Nonlinear Dynamics and Systems Theory, **13** (1) (2013) 69–78

On the Approximate Controllability of Fractional Order Control Systems with Delay

S. Kumar * and N. Sukavanam

Department of Mathematics, Indian Institute of Technology, Roorkee (IITR), Roorkee-247667, India.

Received: December 14, 2011; Revised: January 30, 2013

Abstract: In this paper, sufficient conditions of the approximate controllability for a class of fractional order semilinear control systems with bounded delay are established. To illustrate the theory an example is given.

Keywords: *fractional order system; semilinear delay systems; mild solution; reachable set; approximate controllability.*

Mathematics Subject Classification (2010): 34H05, 34K30, 34K37, 47D60, 93B05.

1 Introduction

Let V and $\hat{V}$ be real Hilbert spaces. Also, let $Z = L_2([0,\tau];V)$ and $Y = L_2([0,\tau];\hat{V})$ be the corresponding function spaces defined on $[0,\tau]$. Let $C([-h,0],V)$ be the Banach space of all continuous functions from $[-h,0]$ to V with the supremum norm.

Consider the following fractional order semilinear control system with bounded delay

$$\left.\begin{array}{ll} {}^C D_t^\alpha x(t) = Ax(t) + Bu(t) + f(t,x_t), & t \in]0,\tau]; \\ x(t) = \varphi(t), & t \in [-h,0]. \end{array}\right\} \tag{1}$$

Here ${}^C D_t^\alpha$ is the Caputo fractional derivative of order α, where $1/2 < \alpha < 1$; the state $x(\cdot)$ takes its values in the space V; $A : D(A) \subseteq V \to V$ is a closed linear operator with dense domain $D(A)$ generating a C_0-semigroup $T(t)$; the control function $u(\cdot)$ takes its values in $\hat{V}$. The operator B is a bounded linear operator from $\hat{V}$ to V; $f : [0,\tau] \times C([-h,0],V) \to V$ is a continuous function and φ is the element of $C([-h,0];V)$.

* Corresponding author: mailto:mathdma@gmail.com

The investigation of the theory of fractional calculus have been started about three decades before. Fractional order differential equations are generalizations of ordinary differential equations to an arbitrary (noninteger) order. Fractional order nonlinear equations are abstract formulations for many problems arising in engineering, physics and many other fields. In particular, the fractional calculus is used in diffusion process, electrical science, electrochemistry, viscoelasticity, control science, electro magnetic theory and several more. For more details one can see [1–6] and the references cited therein. In [7] phase synchronizations in coupled chaotic systems presented by fractional differential equations has been considered. The existence and uniqueness of solutions of a nonlinear multi-variables fractional differential equations have been investigated in [8] by using Schauder's fixed points theorems and Global contraction mapping theory.

The problems of optimal control [9, 10] and various type of controllability like exact controllability [11–13], boundary controllability [14] and the approximate controllability [15, 16] of fractional order systems have been studied in the area of control theory in infinite dimension spaces.

To prove exact controllability and the boundary controllability, the main tool used by the authors is to convert the controllability problem into a fixed point problem together with the assumption that the controllability operator has an induced inverse on a quotient space. In [12–14], to prove the controllability results for fractional order semilinear systems authors made an assumption that the semigroup associated with the linear part is compact. However, if the operator B is compact or C_0-semigroup $T(t)$ is compact then the controllability operator is also compact and hence inverse of it does not exist if the state space V is infinite dimensional [17]. Thus, the concept of exact controllability is too strong in infinite dimensional space and the approximate controllability is more appropriate for these control systems.

