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Existence of Solutions for m-Point Boundary Value Problem with p-Laplacian on Time Scales

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Abstract: We consider the existence of positive solutions for a class of second-order m-point boundary value problem with p-Laplacian on time scales. By using Avery-Peterson's fixed point theorem, sufficent conditions for the positive solutions are established. Meanwhile an example is worked out to illustrate the main result.

Keywords: *m-point boundary value problems; p-Laplacian operator; positive solutions; fixed point theorems; time scales.*

Mathematics Subject Classification (2010): 39A10, 34B15, 34B16.

1 Introduction

Calculus on time scales was introduced by Hilger (see [6]), as a theory which is undergoing rapid development as it provides a unifying structure for the study of differential equations in the continuous case and the study of difference equations in the discrete case. Some preliminary definitions and theorems on time scales can be found in books [3, 4] which are excellent references for calculus of time scales. Also, there is much attention paid to the study of multipoint boundary value problem (see [1, 2, 7–13]).

In [5] the following m-point boundary value problem on time scales was studied

$$u^{\Delta\nabla}(t) + q(t)f(u(t)) = 0, \quad t \in [0,T]_{\mathbb{T}},$$
$$u^{\Delta}(0) = \sum_{i=1}^{m-2} b_i u^{\Delta}(\xi_i), \quad u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i),$$

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where $a_i, b_i \geq 0$ $(i = 1, 2, ..., m-2)$, and $\xi_i \in (0, \rho(T))_{\mathbb{T}}$ with $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < \rho(T)$. And the existence of at least two positive solutions of the above problem was established by means of a fixed point theorem in a cone.

Zhao and Ge [13] studied the following m-point boundary value problem on time scales

$$(\phi_p(u^{\triangle}))^{\nabla}(t) + h(t)f(t, u(t), u^{\Delta}(t)) = 0, \quad t \in (0, \infty)_{\mathbb{T}},$$
$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \quad u^{\Delta}(+\infty) = \sum_{i=1}^{m-2} \beta_i u^{\Delta}(\eta_i),$$

where $\alpha_i, \beta_i \geq 0$ $(i = 1, 2, ..., m-2)$, and $\eta_i \in (0, \infty)_{\mathbb{T}}$ with $\sigma(0) < \eta_1 < \eta_2 < ... < \eta_{m-2} < +\infty$. They established new criteria for the existence of at least three unbounded positive solutions by using Avery-Peterson's fixed point theorem.

Ji, Bai and Ge [7] studied the following singular multipoint boundary value problem on time scales

$$(\phi_p(u'))'(t) + a(t)f(u(t)) = 0, \quad t \in (0, 1),$$
$$u'(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i),$$

where $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1$, $0 < \eta_1 < \eta_2 < ... < \eta_{m-2} < 1$, $\xi_i < \eta_i$, $\alpha_i > 0$ for $i = 1, 2, ..., m-2$. By using fixed point index theory and the Legget-Williams fixed point theorem, sufficent conditions for the existence of countably many positive solutions are established.

Sun, Wang and Fan [10] studied the nonlocal boundary value problem with p-Laplacian of the form

$$(\phi_p(u^{\triangle}))^{\nabla}(t) + h(t)f(t, u(t)) = 0, \quad t \in [t_1, t_m]_{\mathbb{T}},$$
$$u^{\Delta}(t_1) - \sum_{j=1}^{n} \theta_j u^{\Delta}(\eta_j) - \sum_{i=1}^{m-2} \varepsilon_i u(\xi_i) = 0, \quad u^{\Delta}(t_m) = 0,$$

where $0 \leq t_1 \leq \xi_1 \leq \xi_2 \leq ... \leq \xi_{m-2} \leq t_m$ and $0 \leq t_1 \leq \eta_1 \leq \eta_2 \leq ... \leq \eta_{m-2} \leq t_m$ and $\varepsilon_i > 0$, $\theta_i \geq 0$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. By using the Four functionals fixed point theorem and Five Functionals fixed point theorem, they obtained the existence criteria of at least one positive solution and three positive solutions.

Inspired by the mentioned works, in this paper we consider the following m-point boundary value problem (BVP) with p-Laplacian

$$(\phi_p(x^{\triangle}))^{\nabla}(t) + h(t)f(t, x(t), x^{\Delta}(t)) = 0, \quad t \in [0, 1]_{\mathbb{T}}, \tag{1}$$
$$x^{\Delta}(0) - \sum_{i=1}^{m-2} \alpha_i x(\xi_i) = 0, \quad x^{\Delta}(1) + \sum_{i=1}^{m-2} \alpha_i x(\eta_i) = 0, \tag{2}$$

where $\mathbb{T}$ is a time scale, $\phi_p(s) = |s|^{p-2}s$ for $p > 1$, $(\phi_p)^{-1}(s) = \phi_q(s)$, and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

We assume that the following conditions are satisfied:

(H1) $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < \rho(1)$, $0 < \eta_1 < \eta_2 < ... < \eta_{m-2} < \rho(1)$, $\xi_i < \eta_i$, $\alpha_i > 0$ for $i = 1, 2, ..., m-2$, $\sum_{i=1}^{m-2} \alpha_i \xi_i < 1$ and $[\sum_{i=1}^{m-2} \alpha_i (1-\xi_i)]^2 + \sum_{i=1}^{m-2} \alpha_i (1-\xi_i) < 1$,

(H2) $f \in \mathcal{C}([0,1]_{\mathbb{T}} \times [0,\infty) \times (-\infty,\infty), (0,\infty))$,

(H3) $h \in \mathcal{C}_{ld}([0,1]_{\mathbb{T}}, [0,\infty))$.

By using Avery-Peterson fixed point theorem, we establish the existence of at least three positive solutions for the BVP (1)-(2). The remainder of this paper is organized as follows. Section 2 is devoted to some preliminary lemmas. We give and prove our main result in Section 3.

