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Asymptotic Estimates Related to an Integro Differential Equation

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Abstract: The paper deals with an integrodifferential operator which models numerous phenomena in superconductivity, in biology and in viscoelasticity. Initial-boundary value problems with Neumann, Dirichlet and mixed boundary conditions are analyzed. An asymptotic analysis is achieved proving that for large t, the influences of the initial data vanish, while the effects of boundary disturbances are everywhere bounded.

Keywords: *initial-boundary problems for higher order parabolic equations; Laplace transform; superconductivity; FitzHugh-Nagumo model.*

Mathematics Subject Classification (2010): 44A10, 35K57, 35A08, 35K35.

1 Introduction

If $u = u(x,t)$, let us consider the following integrodifferential equation

$$\mathcal{L}u \equiv \; u_t - \varepsilon u_{xx} + au + b\int_0^t e^{-\beta(t-\tau)}\, u(x,\tau)\, d\tau \; = \; F(x,t,u), \tag{1}$$

where ε, a, b, β are positive constants, x denotes the direction of propagation and t is the time. According to the meaning of $F(x,t,u)$, equation (1) describes the evolution of several linear or non linear physical models. For instance, when $F = f(x,t)$, (1) is related to the following linear phenomena:

- motions of viscoelastic fluids or solids [1–4];

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- heat conduction at low temperature [5–7],
- sound propagation in viscous gases [8].

When $F = F(x,t,u)$, some non linear phenomena involve equation (1) both in superconductivity and biology.

• *Superconductivity* – Let u be the difference between the wave functions phases of two superconductors in a Josephson junction. The equation describing tunnel effects is the following one:

$$\varepsilon u_{xxt} - u_{tt} + u_{xx} - \alpha u_t = \sin u - \gamma, \tag{2}$$

where constant γ is a forcing term proportional to a bias current, while the $\varepsilon - term$ and the $\alpha - term$ account for the dissipative normal electron current flow, respectively along and across the junction [9, 10].

Equation (2) can be obtained by (1) as soon as one assumes

$$a = \alpha - \frac{1}{\varepsilon}, \qquad b = -\frac{a}{\varepsilon}, \qquad \beta = \frac{1}{\varepsilon}, \tag{3}$$

and F is such that

$$F(x,t,u) = -\int_0^t e^{-\frac{1}{\varepsilon}(t-\tau)} \, [\, sen\, u(x,\tau) - \gamma \,] \, d\tau. \tag{4}$$

Besides, when the case of an exponentially shaped Josephson junction (ESJJ) is considered, the evolution of the phase inside this junction is described by the third order equation:

$$(\partial_{xx} - \lambda \partial_x)(\varepsilon u_t + u) - \partial_t(u_t + \alpha u) = \sin u - \gamma, \tag{5}$$

where λ is a positive constant generally less than one and the terms λu_{xt} and λu_x represent the current due to the tapering junction. In particular λu_x corresponds to a geometrical force driving the fluxons from the wide edge to the narrow edge. [10–12] An (ESJJ) provides several advantages with respect to a rectangular junction ([14] and reference therein). For instance, in [11] it has been proved that it is possible to obtain a voltage which is not chaotic anymore, but rather periodic excluding, in this way, some among the possible causes of large spectral width. It is also proved that the problem of trapped flux can be avoided. Numerous applications and devices involve Josephson junctions, for example SQUIDs which are very versatile and can be used in a lot of fields. (see f.i. [15] and references therein).

Moreover, if $u = e^{\lambda x/2}\, \overline{u}$, (5) turns into an equation like (2) and hence into (1).

• *Biology* – Let us consider the FitzHugh-Nagumo system (FHN) which models the propagation of nerve impulses. [16]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \dfrac{\partial^2 u}{\partial x^2} - v + f(u), \\[2ex] \dfrac{\partial v}{\partial t} = b\, u - \beta\, v\,. \end{cases} \tag{6}$$

Here, $u\,(x,t\,)$ models the transmembrane voltage of a nerve axon at a distance x and time t, while $v\,(x,t\,)$ is an auxiliary variable acting as a recovery variable. Besides, the function $f(u)$ has the qualitative form of a cubic polynomial

$$f(u) \,=\, -\,a\,u \,+\, \varphi(u) \quad with \quad \varphi \,=\, u^2\,(\,a+1\,-u\,), \tag{7}$$

while ε, b, β are non negative and the parameter a, representing the threshold constant, is generally $0 < a < 1$. (see f.i. [17] and references therein)

