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Nonlinear Dynamic Inequalities and Stability of Quasilinear Systems on Time Scales

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Abstract: In this paper a novel nonlinear integral inequality on time scale is proposed. This inequality is applied to analyze stability of zero solution of quasilinear dynamic equations on time scale. Also, stability conditions are established for a wide class of nonlinearities in the system of dynamic equations.

Keywords: *dynamic equations on time scales; nonlinear inequalities; stability; asymptotic stability.*

Mathematics Subject Classification (2010): 34A34, 34A40, 34D20, 39A13, 39A11.

1 Introduction

The method of integral inequalities of motion stability theory (see [8,9] and bibliography therein) has been developed in terms of linear and nonlinear integral inequalities treated in numerous papers (see [2, 14] and bibliography therein). Appearance of dynamic equations on time scale [6] gave an impetus to the investigations in the theory of dynamic integral inequalities (see [3] and bibliography therein). The inequalities of Gronwall - Bellman type established by now and some types of nonlinear inequalities (see [4]) have been applied in the stability analysis of solutions to dynamic equations on time scale. It is of interest to further generalize nonlinear dynamic inequality of Stakhursky type (see [4, 10, 13]) for dynamic equations in the case of arbitrary real nonlinearity exponent larger than one. Such generalization makes possible the analysis of various types of stability of zero solution for a new class of quasilinear dynamic equations.

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In the present paper a new nonlinear dynamic integral inequality is obtained in view of the results of [10]. The new inequality is applied to establish sufficient conditions for stability, uniform stability and asymptotic stability of trivial solutions to a class of quasilinear dynamic equations. All the necessary information from the mathematical analysis on time scale can be found in monographs [3, 7] or paper [4] and so is omitted here.

2 Statement of the Problem

Consider a quasilinear dynamic equation of the type

$$x^{\Delta} = A(t)x + f(t,x), \quad f(t,0) = 0, \tag{1}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{T}$, and the matrix-valued function $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ and the vector-function $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following hypotheses:

(H_1) functions $A(t)$ and $f(t,x)$ are rd-continuous and $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$;

(H_2) function $f(t,x)$ satisfies Lipschitz condition with respect to spatial variable in $\mathbb{R}^n$, i.e. there exists $L > 0$ such that

$$\|f(t,x_1) - f(t,x_2)\| \le L\|x_1 - x_2\|, \quad \text{for all} \quad (t,x_1),(t,x_2) \in \mathbb{T}\times\mathbb{R}^n; \tag{2}$$

(H_3) there exist functions $\alpha(t), \varphi(t), \psi(t) \in \mathbb{C}_{rd}(\mathbb{T}, \mathbb{R}_+)$ and a constant $m > 1$ such that:

(a) $\|f(t,x)\| \le \alpha(t)\|x\|^m$;

(b) $\|e_A(t,t_0)\| \le \varphi(t)\psi(t_0)$,

for all $t \ge t_0$, belonging to $\mathbb{T}$, and $x \in \mathbb{R}^n$, where $e_A(t,t_0)$ denotes the matrix exponential function [3] of the linear dynamic equation: $x^{\Delta} = A(t)x$.

It should be noted that the conditions of hypotheses (H_1) and (H_2) ensure existence and uniqueness of solution for the dynamic equation with given initial conditions. Further, under hypotheses (H_1) — (H_3), we investigate the problem on stability, uniform stability and asymptotic stability of zero solution for dynamic equation (1). For quasilinear systems of ordinary differential equations of (1) type the conditions similar to condition (a) for integer nonlinearity exponents have been considered in a number of papers (see [5], p.266-270, [7], [12] and bibliography therein).

3 Generalized Nonlinear Dynamic Inequality

Nonlinear dynamic inequality has been a subject of investigation in paper [4] for the integer nonlinearity exponents larger than one. Here we deal with a more general situation.

Let $\mu(t)$ be a graininess function on the time scale $\mathbb{T}$. The following assertion holds.

