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Existence and Uniqueness of a Solution of Fisher-KKP Type Reaction Diffusion Equation

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Abstract: In this paper we prove the existence and uniqueness of a strong solution of a Fisher-KKP type reaction diffusion equation with Dirichlet boundary conditions using the method of semidiscretization.

Keywords: *method of semidiscretization; reaction diffusion equation; strong solution; A priori estimate.*

Mathematics Subject Classification (2010): 35K57, 65N40, 35B45, 35D35.

1 Introduction

In this paper we concerned with the following reaction diffusion equation of KPP-Fisher type with Dirichlet boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ku(t,x)[1-u(t,x)] + f(t,x), \quad t \in (0,T], \quad x \in (0,\pi), \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in (0,\pi), \tag{2}$$

$$u(0,t) = u(\pi,t) = 0, \quad t \in (0,T], \tag{3}$$

where k is a positive constant and $u_0 \in L_2(0,\pi)$.

Since 1930, various classical types of initial boundary value problem have been investigated by many authors using the method of semidiscretization; see for instance [11,15,16] and references therein.

The method of semidescretization in time is a very efficient tool in the study of an approximate solution and its convergence to the solution of the problem. In this

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method we replace the time derivative by the corresponding difference quotients giving rise to a system of time independent operator equations. With the help of the theory of semigroups, these systems are guaranteed to have unique solutions. An approximate solution to the given problem is defined in terms of the solutions of these time independent systems. After proving a priori estimates for the approximate solution, the convergence of the approximate solution to a unique strong solution is established.

In this paper my aim is to apply the method of semidiscretization to a reaction diffusion equation of KPP-Fisher type with Dirichlet boundary conditions. Fisher-KKP equations are most simple case of nonlinear reaction diffusion equation that was first shown to have traveling wave front by Fisher [18].

This work is motivated by the work of Fisher [18], in which he has considered the Fisher-KKP type reaction diffusion equation:

$$\frac{\partial u}{\partial t} = ru(t,x)\left[1 - \frac{u(t,x)}{K}\right] + D\frac{\partial^2 u}{\partial x^2},$$

where r and D are positive parameters.

Dubey [3], has established the existence and uniqueness of a strong solution for the following nonlinear nonlocal functional differential equation in a Banach X, using the method of semidiscretization:

$$\begin{aligned} u'(t) + Au(t) &= f(t, u(t), u_t), \quad t \in (0, T], \\ h(u_0) &= \phi \quad \text{on} \quad [-\tau, 0], \end{aligned}$$

where $0 < T < \infty$, $\phi \in C_0 := C([-\tau,0];X)$, $\tau > 0$, the nonlinear operator A is singlevalued and m-accretive defined from the domain $D(A) \subset X$ into X, the nonlinear map f is defined from $[0,T] \times X \times C_0 := C([-\tau,0];X)$ into X, the map h is defined from C_0 into C_0. For $u \in C_T := C([-\tau,T];X)$, function $u_t \in C_0$ is given by $u_t(s) = u(t+s)$ for $s \in [-\tau, 0]$. Here $C_t := C([-\tau,t];X)$ for $t \in [0,T]$ is the Banach space of all continuous functions from $[-\tau, t]$ into X endowed with the supremum norm

$$\|\phi\|_t = \sup_{-\tau \le \eta \le t} \|\phi(\eta)\|, \quad \phi \in C_t.$$

Bouziani, Merchri [17] and Lakoud, Chaoui [14] have applied the method of semidiscretization to integrodifferential equations, and prove the existence and uniqueness of a weak solution. For the application of method of semidiscretization to delayed cooperation diffusion system with Dirichlet boundary conditions, we refer readers to [19]. For the more applications of Rothe method to integrodifferential equations, parabolic problems, hyperbolic problems, we refer readers to [9, 10, 12, 13] and references therein.

By literature, it is clear that method of semidiscretization is applicable in many physical, mathematical, biological problems modeled by partial differential equations.

The plan of the rest paper is as follows. In Section 2, we state some basic results and definitions that will be used in the next sections. In Section 3, we state the main result. In the last section, we state and prove all the lemmas that are required to prove the main result and at the end of this section, we prove the main result.

