi(0)|)$*, then the zero solution is uniformly asymptotically stable.*

A counterexample was given by Geza Makay in 1991 and a more complete one by Junji Kato in 1992, exactly 100 years after the publication of Liapunov's famous work. Details and a history on these are found in [7], pp. 264–293. The conjecture is on p. 269. But something close to it is true. In the 1950's Krasovskii had noted that in most applications the function W_2 was actually replaced by something a bit more severe: $V(t, \phi) \le W_4(\phi(0)| + W_5(|||\phi|||)$, where $|||\cdot|||$ is the L^2-norm on $\phi : [-r, 0] \to \Re^n$. He had obtained an asymptotic stability conclusion under the additional assumption that $f(t, x_t)$ is bounded for x_t bounded, the old condition of Marachkoff from 1942.

In 1978, T. A. proved that Krasovskii's result was true without the Marachkoff condition and that the conclusion is actually uniform asymptotic stability, just as in the original conjecture.

Paper [6] introduced the idea of playing W_3 against W_2 which proved to be very successful in many problems.

[6] Burton, T. A. Uniform asymptotic stability in functional differential equations. *Proc. Amer. Math. Soc.* **68** (1978) 195–199.

It opened a way to further results, particularly by Laszlo Hatvani, Tingxiu Wang, and Bo Zhang. A summary is found on pp. 264-293 of:

[7] Burton, T. A. *Volterra Integral and Differential Equations, Second Edition.* Elsevier, Amsterdam, 2005.

It is an interesting and important problem. However, the counterexamples and the subsequent advances of Hatvani, Wang, and Zhang show that this old conjecture on which so many investigators had worked for so long was very nearly true.

Stability in Functional Differential Equations

His next main project involved Liapunov functional methods for functional differential equations with emphasis on integrodifferential equations. The foundation was laid in five main papers:

[8] Burton, T. A. Stability theory for Volterra equations. *J. Differential Equations* **32** (1979) 101–118.

[9] Burton, T. A. Stability theory for functional differential equations. *Trans. Amer. Math. Soc.* **255** (1979) 263–275.

[10] Burton, T. A. Stability theory for delay equations. *Funkcialaj Ekvacioj* **22** (1979) 67–76.

[11] Burton, T. A. and Mahfoud, W. E. Stability criteria for Volterra equations. *Trans. Amer. Math. Soc.* **279** (1983) 143–174.

[12] Burton, T. A. and Hatvani, L. Stability theorems for nonautonomous functional differential equations by Liapunov functionals. *Tohoku Math. J.* **41** (1989) 65–104.

A unified treatment of existing theory through 1983 is found in

[13] Burton, T. A. *Volterra Integral and Differential Equations.* Academic Press, Orlando, 1983,

which was updated in a second edition [7].

Fixed Point Theory I

The work on stability of integrodifferential equations led naturally to questions of periodic solutions of Volterra integrodifferential equations with infinite delay. This presented special problems in fixed point theory concerning the "sandwich" theorems, asymptotic fixed point theorems, and compactness. Everything depended on two things. There was the need to establish conditions under which solutions are uniformly ultimately bounded and this was done with Liapunov theory. But the translation arguments given by N. Levinson so long ago had no chance of holding for these problems. Everything depended on the study of spaces having unbounded compact sets. It was an interesting study and investigators were constantly on new ground. In such an unexpected way, exactly the same problems occur when we study fractional differential equations. These are developed in [33].

One can trace the evolution of those questions and solutions through the papers:

[14] Burton, T. A. Periodic solutions of nonlinear Volterra equations. *Funkcialaj Ekjvacioj* **27** (1984) 301–317.

[15] Burton, T. A. Periodic solutions of integrodifferential equations. *Proc. London Math. Soc.* **31** (1985) 537–548.

[16] Arino, O. A., Burton, T. A., and Haddock, J. R. Periodic solutions of functional differential equations. *Roy. Soc. Edinburgh Proc. A.* **101A** (1985) 253–271.

All of this was integrated into the existing theory and appeared in:

[17] Burton, T. A. *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*. Academic Press, Orlando, 1985; reprinted by Dover, Mineola, New York, 2005.

But a much more final disposition appeared later in:

[18] Burton, T. A. and Zhang, Bo. Uniform ultimate boundedness and periodicity in functional differential equations. *Tohoku Math. J.* **42** (1990) 93–100.

Liapunov Functionals for Integral Equations I

The work on stability and periodic solutions of integrodifferential equations provided a background and insights to attack the old problem of constructing Liapunov functionals for integral equations. The direct method of Liapunov had been applied very successfully for ordinary, functional, and partial differential equations, as well as related areas such as control theory. In all of these problems, the elementary technique of uniting the Liapunov functional with the differential equation was achieved by means of the chain rule, extended in a reasonable way to non-elementary cases. If an integral equation could be differentiated to obtain an integrodifferential equation, then the direct method could be readily applied. In 1992, exactly 100 years after Liapunov's famous paper, T. A. constructed the first successful Liapunov functionals for integral equations and united them in a simple way to the integral equation. The work was presented at the centennial celebration of Liapunov's paper held in Tampa, Florida and sponsored by one of the great investigators of the direct method, V. Laksmikantham, and the International Federation of Nonlinear Analysts. It appeared in the conference proceedings:

[19] Burton, T. A. Examples of Lyapunov functionals for non-differentiated equations. *Proc. First World Congress Nonlinear Analysts*, 1992. V. Lakshmikantham, ed. Walter de Gruyter publisher, 1996, New York. pp. 1203–1214.