Lemma 3.1 *Assume that the functions $a(t), b(t)$ are positive rd-continuous on $\mathbb{T}$, the function $h(t)$ is nonnegative rd-continuous on $\mathbb{T}$ and $m > 1$ is a real number. If the function $\frac{a(t)}{b(t)}$ is non-decreasing on $\mathbb{T}$, for any function $u(t)$, satisfying the inequality*

$$u(t) \le a(t) + b(t)\int_{t_0}^{t} h(s)u^m(s)\Delta s, \quad t \ge t_0, \tag{3}$$

the estimate

$$u(t) \le \frac{a(t)}{\left[1 + \int\limits_{t_0}^{t} \frac{a^{m-1}(\sigma(s)) - (a(\sigma(s)) + \mu(s)b(\sigma(s))a^m(s)h(s))^{m-1}}{\mu(s)(a(\sigma(s)) + \mu(s)b(\sigma(s))a^m(s)h(s))^{m-1}} \Delta s\right]^{\frac{1}{m-1}}} \tag{4}$$

is valid on the interval $[t_0, \widetilde{t})$*, where* $\widetilde{t}$ *is the first point from the interval* $[t_0, +\infty) \cap \mathbb{T}$*, at which the denominator base number in the right-hand part of inequality* (4) *becomes non-positive.*

Proof. Assume that the function $u(t)$ satisfies inequality (3) which is written as

$$u(t) \le a(t)\Big(1 + \frac{b(t)}{a(t)} \int\limits_{t_0}^{t} h(s)u^m(s)\Delta s\Big) = a(t)w(t), \quad \text{for all} \quad t \ge t_0.$$

According to the rule of Δ-differentiation of a product of two functions, we have for $w(t)$:

$$w^{\Delta}(t) = \Big(\frac{b(t)}{a(t)}\Big)^{\Delta} \int\limits_{t_0}^{t} h(s)u^m(s)\Delta s + \Big(\frac{b(t)}{a(t)}\Big)^{\sigma} h(t)u^m(t) \le \frac{b(\sigma(t))}{a(\sigma(t))} h(t)u^m(t),$$

due to the function $b(t)/a(t)$ decreasing. Further

$$w^{\Delta}(t) \le \frac{b(\sigma(t))}{a(\sigma(t))} h(t)u^m(t) \le \frac{b(\sigma(t))}{a(\sigma(t))} h(t)a^m(t)w^m(t) = r(t)w^m(t),$$

for all $t \ge t_0$. Consider the dynamic comparison equation

$$v^{\Delta}(t) = r(t)v^m(t) \tag{5}$$

and study the behavior of its solution starting from the point $v(t_0) = 1 + \varepsilon$, where $\varepsilon > 0$ is a sufficiently small number. To this end in (5) we make the change of variable $\xi(t) = v^{1-m}(t)$, and by definition of Δ-derivative of a function obtain

$$\begin{aligned} \xi^{\Delta}(t) &= \frac{\xi(\sigma(t)) - \xi(t)}{\mu(t)} = \frac{v^{1-m}(\sigma(t)) - v^{1-m}(t)}{\mu(t)} = \\ &= \frac{(v(t) + \mu(t)v^{\Delta}(t))^{1-m} - v^{1-m}(t)}{\mu(t)} = \frac{v^{1-m}(t)}{\mu(t)}\Big(\Big(1 + \mu(t)\frac{v^{\Delta}(t)}{v(t)}\Big)^{1-m} - 1\Big) = \\ &= \frac{v^{1-m}(t)}{\mu(t)}\Big((1 + \mu(t)r(t)v^{1-m}(t))^{1-m} - 1\Big) = \frac{\xi(t)}{\mu(t)}\Big(\Big(1 + \frac{\mu(t)r(t)}{\xi(t)}\Big)^{1-m} - 1\Big) \equiv \\ &\equiv F(t, \xi), \quad \xi(t_0) = (1+\varepsilon)^{1-m}. \end{aligned}$$

Besides, it is assumed that the expression $\frac{\xi(\sigma(t)) - \xi(t)}{\mu(t)}$ in the case $\mu(t) = 0$ is equal to the limit $\lim\limits_{\tau \to 0} \frac{\xi(t+\tau) - \xi(t)}{\tau}$. Further we find that

$$\frac{\partial F(t, \xi)}{\partial \xi} = \frac{1}{\mu(t)} \left(\frac{1 + \frac{m\mu(t)r(t)}{\xi} - \Big(1 + \frac{\mu(t)r(t)}{\xi}\Big)^m}{\Big(1 + \frac{\mu(t)r(t)}{\xi}\Big)^m} \right) \le 0,$$

for all $t \in [t_0, +\infty)$, i.e. the function $F(t, \cdot)$ does not increase on the set $(0, +\infty)$. Since $\xi(t) \in (0, 1)$ for all $t \in [t_0, +\infty)$ (due to connection with the function $v(t)$), for the indicated values of t the chain of inequalities holds true