2 Preliminaries

To prove the main result in this paper, we will employ several lemmas. These lemmas are based on the BVP

$$(\phi_p(x^{\triangle}))^{\nabla}(t) + y(t) = 0, \quad t \in [0,1]_{\mathbb{T}}, \tag{3}$$

$$x^{\Delta}(0) - \sum_{i=1}^{m-2} \alpha_i x(\xi_i) = 0, \quad x^{\Delta}(1) + \sum_{i=1}^{m-2} \alpha_i x(\eta_i) = 0. \tag{4}$$

Lemma 2.1 *Let $(H1)-(H3)$ hold. Then for $y \in \mathcal{C}_{ld}[0,1]_{\mathbb{T}}$, the BVP (3)-(4) has the unique solution*

$$x(t) = \frac{\phi_q(A_x) + \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{1} \phi_q(A_x - \int_0^s y(\tau)\nabla\tau)\triangle s}{\sum_{i=1}^{m-2} \alpha_i}$$

$$- \int_t^1 \phi_q(A_x - \int_0^s y(\tau)\nabla\tau)\triangle s, \tag{5}$$

where A_x satisfies

$$\phi_q(A_x) + \phi_q\left(A_x - \int_0^1 y(s)\nabla s\right)$$

$$+ \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{\eta_i} \phi_q\left(A_x - \int_0^s y(\tau)\nabla\tau\right)\triangle s = 0. \tag{6}$$

Moreover, there exists a unique $A_x \in (0, \int_0^1 y(s)\nabla s)$ satisfying (6).

Proof. Integrating (3) from 0 to t, we have

$$x^{\triangle}(t) = \phi_q\left(\phi_p(x^{\triangle}(0)) - \int_0^t y(s)\nabla s\right). \tag{7}$$

Integrating (7) from t to 1, we get

$$x(t) = x(1) - \int_t^1 \phi_q\left(A_x - \int_0^s y(\tau)\nabla\tau\right)\triangle s, \tag{8}$$

where $A_x = \phi_p(x^{\triangle}(0))$. Setting $t = \xi_i$ in (8) we have

$$x(\xi_i) = x(1) - \int_{\xi_i}^1 \phi_q\left(A_x - \int_0^s y(\tau)\nabla\tau\right)\triangle s, \quad i = 1, 2, 3, ..., m-2$$

and

$$\sum_{i=1}^{m-2}\alpha_i x(\xi_i) = \sum_{i=1}^{m-2}\alpha_i x(1) - \sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^1 \phi_q\left(A_x - \int_0^s y(\tau)\nabla\tau\right)\triangle s$$

then

$$x(1) = \frac{\phi_q(A_x) + \sum\limits_{i=1}^{m-2}\alpha_i \int_{\xi_i}^1 \phi_q\left(A_x - \int_0^s y(\tau)\nabla\tau\right)\triangle s}{\sum\limits_{i=1}^{m-2}\alpha_i}. \tag{9}$$

Substituting (9) into (8) we see that $x(t)$ satisfies (5) on $[0,1]_{\mathbb{T}}$. (4) boundary conditions satisfy

$$x^{\triangle}(0) + x^{\triangle}(1) = \sum_{i=1}^{m-2}\alpha_i x(\xi_i) - \sum_{i=1}^{m-2}\alpha_i x(\eta_i)$$

$$\phi_q(A_x) + \phi_q\left(A_x - \int_0^1 y(s)\nabla s\right) = \sum_{i=1}^{m-2}\alpha_i(x(\xi_i) - x(\eta_i))$$

$$= \sum_{i=1}^{m-2}\alpha_i\left(-\int_{\xi_i}^1 \phi_q\left(A_x - \int_0^s y(\tau)\nabla\tau\right)\triangle s + \int_{\eta_i}^1 \phi_q\left(A_x - \int_0^s y(\tau)\nabla\tau\right)\triangle s\right)$$
$$= -\sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^{\eta_i} \phi_q\left(A_x - \int_0^s y(\tau)\nabla\tau\right)\triangle s.$$

So that BVP (3)-(4) has a solution $x(t)$ where A_x satisfies (6).

For any $x \in \mathcal{C}_{ld}^{\triangle}[0,1]_{\mathbb{T}}$, define

$$H_x(c) = \phi_q(c) + \phi_q\left(c - \int_0^1 h(s)f(s, x(s), x^{\triangle}(s))\nabla s\right)$$

$$+\sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^{\eta_i} \phi_q\left(c - \int_0^s h(\tau)f(\tau, x(\tau), x^{\triangle}(\tau))\nabla\tau\right)\triangle s.$$

Then $H_x : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing. $H_x(0) < 0$,
$H_x\left(\int_0^1 h(s)f(s,x(s),x^{\triangle}(s))\nabla s\right) > 0$, imply the existence of a unique
$c = A_x \in (0, \int_0^1 h(s)f(s,x(s),x^{\triangle}(s))\nabla s)$ such that $H_x(A_x) = 0$. $\square$

Lemma 2.2 *If $(H1)-(H3)$ hold, then for $x \in \mathcal{C}_{ld}^{\triangle}[0,1]_{\mathbb{T}}$, the unique solution $x(t)$ of BVP (3)-(4) has the following properties:*
(i) $x(t)$ is concave on $[0,1]_{\mathbb{T}}$,
(ii) $x(t) > 0$.

Proof. Suppose that $x(t)$ is a solution of BVP (3)-(4), then
(i) $(\phi_p(x^{\triangle}))^{\nabla}(t) = -h(t)f(t,x(t),x^{\triangle}(t)) \le 0$, $\phi_p(x^{\triangle})$ is nonincreasing so that $x^{\triangle}(t)$ is nonincreasing. This implies that $x(t)$ is concave.
(ii) We have $x^{\triangle}(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i) = \phi_q\left(A_x\right) > 0$ and
$x^{\triangle}(1) = \phi_q\left(A_x - \int_0^1 h(s)f(s,x(s),x^{\triangle}(s))\nabla s\right) < 0$. Furthermore, we get

$$\alpha_1 x(\xi_1) - \alpha_1 x(0) = \alpha_1 \int_0^{\xi_1} x^{\triangle}(s)\Delta s \le \alpha_1 \xi_1 x^{\triangle}(0) = \alpha_1 \xi_1 \sum_{i=1}^{m-2} \alpha_i x(\xi_i)$$

$$\alpha_2 x(\xi_2) - \alpha_2 x(0) = \alpha_2 \int_0^{\xi_2} x^{\triangle}(s)\Delta s \le \alpha_2 \xi_2 x^{\triangle}(0) = \alpha_2 \xi_2 \sum_{i=1}^{m-2} \alpha_i x(\xi_i).$$