Denoting by v_0 the initial value of v, system (6) (7) can be given the form of the integrodifferential equation (1) as soon as one puts:

$$F(x,t,u) \,=\, \varphi(u) \,-\, v_0(x)\,e^{-\,\beta\,t}\,. \tag{8}$$

In this paper, initial value problems with Neumann, Dirichlet and mixed boundary conditions for (1) are considered. By means of properties of the fundamental solution $K_0(x,t)$ of the operator $\mathcal{L}$, appropriate estimates are obtained. The function $K_0(x,t)$ has already been determined and analyzed in [18] and an analysis related to a Neumann boundary problem has been conducted in [19]. The aim of this paper is an asymptotic analysis for the initial boundary value problem both with Dirichlet conditions and with mixed conditions. These cases involve x-derivative of theta functions $\theta(x,t)$ and $\theta^*(x,t)$ which are determined in Section (3). So, effects of boundary perturbations can be evaluated by means of a well known theorem on asymptotic behavior of convolutions. As an example, according to the equivalence between operator $\mathcal{L}$ and the FHN system, an estimate of the solution related to the reaction-diffusion system (6) is obtained proving that, for large t, effects determined by boundary disturbance are bounded.

2 Some Models of Superconductivity and Biology

Let T be an arbitrary positive constant and

$$\Omega_T \,\equiv \{\,(x,t) : \,0 \,\leq\, x \,\leq L\,;\; 0 < t \leq T.$$

(I) A first example is related to *Neumann* boundary conditions (NBC)

$$\begin{cases} \mathcal{L}\,u \,=\, F(x,t,u), & (x,t) \in \Omega_T, \\ u(x,0) \,= u_0(x), & x \,\in [0,L], \\ u_x(0,t) \,=\, \psi_1(t), \qquad u_x(L,t) \,=\, \psi_2(t), & 0 < t \leq T. \end{cases} \tag{9}$$

In superconductivity, this problem occurs when the magnetic field, proportional to the phase gradient, is assigned [20,21]. In mathematical biology, it can refer to a two-species reaction diffusion system subjected to flux boundary conditions [16]. The same conditions are present in case of pacemakers [22] and are applied also to study distributed (FHN) systems [23] or to solve FHN systems by means of numerical calculations [24].

(II) Another example concerns *Dirichlet* boundary conditions (DBC)

$$\begin{cases} \mathcal{L}\,u \,=\, F(x,t,u), & (x,t) \in \Omega_T, \\ u(x,0) \,= u_0(x), & x \,\in [0,L], \\ u(0,t) \,=\, g_1(t), \qquad u(L,t) \,=\, g_2(t), & 0 < t \leq T. \end{cases} \tag{10}$$

In superconductivity, $(10)_3$ refer to the phase boundary specifications [12–14]. In excitable systems these conditions occur when the behavior of a single dendrite has to be determined and the voltage level is fixed [22] or when the pulse propagation in a continuum of heart cells is studied [22, 25]. Besides, the Dirichlet problem is also considered to determine universal attractors both for Hodgkin-Huxley equations and for FHN systems, [26] and for stability analysis and asymptotic behavior of reaction-diffusion systems solutions, [27–31], or in hyperbolic diffusion [32].

(III) At last, *mixed* boundary conditions (MBC) as

$$\begin{cases} \mathcal{L}\, u \; = F(x,t,u), & (x,t) \in \Omega_T, \\ u(x,0) \; = u_0(x), & x \in [0,L], \\ u(0,t) \; = h_1(t), \qquad u_x(L,t) \; = h_2(t), & 0 < t \le T, \end{cases} \tag{11}$$

occur in many physical examples both in superconductivity (see,f.i. [33] and references therein) and in biology, as shown in [16, 22]. In particular, in [34], mixed boundary conditions are considered in order to give qualitative information concerning both the threshold problem and the asymptotic behavior of large solutions for the FHN system.