$$F(t, 0) \geq F(t, \xi(t)) \geq F(t, 1) > F(t, \infty). \tag{6}$$

We find that $F(t, 1) = \frac{(1+\mu(t)r(t))^{1-m}-1}{\mu(t)}$. It is easy to verify that the function $F(t, \xi)$ satisfies all conditions of the theorem on existence and uniqueness of solution to Cauchy problem for dynamic equation on time scale (see [6]). Therefore, the Cauchy problem

$$\xi^{\Delta}(t) = F(t, \xi(t)), \quad \xi(t_0) = (1 + \varepsilon)^{1-m}$$

possesses the only solution $\xi(t)$, which can be presented in the integral form

$$\xi(t) = (1 + \varepsilon)^{1-m} + \int_{t_0}^{t} F(s, \xi(s))\Delta s. \tag{7}$$

Further, using formula (7) and inequalities (6) we arrive at the estimate

$$\begin{aligned} \xi(t) = (1 + \varepsilon)^{1-m} + \int_{t_0}^{t} F(s, \xi(s))\Delta s \geq (1 + \varepsilon)^{1-m} + \int_{t_0}^{t} F(s, 1)\Delta s = \\ = (1 + \varepsilon)^{1-m} + \int_{t_0}^{t} \frac{(1 + \mu(s)r(s))^{1-m} - 1}{\mu(s)}\Delta s, \end{aligned} \tag{8}$$

which is valid for all $t \in [t_0, \widetilde{t})$. For the values of t from the scale $\mathbb{T}$ the expression in the right-hand part of inequality (8) is positive by Lemma 3.1, and therefore, inequality (8) is equivalent to the inequality

$$v(t) = v(t; t_0, 1 + \varepsilon) \leq \left((1 + \varepsilon)^{1-m} + \int_{t_0}^{t} \frac{(1 + \mu(s)r(s))^{1-m} - 1}{\mu(s)}\Delta s \right)^{\frac{1}{1-m}},$$

for all $t \in [t_0, \widetilde{t})$.

In view of the comparison principle [6] and the passage to the limit for $\varepsilon \to 0$ we get inequality (4). Lemma 3.1 is proved. □

Designate $h(t) = \psi(\sigma(t))\varphi^m(t)\alpha(t)$,

$$D(t, a, \rho) = \int_{a}^{t} \frac{1}{\mu(s)} \left(1 - \frac{1}{(1 + \mu(s)h(s)\psi^{m-1}(a)\rho^{m-1})^{m-1}} \right) \Delta s.$$

The following lemma provides estimate of solution to equation (1) by means of inequality (4).

Lemma 3.2 *For equation* (1) *let hypotheses* (H_1) — (H_3) *be satisfied. Then for arbitrary* $t_0 \in \mathbb{T}$ *and* $x_0 \in \mathbb{R}^n$ *the following estimate of solution* $x(t; t_0, x_0)$ *to equation* (1) *holds*

$$\|x(t; t_0, x_0)\| \leq \varphi(t)\psi(t_0)\|x_0\| \Big[1 - D(t, t_0, \|x_0\|) \Big]^{1/1-m}, \tag{9}$$

for all $t \in [t_0, +\infty) \cap \mathbb{T}$, *for which* $D(t, t_0, \|x_0\|) < 1$.

Proof. As noted, hypotheses (H_1) — (H_2) ensure the existence and uniqueness of solution $x(t;t_0,x_0)$ to equation (1) found by the Cauchy formula [6]:

$$x(t;t_0,x_0) = e_A(t,t_0)x_0 + \int_{t_0}^{t} e_A(t,\sigma(s))f(s,x(s;t_0,x_0))\Delta s, \tag{10}$$

where the integration is made on the scale $\mathbb{T}$ within the limits from t_0 to t. From (10) and hypothesis (H_3) we have the estimate of the norm $x(t;t_0,x_0)$ (further denoted as $x(t)$)

$$\|x(t)\| \le \varphi(t)\psi(t_0)\|x_0\| + \int_{t_0}^{t} \varphi(t)\psi(\sigma(s))\alpha(s)\|x(s)\|^m \Delta s.$$

Having designated $u(t) = \frac{\|x(t)\|}{\varphi(t)}$, $a(t) = \psi(t_0)\|x_0\|$, we get the inequality

$$u(t) \le a(t) + \int_{t_0}^{t} h(s)u^m(s)\Delta s, \quad t \ge t_0.$$

Since the functions in this inequality satisfy all conditions of Lemma 3.1, we get the estimate

$$u(t) \le a(t)\Big(1 - D(t,t_0,\|x_0\|)\Big)^{1/1-m},$$

which is valid for all t, such that $D(t,t_0,\|x_0\|) < 1$. Lemma 3.2 is proved. □

4 Stability Analysis of Quasilinear System.

In this section sufficient conditions of stability, uniform stability and asymptotic stability of zero solution to dynamic equations of (1) type are established in terms of generalized nonlinear dynamic inequality.