2 Preliminaries

We define

$$B_R(0) = \{u \in L^2(0,\pi) : \|u\| \le R\}.$$

Now we define a function $F : (0,T] \times B_R(0) \to B_R(0)$ by

$$F(t,\chi)(x) = k\chi[1-\chi](x) + f(t,x).$$

Consider that $H := L^2[0,\pi]$ is the real Hilbert space of all real-valued square-integrable functions on the interval $[0,\pi]$, let the linear operator A be defined by

$$D(A) := \{u \in H : u'' \in H, u(0) = u(\pi) = 0\}, \quad Au = -u''.$$

Then we know that $-A$ is the infinitesimal generator of a C_0-semigroup $S(t),\ t \geq 0$ of contractions in H.

If we identify $u : (0,T] \to H$, by $u(t)(x) = u(t,x)$, and $f : (0,T] \to H$ by $f(t)(x) = f(t,x)$, then (1)-(3) reduce to:

$$\frac{\partial u(t)}{\partial t} + Au(t) = F(t,u(t)), \tag{4}$$

$$u(0) = u_0. \tag{5}$$

Lemma 2.1 *There exists a constant $L_F(R) > 0$ such that*

$$\|F(t,\chi_1) - F(t,\chi_2)\| \leq L_F(R)\|\chi_1 - \chi_2\|,$$

for all $\chi_1, \chi_2 \in B_R(0), \quad t \in (0,T]$.

Proof. Now for any $\chi_1, \chi_2 \in B_R(0)$ and $t \in (0,T]$, we have

$$\begin{aligned}
&\|F(t,\chi_1) - F(t,\chi_2)\|_2^2 \\
&= \int_0^\pi |F(t,\chi_1)(x) - F(t,\chi_2)(x)|^2 dx \\
&= \int_0^\pi |k\chi_1(1-\chi_1)(x) - k\chi_2(1-\chi_2)(x)|^2 dx \\
&\leq k^2 \int_0^\pi (|\chi_1(x) - \chi_2(x)|^2 + |\chi_2^2(x) - \chi_1^2(x)|^2) dx \\
&\leq k^2 \int_0^\pi |\chi_1(x) - \chi_2(x)|^2 (1 + |\chi_1(x) + \chi_2(x)|^2) dx \\
&\leq k^2 \int_0^\pi |\chi_1(x) - \chi_2(x)|^2 dx \int_0^\pi (1 + |\chi_1(x) + \chi_2(x)|^2) dx \\
&\leq k^2 \|\chi_1 - \chi_2\|_2^2 (\pi + \|\chi_1 + \chi_2\|^2) \\
&\leq k^2(\pi + 2R^2)\|\chi_1 - \chi_2\|_2^2.
\end{aligned}$$

This implies that

$$\|F(t,\chi_1) - F(t,\chi_2)\|_2 \leq L'_F(R)\|\chi_1 - \chi_2\|_2,$$

where $L'_F(R) = k\sqrt{\pi + 2R^2}$. □

Lemma 2.2 *If f satisfies a Lipschitz-like condition, i.e., there exists a constant $k_1 > 0$ such that*

$$\|f(t) - f(s)\| \leq k_1 \mid t - s \mid, \quad \forall t, s \in (0,T],$$

then F also satisfies a Lipschitz condition in $(0,T]$, i.e.,

$$\|F(t,\chi) - F(s,\chi)\| \leq k_1 \mid t - s \mid, \quad \forall t, s \in (0,T].$$

Remark 2.1 From Lemma 2.1 and Lemma 2.2, we conclude that F satisfies a local Lipschitz condition, i.e., there exists a constant $L_F(R) > 0$ such that

$$\|F(t,\chi_1) - F(s,\chi_2)\| \leq L_F(R)[|t-s| + \|\chi_1 - \chi_2\|_2], \quad \forall t,s \in (0,T], \quad \forall \chi_1,\chi_2 \in B_R(0).$$

Definition 2.1 Let X be a Banach space and let X^* be its dual. For every $x \in X$ we define the duality map J as:

$$J(x) = \{x^* : \ x^* \in X^* \quad \text{and} \quad (x^*, x) = \|x\|^2 = \|x^*\|^2\},$$

where (x^*, x) denotes the value of x^* at x.

Lemma 2.3 ([1], Theorem 1.4.3) *If $-A$ is the infinitesimal generator of a C_0-semigroup of contractions then A is m-accretive, i.e.,*

$$(Au, J(u)) \geq 0 \quad \text{for} \ \ u \in D(A),$$

where J is the duality mapping and $R(I + \lambda A) = X$ for $\lambda > 0$, I is the identity operator on X and $R(.)$ is the range of an operator.

