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Stabilisation of a Class of Underactuated System with Tree Structure Using Backstepping Approach

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Abstract: In this paper, a method for transforming the structure of a class of underactuated mechanical system from tree to chain structure through a change of coordinates and control law is proposed. The main goal of this transformation is to allow apply control design methodologies suited to the chain structure, namely, the feedback linearization and backstepping. The effectiveness of the proposed transformation is shown via an example of underactuated system that initially possesses a tree structure and to which backstepping control was applied. However, the designed control law presents a singularity that decreases the stability domain. In order to make the latter global, a hybrid control strategy is adopted allowing to switch the control near the singularities. The stability proof and simulation results for using the hybrid switching are given.

Keywords: *underactuated mechanical system; CFD; tree structure; chain structure; systematic backstepping; Tora system; singularity; switching control.*

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1 Introduction

Recently, there has been renewed and extensive research interest in the control of underactuated mechanical systems due to their broad applications and due to the large number of open theoretical problems they present. Many real-life control mechanical systems including aircrafts, helicopters, spacecrafts, underwater vehicles, mobile robots, walking robots and flexible systems are examples of underactuated systems. Underactuated systems are systems that have fewer actuators than configuration variables. This limitation in actuators makes the control design for these systems rather complicated.

As a result, an underactuated system presents challenges which are not found in a system with full control. For instance, controllability, at least locally, is not easy to determine. Most underactuated systems are not fully feedback linearizable, and smooth feedback stabilization to a single equilibrium point is not possible [10]. Furthermore, there is no general theory that allows the systematic analysis and control design for all underactuated systems so that, most of time these systems have to be dealt with on a case by case basis [15]. Consequently, different control strategies have been proposed in the literature, among them there is the backstepping and forwarding control in [31], [14], energy and passivity based control in [12], [17], sliding mode control [5], [9] and observation [23], hybrid and switched control in [28], [41], intelligent and fuzzy control in [38], [22] just to mention a few.

In [31], underactuated systems are classified into three types according to their control flow diagram (CFD) which reflects the way generalized forces are transmitted through components, namely, the chain, tree and isolated point structures. Additionally, the author proposes a control design strategy for systems with chain structure. However, the control design issue for other structures is still an open problem.

In this paper, based on the observation that the CFD of a given system is not invariant under change of coordinate, we will show that a subclass of tree structure can be transformed in a chain structure so that the strategy of control for chain structure can be applied. However, as a result of this transformation, one assumption that was laid in the control scheme is satisfied only on a certain domain rather than on the whole space. As a consequence, a singularity in the control law appears which limits the bassin of attraction. To make this stability global, we propose a hybrid control allowing to switch through these singularities.

Others strategies and viewpoints for dealing with singularities involve the use of nilpotent approximations like in [36] and [26].

The outline of the paper is as follows. In Section 2, a standard model for underactuated systems is presented. Next, in Section 3, definitions of the CFD, the chain and the tree structure are given. In Section 4, the main result on the transformation of the structure of an underactuated system from tree to chain is presented. In Section 5, the proposed design procedure is applied to stabilize the so-called Tora system. Finally, the hybrid control that permits to switch near the singularities is presented.

2 Dynamics of Underactuated Systems

It is well-known that classical Lagrangian mechanics provides dynamical model of underactuated systems. In this paper, we consider mechanical systems with configuration vector $q \in Q$, which is an n-dimensional manifold, and with a Lagrangian:

$$\mathcal{L} = K - V = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q), \tag{1}$$

where K is the kinetic energy, V(q) is the potential energy and $M(q)$ is the inertia matrix of the system which is symmetric and positive definite.

The Euler-Lagrange equation of motion is given by:

$$M(q)\ddot{q} + H(q, \dot{q}) = F(q)u, \tag{2}$$

where $H(q, \dot{q})$ contains Coriolis, centrifugal and gravity terms and $F(q)$ is identity matrix. Suppose that $q = col(q_1, q_2) \in Q_1 \times Q_2$ where the dimension of the manifold Q_i is denoted by $n_i = dim(Q_i)$ for $i = 1, 2$ and $n_1 + n_2 = n$; then, the system (2) can be written as:

$$\begin{aligned} m_{11}(q)\ddot{q}_1 + m_{12}(q)\ddot{q}_2 + h_1(q, \dot{q}) = \tau_1, \\ m_{21}(q)\ddot{q}_1 + m_{22}(q)\ddot{q}_2 + h_2(q, \dot{q}) = \tau_2. \end{aligned} \tag{3}$$

The τ_i's are the control inputs satisfying the conditions of either one of the following actuation modes:

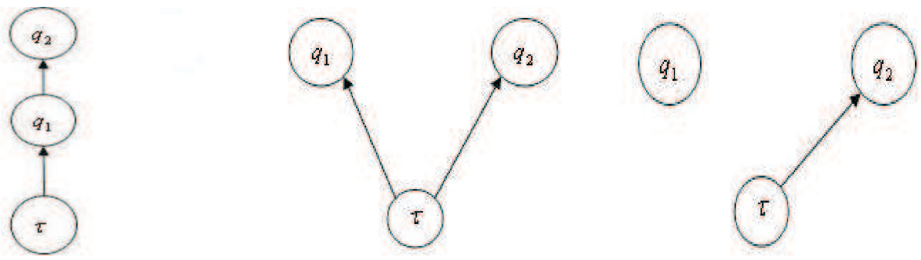
A1) $\tau = \tau_2 \in \Re^{n_2}$ is the control input and $\tau_1 \equiv 0$;

A2) $\tau = \tau_1 \in \Re^{n_1}$ is the control input and $\tau_2 \equiv 0$.

In both of the above cases, system (3) is an underactuated system. The actuation modes A1 and A2 are important due to their applications in robotics. The Acrobot [33], the Tora system [39] are actuated according to mode A1, while the Pendubot [34] and, the cart-pole system [25] are actuated according to mode A2.