Theorem 4.1 *If for equation* (1) *for all $s \ge t_0$ there exists $K(s)$ such that $\varphi(t) \le K(s)$ for all $t \ge s \ge t_0$ and*

$$\widetilde{D}(t_0,\rho) = \lim_{t\to\infty} D(t,t_0,\rho) < \infty, \tag{11}$$

for all $t_0 \in \mathbb{T}$ and $\rho > 0$, the solution $x = 0$ of equation (1) *is stable.*

Proof. We study properties of the function $D(t,a,\rho)$, defined above. Direct computation gives that the function $D(t,a,\cdot)$ increases on the set $\mathbb{R}_+$, uniformly in t and a. Consequently, the function $\widetilde{D}(a,\cdot)$ from (11) does not decrease on $\mathbb{R}_+$, uniformly in a, and, moreover, $\widetilde{D}(a,0) = 0$. Then, for some $\lambda \in (0,1)$ the equation $\widetilde{D}(a,\rho) = \lambda$ possesses the largest solution $\rho = \rho_\lambda(a)$ for all $a \in \mathbb{T}$. We consider λ_1 to be the largest of the mentioned ones.

Then consider the function $G(t,a,\rho) = \rho\Big(1 - D(t,a,\rho)\Big)^{\frac{1}{1-m}}$. We find that

$$\frac{\partial G}{\partial \rho} = \Big(1 - D\Big)^{\frac{m}{m-1}}\Big(1 - D + \frac{\rho}{m-1}\cdot\frac{\partial D}{\partial \rho}\Big) = \Big(1 - D\Big)^{\frac{m}{m-1}}(1 - G_1), \tag{12}$$

where $G_1(t,a,\rho) = D(t,a,\rho) - \frac{\rho}{m-1} \cdot \frac{\partial D(t,a,\rho)}{\partial \rho}$. Having computed the derivative $\frac{\partial G_1}{\partial \rho}$, we make sure that the function $G_1(t,a,\cdot)$ does not decrease on $\mathbb{R}_+$, uniformly in t and a as well.

It can be easily seen that the function $\widetilde{G}_1(a,\rho) = \widetilde{D}(a,\rho) - \frac{\rho}{m-1} \cdot \frac{\partial \widetilde{D}(a,\rho)}{\partial \rho}$ does not decrease in the second argument on the set $\mathbb{R}_+$, uniformly in a, and $\widetilde{G}_1(a,0) = 0$. Then, there exists the largest value ω_1 of the parameter ω from the interval $(0,1]$, such that the equation $\widetilde{G}_1(a,\rho) = \omega$ possesses the largest solution $\rho = \rho_\omega(a)$ for all $a \in \mathbb{T}$.

Also, for the derivative $\frac{\partial \widetilde{G}}{\partial \rho}$ of the function $\widetilde{G}(a,\rho) = \rho\Big(1 - \widetilde{D}(a,\rho)\Big)^{\frac{1}{1-m}}$ make sure that an equality similar to (12) takes place

$$\frac{\partial \widetilde{G}}{\partial \rho} = \Big(1 - \widetilde{D}\Big)^{\frac{m}{m-1}} \Big(1 - \widetilde{D} + \frac{\rho}{m-1} \cdot \frac{\partial \widetilde{D}}{\partial \rho}\Big) = \Big(1 - \widetilde{D}\Big)^{\frac{m}{m-1}} (1 - \widetilde{G}_1). \tag{13}$$

Proceeding from the above arguments we find that on the set $\rho \in (0, \rho_{\omega_1}(a)]$ the derivative $\frac{\partial \widetilde{G}}{\partial \rho}$ is nonnegative for all $a \in \mathbb{T}$, and hence, the function $\widetilde{G}(a,\cdot)$ is nondecreasing.

Now let us choose some $\varepsilon > 0$ and $t_0 \in \mathbb{T}$. Designate by ξ_1 the largest value of the parameter ξ from the interval $(0, \varepsilon/\psi(t_0)K(t_0))$, such that the equation $\widetilde{G}(a,\rho) = \xi$ possesses the largest solution $\rho = \rho_\xi(a)$, not larger than $\rho_{\omega_1}(a)$ for all $a \in \mathbb{T}$. Set $\delta = \min\{\rho_{\lambda_1}(t_0), \rho_{\xi_1}(t_0)\}$ and show that if $\|x_0\| < \delta$, then $\|x(t,t_0,x_0)\| < \varepsilon$, for all $t \geq t_0$.