If we continue like this, we have

$$\begin{aligned}\alpha_{m-2} x(\xi_{m-2}) - \alpha_{m-2} x(0) &= \alpha_{m-2} \int_0^{\xi_{m-2}} x^{\triangle}(s)\Delta s \le \alpha_{m-2}\xi_{m-2} x^{\triangle}(0) \\ &= \alpha_{m-2}\xi_{m-2} \sum_{i=1}^{m-2} \alpha_i x(\xi_i).\end{aligned}$$

Using $(H1)$, we obtain

$$\sum_{i=1}^{m-2} \alpha_i x(\xi_i) - \sum_{i=1}^{m-2} \alpha_i x(0) \le \sum_{i=1}^{m-2} \alpha_i x(\xi_i) \sum_{i=1}^{m-2} \alpha_i \xi_i < \sum_{i=1}^{m-2} \alpha_i x(\xi_i),$$

which implies that $x(0) > 0$. Similarly,

$$\alpha_1 x(1) - \alpha_1 x(\eta_1) = \alpha_1 \int_{\eta_1}^1 x^{\triangle}(s)\Delta s \ge \alpha_1(1-\eta_1)x^{\triangle}(1) = -\alpha_1(1-\eta_1) \sum_{i=1}^{m-2} \alpha_i x(\eta_i),$$

$$\alpha_2 x(1) - \alpha_2 x(\eta_2) = \alpha_2 \int_{\eta_2}^1 x^{\triangle}(s)\Delta s \ge \alpha_2(1-\eta_2)x^{\triangle}(1) = -\alpha_2(1-\eta_2) \sum_{i=1}^{m-2} \alpha_i x(\eta_i).$$

If we continue like this, we have

$$\begin{aligned}\alpha_{m-2}x(1)-\alpha_{m-2}x(\eta_{m-2}) &= \alpha_{m-2}\int_{\eta_{m-2}}^{1}x^{\Delta}(s)\Delta s \geq \alpha_{m-2}(1-\eta_{m-2})x^{\Delta}(1)\\ &= -\alpha_{m-2}(1-\eta_{m-2})\sum_{i=1}^{m-2}\alpha_i x(\eta_i).\end{aligned}$$

Using (H1), we have $\sum_{i=1}^{m-2}\alpha_i x(1)>0$, $x(1)>0$. Therefore, we get $x(t)>0, t\in[0,1]_{\mathbb{T}}$. □

Let $E=\mathcal{C}_{ld}^{\triangle}[0,1]_{\mathbb{T}}$, then E is a Banach space with the norm

$$\|x\|=\max\{\sup_{t\in[0,1]_{\mathbb{T}}}|x(t)|,\sup_{t\in[0,1]_{\mathbb{T}}}|x^{\Delta}(t)|\}$$

and choose the cone $P\subset E$ denoted by

$$P=\{x\in E: x(t)\geq 0, x^{\Delta}(0)=\sum_{i=1}^{m-2}\alpha_i x(\xi_i),\quad x(t)\quad is\quad concave\quad on\quad [0,1]_{\mathbb{T}}\}.$$

Define the operator $T:P\to E$ by

$$\begin{aligned}Tx(t) &= \frac{\phi_q(A_x)+\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{1}\phi_q(A_x-\int_0^s h(\tau)f(\tau,x(\tau),x^{\triangle}(\tau))\nabla\tau)\triangle s}{\sum_{i=1}^{m-2}\alpha_i}\\ &\quad -\int_t^1\phi_q(A_x-\int_0^s h(\tau)f(\tau,x(\tau),x^{\triangle}(\tau))\nabla\tau)\triangle s. \qquad (10)\end{aligned}$$

Lemma 2.3 *If (H1) holds, then* $\sup_{t\in[0,1]_{\mathbb{T}}}x(t)\leq M\sup_{t\in[0,1]_{\mathbb{T}}}|x^{\Delta}(t)|$ *for* $x\in P$, *where*

$$M=1+\frac{1}{\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)}. \qquad (11)$$

Proof. For $x\in P$, one arrives at

$$x(1)-x(0)\leq\frac{x(\xi_i)-x(0)}{\xi_i}.$$

Hence,

$$\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)x(0)\leq\sum_{i=1}^{m-2}\alpha_i x(\xi_i).$$

By $x^{\triangle}(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i)$, we get

$$x(0) \leq \frac{1}{\sum_{i=1}^{m-2} \alpha_i (1-\xi_i)} x^{\triangle}(0).$$

Hence

$$\begin{aligned} x(t) &= \int_0^t x^{\triangle}(s) \triangle s + x(0) \\ &\leq tx^{\triangle}(0) + x(0) \\ &\leq tx^{\triangle}(0) + \frac{1}{\sum_{i=1}^{m-2} \alpha_i (1-\xi_i)} x^{\triangle}(0) \\ &\leq [1 + \frac{1}{\sum_{i=1}^{m-2} \alpha_i (1-\xi_i)}] x^{\triangle}(0) \\ &= Mx^{\triangle}(0), \end{aligned}$$

i.e,

$$\sup_{t\in[0,1]_{\mathbb{T}}} x(t) \leq Mx^{\triangle}(0) = M \sup_{t\in[0,1]_{\mathbb{T}}} x^{\triangle}(t) \leq M \sup_{t\in[0,1]_{\mathbb{T}}} |x^{\triangle}(t)|.$$

The proof is complete. □

From Lemma 2.3, we obtain

$$\begin{aligned} \|x\| &= \max\{ \sup_{t\in[0,1]_{\mathbb{T}}} |x(t)|, \quad \sup_{t\in[0,1]_{\mathbb{T}}} |x^{\triangle}(t)| \} \\ &\leq \max\{ M \sup_{t\in[0,1]_{\mathbb{T}}} |x^{\triangle}(t)|, \quad \sup_{t\in[0,1]_{\mathbb{T}}} |x^{\triangle}(t)| \} \\ &\leq M \sup_{t\in[0,1]_{\mathbb{T}}} |x^{\triangle}(t)|. \end{aligned}$$