3 Control Flow Diagram

In [31] a Control Flow Diagram (CFD) is constructed for each mechanism to represent the interaction forces among the degrees of freedom. Each CFD will be comprised of three possible sructutres: chain (Figure 1(a)), tree (Figure 1(b)) or isolated point (Figure 1(c)).



(a) (b) (c)

Figure 1: CFD structures for an underactuated system with 2 degrees of freedom.

In terms of these structures a precise definition of the degree of complexity is given. It was shown that the chain structure is the least complex, where both, feedback linearization technique [20] and backstepping strategy [29] can be applied. A system with tree structure is more difficult to control since we need to control certain configuration variables in parallel; that is, one control input must control more than one degree of freedom simultaneously. For systems with isolated points, certain control goal are difficult to

achieve because the control input have no influence on some variables at certain states. Control design for the last two classes is currently under investigation.
From the above discussions, it is clear that if we can transform the tree structure (or at least a subclass of tree structure) to a chain structure, then we will considerably simplify the control design for this class of systems.

4 Systems with Chain Structure

The configuration variables in a chain structure affect each other in a serial way. The most general representation of this serial connection is a triangular form given by Seto and Baillieul in [31]:

$$\begin{aligned} \ddot{q}_i &= N_i(q_1, \cdots, q_{i+1}, \dot{q}_1, \cdots, \dot{q}_{i+1}), \quad i = 1, \cdots, n-1, \\ \ddot{q}_n &= N_n(q, \dot{q}) + G(q, \dot{q})u, \end{aligned} \tag{4}$$

where $G(q, \dot{q}) \neq 0$, $N_i(.) i = 1, \cdots, n-1$ are smooths functions and either $\frac{\partial N_i}{\partial \dot{q}_{i+1}} \neq 0$ or $\frac{\partial N_i}{\partial \dot{q}_{i+1}} = 0$ but $\frac{\partial N_i}{\partial q_{i+1}} \neq 0 \quad \forall (q, \dot{q}) \in \Re^{2n}$.

The former condition ensures the conrollability of the system while the latter one ensures the connection between the degrees of freedom.

Note that the chain structure proposed here is different from the chained form systems studied in [1], generally represented by the following configuration:

$$\begin{aligned} \ddot{\xi}_1 &= u_1, \\ \ddot{\xi}_2 &= u_2, \\ \ddot{\xi}_3 &= \xi_2 u_1. \end{aligned} \tag{5}$$

In [31], Seto and Baillieul propose a systematic backstepping control strategy which globally asymptotically stabilize systems in chain structure (4). However, few underatuated systems are naturally in this form, the only examples we found are the mass sliding on a cart system [31] and the robot with joint elasticity [7]. Most of the underatuated systems are either in tree structure as the Acrobot, the Tora system, the Inverted pendulum, or in isolated point as the Ball and Beam system [16], as far as simple systems with two degrees of freedom are considered. As there is no systematic procedure for dealing with tree structure and isolated point, such structures are generally studied on a case by case basis.

In the next section, we propose to transform a subclass with tree structure into a chain structure so that the well established backstepping design procedure associated with chain structure can be applied.

5 Transformation from Tree Structure to Chain Structure

The construction of CFD for a given system depends on its coordinates, specially on the choice of generalized coordinates and the external forces. Thus, the CFD is not invariant under coordinate transformation. This simple observation leads us to search for a change of coordinates in order to transform the CFD. Thus, we consider underactuated systems satisfying the following assumptions:

Assumptions 1

B1) q_2 *is the actuated variable (case A1).*

B2) the considered system possesses a kinetic symmetry property, that is the inertia matrix depends only on the variable q_2 *so that* $M(q) = M(q_2)$.

B3) the quantity $m_{11}^{-1}(q_2)m_{12}(q_2)$ *is integrable.*

It is important to note that these assumptions are satisfied by a broad class of underactuated systems. Our main result is presented in the next theorem.

Theorem 5.1 *Assumming that Assumptions B1)-B3) hold, then an underactuated system with tree structure can be transformed in a system with chain structure.*

Proof. The proof can be broken down in two parts: first, we will show how an underactuated system can be partially linearized. Next, we will show how the linearized system can be expressed under a chain form.

In [32], Spong shows that all underatuated systems can be partially linearized using the following change of control law:

$$\tau = \alpha(q)u + \beta(q, \dot{q}) \tag{6}$$

which transforms the dynamics of (3) into

$$\begin{aligned} \dot{q}_1 &= p_1, \\ \dot{p}_1 &= f(q,p) + g_0(q)u, \\ \dot{q}_2 &= p_2, \\ \dot{p}_2 &= u, \end{aligned} \tag{7}$$

where $\alpha(q)$ is an $m \times m$ positive definite symmetric matrix and

$$g_0(q) = -m_{11}^{-1}(q)m_{12}(q).$$

In fact, from the first line of (3), for $\tau_1 = 0$ we have

$$\ddot{q}_1 = -m_{11}^{-1}(q)h_1(q, \dot{q}) - m_{11}^{-1}(q)m_{12}\ddot{q}_2$$

which yields the expression for $g_0(q)$. Substituting this in the second line of (3), we get

$$(m_{22}(q) - m_{21}(q)m_{11}^{-1}(q)m_{12}(q))\ddot{q}_2 + h_2(q, \dot{q}) - m_{21}(q)m_{11}^{-1}(q)h_1(q, \dot{q}) = \tau$$

thus, defining

$$\begin{aligned} \alpha(q) &= m_{22}(q) - m_{21}(q)m_{11}^{-1}(q)m_{12}(q), \\ \beta(q, \dot{q}) &= h_2(q, \dot{q}) - m_{21}(q)m_{11}^{-1}(q)h_1(q, \dot{q}), \end{aligned}$$

and observing that $\alpha(q)$ is positive definite and symmetric complete the first part of the proof.

However, after applying this change of control law, the new control input u appears both in linear and nonlinear subsystems. This means that (7) has a tree structure. The idea is to decouple the linear and nonlinear subsystems so that the system (7) will be in a triangular form.