When $F = f(x,t)$ is a linear function, problems (9)-(11) can be solved by Laplace transformation with respect to t. Let $z(x,t)$ be an arbitrary function admitting Laplace transform $\hat{z}(x,s)$

$$\hat{z}(x,s) = \int_0^\infty e^{-st}\, z(x,t)\, dt \; = \mathcal{L}_t\, z. \tag{12}$$

Referring to the parameters $a,\ \beta,\ b,\ \varepsilon$ of the operator $\mathcal{L}$, if

$$\sigma^2 \; = s + a + \frac{b}{s+\beta}, \qquad \tilde{\sigma}^2 \; = \sigma^2/\varepsilon, \tag{13}$$

we denote by $\theta(x,s)$ and $\theta^*(x,s)$ the following Laplace transforms:

$$\hat{\theta}\,(\,y,\tilde{\sigma}) \; = \frac{\cosh[\,\tilde{\sigma}\;(L-y)\,]}{2\;\varepsilon\;\tilde{\sigma}\;\;\sinh(\,\tilde{\sigma}\,L\,)} \; = \tag{14}$$

$$= \frac{1}{2\,\sqrt{\varepsilon}\;\sigma}\left\{ e^{-\frac{y}{\sqrt{\varepsilon}}\;\sigma} + \sum_{n=1}^{\infty}\left[e^{-\frac{2nL+y}{\sqrt{\varepsilon}}\;\sigma} + e^{-\frac{2nL-y}{\sqrt{\varepsilon}}\;\sigma}\right]\right\},$$

$$\hat{\theta}^*\,(\,y,\tilde{\sigma}) \; = \frac{\sinh[\,\tilde{\sigma}\;(L-y)\,]}{2\;\varepsilon\;\tilde{\sigma}\;\;\cosh(\,\tilde{\sigma}\,L\,)} \; = \tag{15}$$

$$= \frac{1}{2\sqrt{\varepsilon}\sigma}\left\{ e^{-\frac{y}{\sqrt{\varepsilon}}\sigma} + 2\sum_{n=1}^{\infty}\left(e^{-\frac{4nL+y}{\sqrt{\varepsilon}}\;\sigma} + e^{-\frac{4nL-y}{\sqrt{\varepsilon}}\;\sigma}\right) - \sum_{n=1}^{\infty}\left(e^{-\frac{2nL+y}{\sqrt{\varepsilon}}\;\sigma} + e^{-\frac{2nL-y}{\sqrt{\varepsilon}}\;\sigma}\right)\right\}.$$

Then the Laplace transform solutions of the linear problems (9)-(11) can be obtained by means of standard techniques and it results:

- Formal solution for initial boundary problem with (NBC)

$$\begin{aligned} \hat{u}(x,s) = & \int_0^L [\,\hat{\theta}\,(\,|x-\xi|,\,s\,) + \;\hat{\theta}\,(\,|x+\xi|,\,s\,)\,]\;[\,u_0(\,\xi\,) + \hat{f}(\,\xi,s)\,]\,d\xi \\ & - \;2\;\varepsilon\;\hat{\psi}_1\,(s)\;\hat{\theta}(x,s) + 2\;\varepsilon\;\hat{\psi}_2\,(s)\;\hat{\theta}\,(x-L,s\,). \end{aligned} \tag{16}$$

• Formal solution for (DBC)

$$\hat{u}(x,s) = \int_0^L [\,\hat{\theta}\,(\,|x-\xi|,\,s\,) - \hat{\theta}\,(\,x+\xi,\,s\,)\,]\,[\,u_0(\,\xi\,) + \hat{f}(\,\xi,s)\,]\,d\xi - \\ - \; 2\,\varepsilon\,\hat{g}_1\,(s)\;\hat{\theta}_x(x,s) + 2\,\varepsilon\,\hat{g}_2\,(s)\;\hat{\theta}_x\,(x-L,s\,). \tag{17}$$

• Formal solution for (MBC)

$$\hat{u}(x,s) = \int_0^L [\,\hat{\theta}^*\,(\,x+\xi,\,s\,) - \hat{\theta}^*\,(\,|x-\xi|,\,s\,)\,]\,[\,u_0(\,\xi\,) + \hat{f}(\,\xi,s)\,]\,d\xi + \\ - \; 2\,\varepsilon\,\hat{h}_1\,(s)\;\hat{\theta}^*_x(x,s) + 2\,\varepsilon\hat{h}_2\,(s)\;\hat{\theta}^*\,(L-x,s\,). \tag{18}$$

3 $K_0(x,t)$ and $\theta(x,t)$ Properties

The Neumann boundary value problem has already been solved in [19]. Let us consider now cases (II) and (III).