The approximate controllability of the systems of integer order ($\alpha = 1, 2$) has been proved in [18–23] and the references therein. To show the results on the approximate controllability a relation between the reachable set of a semilinear system and that of the corresponding linear system is proved. In [15] Sakthivel et al. proved the approximate controllability by assuming that the C_0-semigroup $T(t)$ is compact and the nonlinear function is continuous and uniformly bounded. Sukavanam et al. [16] proved the approximate controllability for a class of semilinear delayed control system of fractional order by assuming that the corresponding linear system is approximately controllable and nonlinear function satisfies the Lipschitz condition. Recently, Kumar et al. [24] established sufficient conditions for the approximate controllability of a class of semilinear delay control systems of fractional order by using Schauder's fixed point theorem and the compactness of the C_0-semigroup together with the Lipschitz continuity of nonlinear term.

In this paper, sufficient conditions for the approximate controllability of fractional order semilinear control system (1) are established.

The paper is organized as follows: in Section 2, we present some necessary preliminaries. The approximate controllability of semilinear system (1) is proved in Section 3. In Section 4, an example is given to illustrate the theory.

2 Preliminaries

This section is devoted to the basic definitions and lemma, which are useful for further development.

Definition 2.1 A real function $f(t)$ is said to be in the space C_α, $\alpha \in \mathbb{R}$ if there exists a real number $p > \alpha$, such that $f(t) = t^p g(t)$, where $g \in C[0, \infty[$ and it is said to be in the space C_α^m iff $f^{(m)} \in C_\alpha$, $m \in N$.

Definition 2.2 The Riemann-Liouville fractional integral operator of order $\beta > 0$ of function $f \in C_\alpha$, $\alpha \geq -1$ is defined as

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds,$$

where Γ is the Euler gamma function.

Definition 2.3 If the function $f \in C_{-1}^m$ and m is a positive integer then we can define the fractional derivative of $f(t)$ in the Caputo sense as

$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^m(s) ds, \quad m-1 \leq \alpha < m.$$

Definition 2.4 [25] A function $x(\cdot) \in C([-h, \tau]; V)$ is said to be the mild solution of (1) if it satisfies

$$x(t) = \begin{cases} S_\alpha(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bu(s) + f(s, x_s)] ds, & t \in [0, \tau], \\ \varphi(t), & t \in [-h, 0], \end{cases}$$

where $S_\alpha(t)x = \int_0^\infty \phi_\alpha(\theta) T(t^\alpha \theta) x d\theta$ and $T_\alpha(t)x = \alpha \int_0^\infty \theta \phi_\alpha(\theta) T(t^\alpha \theta) x d\theta$. Here $\phi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-1/\alpha} \psi_\alpha(\theta^{-1/\alpha})$ is the probability density function defined on $(0, \infty)$, that is $\phi_\alpha(\theta) \geq 0$, and $\int_0^\infty \phi_\alpha(\theta) d\theta = 1$. Also the term $\psi_\alpha(\theta)$ is defined as $\psi_\alpha(\theta) = \frac{1}{\pi} \Sigma_{n=1}^\infty (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha)$, $\theta \in (0, \infty)$.

Define the solution mapping Φ from Z to $C([0, \tau]; V)$ as

$$(\Phi u)(t) = x(t).$$

Definition 2.5 The set $K_\tau(f) = \{x(\tau) \in V : \ x(t) \text{ is a mild solution of (1)}\}$ is called the reachable set of the system (1).

Definition 2.6 Let $x(\tau)$ be the state value of system (1) at time τ corresponding to the control u. The system (1) is said to be approximately controllable in time interval $[0, \tau]$, if for every desired final state ξ and $\epsilon > 0$ there exists a control function $u(\cdot) \in Y$ such that the mild solution $x(t)$ of (1) satisfies

$$\|x(\tau) - \xi\| < \epsilon.$$

The system (1) is said to be approximately controllable on $[0, \tau]$ iff $\overline{K_\tau(f)} = V$, where $\overline{K_\tau(f)}$ denotes the closure of $K_\tau(f)$.