According to [25], an underactuated system which satisfies the preceding assumptions can be transformed in a strict feedback normal form. In fact, the following change of coordinates:

$$\begin{aligned} q_r &= q_1 + \gamma(q_2), \\ p_r &= m_{11}(q_2)p_1 + m_{12}(q_2)p_2 := \frac{\partial \mathcal{L}}{\partial \dot{q}_1}, \end{aligned} \tag{8}$$

transforms the dynamics of the system (7) into a cascade nonlinear system in strict feedback form:

$$\begin{aligned} \dot{q}_r &= m_{11}^{-1}(q_2)p_r, \\ \dot{p}_r &= g(q_r, q_2), \\ \dot{q}_2 &= p_2, \\ \dot{p}_2 &= u, \end{aligned} \tag{9}$$

where

$$\gamma(q_2) = \int_0^{q_2} m_{11}^{-1}(\theta) m_{12}(\theta)\, d\theta, \quad g(q_r, q_2) = -\frac{\partial V(q)}{\partial q_1}.$$

The so obtained system is also in a triangular form. More precisely, in a chain structure, since the control appears in the last equation and each variable affects the other in a serial way. Hence, the tree structure is transformed in a chain structure.

Remark 5.1 For case A2 (i.e. q_2 is not actuated) there is an other change of coordinates to transform the initial system but the obtained normal form is not in strict feedback form. It means that some tree structure could not be transformed in chain structure as the cart pole system, the pendubot, the rotating pendulum and others.

In the next section, we will illustrate this procedure design by an example.

6 Application

The problem of controlling the Tora (Translational oscillator with rotational actuator) system was introduced first by Wan, Brenstein and Coppola at the University of Michigan [39] and has attracted much attention of control theorists recently; since it exhibits nonlinear interaction between the translational and rotational motions. As a result, it has been extensively used as a benchmark for nonlinear controllers for cascade systems; namely for passivity based approaches [19], integrator backstepping procedure [39], sliding mode and robust controllers [24], dynamic surface control [27], Tensor product distributed compensation and linear matrix inequality based controller [3], speed gradient [13] and even fuzzy controller with [18]. In the best of our knowledge, this work is the first one where a switched control is applied to the Tora system. As a matter of fact, this constitutes the second contribution of the present paper.

The Tora system, depicted in Figure 2, consists of a platform that can oscillate without damping in the horizontal plane. On the platform a rotating eccentric mass is actuated by a DC motor whose motion applies a force to the platform which can be used to damp the translational oscillations. Assuming that the motor torque is the control variable, our task is to find a control law that stabilizes both rotation and translation to the rest. This implies the Tora is an underactuated mechanical system.

Note that this system possesses 2 degrees of freedom (q_1, q_2) where q_1 is the unactuated variable and q_2 is the actuated one. The Euler-Lagrange equation of motion for the

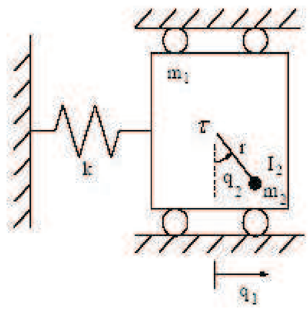


Figure 2: The Tora system.

Tora system is given by:

$$\begin{aligned}(m_1+m_2)\ddot{q}_1 + m_2 r\cos(q_2)\ddot{q}_2 - m_2 r\sin(q_2)\dot{q}_2^2 + kq_1 &= 0,\\ m_2 r\cos(q_2)\ddot{q}_1 + (m_2r^2+I_2)\ddot{q}_2 + m_2 g r\sin(q_2) &= \tau,\end{aligned} \tag{10}$$

where m_1 is the mass of the cart, m_2 is the mass of the eccentric mass, r is the radius of the rotation, k is the spring constant, g is the gravity acceleration and τ is the torque input.

The system (10) can be rewritten as:

$$\begin{aligned}\ddot{q}_1 &= \frac{1}{detM(q_2)}(-m_2r\cos(q_2)\tau + gm_2^2r_2^2\cos(q_2)\sin(q_2)\\ &\quad -(m_2r^2+I_2)(kq_1 - m_2r\sin(q_2)\dot{q}_2^2)),\\ \ddot{q}_2 &= \frac{1}{detM(q_2)}((m_1+m_2)\tau - (m_1+m_2)m_2gr\sin(q_2)\\ &\quad +m_2r\cos(q_2)(kq_1 - m_2r\sin(q_2)\dot{q}_2^2)),\end{aligned} \tag{11}$$

with $detM(q_2) = (m_1+m_2)(m_2r^2+I_2) - (m_2r\cos(q_2))^2$.

The associated CFD to (11) is given by Figure 3 which is in tree structure. After a partial linearization using change of control input:

$$\tau = \alpha(q)u + \beta(q,\dot{q}) \tag{12}$$

with

$$\begin{aligned}\alpha(q_2) &= (m_2r^2+I_2) - \frac{(m_2r\cos(q_2))^2}{m_1+m_2} \quad \forall q_2 \in [-\pi,\pi],\\ \beta(q,\dot{q}) &= m_2gr\sin(q_2) - \frac{m_2r\cos(q_2)}{m_1+m_2}(kq_1 - m_2r\sin(q_2)\dot{q}_2^2).\end{aligned}$$

The dynamics of the Tora becomes

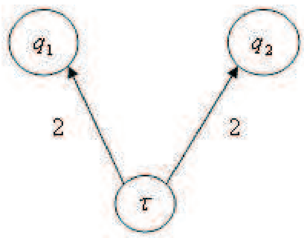


Figure 3: Tora system CFD.

$$\begin{aligned} \dot{q}_1 &= p_1, \\ \dot{p}_1 &= f_0(q,p) + g_0(q)u, \\ \dot{q}_2 &= p_2, \\ \dot{p}_2 &= u, \end{aligned} \tag{13}$$

with

$$f_0(q,p) = \frac{(m_2 r \sin(q_2))p_2 - kq_1}{m_1 + m_2}, \quad g_0 = \frac{m_2 r \cos(q_2)}{m_1 + m_2}.$$