Let $K_0(x,t)$ be the fundamental solution of the linear operator $\mathcal{L}$ defined in (1). It has already been determined in [18] and one has:

$$K_0(r,t) = \frac{1}{2\sqrt{\pi\varepsilon}}\left[\frac{e^{-\frac{r^2}{4t}-a\,t}}{\sqrt{t}} - \sqrt{b}\int_0^t \frac{e^{-\frac{r^2}{4y}-ay}}{\sqrt{t-y}}\,e^{-\beta(\,t-y\,)}J_1(2\sqrt{by\,(t-y)}\,)\,dy\right], \tag{19}$$

where $r = |x|\,/\sqrt{\varepsilon}$ and $J_n(z)$ is the Bessel function of first kind. Function K_0 has the same basic properties of the fundamental solution of the heat equation, and in the half-plane $\Re e\, s > max(-a,\,-\beta\,)$ it results:

$$\mathcal{L}_t\;K_0 \equiv \int_0^\infty e^{-st}\;K_0\,(r,t)\;dt = \frac{e^{-\,r\,\sigma}}{2\,\sqrt{\varepsilon}\,\sigma}, \tag{20}$$

where σ is defined in $(13)_1$.

Among other properties, in [18] the following estimates have been proved:

$$\int_\Re |\,K_0(x-\xi,t)|d\xi \le e^{-\,at} + \sqrt{b}\,\pi t e^{-\,\omega\,t} \qquad \int_0^t d\tau\int_\Re |K_0(x-\xi,t)|\;d\xi \le \beta_0, \tag{21}$$

$$|K_0| \le \frac{e^{-\frac{r^2}{4t}}}{2\,\sqrt{\pi\varepsilon t}}\,[\,e^{-\,at} + bt\,E(t)\,], \tag{22}$$

where constants $\omega,\ \beta_0$ and $E(t)$ are given by:

$$\omega = min(a,\beta), \qquad \beta_0 = \frac{1}{a} + \pi\sqrt{b}\,\frac{a+\beta}{2(a\beta)^{3/2}}, \tag{23}$$

$$E(t) = \frac{e^{-\,\beta t} - e^{-\,at}}{a-\beta} > 0.$$

Moreover, denoting by

$$K_i(r,t) = \int_0^t e^{-\,\beta\,(\,t-\tau)}\;K_{i-1}\,(x,\tau\,)\,d\tau \qquad (i=1,2) \tag{24}$$

kernels $K_1(x,t)$ and $K_2(x,t)$ have the same properties of $K_0(x,t)$. Hence, the following theorem holds [18]:

Theorem 3.1 *For all the positive constants a, b, ε, β it results:*

$$\int_{\Re} |K_1| \; d\xi \leq \; E(t); \qquad \int_0^t d\tau \int_{\Re} |K_1| \; d\xi \leq \beta_1, \tag{25}$$

$$\int_{\Re} |K_2(x-\xi,t)| \; d\xi \; \leq \; t\, E(t), \tag{26}$$

where $\beta_1 \; = \; (a\,\beta)^{-1}$.

In order to obtain inverse formulae of (17) and (18), let us apply (20) to (14)(15). Then, one deduces the following functions which are similar to *theta functions*:

$$\begin{aligned} \theta(x,t) \; &= K_0(x,t) \; + \sum_{n=1}^{\infty} \; [\, K_0(x\,+2nL,\,t) \,+\, K_0(x-2nL,\,t)\,] \\ &= \sum_{n=-\infty}^{\infty} \; K_0(x\,+2nL,\,t), \end{aligned} \tag{27}$$

$$\theta^*(x,t) \; = \; 2 \sum_{n=-\infty}^{\infty} K_0(x\,+4nL,\,t) \; - \; \sum_{n=-\infty}^{\infty} K_0(x\,+2nL,\,t). \tag{28}$$