Lemma 2.4 *For $x \in \mathcal{C}_{ld}^{\triangle}[0,1]_{\mathbb{T}}$, let A_x satisfy (6) corresponding to x. Suppose that $(H1)-(H3)$ hold, then $A_x : \mathcal{C}_{ld}^{\triangle}[0,1]_{\mathbb{T}} \longrightarrow \mathbb{R}$ is continuous about x.*

Proof. Suppose $\{x_n\} \in \mathcal{C}_{ld}^{\triangle}[0,1]_{\mathbb{T}}$ with $x_n \longrightarrow x_0 \in \mathcal{C}_{ld}^{\triangle}[0,1]_{\mathbb{T}}$, then there exists r_0 such that

$$\max\{\|x_0\|, \sup_{n\in\mathbb{N}-\{0\}} \|x_n\|\} < r_0.$$

Let A_n $(n = 0, 1, ...)$ be constants decided by (6) corresponding to x_n $(n = 0, 1, 2, ...)$. By $(H2)$, we get that $f(t, u, v)$ is bounded on $[0, 1]_{\mathbb{T}} \times [0, r_0]^2$. Set

$$B_{r_0} = \sup\{f(t, u, v) : (t, u, v) \in [0, 1]_{\mathbb{T}} \times [0, r_0]^2\}.$$

Since

$$\int_0^1 h(s)f(s,x(s),x^{\Delta}(s))\Delta s \le B_{r_0}\int_0^1 h(s)\Delta s = B_{r_0}\Lambda,$$

where $\Lambda = \int_0^1 h(s)\Delta s$, $A_n \in [0, \int_0^1 h(s)f(s,x(s),x^{\Delta}(s))\Delta s] \subseteq [0, B_{r_0}\Lambda]$, which means $\{A_n\}$ is bounded. Suppose that sequence $\{A_n\}$ does not convergence, then there exist two subsequences $\{A^{(1)}_{n_k}\}$, $\{A^{(2)}_{n_k}\}$ of $\{A_n\}$ with $A^{(1)}_{n_k} \longrightarrow c_1$, $A^{(2)}_{n_k} \longrightarrow c_2$, and $c_1 \ne c_2$. Combining $(H2)$ and using the Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}
\phi_q(c_1) &= -\lim_{n_k\to+\infty}\phi_q(A^{(1)}_{n_k} - \int_0^1 h(s)f(s,x_{n_k}(s),x^{\Delta}_{n_k}(s))\nabla s) \\
&\quad -\lim_{n_k\to+\infty}\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{\eta_i}\phi_q(A^{(1)}_{n_k} - \int_0^s h(\tau)f(\tau,x_{n_k}(\tau),x^{\Delta}_{n_k}(\tau))\nabla\tau)\Delta s \\
&= -\phi_q(\lim_{n_k\to+\infty}A^{(1)}_{n_k} - \lim_{n_k\to+\infty}\int_0^1 h(s)f(s,x_{n_k}(s),x^{\Delta}_{n_k}(s))\nabla s \\
&\quad -\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{\eta_i}\phi_q(\lim_{n_k\to+\infty}A^{(1)}_{n_k} - \lim_{n_k\to+\infty}\int_0^s h(\tau)f(\tau,x_{n_k}(\tau),x^{\Delta}_{n_k}(\tau))\nabla\tau)\Delta s \\
&= -\phi_q(c_1 - \int_0^1 h(s)f(s,x_0(s),x^{\Delta}_0(s))\nabla s \\
&\quad -\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{\eta_i}\phi_q(c_1 - \int_0^s h(\tau)f(\tau,x_0(\tau),x^{\Delta}_0(\tau))\nabla\tau)\Delta s.
\end{aligned}$$

Since sequence $\{A_n\}$ is unique, we get $c_1 = A_0$. Similarly $c_2 = A_0$. So $c_1 = c_2$, which is a contradiction. Therefore $A_n \longrightarrow A_0$ for $x_n \longrightarrow x_0$, which means $A_x : \mathcal{C}^{\triangle}_{ld}[0,1]_{\mathbb{T}} \longrightarrow \mathbb{R}$ is continuous. The proof is complete. $\square$

Lemma 2.5 *Suppose that* $(H1)-(H3)$ *hold, then* $T : P \longrightarrow P$ *is completely continuous.*

Proof. We divide the proof into three steps.

Step 1. We show that $TP \subset P$. For $x \in P$, by $(H1)-(H3)$, we have $(Tx)(t) \ge 0$ and $(Tx)^{\triangle}(0) = \sum_{i=1}^{m-2}\alpha_i(Tx)(\xi_i)$.

If $t \in [0,1]_{\mathbb{T}}$ is left scattered, then

$$(Tx)^{\Delta\nabla}(t) = \frac{(Tx)^{\Delta}(t) - (Tx)^{\Delta}(\rho(t))}{t-\rho(t)} \le 0$$

on $t \in [0,1]_{\mathbb{T}}$. If $t \in [0,1]_{\mathbb{T}}$ is left dense, then

$$(Tx)^{\Delta\nabla}(t) = \lim_{s\to t}\frac{(Tx)^{\Delta}(t) - (Tx)^{\Delta}(s)}{t-s} \le 0$$

on $t \in [0,1]_{\mathbb{T}}$. Hence Tx is nonnegative, concave on $[0,1]_{\mathbb{T}}$, i.e., $TP \subset P$.