Lemma 2.1 *[25] For any fixed $t \geq 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are bounded linear operators, that is, for any $x \in V$, $\|S_\alpha(t)x\| \leq M\|x\|$ and $\|T_\alpha(t)x\| \leq \frac{M\alpha}{\Gamma(1+\alpha)}\|x\|$, where M is a constant such that $\|T(t)\| \leq M$, for all $t \in [0, \tau]$.*

Define a continuous linear operator $\mathcal{L}$ from Z to $C([0,\tau];V)$ by

$$\mathcal{L}p = \int_0^\tau (\tau - s)^{\alpha-1} T_\alpha(\tau - s)p(s)ds, \text{ for } p(\cdot) \in Z.$$

Assumption: We need the following hypotheses to prove our results:
(H1) The nonlinear operator $f(t,x)$ satisfies the Lipschitz condition i.e. there exists a positive constant l such that

$$\|f(t,x) - f(t,y)\|_V \leq l\|x - y\|_V, \text{ for all } x, y \in V \text{ and } t \in [0,\tau],$$

and $\|f(t,0)\|_V \leq l_f$.
(H2) For any given $\epsilon > 0$, and $p(\cdot) \in Z$, there exists some $u(\cdot) \in Y$ such that

$$\|\mathcal{L}p - \mathcal{L}Bu\|_V < \epsilon.$$

(H3) $\|Bu(\cdot)\|_Z \leq \lambda\|p(\cdot)\|_Z$, where λ is a positive constant independent of $p(\cdot)$.
(H4) The constant λ satisfies $\frac{M\alpha\tau^\alpha\lambda l}{\Gamma(1+\alpha)\sqrt{2\alpha-1}} \exp\left(\frac{Ml\tau^\alpha}{\Gamma(1+\alpha)}\right) < 1$.

3 Controllability Results

In this section, we prove the approximate controllability for a class of fractional order semilinear control system (1) with bounded delay.

Lemma 3.1 *Under hypotheses (H1) the solution mapping* $(\Phi u)(\cdot)$ *satisfies*

$$\|(\Phi u)(t)\|_V \leq K \exp\left(\frac{Ml\tau^\alpha}{\Gamma(1+\alpha)}\right),$$

where $K = M\left[\|\varphi(0)\| + \frac{\alpha}{\Gamma(1+\alpha)}\sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}}\|Bu\|_Z + \frac{l_f\tau^\alpha}{\Gamma(1+\alpha)}\right]$.
Let $u_1(\cdot)$ *and* $u_2(\cdot)$ *be in* Y. *Then*

$$\|x_1 - x_2\|_Z \leq \frac{M\alpha\tau^\alpha}{\Gamma(1+\alpha)\sqrt{2\alpha-1}} \exp\left(\frac{Ml\tau^\alpha}{\Gamma(1+\alpha)}\right)\|Bu_1(\cdot) - Bu_2(\cdot)\|_Z,$$

where $x_n(t) = (\Phi u_n)(t),\ n = 1, 2, \cdots$.

Proof. The solution mapping $(\Phi u)(t) = x(t)$ is given by

$$x(t) = x_t(0) = S_\alpha(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bu(s) + f(s, x_s)]ds.$$

Taking the norm on both sides, we have

$$\begin{aligned}\|x_t\|_V &= \|S_\alpha(t)\|\|\varphi(0)\| + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)\|\|Bu(s)+f(s,x_s)\|ds\\ &\le M\|\varphi(0)\| + \frac{M\alpha}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\|Bu(s)\|ds\\ &\quad + \frac{M\alpha}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x_s)\|ds,\\ &\le M\|\varphi(0)\| + \frac{M\alpha}{\Gamma(1+\alpha)}\sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}}\|Bu\|_Z\\ &\quad + \frac{M\alpha l}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\|x_s\|_C ds + \frac{Ml_f\tau^\alpha}{\Gamma(1+\alpha)}.\\ &\le K + \frac{M\alpha l}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\|x_s\|_C ds.\end{aligned}$$