Some of the properties of function $\theta(x,t)$ have already been evaluated in [19]. Precisely, denoting by $C = 2\varepsilon\; \pi^2/(\,6\,eL^2\,)$ and letting

$$C_0 \; = \; \frac{1}{2\sqrt{\varepsilon\,\omega}} \; + \; \frac{b\;\omega^{-\,3/2}}{4\sqrt{\varepsilon}\;|a-\beta|} \left[1 \; + \; \frac{C}{b}\,|a-\beta| \; + \frac{3\,C}{2\,\omega} \; \right], \tag{29}$$

the $\theta(x,t)$ function, defined in (27), satisfies the following inequalities:

$$\int_0^L |\theta(|x-\xi|,\,t)| \; d\xi \leq \; (1 \; + \; \sqrt{b}\,\pi\, t\,) \; e^{-\omega\, t}\,, \tag{30}$$

$$\int_0^t d\tau \int_0^L |\theta(|x-\xi|,\,t)| \; d\xi \leq \; \beta_0; \qquad \int_0^{\infty} \; |\theta(x,\tau)| \; d\tau \; \leq \; C_0, \tag{31}$$

and, it results:

$$\lim_{t\to\infty} \theta(x,t) \; = \; 0; \qquad \lim_{t\to\infty} \int_0^t \theta(x,\tau) \; d\tau \; = \frac{1}{2\,\varepsilon\;\sigma_0} \; \frac{\cosh\sigma_0 \; (L-x)}{\sinh\,(\sigma_0\, L).}, \tag{32}$$

where $\sigma_0 = \sqrt{\left(a \; + \dfrac{b}{\beta} \right) \dfrac{1}{\varepsilon}}$.

Furthermore, as for $\frac{\partial\theta}{\partial x}$, from (19), it is well-rendered that the x derivative of the integral term vanishes for $x \; \to \; 0$, while the first term represents the derivative with respect to x of the fundamental solution related to the heat equation. So, by means of classic theorems (see,f.i. [35] p. 60), conditions $(10)_3$ are surely satisfied.

Moreover, one has:

$$\lim_{t\to\infty}\int_0^t \theta_x(x,\tau)\; d\tau \;=\frac{1}{2\,\varepsilon}\;\;\frac{\sinh\sigma_0\;(x-L)}{\sinh\,(\sigma_0\,L)},$$
$$\lim_{t\to\infty}\int_0^t \theta^*_x(x,\tau)\; d\tau \;=-\frac{1}{2\,\varepsilon}\;\;\frac{\cosh\sigma_0\;(L-x)}{\cosh\,(\sigma_0\,L)}. \tag{33}$$

4 Asymptotic Behaviours

When the source term $F = f(x,t)$ is a prefixed function depending only on x and t, then, initial boundary value problems (10) (11) are linear and can be solved explicitly. Moreover, when $F = F(x,t,u)$ depends also on the unknown function $u(x,t)$, then these problems admit integral differential formulations and one has:

- Integro differential equation for problem (10) (DBC):

$$u(\,x,\,t\,)\;=\;\int_0^L\,[\theta\,(|x-\xi|,\,t)\,-\theta(x+\xi,\,t)\,]\;u_0(\xi)\;d\xi\;-$$
$$2\,\varepsilon\int_0^t\theta_x\,(x,\,t-\tau)\;\;g_1(\tau)\;d\tau\;+\;2\;\varepsilon\int_0^t\theta_x\,(x-L,\,t-\tau)\;\;g_2(\tau)\;d\tau \tag{34}$$
$$+\int_0^t d\tau\int_0^L\,[\,\theta\,(|x-\xi|,\,t-\tau)-\theta(x+\xi,\,t-\tau)]\;\;F\,(\,\xi,\tau,\,u(x,\tau))\;\;d\xi.$$

- Integro differential equation for (11) (MBC):

$$u(\,x,\,t\,)\;=\;\int_0^L\,[\theta^*\,(|x-\xi|,\,t)\,-\theta^*(x+\xi,\,t)\,]\;u_0(\xi)\;d\xi\;-$$
$$2\,\varepsilon\int_0^t\theta^*_x\,(x,\,t-\tau)\;\;h_1(\tau)\;d\tau\;+\;2\;\varepsilon\int_0^t\theta^*\,(L-x,\,t-\tau)\;\;h_2(\tau)\;d\tau \tag{35}$$
$$+\int_0^t d\tau\int_0^L\,[\,\theta^*\,(|x-\xi|,\,t-\tau)-\theta^*(x+\xi,\,t-\tau)]\;\;F\,(\,\xi,\tau,\,u(x,\tau))\;\;d\xi.$$