Note that $M(q) = M(q_2)$, that the Tora system is actuated according to mode A1 and the function $\gamma(q_2)$ can be calculated explicitly as

$$\gamma(q_2) = \int_0^{q_2} \frac{m_2 r \cos(\theta)}{m_1 + m_2} \, d\theta = \frac{m_2 r \sin(q_2)}{m_1 + m_2}$$

so all the assumptions B1-B3 are verified. Thus, the global change of coordinates:

$$\begin{aligned} q_r &= q_1 + \frac{m_2 r \sin(q_2)}{m_1 + m_2}, \\ p_r &= (m_1 + m_2)p_1 + m_2 r \cos(q_2) p_2, \end{aligned} \tag{14}$$

transforms the dynamics of the Tora system into cascade nonlinear system in strict feedback form:

$$\begin{aligned} \dot{q}_r &= \frac{1}{(m_1 + m_2)} p_r, \\ \dot{p}_r &= -kq_r + k\gamma(q_2), \\ \dot{q}_2 &= p_2, \\ \dot{p}_2 &= u. \end{aligned} \tag{15}$$

The system (15) can be written as:

$$\begin{aligned} \ddot{q}_r &= -\frac{k}{m_1 + m_2} q_r + \frac{k m_2 r}{(m_1 + m_2)^2} \sin(q_2), \\ \ddot{q}_2 &= u, \end{aligned} \tag{16}$$

which is in the form of a chain structure. The associated CFD to (16) is given by Figure 4 Hence, the change of control (12) and the coordinates transformation (14) transform

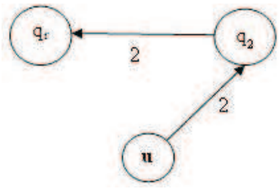


Figure 4: CFD of the transformed Tora system.

the tree structure of the Tora system into a chain structure.

Next, as the Tora is now expressed as a chain structure, we can then apply the procedure proposed by Seto and Baillieul in [31], to design the control law that globally asymptotically stabilizes the system. In order to apply this procedure, one must first verify the following assumptions:

Assumptions 2

C1) $N_i(0) = 0, i = 1, \cdots, n.$

C2) *For each* $i = 1, \cdots, n-1, N_i(.)$ *are smooth functions with bounded states* $q_1, \cdots, q_i, \dot{q}_1, \cdots, \dot{q}_i$, *the boundedness of the function* N_i *implies the boundedness of the states* q_{i+1} *and* $\dot{q}_{i+1}$.

C3) *Either* $\frac{\partial N_i}{\partial \dot{q}_{i+1}} \neq 0$ *or* $\frac{\partial N_i}{\partial \dot{q}_{i+1}} = 0$ *but* $\frac{\partial N_i}{\partial q_{i+1}} \neq 0 \quad \forall (q, \dot{q}) \in \Re^{2n}$.

C4) *For any* $\frac{\partial N_i}{\partial \dot{q}_{i+1}} \neq 0$ *, the nonlinear system* $N_i(0, \cdots, 0, q_{i+1}, 0, \cdots, 0, \dot{q}_{i+1}) = 0$ *is globally asmptotically stable at the origin, or when* $\frac{\partial N_i}{\partial \dot{q}_{i+1}} = 0$ *but* $\frac{\partial N_i}{\partial q_{i+1}} \neq 0$, *the nonlinear system* $N_i(0, \cdots, 0, q_{i+1}, 0, \cdots, 0) = 0$ *is globally asmptotically stable at the origin.*

Assumption C1 is a necessary condition for the origin to be an equilibrium point of the closed loop system. C2 is necessary to avoid the peaking phenomenon, C3 ensures the connection between degrees of freedom of the system and C4 is equivalent to the condition on the global asymptotic stability of the zero dynamics.

Then, the procedure is defined as follows. Let $\bar{q}_1 = [q_1, \dot{q}_1]^T$, $b = [0, 1]^T$, P is a positive definite matrix with all elements being positive and N_i, N_n and G are variables defined in (4). The sequences e_i, G_i and W_i are defined as:

$$e_1 = \bar{q}_1^T P b, \qquad G_1 = 1, \qquad W_1 = 0,$$

for $i = 1, \cdots, n-1$,

$$\left.\begin{array}{l} e_{i+1} = G_i N_i + W_i + k_i e_i, \\ G_{i+1} = \frac{\partial N_i}{\partial \dot{q}_{i+1}} G_i, \\ W_{i+1} = \sum_{j=1}^{i+1} \frac{\partial e_{i+1}}{\partial q_j} \dot{q}_j + \sum_{j=1}^{i} \frac{\partial e_{i+1}}{\partial \dot{q}_j} N_j + e_i, \end{array}\right\} if \frac{\partial N_i}{\partial \dot{q}_{i+1}} \neq 0; \tag{17}$$

$$\left.\begin{array}{l} e_{i+1} = G_{i+1}\dot{q}_{i+1} + W_{(i+1)1} + k_{(i+1)1}e_{(i+1)1}, \\ e_{(i+1)1} = G_iN_i + W_i + k_ie_i, \\ G_{i+1} = \frac{\partial N_i}{\partial q_{i+1}}G_i, \\ W_{i+1} = \sum_{j=1}^{i+1}\frac{\partial e_{i+1}}{\partial q_j}\dot{q}_j + \sum_{j=1}^{i}\frac{\partial e_{i+1}}{\partial \dot{q}_j}N_j + e_{(i+1)1}, \\ W_{(i+1)1} = \sum_{j=1}^{i}(\frac{\partial e_{(i+1)1}}{\partial q_j}\dot{q}_j + \frac{\partial e_{(i+1)1}}{\partial \dot{q}_j}N_j) + e_i, \end{array}\right\} if\, \frac{\partial N_i}{\partial \dot{q}_{i+1}} = 0;$$

and $k_{(i+1)1}$, k_i, $i = 1, \cdots n-1$, k_n are positive constants.

The control law is chosen according to the following theorem.