Step 2. We show that $T : P \longrightarrow P$ is continuous. Let $x_n \longrightarrow x$ as $n \longrightarrow +\infty$ in P, then there exists r_0 such that

$$\max\{\|x\|, \sup_{n\in\mathbb{N}-\{0\}} \|x_n\|\} < r_0.$$

By $(H2)$, we get that $f(t,u,v)$ is bounded on $[0,1]_{\mathbb{T}} \times [0,r_0]^2$. Set

$$B_{r_0} = \sup\{f(t,u,v) : (t,u,v) \in [0,1] \times [0,r_0]^2\}.$$

We get

$$\begin{aligned}
& |\phi_p((Tx_n)^{\Delta}(t)) - \phi_p((Tx)^{\Delta}(t))| \\
= \;& |A_{x_n} - \int_0^t h(s)f(s,x_n(s),x_n^{\Delta}(s))\nabla s - A_x - \int_0^t h(s)f(s,x(s),x^{\Delta}(s))\nabla s| \\
\leq \;& |A_{x_n} - A_x| + \int_0^t h(s)|f(s,x_n(s),x_n^{\Delta}(s)) - f(s,x(s),x^{\Delta}(s))|\nabla s \\
\leq \;& |A_{x_n} - A_x| + 2B_{r_0}\Lambda = 2B_{r_0}\Lambda + 2B_{r_0}\Lambda = 4B_{r_0}\Lambda.
\end{aligned}$$

Therefore by the Lebesgue's dominated convergence theorem, we have

$$|\phi_p((Tx_n)^{\Delta}(t)) - \phi_p((Tx)^{\Delta}(t))| \longrightarrow 0 \;as\; n \longrightarrow +\infty.$$

By using Lemma 2.3 we get

$$0 \leq \|(Tx_n)(t) - (Tx)(t)\| \leq M \sup_{t\in[0,1]_{\mathbb{T}}} |(Tx_n)^{\Delta}(t) - (Tx)^{\Delta}(t)| \longrightarrow 0 \;as\; n \longrightarrow +\infty.$$

Hence T is continuous.

Step 3. We show that $T : P \longrightarrow P$ is relatively compact. Let Ω be any bounded set of P. Then there exists $L > 0$ such that $\|x\| \leq L$ for all $x \in \Omega$. Set

$$B_L = \sup\{f(t,u,v) : (t,u,v) \in [0,1] \times [0,r_0]^2\}.$$

For $x \in \Omega$, we have

$$\begin{aligned}
\|Tx\| &= \max\{\sup_{t\in[0,1]_{\mathbb{T}}} Tx(t), \sup_{t\in[0,1]_{\mathbb{T}}} |(Tx)^{\Delta}(t)|\} \\
&\leq M(Tx)^{\triangle}(0) \\
&\leq M\phi_q(A_x) \leq M\phi_q(B_L\Lambda).
\end{aligned}$$

Hence $T\Omega$ is uniformly bounded.

Now we show that $T\Omega$ is locally equicontinuous on $[0,1]_{\mathbb{T}}$. For $t_1, t_2 \in [0,1]_{\mathbb{T}}$ and $x \in \Omega$, we may assume that $t_2 > t_1$.

$$\begin{aligned}
& |\phi_p((Tx)^{\Delta}(t_1)) - \phi_p((Tx)^{\Delta}(t_2))| \\
= \;& |A_x - \int_0^{t_1} h(s)f(s,x(s),x^{\Delta}(s))\nabla s - A_x + \int_0^{t_2} h(s)f(s,x(s),x^{\Delta}(s))\nabla s|.
\end{aligned}$$

Hence,

$$|\phi_p((Tx)^{\Delta}(t_1)) - \phi_p((Tx)^{\Delta}(t_2))| \longrightarrow 0 \text{ as } t_1 \longrightarrow t_2.$$

Since

$$\sup_{t\in[0,1]_{\mathbb{T}}} |(Tx)^{\Delta}(t_1) - (Tx)^{\Delta}(t_2)| \longrightarrow 0 \text{ as } t_1 \longrightarrow t_2,$$

we get

$$\|(Tx)(t_1) - (Tx)(t_2)\| \longrightarrow 0 \text{ as } t_1 \longrightarrow t_2.$$

Hence $T\Omega$ is locally equicontinuous on $[0,1]_{\mathbb{T}}$. From step $1-3$, we get $T : P \longrightarrow P$ is completely continuous. The proof is complete. $\square$

3 Existence of Three Positive Solutions

Let γ and θ be nonnegative continuous convex functionals on a cone P, α be nonnegative continuous concave functional on P and ψ be nonnegative continuous functional on P. Then for positive real numbers a, b, c and d, we define the following convex sets

$$P(\gamma, d) = \{x \in P : \gamma(x) < d\},$$

$$P(\gamma, \alpha, b, d) = \{x \in P : b \le \alpha(x),\ \gamma(x) \le d\},$$

$$P(\gamma, \theta, \alpha, b, c, d) = \{x \in P : b \le \alpha(x),\ \theta(x) \le c,\ \gamma(x) \le d\},$$

$$R(\gamma, \psi, a, d) = \{x \in P : a \le \psi(x),\ \gamma(x) \le d\}.$$

Theorem 3.1 *(Avery-Peterson's Fixed Point Theorem) [13] Let $\mathcal{P}$ be a cone in a real Banach space E. Assume that there exist two positive number M and d, two nonnegative continuous convex functionals γ and θ on P, a nonnegative continuous concave functional α on P and a nonnegative continuous functional ψ on P such that $\psi(\lambda x) \le \lambda\psi(x)$ for all $0 \le \lambda \le 1$ and*

$$\alpha(x) \le \psi(x), \quad \|x\| \le M\gamma(x)$$

for all $x \in \overline{P(\gamma, d)}$. Suppose that $T : \overline{P(\gamma, d)} \longrightarrow \overline{P(\gamma, d)}$ is completely continuous and there exist three positive numbers a, b and c with $a < b$ such that

(S1) $\{x \in P(\gamma, \theta, \alpha, b, c, d) | \alpha(x) > b\} \ne \emptyset$ *and* $\alpha(Tx) > b$ *for* $x \in P(\gamma, \theta, \alpha, b, c, d)$;

(S2) $\alpha(Tx) > b$ *for* $x \in P(\gamma, \alpha, b, d)$ *with* $\theta(Tx) > c$;

(S3) $0 \notin R(\gamma, \psi, a, d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\gamma, \psi, a, d)$ *with* $\psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that

$$\gamma(x_i) \le d, \quad i = 1, 2, 3, \quad \psi(x_1) < a, \quad a < \psi(x_2) \quad \text{with} \quad \alpha(x_2) < b, \quad \alpha(x_3) > b.$$