He was joined the next year by a former doctoral student of Taro Yoshizawa, Tetsuo Furumochi, from Shimane University in Matsue, Japan who came with his family to Southern Illinois University for ten months to develop the theory. A substantial number of papers resulted from the study including:

[**20**] Burton, T. A. and Furumochi, Tetsuo. Periodic solutions of a Volterra equation and robustness. *Nonlinear Analysis* **25** (1995) 1199–1219.

[**21**] Burton, T. A. and Furumochi, Tetsuo. Stability theory for integral equations. *J. Integral Equations and Applications* **6** (1994) 445–477.

The work progressed and a preliminary book was printed for use in a graduate course which he taught at the University of Memphis, Tennessee in the spring of 2009. A pdf file of the preliminary book can be downloaded free of charge at T. A.'s web page, Item 91: **http://www.math.siu.edu/burton/papers.htm**.

[**22**] Burton, T. A. *Liapunov Functionals for Integral Equations.*

The work was preliminary because the method only worked for continuous kernels which was certainly a step up from the earlier requirement that the integral equation be differentiable. It failed to cover so many important real-world problems such as those represented by fractional differential equations with kernel $(t - s)^{q-1}$ for $0 < q < 1$, including many forms of heat equations with $q = 1/2$. Much of the work involved a careful strategy and that is developed in:

[**23**] Burton, T. A. Liapunov functionals, convex kernels, and strategy. *Nonlinear Dynamics and Systems Theory* **10** (2010) 325–337.

We will return to a conclusion later.

Fixed Point Theory II

Sixty years ago Krasnoselskii studied an old paper by Schauder on elliptic partial differential equations and formulated a principle which we formalize as Krasnoselskii's Hypothesis. The inversion of a perturbed differential operator yields the sum of a contraction and a compact map. Accordingly, he formulated a fixed point theorem which was a combination of the contraction mapping principle and Schauder's fixed point theorem. In recent years the idea emerged that Krasnoselskii was advancing an idea which could unify the broad and disconnected area of differential equations. In the mid 1990s T. A. began a study of Krasnoselskii's theorem with a view to discovering the unification. The first result was

[**24**] Burton, T. A. Integral equations, implicit functions, and fixed points. *Proc. Amer. Math. Soc.* **124** (1996) 2383–2390.

That paper introduced the idea of a large contraction which has proved useful in transforming totally nonlinear differential equations into integral equations. It has played a major role in fractional differential equations, as may be seen in [35], below. Krasnoselskii's theorem had a condition which was very difficult to meet. The next three papers circumvented that problem.

[**25**] Burton, T. A. and Kirk, Colleen. A fixed point theorem of Krasnosel'skii-Schaefer type. *Mathematische Nachrichten* **189** (1998) 23–31.

[**26**] Burton, T. A. A fixed-point theorem of Krasnoselskii. *Appl. Math. Lett.* **11** (1998) 85–88.

[**27**] Burton, T. A. Krasnoselskii's inversion principle and fixed points. *Nonlinear Analysis* **30** (1997) 3975–3986.

With this background and in collaboration with Tetsuo Furumochi and Bo Zhang a comprehensive theory of stability by fixed point methods was developed. There were two main advantages over the Liapunov theory. First, with the fixed point theory the conditions were averages, while Liapunov theory was usually point-wise. Next, the construction of a Liapunov function was replaced by the usually simpler fixed point mapping. The first comprehensive paper on the subject with many examples was

[**28**] Burton, T. A. and Furumochi, Tetsuo. Fixed points and problems in stability theory. *Dynamical Systems and Applications* **10** (2001) 89–116.

Five years later the papers were collected, the techniques compared with Liapunov theory, and published in:

[**29**] Burton, T. A. *Stability by Fixed Point Theory for Functional Differential Equations.* Dover, Mineola, New York, 2006.

It had been clearly established that much stability theory of functional differential equations could be established from Krasnoselskii's theory. There was, however, a major problem. The unification which was promised by his theory was entirely missing. In an interesting way, that unification was achieved when the efforts returned to Liapunov theory for integral equations. Thus, the next topic offers the foundation for both the fixed point problem and the quest for Liapunov functionals for integral equations with singular kernels.

Liapunov Theory for Integral Equations II

In 2010 Liapunov theory was advanced to integral equations with singular kernels and, in particular, to fractional differential equations in the papers:

[**30**] Burton, T. A. A Liapunov functional for a singular integral equation. *Nonlinear Analysis* **73** (2010) 3873–3882.