Theorem 6.1 *[31] Under assumptions C1-C4, the system (4) is globally asymptotically stable at the origin if the control law is chosen as*

$$u = -(G_nN_n + w_n + k_ne_n)(G_nG)^{-1}. \tag{18}$$

The application of the above control scheme to the Tora system leads to the following control law:

$$u_{nL} = -\frac{(m_1+m_2)^2}{k\cos(q_2)}(c_1\dot{q}_r + \frac{k}{(m_1+m_2)^2}\dot{q}_2(c_2\cos(q_2) - \dot{q}_2) + c_3q_r + c_4\sin(q_2)), \tag{19}$$

where c_1, c_2, c_3, c_4 are positive constants. Clearly the obtained control law is simple and easy to implement. In addition, the rate of convergence can be controlled by adjusting the gain constants c_i.

Nevertheless, this control is valid for any $q_2 \neq (2k+1)\pi/2$. This is a consequence of the fact that assumption C3, is not always verified $\forall(q,\dot{q}) \in \Re^{2n}$, since for the Tora system $\frac{\partial N_i}{\partial q_{i+1}} \neq 0$ only for $q_2 \neq (2k+1)\pi/2$.

This means that the control has singularities that make the bassin of attraction not the entire space and hence the stability is not global.

One solution to avoid divergence of the states is to adjust the gains such that the trajectories are kept near the equilibrium. However, keeping trajectories near the equilibrium will imply little effort but will induce large settling time. Moreover, if the initial conditions of q_2 are chosen greater or equal to $\pi/2$, the states and the control will diverge due to the singularity; therefore, this solution must be discarded.

In the next section, we present a solution to make the asymptotic stability global; i.e. a control system that is valid for any initial conditions.

7 Switching Through Singularities

The idea is to use a hybrid control law which switches between the designed control law (19) away from singularities and another control law that will be designed close the singularities. Control techniques based on switching between different controllers have been applied extensively in recent years [35, 40, 41]. The importance of such control stems from the existence of systems that cannot be asymptotically stabilized by a single continuous feedback control law.

Now, we must design the second control law and the procedure we used is very simple. The idea is to use the Jacobian linearized system around the singularity point to calculate a linear control law that will be applied near singularities. Once the trajectories go through the neighborhood of singularities, we come back to the nonlinear control law to achieve global asymptotic stabilization of all the states.

7.1 Expression of the linear control law

The linearized model of the Tora system around $(q_r, p_r, q_2, p_2) = (0, 0, \pi/2, 0)$ is given by:

$$\begin{aligned} \dot{\delta q_r} &= \frac{1}{(m_1 + m_2)}\delta p_r, \\ \dot{\delta p_r} &= -k\delta q_r, \\ \dot{\delta q_2} &= \delta p_2, \\ \dot{\delta p_2} &= \delta u. \end{aligned} \tag{20}$$

The new problem that appears now is that the subsystem $(\delta q_r, \delta p_r)$ is not controllable; fortunately, it is stable. Due to Borckett in [4], if the uncontrollable modes are stable, the whole system can still be stabilized.

The linear control law is given by:

$$u_L = -Kx, \tag{21}$$

where $x = [\delta q_2, \delta p_2]^T$ and $K = [K_1 \quad K_2]$ is a matrix gain fixed either by LQR or by pole placement approaches.

Remark 7.1 Note that, even the uncontrolled modes of the linearized system around the singularity point are stable, it does not mean that the whole system is stable. Indeed, if any control is applied to the Tora system, all trajectories will go to infinity since $\frac{1}{cosq_2}$ becomes very large.

The application in simulation of this switched control to the Tora system with the parameters $m_1 = 10kg, m_2 = 1kg, k = 5N/m, r = 1m, I = 1kg/m$, shows the effectiveness of the proposed procedure, see Figure 5. In fact, even for hard initial conditions like the singularity point $q_2 = \pi/2$ (Figure 6) or a far initial point $q_2 = \pi$ (Figure 7), the proposed control law still stabilizes the system. The switch from one control to the other is orchestrated by the state q_2, so that, while $|q_2|$ is out of the interval $\frac{\pi}{2} \pm e$, the nonlinear control u_{nL} is applied and when $|q_2|$ goes through this interval, we switch to the linear control u_L)Figure 8). The size of this interval is directly related to the control effort. In fact, we have noted that small value of e (around 0.2 or 0.3) (Figure 9) leads to more important effort than larger value of e (like 0.5 or 0.6) (Figure 5). This is due to the fact that with a large interval, we do not allow $cos(q_2)$ to become too small in order to avoid great value for u_{nl}.

7.2 Stability proof of the hybrid control

Mathematically, a switched system can be described by a differential equation of the form:

$$\dot{x} = f_p(x), \qquad p \in \mathcal{P}, \tag{22}$$

where $\mathcal{P}$ is an index set and let $\sigma(t) = p = \{1, 2\}$ be a switching signal. We are assuming here that the individual subsystems have the origin as a common equilibrium point $f_p(0) = 0$.

Remark 7.2 A necessary condition for asymptotic stability under arbitrary switching is that all of the individual subsystems are asymptotically stable. However, this condition is not sufficient [21]. Nevertheless, if switching among asymptotically stable subsystems is slow enough, one would intuitively expect a stable response.

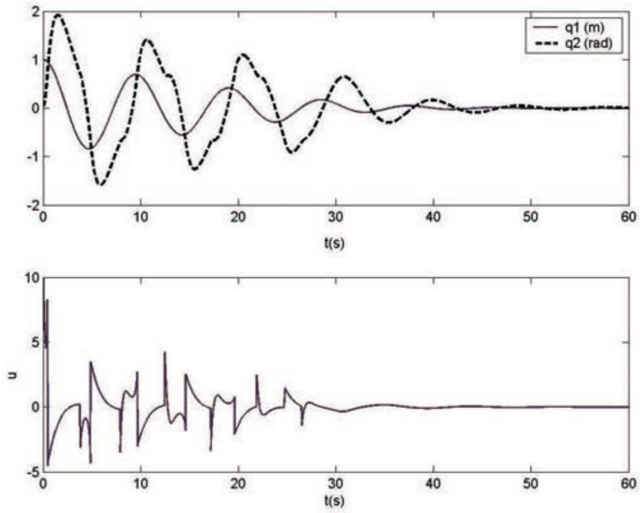


Figure 5: States trajectories and control input of the Tora system for the initial condition $(q_1, q_2, p_1, p_2) = (1, 0, 0, 0)$.