By the condition of the theorem for all $t \geq t_0$ from the scale we have

$$D(t,t_0,\|x_0\|) \leq \lim_{t\to\infty} D(t,t_0,\|x_0\|) = \widetilde{D}(t_0,\|x_0\|). \tag{14}$$

Since it is proved that the function $\widetilde{D}(a,\cdot)$ does not decrease on $\mathbb{R}_+$, inequalities (14) can be continued as

$$D(t,t_0,\|x_0\|) \leq \widetilde{D}(t_0,\delta) \leq \widetilde{D}(t_0,\rho_{\lambda_1}(t_0)) = \lambda_1 < 1. \tag{15}$$

From (15) we conclude that by Lemma 3.2 for all $t \geq t_0$ from the scale estimate (9) is valid. Using (9), the established properties of functions D, $\widetilde{D}$, $\widetilde{G}$ and the method of choosing of δ, we arrive at the estimates

$$\begin{gathered}
\|x(t;t_0,x_0)\| \leq \varphi(t)\psi(t_0)\|x_0\|\Big[1 - D(t,t_0,\|x_0\|)\Big]^{1/1-m} \leq \\
\leq K(t_0)\psi(t_0)\|x_0\|\Big[1 - D(t,t_0,\|x_0\|)\Big]^{1/1-m} \leq \\
\leq K(t_0)\psi(t_0)\|x_0\|\Big[1 - \widetilde{D}(t_0,\|x_0\|)\Big]^{1/1-m} = K(t_0)\psi(t_0)\widetilde{G}(t_0,\|x_0\|) \leq \\
\leq K(t_0)\psi(t_0)\widetilde{G}(t_0,\delta) \leq K(t_0)\psi(t_0)\widetilde{G}(t_0,\rho_{\xi_1}(t_0)) = K(t_0)\psi(t_0)\xi_1 \leq \\
\leq K(t_0)\psi(t_0)\xi < K(t_0)\psi(t_0)\frac{\varepsilon}{K(t_0)\psi(t_0)} = \varepsilon,
\end{gathered}$$

which are valid for all $t \geq t_0$ from the scale. This completes the proof. □

Theorem 4.2 *If for equation* (1) *there exist a positive constant K_1 and a continuous nondecreasing function $K_2(\rho)$ such that $\varphi(t)\psi(s) \leq K_1$ for all $t \geq s \geq t_0$ and*

$$\widetilde{D}(s,\rho) = \lim_{t\to\infty} D(t,s,\rho) \leq K_2(\rho),$$

for all $s \geq t_0$ $\rho > 0$, then solution $x = 0$ of equation (1) *is uniformly stable.*

Proof. Let $\varepsilon > 0$, $t_0 \in \mathbb{T}$. Due to the properties of function $K_2(\rho)$ there exists a value of the parameter η from the interval $(0, \frac{1}{2}]$ such that the equation $K_2(\rho) = \eta$ possesses the largest solution $\rho(\eta)$. Designate by η_1 the largest of the mentioned values of parameter η. We set $\delta = \min\{\rho(\eta_1), \varepsilon(2^{\frac{1}{m-1}}k_1)^{-1}\}$ and show that if $\|x_0\| < \delta$, then $\|x(t; t_0, x_0)\| < \varepsilon$, for all $t \geq t_0$.

By the condition of the theorem, for all $t \geq t_0$ from the time scale we have

$$D(t, t_0, \|x_0\|) \leq \lim_{t\to\infty} D(t, t_0, \|x_0\|) \leq K_2(\|x_0\|) < k_2(\delta) << K_2(\rho(\eta_1)) = \eta_1 \leq \frac{1}{2} < 1. \tag{16}$$