This implies that

$$\|x_t\|_C = \sup \|x_t\|_V \le K + \frac{M\alpha l}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\|x_s\|_C ds.$$

Now, using the Gronwall's inequality, we get

$$\|x(t)\| \quad \le \quad K\exp\Big(\frac{Ml\tau^\alpha}{\Gamma(1+\alpha)}\Big).$$

Thus, we have

$$\|(\Phi u)(t)\|_V \le K\exp\Big(\frac{Ml\tau^\alpha}{\Gamma(1+\alpha)}\Big).$$

Let us define $y(\cdot,\varphi) : [-h,\tau] \to V$ as

$$y(t,\varphi) = \begin{cases} \varphi(t), & t\in[-h,0],\\ S_\alpha(t)\varphi(0), & t\in[0,\tau].\end{cases}$$

Let $x(t) = y(t) + z(t)$, $t \in [-h,\tau]$. It is easy to see that $x(\cdot)$ satisfies (1) if and only if $z_0 = 0$ and for $t \in [0,\tau]$, we have

$$z(t) = \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[Bu(s) + f(s, y_s + z_s)]ds.$$

Now, let us take $x_1(\cdot),\ x_2(\cdot) \in V$ and $u_1,\ u_2 \in Y$, then

$$\begin{aligned}&\|(z_1)_t - (z_2)_t\|_V\\ &\quad\le \frac{M\alpha}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\|Bu_1(s) - Bu_2(s)\|ds\\ &\quad\quad + \frac{M\alpha}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y(s)+(z_1)_s) - f(s,y(s)+(z_2)_s)\|ds\\ &\quad\le \frac{M\alpha}{\Gamma(1+\alpha)}\sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}}\|Bu_1 - Bu_2\|_Z\\ &\quad\quad + \frac{M\alpha l}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\|(z_1)_s - (z_2)_s\|_C ds.\end{aligned}$$

Using the Gronwall's inequality, we get

$$\begin{aligned}\sup\|(z_1)_t-(z_2)_t\|_V &= \|(z_1)_t-(z_2)_t\|_C\\ &\le \frac{M\alpha}{\Gamma(1+\alpha)}\sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}}\exp\Big(\frac{Ml\tau^\alpha}{\Gamma(1+\alpha)}\Big)\|Bu_1-Bu_2\|_Z.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|x_1-x_2\|_Z &= \Big(\int_0^\tau \|x_1(s)-x_2(s)\|_V^2ds\Big)^{1/2}\\ &= \Big(\int_0^\tau \|z_1(s)-z_2(s)\|_V^2ds\Big)^{1/2}\\ &\le \frac{M\alpha\tau^\alpha}{\Gamma(1+\alpha)\sqrt{2\alpha-1}}\exp\Big(\frac{Ml\tau^\alpha}{\Gamma(1+\alpha)}\Big)\|Bu_1(\cdot)-Bu_2(\cdot)\|_Z.\end{aligned}$$

This completes the proof of lemma.

Theorem 3.1 *Under hypotheses (H1)–(H4) the fractional order semilinear control system (1) is approximately controllable.*

Proof. Since the domain $D(A)$ of the operator A is dense in Z, it is sufficient to prove that $D(A)\subset\overline{K_\tau(f)}$. For this, let us take $\xi\in D(A)$, then for any given $\epsilon>0$, there exists a control function $u_\epsilon(\cdot)\in Y$ such that

$$\|\xi-S_\alpha(\tau)\varphi(0)-\mathcal{L}f(s,x_\epsilon(s))-\mathcal{L}Bu_\epsilon\|<\epsilon,$$

where $x_\epsilon(t)=(\Phi u_\epsilon)(t)$ satisfies

$$x_\epsilon(t)=S_\alpha(t)\varphi(0)+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)[Bu_\epsilon(s)+f(s,(x_\epsilon)_s)]ds.$$

Now, we construct a sequence recursively as follows.