Set

$$\Omega = \int_w^{\nu} h(\tau)\nabla\tau,$$

and define the maps

$$\gamma(x) = \sup_{t\in[0,1]_{\mathbb{T}}} |x^{\Delta}(t)|, \ \psi(x) = \theta(x) = \sup_{t\in[0,1]_{\mathbb{T}}} x(t), \ \alpha(x) = \min_{t\in[w,v]_{\mathbb{T}}} x(t). \tag{12}$$

Theorem 3.2 *Assume* $(H1)-(H3)$ *hold. Let*

$$\frac{2b}{w}\frac{1}{\sum_{i=1}^{m-2}\alpha_i}\left[\sum_{i=1}^{m-2}\alpha_i+1-\sum_{i=1}^{m-2}\alpha_i\xi_i\right]<c<d,$$

$$\max\left\{\xi_i,\frac{1}{\sum_{i=1}^{m-2}\alpha_i}\left[2\sum_{i=1}^{m-2}\alpha_i-\sum_{i=1}^{m-2}\alpha_i\xi_i+\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)\sum_{i=1}^{m-2}\alpha_i(\eta_i-\xi_i)-1\right],\right.$$

$$\left.\frac{2b}{c}\frac{1}{\sum_{i=1}^{m-2}\alpha_i}\left[\sum_{i=1}^{m-2}\alpha_i+1-\sum_{i=1}^{m-2}\alpha_i\xi_i\right]\right\}<w<\nu<\frac{1}{2}$$

and suppose that f *satisfies the following conditions*

(A1) $f(t,u,v)\le\frac{1}{2\Lambda}\phi_p(d)$ *for* $(t,u,v)\in[0,1]_{\mathbb{T}}\times[0,Md]\times[0,d]$;

(A2) $f(t,u,v)>\frac{1}{\Omega}\phi_p(\frac{b}{A})$ *for* $(t,u,v)\in[w,v]_{\mathbb{T}}\times[b,c]\times[0,d]$;

(A3) $f(t,u,v)<\frac{1}{2\Lambda}\phi_p(\frac{a}{M})$ *for* $(t,u,v)\in[0,1]_{\mathbb{T}}\times[0,a]\times[0,d]$;

where M, Λ *are defined as in (11) and Lemma* 2.4 *respectively, and*
$A=\frac{1}{\sum\limits_{i=1}^{m-2}\alpha_i}\left[\left(1+\sum_{i=1}^{m-2}\alpha_i(w-\xi_i)\right)\frac{1}{2+\sum_{i=1}^{m-2}\alpha_i(\eta_i-\xi_i)}-\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)\right].$
Then the BVP (1)-(2) has at least three positive solutions $x_1\,x_2$ *and* x_3 *such that*

$$\gamma(x_i)\le d,\ i=1,2,3,\quad \psi(x_1)<a,\quad a<\psi(x_2)\ \textit{with}\ \alpha(x_2)<b,\ \alpha(x_3)>b.$$

Proof. The boundary value problem (1)-(2) has a solution $x=x(t)$ if and only if x solves the operator equation $x=Tx$. Thus we set out to verify that the operator T satisfies Avery-Peterson's fixed point theorem which will prove the existence of three fixed point of T. Now the proof is divided into four steps.

Step 1 : We will show that $(A1)$ implies that

$$T:\overline{P(\gamma,d)}\longrightarrow\overline{P(\gamma,d)}.$$

For $x\in\overline{P(\gamma,d)}$, there is $\gamma(x)=\sup_{t\in[0,1]_{\mathbb{T}}}|x^{\Delta}(t)|\le d$. From Lemma 2.3,

$$\sup_{t\in[0,1]_{\mathbb{T}}}x(t)\le M\sup_{t\in[0,1]_T}|x^{\Delta}(t)|\le Md,$$

then the condition $(A1)$ implies

$$f(t,x(t),x^{\Delta}(t))\le\frac{\phi_p(d)}{2\Lambda}.$$

On the other hand, for $x \in P$, we get

$$\begin{aligned}
\gamma(Tx) &= \sup_{t\in[0,1]_{\mathbb{T}}} |(Tx)^{\Delta}(t)| \\
&= \sup_{t\in[0,1]_{\mathbb{T}}} |\phi_q(A_x - \int_0^t h(s)f(s,x(s),x^{\Delta}(s))\nabla s)| \\
&\le \phi_q(A_x + \int_0^1 h(s)f(s,x(s),x^{\Delta}(s))\nabla s) \\
&\le \phi_q(2\int_0^1 h(s)f(s,x(s),x^{\Delta}(s))\nabla s) \\
&\le \phi_q(\frac{\phi_p(d)}{\Lambda}\int_0^1 h(s)\nabla s) = d.
\end{aligned}$$

Step 2. We show that condition $(S1)$ in Theorem 3.1 holds. We take

$$x(t) = \frac{c}{2}\frac{1-\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i\xi_i+\sum_{i=1}^{m-2}\alpha_i}[\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\xi_i}t+1]$$

for $t \in [0,1]_{\mathbb{T}}$. By (12), we get
$\gamma(x) = \sup_{t\in[0,1]_{\mathbb{T}}} |x^{\Delta}(t)| = \frac{c}{2}\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\xi_i+\sum_{i=1}^{m-2}\alpha_i} < d$,
$\psi(x) = \theta(x) = \sup_{t\in[0,1]_{\mathbb{T}}} x(t) = x(1) = \frac{c}{2} < c$,
$\alpha(x) = \min_{t\in[w,v]_{\mathbb{T}}} x(t) = x(w) > b$.

Hence $\{x \in P(\gamma,\theta,\alpha,b,c,d : \alpha(x) > b\} \neq \emptyset$.