From (16) we conclude that estimate (9) is fulfilled for all $t \geq t_0$ from the time scale. Therefore,

$$\|x(t; t_0, x_0)\| \leq \varphi(t)\psi(t_0)\|x_0\|\Big[1 - D(t, t_0, \|x_0\|)\Big]^{1/1-m} \leq$$

$$\leq K_1\|x_0\|\Big[1 - D(t, t_0, \|x_0\|)\Big]^{1/1-m} \leq K_1\|x_0\|\Big[1 - \widetilde{D}(t_0, \|x_0\|)\Big]^{1/1-m} \leq$$

$$\leq K_1\|x_0\|\Big(1 - K_2(\|x_0\|)\Big)^{1/1-m} < K_1\delta 2^{\frac{1}{m-1}} \leq \varepsilon,$$

for all $t \geq t_0$ from the scale. The theorem is proved. □

Theorem 4.3 *If for equation* (1) *the conditions*

$$\widetilde{D}(s, \rho) = \lim_{t\to\infty} D(t, s, \rho) < \infty,$$

are satisfied for all $s \geq t_0$ and $\rho > 0$, and $\lim_{t\to\infty} \varphi(t) = 0$, then the solution $x = 0$ of equation (1) *is asymptotically stable. Besides, the domain of attraction of solution $x = 0$ contains a sphere $B(0, \rho_\lambda(t_0))$, where $\rho_\lambda(t_0)$ is the largest solution of the equation $\widetilde{D}(t_0, \rho) = \lambda$, $\lambda \in (0, 1)$.*

Proof. Let $\varepsilon > 0$, $t_0 \in \mathbb{T}$. Since the value of function $\varphi(t)$ vanishes for $t \to \infty$, the function is bounded. Then, by Theorem 4.2 the solution $x = 0$ of equation (1) is stable. Let us show that there exists a $\delta_0 > 0$ such that if $\|x_0\| < \delta_0$, then the limit equality $\lim_{t\to+\infty} \|x(t; t_0, x_0)\| = 0$ holds true. It can be easily verified that the function $D(t, s, \rho)$ increases in the last variable on $\mathbb{R}_+$. Therefore, the function $\widetilde{D}(s, \rho)$ does not decrease in ρ on $\mathbb{R}_+$. Then, there exists a $\lambda \in (0, 1)$, for which the equation $\widetilde{D}(a, \rho) = \lambda$ possesses the largest solution designated by $\rho_\lambda(a)$. We set $\delta_0 = \rho_\lambda(t_0)$, then for all $t \geq t_0$ and $\|x_0\| < \delta_0$ the following inequalities hold true

$$D(t, t_0, \|x_0\|) \leq \widetilde{D}(t_0, \|x_0\|) \leq \widetilde{D}(t_0, \delta_0) = \lambda < 1.$$

By Lemma 3.2 for solution $x(t; t_0, x_0)$ of equation (1) estimate (9) is valid. Using the above inequality and inequality (9) we get

$$\|x(t; t_0, x_0)\| \leq \varphi(t)\psi(t_0)\|x_0\|\Big[1 - D(t, t_0, \|x_0\|)\Big]^{1/1-m} \leq$$

$$\leq \varphi(t)\psi(t_0)\|x_0\|\Big[1 - \widetilde{D}(t_0, \|x_0\|)\Big]^{1/1-m} <$$

$$< \varphi(t)\psi(t_0)\delta_0\Big[1 - \widetilde{D}(t_0, \delta_0)\Big]^{1/1-m} \to 0,$$

whenever $t \to \infty$.

Thus, the neighborhood of the point $x = 0$ with the radius $\rho_\lambda(t_0)$ is contained in the domain of attraction of the solution $x = 0$ of equation (1). $\square$

5 Applications

Consider system of dynamic equations (1) on time scale, satisfying hypotheses (H_1) — (H_3) for any real $m > 1$, for the following values of functions $\varphi(t)$, $\psi(t)$, $\alpha(t)$:

$$\varphi(t) = Me_\lambda(t, 0), \quad \psi(t) = e_\lambda(0, t), \quad \alpha(t) = Ae_\gamma(t, 0). \tag{17}$$

Here A and M are positive constants and the real numbers λ and γ satisfy positive regressivity conditions [3]

$$1 + \mu(t)\lambda > 0, \quad 1 + \mu(t)\gamma > 0, \quad \text{for all} \quad t \in \mathbb{T}. \tag{18}$$

Assume that the scale $\mathbb{T}$ has a bounded graininess function $\mu(t)$ (i.e. there exists $\mu^* \geq 0$ v $\mu(t) \leq \mu^*$ for all $t \in \mathbb{T}$) and for arbitrary integrable function $f(t)$ and any scale segment $\langle a, b\rangle$ the representation

$$\int_a^b f(t)\Delta t = \sum_i \int_{a_i}^{b_i} f(t)dt + \sum_k f(t_k)\mu(t_k), \tag{19}$$

is valid, where the segments $\langle a_i, b_i\rangle$ and the points t_k belong to $\langle a, b\rangle$.