Assume that $u_1(\cdot)\in Y$ is arbitrarily given. By hypothesis (H2), there exists some $u_2(\cdot)\in Y$ such that

$$\|\xi-S_\alpha(\tau)\varphi(0)-\mathcal{L}f(s,(x_1)_s)-\mathcal{L}Bu_2\|<\frac{\epsilon}{2^2}, \tag{2}$$

where $x_1(t)=(\Phi u_1)(t)$, for all $t\in[0,\tau]$.

For $u_2(\cdot)\in Y$ thus obtained, we determine $w_2(\cdot)\in Y$ by hypotheses (H2) and (H3) such that

$$\|\mathcal{L}[f(s,(x_2)_s)-f(s,(x_1)_s)]-\mathcal{L}Bw_2\|<\frac{\epsilon}{2^3}, \tag{3}$$

and by Lemma 3.1, we have

$$\begin{aligned}\|Bw_2(\cdot)\|_{L_2([0,\tau];V)} &\le \lambda\|f(s,(x_2)_s)-f(s,(x_1)_s)\|_Z\\ &\le \lambda\Big(\int_0^\tau\|f(s,(x_2)_s)-f(s,(x_1)_s)\|_V^2ds\Big)^{1/2}\\ &\le \lambda l\Big(\int_0^\tau\|(x_2)_s-(x_1)_s\|_V^2ds\Big)^{1/2}\\ &\le \lambda l\|x_2-x_1\|_Z\\ &\le \frac{M\alpha\tau^\alpha\lambda l}{\Gamma(1+\alpha)\sqrt{2\alpha-1}}\exp\Big(\frac{Ml\tau^\alpha}{\Gamma(1+\alpha)}\Big)\|Bu_1(\cdot)-Bu_2(\cdot)\|_Z,\end{aligned}$$

where $x_n(t) = (\Phi u_n)(t)$, $n = 1, 2$, for all $t \in [0, \tau]$.

Thus, we may define $u_3 = u_2 - w_2$ in Y which has the following property

$$\begin{aligned}\|\xi - S_\alpha(\tau)\varphi(0) \quad &- \quad \mathcal{L}f(s,(x_2)_s) - \mathcal{L}Bu_3\| \\ &\leq \quad \|\xi - S_\alpha(\tau)\varphi(0) - \mathcal{L}f(s,(x_1)_s) - \mathcal{L}Bu_2 \\ &\quad\quad + \mathcal{L}Bw_2 - \mathcal{L}[f(s,(x_2)_s) - f(s,(x_1)_s)]\| \\ &\leq \quad \|\xi - S_\alpha(\tau)\varphi(0) - \mathcal{L}f(s,(x_1)_s) - \mathcal{L}Bu_2\| \\ &\quad\quad + \|\mathcal{L}Bw_2 - \mathcal{L}[f(s,(x_2)_s) - f(s,(x_1)_s)]\|.\end{aligned}$$

Using Eq. (2) and (3), we get

$$\|\xi - S_\alpha(\tau)\varphi(0) - \mathcal{L}f(s,(x_2)_s) - \mathcal{L}Bu_3\| \leq \left(\frac{1}{2^2} + \frac{1}{2^3}\right)\epsilon.$$

Mathematical induction implies that there exists a sequence $u_n(\cdot) \in Y$ such that

$$\|\xi - S_\alpha(\tau)\varphi(0) - \mathcal{L}f(s,(x_n)_s) - \mathcal{L}Bu_{n+1}\| \leq \left(\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}}\right)\epsilon, \tag{4}$$

where $x_n(t) = (\Phi u_n)(t)$, $n = 1, 2, \cdots$, for all $t \in [0, \tau]$ and

$$\begin{aligned}\|Bu_{n+1}(\cdot) \quad &- \quad Bu_n(\cdot)\|_Z \\ &\leq \quad \frac{M\alpha\tau^\alpha\lambda l}{\Gamma(1+\alpha)\sqrt{2\alpha-1}}\exp\left(\frac{Ml\tau^\alpha}{\Gamma(1+\alpha)}\right)\|Bu_n(\cdot) - Bu_{n-1}(\cdot)\|_Z.\end{aligned}$$