It is easy to see that if the family of systems (22) has a common Lyapunov function V such that $\nabla V(x) f_p(x) < 0$ for all $x \neq 0$ and all $p \in \mathcal{P}$, then the switched system is asymptotically stable for any switched signal σ [21]. Hence, one possible approach to prove the stability of the hybrid system is to find a common Lyapunov function for the family (22). If we can not find such function, one tool for proving stability in such cases employs multiple Lyapunov functions (see [2], [8] and the references therein). Since the individual subsystems in the family (22) are assumed to be asymptotically stable, there is a family of Lyapunov functions $[V_p : p \in \mathcal{P}]$ such that the value of V_p decreases on each interval where the $p-th$ subsystem is active. Then, the switched system is globally asymptotically stable if for every p the value of V_p at the end of each such interval exceeds the value at the end of the next interval on which the $p-th$ subsystem term is active [21]

For the Tora system, these functions are given by:
$V_{nL} = \frac{1}{2}\bar{q}_1^T P \bar{q}_1 + \frac{1}{2}e_{21}^2 + \frac{1}{2}e_2^2$ for the nonlinear subsystem,
$V_L = \frac{1}{2}\tilde{x}^T R \tilde{x}$ for the linearized subsystem,
where $\bar{q}_1$, P, e_{21} and e_2 are variables defined in the sequences of the control scheme (17), $\tilde{x} = (\delta q_r, \delta p_r, \delta q_2, \delta p_2)$ is the vector of coordinates of the linearized system and R is a symmetric positive definite matrix.

In a previous work [6], we give the proof that V_{nL} is a Lyapunov function for the nonlinear subsystem under u_{nL} control. We first recall briefly this proof and then give the one related to the linearized subsystem under u_L control.

In [31], the authors did not give the proof of Theorem 6.1 and refer the reader to the proof given for the adaptive case for system with parametric uncertainties in [30]. Moreover, the proof there is given only for the control derived from the first sequences in (17). We propose to give the proof of Theorem 6.1 for system with no parametric uncertainties and for the case when the control is derived from the second sequences in

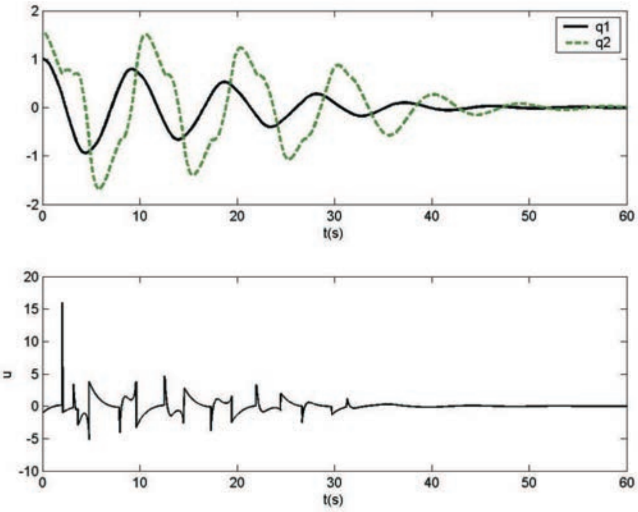


Figure 6: States trajectories and control input of the Tora system for the initial condition $(q_1, q_2, p_1, p_2) = (1, \frac{\pi}{2}, 0, 0)$.

(17) since for the Tora system $\frac{\partial N_i}{\partial \dot{q}_{i+1}} = 0$ but $\frac{\partial N_i}{\partial q_{i+1}} \neq 0$.

Proof. As the Tora system possesses two degrees of freedom, we limit the proof to the case $n = 2$. For each degree of freedom q_i, q_{i+1} can be considered as a "control variable" which governs the behavior of q_i. Hence, we determine a reference position q_{r2} for q_2 such that when $q_2 \to q_{r2}$, q_1 will behave as desired.

Step 1 $i = 1$.

When $\frac{\partial N_1}{\partial \dot{q}_2} = 0$ and $\frac{\partial N_1}{\partial q_2} \neq 0$, we obtain the differential equation

$$\ddot{q}_1 = N_1(q_1, q_2, \dot{q}_1) \tag{23}$$

and define a reference position q_{r2} as $q_{r2} = q_2 - N_1 - k_1 q_1 - k_2 \dot{q}_1$. The error between the reference and the actual position is given by

$e_{21} = q_2 - q_{r2} = N_1 + k_1 q_1 + k_2 \dot{q}_1 \Rightarrow N_1 = e_{21} - k_1 q_1 - k_2 \dot{q}_1$. Define

$$\bar{q}_1 = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ -k_1 & -k_2 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where k_1 and k_2 are chosen such that $\ddot{q}_1 + k_2 \dot{q}_1 + k_1 q_1 = 0$ is asymptotically stable at $(q_1, \dot{q}_1) = (0, 0)$. This implies the existence of a positive definite matrix P such that $A^T P + PA = -Q < 0$. Applying the above definitions to (23), we get