Applying Theorem 4.3 one can easily establish additional conditions, which the constants λ and γ must satisfy to, so that the solution $x = 0$ of equation (1) be asymptotically stable under assumptions (17) and (18). Such result is contained in Corollary 5.1. Recall that for any function $F = F(\mu(t))$ under consideration it is assumed that $F(0) = \lim\limits_{\mu \to 0} F(\mu)$ if the value $F(0)$ is not defined.

Corollary 5.1 *Let equation* (1) *satisfy assumptions (H_1) — (H_3), and the functions $\varphi(t)$, $\psi(t)$ and $\alpha(t)$ from these assumptions, in their turn, satisfy assumptions* (17) *and* (18). *Then, if there exist positive constants $\delta_1, \delta_2, \delta_3$ such that for all $t \in \mathbb{T}$ the following conditions are fulfilled:*

1) $\ln(1 + \mu(t)\lambda)^{\frac{1}{\mu(t)}} \leq -\delta_1$;

2) $\ln\Big((1 + \mu(t)\lambda)^{m-1}(1 + \mu(t)\gamma)\Big)^{\frac{1}{\mu(t)}} \leq -\delta_2$;

3) $1 + \mu(t)\lambda \geq \delta_3$,

then the solution $x = 0$ of equation (1) *is asymptotically stable.*

Proof. Since by the definition of an exponential function $\varphi(t) = Me_\lambda(t, 0) = Mexp\Big\{\int_0^t \frac{Log(1+\mu(s)\lambda)}{\mu(s)}\Delta s\Big\}$, where Log is the main logarithmic function (if $\mu(s) = 0$, then the integrand equals to λ by definition), due to (18) we have $\varphi(t) = Me_\lambda(t, 0) = Mexp\Big\{\int_0^t \frac{\ln(1+\mu(s)\lambda)}{\mu(s)}\Delta s\Big\}$. According to condition 1) of Corollary 5.1 we find that

$\varphi(t) \leq Me^{-t\delta_1} \to 0$, for $t \to \infty$. Consider further the integrand $R(t,a,\rho) = \frac{1}{\mu(t)}\Big(1 - \frac{1}{(1+\mu(t)h(t)\psi^{m-1}(a)\rho^{m-1})^{m-1}}\Big)$ of the integral in the expression for $D(t,a,\rho)$. If the inequality $\mu(t)h(t)\psi^{m-1}(a)\rho^{m-1} < 1$ is fulfilled, then the function $R(t,a,\rho)$ can be presented in the form of a sum of the convergent series

$$
\begin{gathered}
R(t,a,\rho) = \frac{1}{\mu(t)}\Big(1 - \sum_{k=0}^{\infty} \frac{(1-m)(-m)\cdot\ldots\cdot(2-m-k)}{k!}\times \\
\times\Big(\mu(t)h(t)\psi^{m-1}(a)\rho^{m-1}\Big)^k\Big) = -\sum_{k=1}^{\infty}\frac{(1-m)(-m)\cdot\ldots\cdot(2-m-k)}{k!}\times \\
\times\mu^{k-1}(t)(h(t)\psi^{m-1}(a)\rho^{m-1})^k = \sum_{k=1}^{\infty} r_k(t).
\end{gathered}
\tag{20}
$$

We shall establish conditions under which the series in (20) converges uniformly in t. To this end we find a convergent numerical series majorizing the series in (20). In view of assumptions (17) and (18) and the properties of exponential functions we find that

$$
\begin{gathered}
|\mu^{k-1}(t)h^k(t)| = |\mu^{k-1}(t)\psi^k(\sigma(t))\varphi^{km}(t)\alpha^k(t)| = \\
= A^kM^{km}\mu^{k-1}(t)e_\lambda^k(0,\sigma(t))e_\lambda^{km}(t,0)e_\gamma^k(t,0) = A^kM^{km}\mu^{k-1}(t)\times \\
\times\frac{e_\lambda^{km}(t,0)e_\gamma^k(t,0)}{e_\lambda^k(t,0)e_\lambda^k(\sigma(t),t)} = A^kM^{km}\frac{\mu^{k-1}(t)e_\beta(t,0)}{(1+\mu(t)\lambda)^k},
\end{gathered}
$$

where $\beta(t) = \frac{1}{\mu(t)}\Big((1+\mu(t)\lambda)^{k(m-1)}(1+\mu(t)\gamma)^k - 1\Big)$. We estimate the expression obtained for $|\mu^{k-1}(t)h^k(t)|$ in view of conditions 1) – 2) of Corollary 5.1:

$$
\begin{gathered}
|\mu^{k-1}(t)h^k(t)| \leq A^kM^{km}\frac{\mu^{k-1}(t)}{(1+\mu(t)\lambda)^k}\times \\
\times exp\Big\{\int_0^t \frac{\ln(1+\mu(s)\lambda)^{k(m-1)}(1+\mu(s)\gamma)^k}{\mu(s)}\Delta s\Big\} \leq \\
\leq A^kM^{km}\frac{(\mu^*)^{k-1}}{\delta_3^k}exp\Big\{\int_0^t -k\delta_2\Delta s\Big\} = \frac{A^kM^{km}(\mu^*)^{k-1}}{\delta_3^k}e^{-kt\delta_2},
\end{gathered}
$$

for all $t \in \mathbb{T}$. The obtained estimate implies that when choosing the values of the parameter ρ from sufficiently small neighborhood of zero, the series in (20) is uniformly convergent in t, therefore, by the theorem from [11] the series for $R(t,a,\rho)$ allows the term-by-term integration. As a result, we have

$$
D(t,a,\rho) = \int_a^t R(s,a,\rho)\Delta s = \int_a^t \sum_{k=1}^{\infty} r_k(s)\Delta s = \sum_{k=1}^{\infty}\int_a^t r_k(s)\Delta s.
$$

$$\Big|\int_a^t r_k(s)\Delta s\Big| \le \frac{1}{\delta_3^k k!}A^k M^{km}|(1-m)(-m)\cdot\ldots\cdot(2-m-k)|\times$$

$$\times(\psi^{m-1}(a)\rho^{m-1})^k\int_a^t exp\Big\{k\int_0^s\frac{\ln(1+\mu(\tau)\lambda)^{k(m-1)}(1+\mu(\tau)\gamma)^k}{\mu(s)}\Delta\tau\Big\}\Delta s \le$$

$$\le \widetilde{r}_k\int_a^t \mu^{k-1}(s)e^{-ks\delta_2}\Delta s.$$

The integral $I_k = \int\limits_a^t \mu^{k-1}(s)e^{-ks\delta_2}\Delta s$ is bounded with respect to t. Really, by virtue of (19)

$$I_k = \sum_i\int_{a_i}^{b_i}\mu^{k-1}(s)e^{-ks\delta_2}ds + \sum_j \mu^{k-1}(t_j)e^{-kt_j\delta_2}\mu(t_j),$$

where $(a_i, b_i) \subset (a,t)$, $t_j \in (a,t)$. Further,

$$I_k = \sum_j \mu^{k-1}(t_j)e^{-kt_j\delta_2}\mu(t_j) \le (\mu^*)^{k-1}\sum_j e^{-kt_j\delta_2}\mu(t_j) =$$

$$= (\mu^*)^{k-1}e^{k\mu^*\delta_2}\sum_j e^{-k\delta_2(t_j+\mu^*)}\mu(t_j) \le (\mu^*)^{k-1}e^{k\mu^*\delta_2}\sum_j e^{-k\delta_2(t_j+\mu(t_j))}\mu(t_j) =$$

$$= (\mu^*)^{k-1}e^{k\mu^*\delta_2}\sum_j e^{-k\delta_2\sigma(t_j)}\mu(t_j) \le (\mu^*)^{k-1}e^{k\mu^*\delta_2}\int_0^t e^{-ks\delta_2}ds =$$

$$= \frac{(\mu^*)^{k-1}e^{k\mu^*\delta_2}}{k\delta_2}\Big(e^{-ka\delta_2} - e^{-kt\delta_2}\Big).$$

It can be easily seen that for sufficiently small ρ the series for $D(t,a,\rho)$ is uniformly convergent in t and it can be estimated by a function of a and ρ. Thus, when conditions of Corollary 5.1 are fulfilled, all hypotheses of Theorem 4.3 are fulfilled too, and therefore, the solution $x = 0$ of equation (1) is asymptotically stable. □

Note, that the conditions of asymptotic stability obtained in Corollary 5.1 for zero solution of dynamic equation of certain type cover some known results for $\mathbb{T} = \mathbb{R}$.

6 Conclusion

The results of this paper together with those of paper [4] represent a solution to stability problem of quasilinear equations on time scale via the method of dynamic integral inequalities. In the case when the fundamental matrix of solutions of linear approximation of system (1) can be determined in the explicit form the established sufficient conditions of various types of stability may be of interest for applications. Some results of the development of the method of integral inequalities for dynamic equations of (1) type were the subject of paper [1].

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