Clearly, by hypothesis (H4), the sequence $\{Bu_n;\ n = 1, 2, \cdots\}$ is a Cauchy sequence in the Banach space Z and there exists some $v(\cdot) \in Z$ such that

$$\lim_{n\to\infty} Bu_n(t) = v(t), \text{ in } Z.$$

Therefore for any given $\epsilon > 0$, there exists some integer N_ϵ such that

$$\|\mathcal{L}Bu_{N_\epsilon+1} - \mathcal{L}Bu_{N_\epsilon}\| < \frac{\epsilon}{2}. \tag{5}$$

Hence, we obtain

$$\begin{aligned}\|\xi - S_\alpha(\tau)\varphi(0) \quad &- \quad \mathcal{L}f(s,(x_{N_\epsilon})_s) - \mathcal{L}Bu_{N_\epsilon}\| \\ &\leq \quad \|\xi - S_\alpha(\tau)\varphi(0) - \mathcal{L}f(s,(x_{N_\epsilon})_s) - \mathcal{L}Bu_{N_\epsilon+1}\| \\ &\quad\quad + \|\mathcal{L}Bu_{N_\epsilon+1} - \mathcal{L}Bu_{N_\epsilon}\|,\end{aligned}$$

where $x_{N_\epsilon}(t) = (\Phi u_{N_\epsilon})(t)$, for all $t \in [0, \tau]$. Using Eq. (4) and (5), we get

$$\begin{aligned}\|\xi - S_\alpha(\tau)\varphi(0) - \mathcal{L}f(s,(x_{N_\epsilon})_s) - \mathcal{L}Bu_{N_\epsilon}\| \quad &\leq \quad \left(\frac{1}{2^2} + \cdots + \frac{1}{2^{N_\epsilon+1}}\right)\epsilon + \frac{\epsilon}{2} \\ &\leq \quad \epsilon.\end{aligned}$$

This means that $\xi \in \overline{K_\tau(f)}$. Hence the fractional order semilinear system (1) is approximately controllable on $[0, \tau]$. This completes the proof.

Theorem 3.2 *Suppose that the range of the operator B i.e. $R(B)$ is dense in Z. Then under hypothesis (H1) the semilinear system (1) is approximately controllable.*

Proof. Since the range of the operator B is dense in Z, for any given point $p(\cdot) \in Z$ and every $\delta > 0$, there exists some point $Bu(\cdot) \in R(B)$, where $u(\cdot) \in Y$ such that

$$\|Bu(\cdot) - p(\cdot)\|_Z < \delta \|p(\cdot)\|_Z. \tag{6}$$

Now, we have

$$\begin{aligned} \|\mathcal{L}p - \mathcal{L}Bu\| &\leq \frac{M\alpha}{\Gamma(1+\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} \|p(s) - Bu(s)\| ds \\ &\leq \frac{M\alpha}{\Gamma(1+\alpha)} \sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}} \|p(\cdot) - Bu(\cdot)\|_Z \\ &\leq \frac{M\alpha}{\Gamma(1+\alpha)} \sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}} \delta \|p(\cdot)\|_Z \\ &< \epsilon. \end{aligned}$$

Thus from (6), we have

$$\begin{aligned} \|Bu(\cdot)\|_Z &= \|Bu(\cdot) - p(\cdot) + p(\cdot)\|_Z \\ &\leq \|Bu(\cdot) - p(\cdot)\|_Z + \|p(\cdot)\|_Z \\ &\leq \delta \|p(\cdot)\|_Z + \|p(\cdot)\|_Z \\ &\leq (\delta + 1) \|p(\cdot)\|_Z. \end{aligned}$$

This implies that the conditions (H2) and (H3) are satisfies, if we choose $\delta > 0$ in such a manner that (H4) is verified. Then the approximate controllability of (1) follows from Theorem 3.1.