$$\dot{\bar{q}}_1 = A\bar{q}_1 + be_{21}.$$

Consider the following Lyapunov function

$$V_{11} = \frac{1}{2}(\bar{q}_1^T P \bar{q}_1 + e_{21}^2) \tag{24}$$

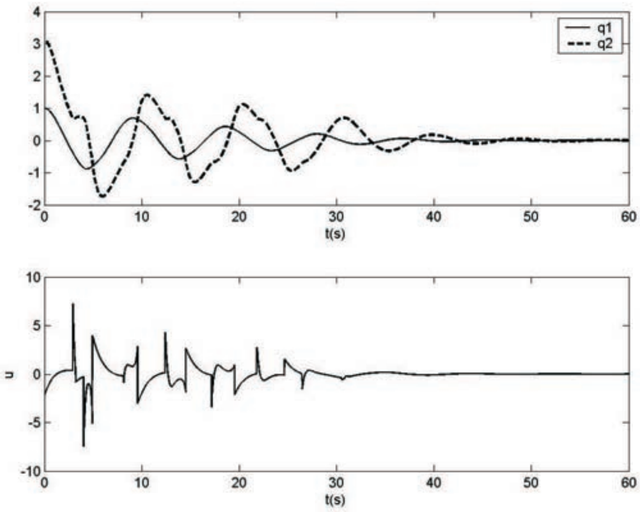


Figure 7: States trajectories and control input of the Tora system for the initial condition $(q_1, q_2, p_1, p_2) = (1, \pi, 0, 0)$.

The time derivative $\dot{V}$ is given by

$$\dot{V}_{11} = -\frac{1}{2}\bar{q}_1^T Q\bar{q}_1 + \bar{q}_1^T Pbe_{21} + e_{21}\dot{e_{21}},$$

if $e_1 = \bar{q}_1^T Pb$ and $\nu_{11} = \frac{1}{2}\bar{q}_1^T Q\bar{q}_1$, then

$$\begin{aligned}
\dot{V}_{11} &= -\nu_{11} + e_{21}(\dot{e}_{21} + e_1) \\
&= -\nu_{11} + e_{21}(\dot{N}_1 + k_1\dot{q}_1 + k_2\ddot{q}_1 + e_1) \\
&= -\nu_{11} + e_{21}(\frac{\partial N_1}{\partial q_1}\dot{q}_1 + \frac{\partial N_1}{\partial q_2}\dot{q}_2 + \frac{\partial N_1}{\partial \dot{q}_1}\ddot{q}_1 + \frac{\partial N_1}{\partial \dot{q}_2}\ddot{q}_2 \\
&\quad +k_1\dot{q}_1 + k_2\ddot{q}_1 + e_1) \\
&= -\nu_{11} + e_{21}((\frac{\partial N_1}{\partial q_1} - k_1)\dot{q}_1 + \frac{\partial e_{21}}{\partial q_2}\dot{q}_2 + (\frac{\partial e_{21}}{\partial \dot{q}_1} - k_2)\ddot{q}_1 \\
&\quad +k_1\dot{q}_1 + k_2\ddot{q}_1 + e_1) \\
&= -\nu_{11} + e_{21}(\underbrace{\frac{\partial N_1}{\partial q_2}}_{\stackrel{def}{=} G_2}\dot{q}_2 + \underbrace{\frac{\partial e_{21}}{\partial q_1}\dot{q}_1 + \frac{\partial e_{12}}{\partial \dot{q}_1}N_1 + e_1)}_{\stackrel{def}{=} W_{21}} \\
&= -\nu_{11} + e_{21}(G_2\dot{q}_2 + W_{21}).
\end{aligned}$$

Note that, we cannot reach u through $\dot{q}_2$ but rather through $\ddot{q}_2$. Hence, we add a step where we determine a reference velocity $\dot{q}_{r2}$ for $\dot{q}_2$ such that $e_{21}(G_2\dot{q}_2 + W_{21})$ is made nonpositive $\dot{q}_{r2} = \dot{q}_2 - G_2\dot{q}_2 - W_{21} - k_{21}e_{21}$.

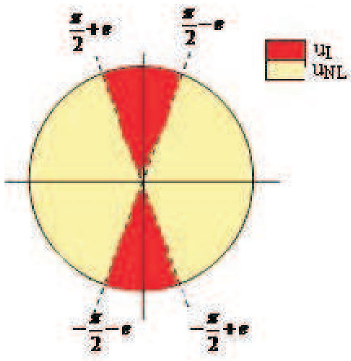


Figure 8: Switching regions for the control.

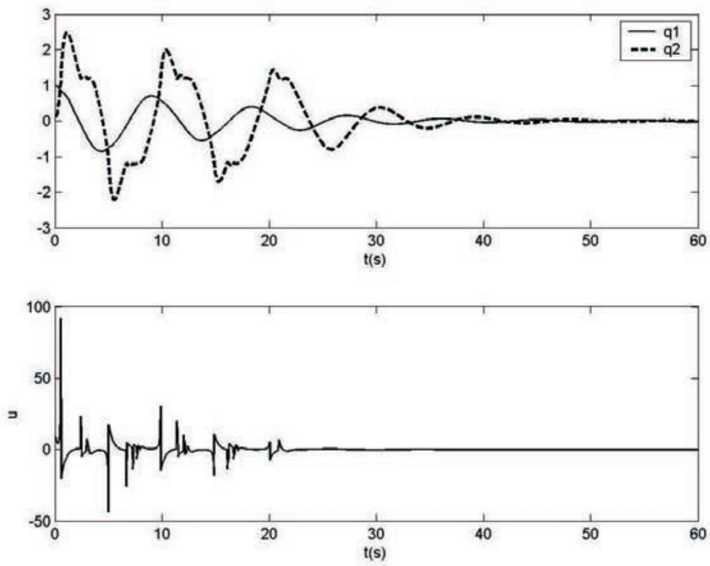


Figure 9: States trajectories and control input of the Tora system for the initial condition $(q_1, q_2, p_1, p_2) = (1, 0, 0, 0)$ and $e = 0.2$.

The error between the reference and the actual velocity is given by
$e_2 = \dot{q}_2 - \dot{q}_{r2} = G_2 \dot{x}_2 + W_{21} + k_{21} e_{21} \Rightarrow G_2 \dot{q}_2 + W_{21} = e_2 - k_{21} e_{21}$, then

$$\begin{aligned}
\dot{V}_{11} &= -\nu_{11} + e_{21}(e_2 - k_{21} e_{21}) \\
&= -\nu_{11} - k_{21} e_{21}^2 + e_{21} e_2 \\
&= -\nu_1 + e_{21} e_2
\end{aligned}$$

with $\nu_1 = \nu_{11} + k_{21} e_{21}^2$.