Lemma 2.4 ([2], Lemma 2.5(a)) *If $-A$ is the infinitesimal generator of a C_0-semigroup of contractions, $X^n \in D(A)$, $n = 1,2,3,.....$, $X^n \to u \in H$ and $\|AX^n\|$ are bounded, then $u \in D(A)$ and $AX^n \rightharpoonup Au$.*

A function $u \in C([0,T],H)$ such that

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(s,u(s))ds, \quad \text{if} \quad t \in [0,T].$$

is called a mild solution of (4)-(5).

By a strong solution of (4)-(5) we mean a function $u \in C([0,T],X)$ such that $u(t) \in D(A)$ for a.e. $t \in [0,T]$, u is differentiable a.e. on $[0,T]$ and

$$u'(t) + Au(t) = F(t,u(t)), \qquad \text{a.e.} \ \ t \in [0,T].$$

3 Main Result

Theorem 3.1 *Under the conditions of Lemma 2.1 and Lemma 2.2, problem (4)-(5) has a unique strong solution on the interval $[0,t_0]$, $0 < t_0 < T$ which can be uniquely continued either on $[0,T]$, or on the maximal interval of existence $[0,t_{max}[$, $0 < t_{max} \leq T$. If $0 < t_{max} < T$, then*

$$\lim_{t \uparrow t_{max}} \|u(t)\| = \infty.$$

We will prove this result by using the method of semidiscretization.

4 Discretization and A priori Estimates

To apply the method of semidiscretization we divide the interval $[0, t_0]$ into the subintervals of length $h_n = \frac{t_0}{n}$ and replace (4) and (5) by the following approximate equations

$$\begin{aligned} \frac{u_j^n - u_{j-1}^n}{h_n} + Au_j^n &= F(t_j^n, u_{j-1}^n), &(6)\\ u_0^n &= u_0. &(7)\end{aligned}$$

Existence of a unique $u_j^n \in H$, satisfying (6) and (7) is a consequence of Lemma 2.3.

Now we construct Rothe's sequence

$$u_n(t) = u_{j-1}^n + \frac{u_j^n - u_{j-1}^n}{h_n}(t - t_j^n), \quad t \in [t_{j-1}^n, t_j^n]. \tag{8}$$

Also, we construct a sequence of step functions:

$$X^n(t) \;=\; \begin{cases} u_0, & \text{if} \quad t = 0, \\ u_j^n, & \text{if} \quad t \in (t_{j-1}^n, t_j^n]. \end{cases} \tag{9}$$

Now we state and prove the following two lemmas which are required to prove the main result.

Lemma 4.1 *There exists a constant C_1 (independent of n, j and h_n) such that $\|u_j^n - u_0\| \le C_1$ (note that here C_1 is a generic constant that may have different value in the same discussion).*

Proof. Substituting $j = 1$ in (6), we get

$$\frac{u_1^n - u_0^n}{h_n} + Au_1^n = F(t_1^n, u_0^n).$$

Subtracting Au_0 from both sides and applying $J(u_1^n - u_0)$ on both sides, we get

$$\begin{aligned}\left(\frac{u_1^n - u_0}{h_n}, J(u_1^n - u_0)\right) + (A(u_1^n - u_0), J(u_1^n - u_0)) = (F(t_1^n, u_0), J(u_1^n - u_0))\\ -(Au_0, J(u_1^n - u_0)).\end{aligned}$$

Using Lemma 2.3 and the definition of duality map, we get

$$\begin{aligned} \frac{1}{h_n}\|u_1^n - u_0\|^2 &\le \|F(t_1^n, u_0)\|\|u_1^n - u_0\| + \|Au_0\|\|u_1^n - u_0\| \\ \implies \|u_1^n - u_0\| &\le h_n[\|F(t_1^n, u_0)\| + \|Au_0\|]. \end{aligned}$$

Using Remark 2.1, we can obtain

$$\begin{aligned} \|F(t_1^n, u_0)\| &\le \|F(t_1^n, u_0) - F(0, u_0)\| + \|F(0, u_0)\| \\ &\le L_F(R)|t_1^n| + \|F(0, u_0)\| \\ &\le L_F(R)t_0 + \|F(0, u_0)\|. \end{aligned}$$

Using the above inequality, we get

$$\begin{aligned}\|u_1^n - u_0\| &\leq h_n[L_F(R)t_0 + \|F(0,u_0)\| + \|Au_0\|] \\ &\leq t_0[L_F(R)t_0 + \|F(0,u_0)\| + \|Au_0\|] = C_1.\end{aligned}$$