4 Example

Let $V = L_2(0, \pi)$ and $A = \frac{\partial^2}{\partial x^2}$ with $D(A)$ consisting of all $y \in V$ with $\frac{\partial^2 y}{\partial x^2}$ and $y(0) = 0 = y(\pi)$. Put $e_n(x) = \sqrt{2/\pi} \sin(nx)$; $0 \leq x \leq \pi$, $n = 1, 2, \cdots$, then $\{e_n,\ n = 1, 2, \cdots\}$ is an orthonormal basis for V and e_n is the eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator A. Then the C_0-semigroup $T(t)$ generated by A has $\exp(\lambda_n t)$ as the eigenvalues and e_n as their corresponding eigenfunctions [26]. Define an infinite-dimensional space $\hat{V}$ by

$$\hat{V} = \left\{ u \mid u = \sum_{n=2}^{\infty} u_n e_n, \text{ with } \sum_{n=2}^{\infty} u_n^2 < \infty \right\}.$$

The norm in $\hat{V}$ is defined by

$$\|u\|_{\hat{V}} = \left(\sum_{n=2}^{\infty} u_n^2 \right)^{1/2}.$$

Define a continuous linear map B from $\hat{V}$ to V as

$$Bu = 2u_2 e_1 + \sum_{n=2}^{\infty} u_n e_n, \text{ for } u = \sum_{n=2}^{\infty} u_n e_n \in \hat{V}. \tag{7}$$

Let us consider the following fractional order semilinear control system of the form

$$
\begin{aligned}
&{}^C D_t^{\alpha} y(t,x) = \frac{\partial^2}{\partial x^2} y(t,x) + Bu(t,x) + f(t, y(t-h,x));\ t \in [0,\tau],\ 0 < x < \pi,\\
&y(t,0) = y(t,\pi) = 0;\ t \in [0,\tau],\\
&y(t,x) = \varphi(t,x);\ t \in [-h,0],
\end{aligned} \tag{8}
$$

where $\varphi(t,x)$ is continuous. The system (8) can be written in the abstract form given by (1). The operator B is defined in (7) and the control function $u(t,x) \in L_2([0,\tau]; \hat{V}) = L_2([0,\tau] \times (0,\pi))$. Here the nonlinear term f is considered as an operator satisfying Hypothesis (H1). If the conditions (H2)-(H4) are satisfied, then the approximate controllability of system (8) follows from Theorem 3.1. For example, if we consider the function f as $f(t,z) = l\|z\|\phi_3(z)$, where $l > 0$ is a constant. The function f satisfies (H1) with Lipschitz constant l.

Conclusion

The approximate controllability for a class of semilinear delay control system of fractional order has been proved provided that it holds for the corresponding linear system. These results hold only for the fractional order such that $1/2 < \alpha < 1$.

Acknowledgment

The first author is thankful to Council of Scientific & Industrial Research (CSIR) for the financial support with grant number 09/143(0621)/2008-EMR-I.

References

[1] Fitt, A.D., Goodwin, A.R.H. and Wakeham, W.A. A fractional differential equation for a MEMS viscometer used in the oil industry. *J. Comput. Appl. Math.* **229** (2009) 373–381.

[2] Diethelm, K. and Freed, A.D. On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity. In: *Scientifice Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties* (F. Keil, W. Mackens, H. Voss, J. Werther, Eds.), Springer-Verlag, Heidelberg, (1999) 217–224.

[3] Hilfer, R. *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[4] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[5] Oldham, K.B. and Spanier, J. *The Fractional Calculus,Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, New York, London, 1974.

[6] Podlubny I. *Fractional Differential Equations*, Academic Press, New York, 1999.