To compensate for e_2, we modify the scalar function V_{11} as $V_1 = V_{11} + \frac{1}{2} e_2^2$. Differentiating V_1, we obtain

$$\begin{aligned}
\dot{V}_1 &= \dot{V}_{11} + e_2 \dot{e}_2 \\
&= -\nu_1 + e_{21} e_2 + e_2 \dot{e}_2 \\
&= -\nu_1 + e_2(\dot{e_2} + e_{21}) \\
&= -\nu_1 + e_2(\underbrace{\frac{\partial e_2}{\partial q_1}\dot{q}_1 + \frac{\partial e_2}{\partial q_2}\dot{q}_2 + \frac{\partial e_2}{\partial \dot{q}_1}\ddot{q}_1 + e_{21}}_{\stackrel{def}{=} W_2} + \underbrace{\frac{\partial e_2}{\partial \dot{q}_2}}_{G_2}\ddot{q}_2) \\
&= -\nu_1 + e_2(G_2 \ddot{q}_2 + W_2) \\
&= -\nu_1 + e_2(G_2(N_2 + Gu) + W_2).
\end{aligned}$$

Finally, the expression of the Lyapunov derivative is

$$\dot{V}_1 = -\nu_1 + e_2(G_2 N_2 + G_2 G u + W_2). \tag{25}$$

In order to make $\dot{V}_1$ nonpositive, it is enough to choose u such that

$$e_2(G_2 N_2 + G_2 G u + W_2) = -k_2 e_2^2. \tag{26}$$

Thus the expression of the control law that globally asymptotically stabilizes the system is given by

$$u = -(G_n N_n + w_n + k_n e_n)(G_n G)^{-1}.$$

Note that, $G_n G$ is invertible since both G_n and G are different from 0 by assumptions ($G \neq 0$ to ensure controllability and $G_n \neq 0$ because of G_n definition in sequences (17) and of assumption C3).

Step 2 $i = 2$.

The final Lyapunov function is given by $V_2 = V_1$ such that $\dot{V}_2 = -\nu_2 - k_2 e_2^2$.
In this work, we take $V_{nL} = V_2$ as the Lyapunov function of the nonlinear subsystem.
Next, as the subsystem (20) is linear, we can choose a Lyapunov function of the form

$$V_L = \frac{1}{2} \tilde{x}^T R \tilde{x}.$$

If the matrix R is chosen diagonal then, V_L can be expressed as:

$$V_L = \frac{1}{2}(R_1 \tilde{x}_1^2 + R_2 \tilde{x}_2^2 + R_3 \tilde{x}_3^2 + R_4 \tilde{x}_4^2).$$

Differentiating V_L, we obtain:

$$\begin{aligned} \dot{V}_L &= R_1\tilde{x}_1\dot{\tilde{x}}_1 + R_2\tilde{x}_2\dot{\tilde{x}}_2 + R_3\tilde{x}_3\dot{\tilde{x}}_3 + R_4\tilde{x}_4\dot{\tilde{x}}_4 \\ &= (\frac{R_1}{m_1+m_2} - R_2k)\tilde{x}_1\tilde{x}_2 + (R_3 - K_1R_4)\tilde{x}_3\tilde{x}_4 - K_2R_4\tilde{x}_4^2. \end{aligned} \tag{27}$$

If the elements of the matrix R are chosen so that the conditions

$$\begin{cases} \frac{R_1}{m_1+m_2} &= R_2k, \\ R_3 &= K_1R_4, \end{cases}$$

are verified. Then

$$\dot{V}_L = -K_2R_4\tilde{x}_4^2.$$

The use of the LaSalle invariance principle finishes the proof.

The analysis of the stability of switched control is very difficult by means of analytical tools, so, often we are bounded to use numerical calculations [11]. The energy profile of the switched control is illustrated in Figure 10.

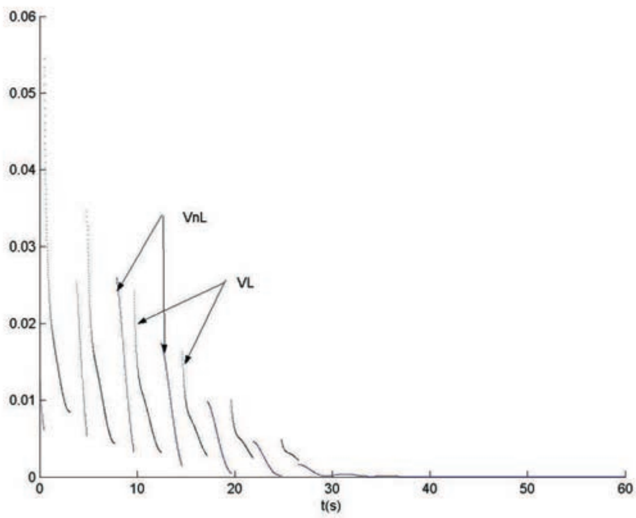


Figure 10: Energy profile of the switched system.

According to this figure, the Lyapunov functions V_{nL} and V_L satisfy the above condition and hence we can conclude that the Tora system is globally asmptotically stable.

8 Conclusion

In this paper, a transformation methodology for a class of underactuated system with tree structure to another underactuated system with chain structure is proposed by using a change of control and coordinates; so that control design strategies pertaining to the last structure can be applied. This transformation is possible under some conditions on integrability, symmetry property and actuation of certain variables that hold for broad applications of underactuated systems such as the Acrobot, Tora, Inertia-wheel pendulum, VTOL aircraft and others. As an illustrating example, the design procedure has been applied to an underactuated system with initially tree structure. However, as the obtained control law contains singularities, a hybrid control scheme that switches between a linear control law, in a neighborhood of the singularities, and a nonlinear one outside of this neighborhood is presented. Simulation results have shown the good performance and effectiveness of the proposed control strategy.

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