To prove this lemma, we will use induction method, for this we assume that

$$\|u_i^n - u_0\| \leq C_1, \quad i = 1, \cdots, j-1.$$

We have to show that

$$\|u_j^n - u_0\| \leq C_1.$$

Subtracting Au_0 from both sides of (6), and applying $J(u_j^n - u_0)$, we get

$$\begin{aligned}&\left(\frac{u_j^n - u_0}{h_n}, J(u_j^n - u_0)\right) + (A(u_j^n - u_0), J(u_j^n - u_0)) \\ &= \left(\frac{u_{j-1}^n - u_0}{h_n}, J(u_j^n - u_0)\right) + (F(t_j^n, u_{j-1}^n), J(u_j^n - u_0)) - (Au_0, J(u_j^n - u_0)).\end{aligned}$$

Using Lemma 2.3 and the definition of duality map, we get

$$\begin{aligned}\frac{1}{h_n}\|u_j^n - u_0\|^2 &\leq \frac{1}{h_n}\|u_{j-1}^n - u_0\|\|u_j^n - u_0\| + \|F(t_j^n, u_{j-1}^n)\|\|u_j^n - u_0\| \\ &\quad + \|Au_0\|\|u_j^n - u_0\| \\ \implies \|u_j^n - u_0\| &\leq \|u_{j-1}^n - u_0\| + h_n[\|F(t_j^n, u_{j-1}^n)\| + \|Au_0\|].\end{aligned}$$

By using induction hypothesis, we obtain

$$\|u_j^n - u_0\| \leq C_1 + t_0[\|F(t_j^n, u_{j-1}^n)\| + \|Au_0\|].$$

Using Remark 2.1, we get

$$\begin{aligned}\|F(t_j^n, u_{j-1}^n)\| &\leq \|F(t_j^n, u_{j-1}^n) - F(0,u_0)\| + \|F(0,u_0)\| \\ &\leq L_F(R)[|t_j^n| + \|u_{j-1}^n - u_0\|] + \|F(0,u_0)\| \\ &\leq L_F(R)[t_0 + C_1] + \|F(0,u_0)\|.\end{aligned}$$

Using the above inequality, we get

$$\|u_j^n - u_0\| \leq C_1 + t_0[L_F(R)(t_0 + C_1) + \|F(0,u_0)\| + Au_0].$$

This completes the proof of the lemma. □

Lemma 4.2 *There exists a constant C_2 (independent of n, j and h_n) such that $\left\|\frac{u_j^n - u_{j-1}^n}{h_n}\right\| \leq C_2$ (note that here C_2 is a generic constant that may have different value in the same discussion).*

Proof. As in the previous lemma, we can show that

$$\left\|\frac{u_1^n - u_0^n}{h_n}\right\| \leq [L_F(R)t_0 + \|F(0,u_0)\| + \|Au_0\|].$$

We will prove this result by induction. For this we assume that

$$\left\|\frac{u_i^n - u_{i-1}^n}{h_n}\right\| \le C_2, \quad i = 1, \cdots, j-1.$$

We have to show that

$$\left\|\frac{u_j^n - u_{j-1}^n}{h_n}\right\| \le C_2.$$

Subtracting from (6) the same equation written for $j-1$, and applying $J(u_j^n - u_{j-1}^n)$ on both sides, we get

$$\begin{aligned}\left(\frac{u_j^n - u_{j-1}^n}{h_n}, J(u_j^n - u_{j-1}^n)\right) \quad &\le \quad \left(\frac{u_{j-1}^n - u_{j-2}^n}{h_n}, J(u_j^n - u_{j-1}^n)\right) \\ &\quad + (F(t_j^n, u_{j-1}^n) - F(t_{j-1}^n, u_{j-2}^n), J(u_j^n - u_{j-1}^n)).\end{aligned}$$

Using Lemma 2.3 and the definition of duality map, we get

$$\left\|\frac{u_j^n - u_{j-1}^n}{h_n}\right\| \le \left\|\frac{u_{j-1}^n - u_{j-2}^n}{h_n}\right\| + \|F(t_j^n, u_{j-1}^n) - F(t_{j-1}^n, u_{j-2}^n)\|.$$

By using induction hypothesis, we get

$$\left\|\frac{u_j^n - u_{j-1}^n}{h_n}\right\| \le C_2 + \|F(t_j^n, u_{j-1}^n) - F(t_{j-1}^n, u_{j-2}^n)\|.$$

By using Remark 2.1, we get

$$\begin{aligned}\|F(t_j^n, u_{j-1}^n) - F(t_{j-1}^n, u_{j-2}^n)\| \quad &\le \quad L_F(R)[t_0 + C_2 h_n] \\ &\le \quad L_F(R)[t_0 + C_2 t_0].\end{aligned}$$

Using the above inequality, we get

$$\left\|\frac{u_j^n - u_{j-1}^n}{h_n}\right\| \le C_2 + L_F(R)[t_0 + C_2 t_0].$$

This completes the proof of the lemma. □

Remark 4.1 By using Lemma 4.1 and Lemma 4.2, we conclude that sequence $\{u^n(t)\}$ is uniformly Lipschitz continuous and $u^n(t) - X^n(t) \to 0$, as $n \to \infty$, $t \in (0, t_0]$.