[7] Erjaee, G.H. and Taghvafard, H. Stability analysis of phase synchronization in coupled chaotic systems presented by fractional differential equations. *Nonlinear Dynamics and Systems Theory.* **11**(2) (2011) 147–154.

[8] Yu, J.M., Luo, Y.W., Zhou, S.B. and Lin, X.R. Existence and uniqueness for nonlinear multi-variables fractional differential equations. *Nonlinear Dynamics and Systems Theory.* **11**(2) (2011) 213–221.

[9] Wang, J. and Zhou, Y. A class of fractional evolution equations and optimal controls. *Nonlinear Analysis: Real World Applications.* **12** (2011) 262-272.

[10] Wang, J. Zhou, Y. and Wei, W. A class of fractional delay nonlinear integrodifferential controlled systems in Banach spaces. *Commun Nonlinear Sci Numer Simulat.* doi:10.1016/j.cnsns.2011.02.003, (2011).

[11] Debbouche, A. and Baleanu, D. Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems. *Computers and Mathematics with Applications.* **62** (2011) 1442–1450.

[12] Tai, Z. and Wang, X. Controllability of fractional-order impulsive neutral functional infinite delay integrodifferential systems in Banach spaces. *Applied Mathematics Letters.* **22** (2009) 1760–1765.

[13] Yan, Z. Controllability of fractional-order partial neutral functional integrodifferential inclusions with infinite delay. *Journal of the Franklin Institute.* **348** (2011) 2156–2173.

[14] Ahmed, H.M. Boundary controllability of nonlinear fractional integrodifferential systems. *Advances in Difference Equations.* **2010** (2010) Article ID 279493.

[15] Sakthivel, R., Ren, Y. and Mahmudov, N.I. On the approximate controllability of semilinear fractional differential systems. *Computers and Mathematics with Applications.* **62** (2011) 1451–1459.

[16] Sukavanam, N. and Kumar, S. Approximate controllability of fractional order semilinear delay systems. *J Optim Theory Appl.* **151** (2011) 373–384.

[17] Triggiani, R. A note on the lack of exact controllability for mild solutions in Banach spaces. *SIAM J. Control Optim.* **15** (1977) 407–411.

[18] Balasubramaniam, P., Park, J.Y. and Muthukumar, P. Approximate controllability of neutral stochastic functional differential systems with infinite delay. *Stochastic Analysis and Applications.* **28** (2010) 389–400.

[19] Henríquez, H.R. and Hernández, E.M. Approximate controllability of second-order distributed implicit functional systems. *Nonlinear Analysis.* **70** (2009) 1023–1039.

[20] Kwun, Y.C., Park, J.Y. and Ryu, J.W. Approximate controllability and controllability for delay volterra system. *Bull. Korean Math. Soc.* **28**(2) (1991) 131–145.

[21] Sukavanam, N. and Tafesse, S. Approximate controllability of a delayed semilinear control system with growing nonlinear term. *Nonlinear analysis.* **74** (2011) 6868–6875.

[22] Tomar, N.K. and Sukavanam, N. Approximate controllability of non-densely defined semilinear delayed control systems. *Nonlinear Studies.* **18**(2) (2011) 229–234.

[23] Zhou, H.X. Approximate controllability for a class of semilinear abstract equations. *SIAM J. Control and Optimization.* **21**(4) (1983) 551–565.

[24] Kumar, S. and Sukavanam, N. Approximate controllability of fractional order semilinear systems with bounded delay, J. Differential Equations (2012): doi:10.1016/j.jde.2012.02.014.

[25] Zhou, Y. and Jiao, F. Existence of mild solutions for fractional neutral evolution equations. *Computers and Mathematics with Applications.* **59** (2010) 1063–1077.

[26] Naito, K. Controllability of semilinear control systems dominated by the linear part. *SIAM J. Control and Optimization.* **25**(3) (1987) 715–722.