Now, if $\mathcal{B}_T$ denotes the Banach space

$$\mathcal{B}_T\;\equiv\;\left\{\,z\,(\,x,t\,)\,:\;z\;\in C\,(\Omega_T),\;\;||\,z\,||\;=\sup_{\Omega_T}\,|\,z\,(\,x,\,t)\,|,\;<\infty\right\} \tag{36}$$

and D is the following set:

$$D\equiv\{(x,t,u):(x,t)\in\Omega_T,-\infty<u<\infty,$$

then, let us assume the source term $F(x,t,u)$ be defined and continuous on D and uniformly Lipschitz continuous in (x,t,u) for each compact subset of Ω_T. Besides, let F be a bounded function for bounded u and there exists a constant C such that:

$$|F(x,t,u_1) - F(x,t,u_2)| \leq C\, |u_1 - u_2|.$$

So, by means of standard methods related to integral equations and owing to basic properties of K_0, it is possible to prove that the mappings defined by (34) (35) are a contraction of $\mathcal{B}_T$ in $\mathcal{B}_T$ and so they admit a unique fixed point $u(x,t) \in \mathcal{B}_T$. [35, 36]

In order to enable a quicker reading, attention will be paid only to the initial boundary value problem with Dirichlet conditions. However, all the following analysis can be applied to the mixed problem,too.

At first, let us consider $g_i = 0$ $(i = 1,2)$ and let

$$\|\, u_0 \,\| = \sup_{0 \leq x \leq L} |\, u_0\,(\,x\,)\,|, \qquad \|\, F \,\| = \sup_{\Omega_T} |\, F\,(\,x,\, t,\, u)\,|.$$

.

In [18] the following theorem has been proved:

Theorem 4.1 *When $g_i = 0$ $(i = 1,2)$, solution (34), for large t, verifies the following estimate:*

$$|u(x,t)| \leq 2\,\left[\, \|\, F \,\|\, \beta_0 + \|\, u_0 \,\|\, (1 + \sqrt{b}\,\pi\, t\,)\, e^{-\omega\, t}\,\right], \tag{37}$$

where $\omega = \min\,(a, \beta)$ and β_0 is defined by $(23)_2$.

As for contributes of boundary data, the well known theorem will be considered [37]:.

Theorem 4.2 *Let $h(t)$ and $\chi(t)$ be two continuous functions on $[0,\infty[$. If they satisfy the following hypotheses*

$$\exists \lim_{t\to\infty} \chi(t) = \chi(\infty), \qquad \exists \lim_{t\to\infty} h(t) = h(\infty), \tag{38}$$

$$\dot{h}(t) \in L_1[0,\infty), \tag{39}$$

then, it results:

$$\lim_{t\to\infty} \int_o^t \chi(t-\tau)\,\dot{h}(\tau)\,d\tau = \chi(\infty)\,[\,h(\infty) - h(0)\,]. \tag{40}$$

According to this, it is possible to state:

Theorem 4.3 *Let g_i $(i = 1,2)$ be two continuous functions converging for $t \to \infty$. In this case one has:*

$$\lim_{t\to\infty} \int_0^t \theta_x\,(x,\tau)\;\, g_i\,(t-\tau)\;\, d\,\tau = g_{i,\infty}\;\, \frac{1}{2\,\varepsilon}\;\, \frac{\sinh \sigma_0\;(x-L)}{\sinh\;\sigma_0\,L}, \tag{41}$$

where $\sigma_0 = \sqrt{\left(a + \dfrac{b}{\beta}\right)\dfrac{1}{\varepsilon}}$.

Proof. Let us apply (40) with $h = \int_0^t \theta_x(x,\tau)d\tau$ and $\chi = g_i$ $(i = 1,2)$. Then, (41) follows by $(33)_1$.