[**31**] Burton, T. A. Fractional differential equations and Lyapunov functionals. *Nonlinear Analysis* **74** (2011) 5648–5662.

[**32**] Becker, L. C., Burton, T. A., and Purnaras, I. K., Singular integral equations, Liapunov functionals, and resolvents. *Nonlinear Analysis* **75** (2012) 3277–3291.

In the second paper the fractional differential equation was inverted to an integral equation which was then transformed into an integral equation with completely monotone kernel, $R(t-s)$, with the property that $0 < R(t) \leq t^{q-1}$ and

$$\int_0^\infty R(s)ds = 1.$$

That last property yielded an integral equation defining a natural mapping that was very fixed point friendly and yielded a number of papers showing qualitative properties of solutions, culminating in:

[**33**] Burton, T. A. and Zhang, Bo. Fractional equations and generalizations of Schauder's and Krasnoselskii's fixed point theorems. *Nonlinear Analysis* **75** (2012) 6485–6495.

With that, the basic Liapunov theory and its relation to fixed point theory seemed to have been laid and a preliminary book was completed and is available on amazon.com in the United States, Europe, and the United Kingdom as:

[**34**] Burton, T. A. *Liapunov Theory for Integral Equations with Singular Kernels and Fractional Differential Equations* (2012), Amazon.co.uk, 379 pages.

In 2009, T.A. participated in a conference at the N. N. Krasovskii Institute of Mathematics and Mechanics in Ekaterinburg, Russia where he presented a basic work on Liapunov theory for integral equations. As a result of that association a manuscript is being translated into the Russian language by Prof. Sergey I. Kumkov. An editing of the translation will be performed by Prof. Nikolay Yurievich Lukoyanov. The translation is to be published by the Autonomous Nonprofit Organization, Izhevsk Institute of Computer Science, Universitetskaya, I, Izhevsk, 426034 Russia.

Fixed Point Theory III

The mapping equation derived in the inversion of the fractional differential equation concerning the new kernel $R(t)$ has proved to be a unifying concept. At the present time scalar fractional differential equations, functional differential equations, and neutral equations are all treated in a unified way using that mapping. It is an intriguing evidence that Krasnoselskii had a very fruitful and general idea. Pursuing that has now become the main project. Paper [35] gives all the details of the transformations, shows the fixed points methods, and illustrates the use of large contractions and the paper [36] employs very different fixed point techniques.

[35] Burton, T. A. and Zhang, Bo. Fixed points and fractional differential equations: Examples. *Fixed Point Theory*, in press.

[36] Burton, T. A. and Zhang, Bo. L^p-solutions of fractional differential equations. *Nonlinear Studies* **19** (2) (2012) 161–177.

3 T.A.'s Doctoral Students and Dissertation Titles

John Graef, *Relaxation and Forced Oscillations in a Second Order Nonlinear Differential Equation*, 1970;
John Haddock, *Some Refinements of Liapunov's Direct Method*, 1970;
John Erhart, *Lyapunov Theory and Perturbations of Differential Equations*, 1970;
Alfredo Somolinos, On the Problem of Lurie and its Generalizations, 1974;
Wadi Mahfoud, *Oscillation, Asymptotic Behavior, and Noncontinuation of Solutions of n^{th} Order Nonlinear Delay Differential Equations*, 1975;
Leigh Becker, *Stability Considerations for Volterra Integrodifferential Equations*, 1979;
Muhammed Islam, *Periodic Solutions of Volterra Integral Equations*, 1985;
Shou Wang, *Stability and Boundedness in Ordinary and Functional Differential Equations*, 1987;
Roger Hering, *Boundedness and Stability in Functional Differential Equations*, 1988;
Tingxiu Wang, *On Uniform Asymptotic Stability of the zero Solution of Functional Differential Equations*, 1991;
Bo Zhang, *Periodic Solutions of Nonlinear Abstract Differential Equations with Infinite Delay*, 1991;
David Dwiggins, *Fixed Point Theory and Periodic Solutions for Differential Equations*, 1993;
Geza Makay, *Boundedness and Periodic Solutions of Functional Differential Equations*, 1993.

4 Exceptional Master's Student

Colleen Kirk, *Neural Networks: Convergence and Stability*, 1995.

5 Journal Editing

T. A. has periodically been an editor of the following journals:
1. *Cubo: A Mathematical Journal*; Chile.
2. *Electronic J. Qualitative Theory of Differential Equations* (jointly founded with Laszlo Hatvani, now honorary editor); Hungary.
3. *Fixed Point Theory*; Romania.
4. *Journal of Fractional Calculus and Applications*; Egypt.
5. *Nonlinear Analysis: TMA*; United Kingdom.
6. *Nonlinear Dynamics and Systems Theory* (editor and honorary editor); Ukraine.
7. *Nonlinear Studies*; United States.
8. *Opuscula Mathematica*; Poland.