Since

$$\begin{aligned}
\phi_q(A_x) &= \phi_q\left(\int_0^1 h(s)f(s,x(s),x^{\Delta}(s))\nabla s - A_x\right) \\
&+ \sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{\eta_i}\phi_q\left(\int_0^s h(\tau)f(\tau,x(\tau),x^{\Delta}(\tau))\nabla\tau - A_x\right)\Delta s \\
&\ge \phi_q\left(\int_0^1 h(s)f(s,x(s),x^{\Delta}(s))\nabla s\right) - \phi_q(A_x) \\
&+ \sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{\eta_i}\phi_q\left(\int_0^s h(\tau)f(\tau,x(\tau),x^{\Delta}(\tau))\nabla\tau\right)\Delta s \\
&- \sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{\eta_i}\phi_q(A_x)\Delta s \\
&\ge \phi_q\left(\int_0^1 h(s)f(s,x(s),x^{\Delta}(s))\nabla s\right) - \phi_q(A_x) \\
&- \sum_{i=1}^{m-2}\alpha_i(\eta_i-\xi_i)\phi_q(A_x),
\end{aligned}$$

we have

$$\left[2+\sum_{i=1}^{m-2}\alpha_i(\eta_i-\xi_i)\right]\phi_q(A_x) \ge \phi_q\left(\int_0^1 h(s)f(s,x(s),x^{\Delta}(s))\nabla s\right).$$

Hence, we get

$$\phi_q(A_x) \geq \frac{1}{2+\sum\limits_{i=1}^{m-2}\alpha_i(\eta_i-\xi_i)}\phi_q\left(\int_0^1 h(s)f(s,x(s),x^{\Delta}(s))\nabla s\right). \tag{13}$$

Case 1. If $\alpha(Tx)=\min\limits_{t\in[w,\nu]_{\mathbb{T}}} Tx(t)=Tx(w)$ holds then from (10), (13) and $(A2)$, we obtain

$$\begin{aligned}
Tx(w) &= \frac{1}{\sum\limits_{i=1}^{m-2}\alpha_i}\left[\phi_q(A_x)+\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{1}\phi_q\left(A_x-\int_0^s h(\tau)f(\tau,x(\tau),x^{\Delta}(\tau))\nabla\tau\right)\Delta s\right]\\
&+ \int_1^w \phi_q(A_x-\int_0^s h(\tau)f(\tau,x(\tau),x^{\Delta}(\tau))\nabla\tau)\Delta s\\
&= \frac{1}{\sum\limits_{i=1}^{m-2}\alpha_i}\left[\phi_q(A_x)+\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{w}\phi_q\left(A_x-\int_0^s h(\tau)f(\tau,x(\tau),x^{\Delta}(\tau))\nabla\tau\right)\Delta s\right]\\
&\geq \frac{1}{\sum\limits_{i=1}^{m-2}\alpha_i}\left[\phi_q(A_x)+\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{w}\phi_q(A_x)\Delta s\right.\\
&- \left.\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{w}\phi_q\left(\int_0^s h(\tau)f(\tau,x(\tau),x^{\Delta}(\tau))\nabla\tau\right)\Delta s\right]\\
&\geq \frac{1}{\sum\limits_{i=1}^{m-2}\alpha_i}\left[\phi_q(A_x)+\sum_{i=1}^{m-2}\alpha_i(w-\xi_i)\phi_q(A_x)\right.\\
&- \left.\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{1}\phi_q\left(\int_0^1 h(\tau)f(\tau,x(\tau),x^{\Delta}(\tau))\nabla\tau\right)\Delta s\right]\\
&= \frac{1}{\sum\limits_{i=1}^{m-2}\alpha_i}\left[\left(1+\sum_{i=1}^{m-2}\alpha_i(w-\xi_i)\right)\phi_q(A_x)-\right.\\
&- \left.\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)\phi_q\left(\int_0^1 h(\tau)f(\tau,x(\tau),x^{\Delta}(\tau))\nabla\tau\right)\right]\\
&\geq \frac{1}{\sum\limits_{i=1}^{m-2}\alpha_i}\left[\left(1+\sum_{i=1}^{m-2}\alpha_i(w-\xi_i)\right)\frac{1}{2+\sum\limits_{i=1}^{m-2}\alpha_i(\eta_i-\xi_i)}\right.
\end{aligned}$$

$$
\begin{aligned}
& - \left.\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)\right]\phi_q\left(\int_0^1 h(\tau)f(\tau,x(\tau),x^{\Delta}(\tau))\nabla\tau\right)\\
&= A\phi_q\left(\int_0^1 h(\tau)f(\tau,x(\tau),x^{\Delta}(\tau))\nabla\tau\right)\\
&> A\phi_q\left(\int_w^{\nu} h(\tau)\frac{1}{\Omega}\phi_p\left(\frac{b}{A}\right)\nabla\tau\right)\\
&= A\frac{b}{A}\phi_q(\frac{1}{\Omega}\int_w^{\nu} h(\tau)\nabla\tau) = b.
\end{aligned}
$$

Thus we get $Tx(w) > b$.

Case 2. If $\alpha(Tx) = \min\limits_{t\in[w,\nu]_{\mathbb{T}}} Tx(t) = Tx(\nu)$ holds then from (10), (13) and $(A2)$, we get

$$
\begin{aligned}
Tx(\nu) \ \geq\ & \frac{1}{\sum_{i=1}^{m-2}\alpha_i}\left[\left(1+\sum_{i=1}^{m-2}\alpha_i(\nu-\xi_i)\right)\frac{1}{2+\sum_{i=1}^{m-2}\alpha_i(\eta_i-\xi_i)}\right.\\
& - \left.\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)\right]\phi_q\left(\int_0^1 h(\tau)f(\tau,x(\tau),x^{\Delta}(\tau))\nabla\tau\right)\\
\geq\ & A\phi_q\left(\int_0^1 h(\tau)f(\tau,x(\tau),x^{\Delta}(\tau))\nabla\tau\right)\\
>\ & A\phi_q\left(\int_w^{\nu} h(\tau)\frac{1}{\Omega}\phi_p\left(\frac{b}{A}\right)\nabla\tau\right) = b.
\end{aligned}
$$

Hence we get $Tx(\nu) > b$.

Therefore we get $\alpha(Tx) > b$ for all $x \in P(\gamma,\theta,\alpha,b,c,d)$. Consequently, condition $(S1)$ in Theorem 3.1 is satisfied.