5 An Example: Estimate for the FitzHugh Nagumo System

When $u(x,t)$ is determined, by means of (6), the $v(x,t)$ component is given by

$$v\,(x,t) \;=\; v_0\, e^{-\,\beta\, t} \;+\; b \int_0^t e^{-\,\beta\,(\,t-\tau\,)}\, u(x,\tau)\, d\tau. \tag{42}$$

To achieve the expression of the solution (u,v), let us denote by $f_1 * f_2$ the convolution

$$f_1(\cdot,t) *\, f_2(\cdot,t) \;=\int_0^t f_1(\cdot,t)\, f_2\;(\cdot,t-\tau)\, d\,\tau.$$

So that, referring to Dirichlet conditions, if

$$G(x,\xi,t) \;=\; \theta\,(\,|x-\xi|,\,t\,) \;-\; \theta\,(\,x+\xi,\,t\,),$$

and denoting by $N(x,t)$ the following known function depending on the data (u_0, v_0, g_1, g_2):

$$N(x,t) \;=\; -2\,\varepsilon\, g_1(t) \,*\, \theta_x(x,t) + \tag{43}$$

$$+\,2\,\varepsilon\; g_2(t) \,*\, \theta_x(x-L,t) \,+ \int_0^L u_0\,(\xi)\; G(x,\xi,t)\, d\xi \;-\; e^{-\,\beta\, t} \;*\int_0^L v_0(\xi)\; G(x,\xi,t)\, d\xi\,,$$

it results:

$$\begin{aligned} v\,(x,t) \;&=\; v_0\, e^{-\,\beta\, t} \;+\; b\, e^{-\,\beta\, t} \;*\; N(x,t) \\ &\quad+\; b\;\; e^{-\,\beta\, t} \;*\int_0^L G\,(\,x,\xi,t-\tau)\;\, *\; \varphi\,[\,\xi,\,\tau,\,u(\xi,\tau)]\;]\}\;\, d\xi\,. \end{aligned} \tag{44}$$

So, the asymptotic effects due to initial disturbances are vanishing, while the effects of the source terms are bounded. Indeed, letting

$$||\,u_0\,|| \;=\; \sup_{0\le\, x\,\le\, L}\; |\,u_0\,(\,x\,)\,|, \qquad ||\,v_0\,|| \;=\; \sup_{0\le\, x\,\le\, L}\; |\,v_0\,(\,x\,)\,|,$$

and

$$||\,\varphi\,|| \;= \sup_{\Omega_T}\, |\,\varphi\,(\,x,\,t,\,u)\,|,$$

by means of (8) (34) and (44) and owing to the estimates $(21)_1$, (25), (26), the following theorem holds:

Theorem 5.1 *For regular solution (u,v) of the (FHN) model, when $g_1 \;= g_2 \;=\; 0$, the following estimates hold:*

$$\begin{cases} |u\,| \;\le 2\,[\,\|u_0\|\;(1+\pi\sqrt{b}\,t)\, e^{-\,\omega\, t} \;+\; \|v_0\|\; E(t) \;+\; \beta_0\;\|\varphi\|\,], \\ |v\,| \;\le \|v_0\|\; e^{-\,\beta\, t} \;+\; 2\,[\,b\,(\,\|u_0\| \;+\; t\;\|v_0\|\,)\, E(t) \;+\; b\,\beta_1\;\|\varphi\|\,]. \end{cases} \tag{45}$$

As for the asymptotic effects of boundary perturbations g_1, g_2 by means of (41), when $u_0 = 0$ and $F = 0$, one has

$$\begin{cases} u = g_{1,\infty} \frac{\sinh \sigma_0 \ (L-x)}{\sinh \ \sigma_0 L} + g_{2,\infty} \frac{\sinh \sigma_0 \, x}{\sinh \ \sigma_0 L} \Big|, \\ v = \dfrac{b}{\beta} \left[g_{1,\infty} \quad \frac{\sinh \sigma_0 \ (L-x)}{\sinh \ \sigma_0 L} + g_{2,\infty} \quad \frac{\sinh \sigma_0 \ (x)}{\sinh \ \sigma_0 L} \right]. \end{cases} \tag{46}$$

6 Remarks

• The paper is concerned with the nonlinear integral equation (1) whose kernel is a Green function with numerous basic properties typical of the diffusion equation.

• Neumann, Dirichlet and mixed boundary conditions are considered, and integro differential formulations of *non linear* problems are obtained.

• The asymptotic behavior for initial boundary value problem with Dirichlet conditions is evaluated, showing that effects due to initial disturbances vanish, while the influences of the source term and boundary perturbations are everywhere bounded.

•The analysis related to Dirichlet conditions can be applied to mixed problem, too. Indeed, like $\theta(x,t)$, also the Green function $\theta^*(x,t)$ defined in (15) depends on the fundamental solution K_0.

• The equivalence among equation (1) and numerous models allow us to apply asymptotic theorems to many other problems related to various physical fields.

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