If we denote that

$$f^n(t) = F(t_j^n, u_{j-1}^n),$$

and using (8) and (9), then (4) reduces to:

$$\frac{d^-}{dt} U^n(t) + AX^n(t) = f^n(t), \quad t \in (0, t_0], \tag{10}$$

where $\frac{d^-}{dt}$ denotes the left derivative in $(0, t_0]$.
Also, for $t \in (0, t_0]$, we have

$$\int_0^t AX^n(s)ds = u_0 - U^n(t) + \int_0^t f^n(s)ds. \tag{11}$$

Next we prove the convergence of U^n to u in $C([0, t_0], H)$.

Lemma 4.3 ([3], Lemma 3.4) *There exists $u \in C([0,t_0],H)$, such that $U^n \to u$ in $C([0,t_0],H)$ as $n \to \infty$. Moreover, u is Lipschitz continuous on $[0,t_0]$.*

Remark 4.2 Clearly $X^n(t) \in D(A)$, for each n. As $u^n(t) - X^n(t) \to 0$ as $n \to \infty$, $X^n(t) \to u(t) \in H$. Also $\|AX^n\|$ are bounded therefore by Lemma 2.4, it is clear that $AX^n \rightharpoonup Au$.

So for every $x^* \in X^*$ and $t \in (0,t_0]$, we have

$$\int_0^t (AX^n(s),x^*)ds = (u_0,x^*) - (U^n(t),x^*) + \int_0^t (f^n(s),x^*)ds.$$

Using Lemma 4.3, Remark 4.2 and the bounded convergence theorem, we obtain as $n \to \infty$,

$$\int_0^t (Au(s),x^*)ds = (u_0,x^*) - (u(t),x^*) + \int_0^t (F(s,u(s)),x^*)ds. \tag{12}$$

As $Au(t)$ is Bochner integrable on $[0,t_0]$, from (12) we have

$$\frac{d}{dt}u(t) + Au(t) = F(t,u(t)), \quad \text{a.e.} \quad t \in (0,t_0]. \tag{13}$$

Clearly $u \in C([0,t_0];H)$ and differentiable a.e. on $(0,t_0]$ with $u(t) \in D(A)$ a.e. on $(0,t_0]$ satisfying (13). Hence u is a strong solution of (6)-(7) on $[0,t_0]$.

Now we show the uniqueness of this strong solution. For this we assume that u_1 and u_2 are two strong solutions of (6)-(7) on the interval $[0,t_0]$. Let $u = u_1 - u_2$

$$\begin{aligned}&\left(\frac{du(t)}{dt}, J(u(t))\right) + (A(u_1(t)-u_2(t)), J(u_1(t)-u_2(t)))\\ &= (F(t,u_1(t)) - F(t,u_2(t)), J(u(t))).\end{aligned}$$

By Lemma 2.3 and by the definition of duality mapping, we get

$$\frac{d}{dt}\|u(t)\|^2 \leq \|F(t,u_1(t)) - F(t,u_2(t))\|\|u(t)\|.$$

Using Lemma 2.1, we get

$$\frac{d}{dt}\|u(t)\|^2 \leq K\|u(t)\|^2.$$

This implies that

$$\|u(t)\|^2 \leq K\int_0^t \|u(t)\|^2 ds.$$

Applying Grownwall's inequality, we get $u \equiv 0$ on $[0,t_0]$. Hence we get a unique strong solution on the interval $[0,t_0]$.

Strong solution u of (6)–(7) on interval $[0,t_0]$ can be extended on the larger interval $[0,t_0+\delta]$, $\delta > 0$ [[1], Theorem 6.2.2]. Continuing this process, we obtain a unique strong solution either on the whole interval or on the maximal interval of existence $[0,t_{max}]$. If $t_{max} < \infty$, then $\lim_{t \uparrow t_{max}} \|u(t)\| = \infty$, otherwise we get contradiction [[1], Theorem 6.1.4].

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