Step 3. We prove that $(S2)$ in Theorem 3.1 holds. Since x is nonnegative and concave on $[0,1]_{\mathbb{T}}$, we obtain

$$
\begin{aligned}
x(w) &= x\left[\frac{\frac{1}{w}(1+t)-1}{\frac{1}{w}(1+t)}\frac{1}{\frac{1}{w}(1+t)-1}+\frac{1}{\frac{1}{w}(1+t)}t\right]\\
&\geq \frac{\frac{1}{w}(1+t)-1}{\frac{1}{w}(1+t)}x\left(\frac{1}{\frac{1}{w}(1+t)-1}\right)+\frac{1}{\frac{1}{w}(1+t)}x(t)\\
&\geq \frac{w}{1+t}x(t) \geq \frac{w}{2}x(t).
\end{aligned}
$$

Therefore $x(w) \geq \frac{w}{2}\sup_{t\in[0,1]_{\mathbb{T}}} x(t) = \frac{w}{2}\theta(x)$. Similarly $x(\nu) \geq \frac{\nu}{2}\theta(x) > \frac{w}{2}\theta(x)$ holds. Hence

$$
\alpha(x) \geq \frac{w}{2}\theta(x), \quad x\in[0,1]_{\mathbb{T}}.
$$

Then we get

$$
\begin{aligned}
\alpha(Tx) &\geq \frac{w}{2}\theta(Tx) > \frac{w}{2}c > \frac{w}{2}\frac{2b(L+1)}{wL}\\
&= b\left(\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}+1\right)\frac{1-\sum_{i=1}^{m-2}\alpha_i\xi_i}{\sum_{i=1}^{m-2}\alpha_i} > b
\end{aligned}
$$

for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > c$.
Step 4. Finally, we prove that $(S3)$ in Theorem 3.1 is satisfied. Since $\psi(0) = 0 < a$, $0 \notin R(\gamma, \psi, a, d)$. Suppose that $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$, then by $(A3)$ and Lemma 2.3 we get

$$\begin{aligned}
\psi(Tx) &= \sup_{t\in[0,1]} Tx(t) \\
&\le M \sup_{t\in[0,1]} |(Tx)^{\Delta}(t)| \\
&\le M\phi_q\left[2\int_0^1 h(s)f(s,x(s),x^{\Delta}(s))\nabla s\right] \\
&< M\phi_q\left[2\int_0^1 h(s)\frac{1}{2\Lambda}\phi_p(\frac{a}{M})\nabla s\right] \\
&< M\frac{a}{M}\phi_q(\frac{1}{\Lambda}\int_0^1 h(s)\nabla s) = a.
\end{aligned}$$

Consequently condition $(S3)$ in Theorem 3.1 holds. From steps $1-4$ together with Theorem 3.1 we get that the boundary value problem (1)-(2) has at least three positive solutions x_1, x_2, x_3 such that

$$\sup_{t\in[0,1]_{\mathbb{T}}} |x_i^{\Delta}(t)| \le d, \quad i = 1,2,3, \quad \sup_{t\in[0,1]_{\mathbb{T}}} x_1(t) < a,$$

$$a < \sup_{t\in[0,1]_{\mathbb{T}}} x_2(t) \ with \min_{t\in[w,\nu]_{\mathbb{T}}} x_2(t) < b, \quad \min_{t\in[w,\nu]_{\mathbb{T}}} x_3(t) > b.$$

The proof is complete. □

Example 3.1 Let $\mathbb{T} = \{\frac{1}{2^n+1} : n \in \mathbb{N}\} \cup \{0,1\}$. Consider the following problem

$$(\phi_3(x^{\Delta}))^{\nabla}(t) + 8f(t,x(t),x^{\Delta}(t)) = 0, \quad t \in [0,1]_{\mathbb{T}}, \tag{14}$$

$$x^{\Delta}(0) = \frac{1}{4}x(\frac{1}{10}) + \frac{1}{6}x(\frac{1}{5}), \quad x^{\Delta}(1) = -\frac{1}{4}x(\frac{1}{3}) - \frac{1}{6}x(\frac{1}{2}), \tag{15}$$

where

$$f(t,u,v) = \begin{cases} t[60u^7 + (\frac{v}{10^3})^4], & u \le 1, \quad 0 \le v, \ v \in \mathbb{T}; \\ t[60 + (\frac{v}{10^3})^4], & u > 1, \quad 0 \le v, \ v \in \mathbb{T}. \end{cases}$$

It is easy to verify that $(H1)-(H3)$ hold. Choose $a = \frac{1}{10}, \ b = 1, \ c = 40, d = 43, \ w = \frac{1}{4}, \ v = \frac{1}{3}$. Then by simple calculations, we can obtain that

$$M = \frac{163}{43}, \ \Lambda = 8, \ A = \frac{4181}{12650}, \ \Omega = \frac{2}{3}.$$

So the nonlinear term f satisfies
$f(t,u,v) \le 60 + (\frac{43}{10^3})^4 = 60.00000342 < \frac{\phi_p(d)}{2\Lambda} = 115.5625$, $(t,u,v) \in [0,1]_{\mathbb{T}} \times [0,163] \times [0,43]$,
$f(t,u,v) \ge 20 > \phi_3(\frac{b}{2A}) = 16.43184338$, $(t,u,v) \in [\frac{1}{4},\frac{1}{3}]_{\mathbb{T}} \times [1,40] \times [0,43]$,

$f(t,u,v) < 60.\frac{1}{10^7} + (\frac{43}{10^3})^4 = 0.000009418801 < \frac{1}{2\Lambda}\phi_p(\frac{a}{M}) = 0.0000434952$, $(t,u,v) \in [0,1]_{\mathbb{T}} \times [0,\frac{1}{10}] \times [0,43]$.

Therefore the conditions in Theorem 3.2 are all satisfied. So BVP (14)-(15) has at least three positive solutions $x_1,\ x_2, x_3$ such that

$$\sup_{t\in[0,1]_{\mathbb{T}}} |x_i^{\Delta}(t)| \leq 43,\ i = 1,2,3,\ \sup_{t\in[0,1]_{\mathbb{T}}} x_1(t) < \frac{1}{10},$$

$$\frac{1}{10} < \sup_{t\in[0,1]_{\mathbb{T}}} x_2(t)\ with\ \min_{t\in[\frac{1}{4},\frac{1}{3}]_{\mathbb{T}}} x_2(t) < 1,\quad \min_{t\in[\frac{1}{4},\frac{1}{3}]_{\mathbb{T}}} x_3(t) > 1.$$

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