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PERSONAGE IN SCIENCE

Academician V.M. Matrosov

In Memoriam

S.N. Vassilyev [1], A.A. Martynyuk [2*] and D.D. Siljak [3]

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This paper is dedicated to the memory of V.M. Matrosov in recognition of the significance of his results in the development of stability theory, his remarkable and versatile talent in research on dynamic systems, as well as the novelty and depth of his contribution to mathematics and world science.

1 Short Biography

V.M. MATROSOV was born on May 8, 1932 in Shipunovo village on the Altay, USSR (now Russian Federation). In 1956, he graduated with distinction from the Kazan Aviation Institute (KAI), the Aircraft Engineering Faculty, and entered the post-graduate studies under the supervision of Professor P.A. Kuzmin at the Chair of Theoretical Mechanics of KAI. He began working at the Institute as a junior professor and advanced to the position of an assistant professor of the Chair of Theoretical Mechanics. In 1959, Matrosov defended his Candidate Thesis (PhD) on the stability of gyroscopic systems. In 1968, Matrosov defended his Doctoral Thesis on the development of new methods of Lyapunov functions in the stability theory of motion. In the same year he became the Head of the Chair of Mathematics, and in 1972 he founded the Chair of Cybernetics which was training mainly the specialists in applied mathematics. At the same time he ran research laboratory at KAI where he carried out his first investigations in applied mathematics.

In 1975, Matrosov and several of his associates were invited by academician G.I. Marchuk, the president of the Siberian Division of the Academy of Sciences of USSR, to move to Irkutsk with the purpose of founding a new academic institute. The institute was formed and Matrosov was its director-organizer.

In 1976, Matrosov was elected the corresponding member, and in 1987 he became the member of the Academy of Sciences of USSR. In 1991, he established and headed the

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Moscow Division of the Institute of Transport Problems of the Academy of Sciences of USSR. In 1996, he founded the Research Center for Stability and Nonlinear Dynamics Research at the A.A. Blagonravov Institute of Machine Building of the Russian Academy of Sciences (RAS) and was at its head until the end of his life (on April 17, 2011). While doing scientific research Matrosov was vigorously engaged in pedagogical activity. He was a professor of Sociology Faculty of M.V. Lomonosov Moscow State University (1998–2003) and held the Chair of Mathematical Cybernetics at Moscow Aviation Institute (2000–2007).

Matrosov passed away on April 17, 2011, at the age of 79. He is survived by his wife Nina, son Ivan, daughter Claudia, and grand children Ivan and Alexander.

2 Scientific Work

2.1 Gyroscopic systems

At the beginning of his academic career Matrosov focused on stability of gyroscopic systems. Using refined modifications of Lyapunov theorems he managed to establish a series of important criteria of equilibrium stability and instability, which are now applied in the theory of gyroscopic and electromechanical systems.

At that time, he also carried out investigations on stability of solutions to nonautonomous differential equations [1]. He showed impossibility of extending to these equations the known Barbashin–Krasovsky theorem on asymptotic stability of motion in terms of Lyapunov function with sign-constant derivative. The obtained results have been applied in many areas of modern nonlinear dynamics.

2.2 Vector Lyapunov functions

Matrosov's investigations on development of the method of Lyapunov functions proved to be of importance for weakening the requirements on Lyapunov functions. At one and the same time as R. Bellman (USA) he introduced the notion of Vector Lyapunov Function (VLF) [2] satisfying the system of differential inequalities of Chaplygin-Wazhevsky type. Matrosov formulated first theorems on stability using VLF [2-4], which provided general criteria for stability of motion. The characteristic features of these results are that the requirements placed on classical Lyapunov functions are replaced by a totality of less strict conditions imposed on some components of VLF.

The new idea of VLF was further developed and applied by numerous researchers in USSR (in present-day Russia, Ukraine, and Kazakhstan), USA, France, Italy, Belgium, Japan, etc. In his turn, Matrosov conducted a profound and versatile investigation of this idea for ordinary differential equations in Banach space with discontinuous unbounded operators (see [5-7] and bibliography therein).

2.3 Mathematical theory of systems

In 1970, Matrosov and his collaborators initiated new models in mathematical system theory (system of processes, generalized structures, etc.). Having analyzed thoroughly the structure of proofs of comparison theorems for differential equations, Matrosov established and proved, in a unique algorithmic form, the principle of comparison for deduction of comparison theorems on dynamic properties of systems of processes [8].

The new results paved the way for comparison theorems involving VLF to be established algorithmically by formulas of the considered dynamic properties of a wide class of systems (see [9–11] and bibliography therein). The development of these algorithms and their computer realizations have provided many new and useful results for study of various models in dynamical systems and control theory [12, 13].

2.4 Methods of construction of vector Lyapunov functions

Matrosov and his associates [14–16] developed three groups of methods of VLF construction. The first group embraces the methods associated with exact exponential estimates based on decomposition-aggregation of multi-interconnected systems. This group also includes a combined method representing a finite iteration process of complex system decomposition and VLF refinement which is associated with the hierarchy of subsystems and VLF as well.

The second group of methods consists of decompositions and further estimation of aggregation of multi-interconnected systems with the application of VLF, which is related to the F.N. Baileys approach (1966).

And finally, the third group of methods is a construction of sub-linear VLF. This construction has turned out to be the most efficient in applications.

2.5 Dynamics and control of aerospace structures

Among numerous applications of Matrosov's scientific results to real engineering systems the investigations of dynamics and control of aerospace structures were of particular importance. The employment of finite iteration process of VLF construction resulted in significant applications. Among them, stability investigations of the first Soviet stratospheric observatory (1975) and orbital astronomic observatory with sub-millimeter telescope BST-1 installed in the space station Salut-6 (1977), which were used successfully for the investigations of thin structure of photosphere of the Sun and other space objects. On the basis of VLF method and other methods of nonlinear analysis these and many other problems of dynamics and control have been studied (see [15, 17–22] and bibliography therein). Under academic supervision of Matrosov a unique complex of packages of applied programs for BESM-6, MVK Elbrus-1K2 and ES EVM was developed to solve the problems of nonlinear dynamics and control theory. This research carried out by Matrosov, his colleagues and research assistants allowed the method of VLF to evolve into a practical tool for scientific and engineering calculations of system dynamics. For the series of investigations on the development of VLF method Matrosov and his associates were given the State Prize in the field of science and technology in 1984.

2.6 Research on multi-package methods

In the late 1970s Matrosov initiated research on multi-package methods of solving the problems of modeling, analysis and optimization of complex systems [23]. Such research was induced by the results on intellectualization of computer systems in terms of the methods of logical generation of alternative solutions to the problems (including computation plans) and multi-criterion estimation of solution preferability. In addition to the logical-and-heuristic approach to the automation of synthesis of theorems of the type of VLF method theorems [9, 12] the other methods were used. They are: the methods of solution of logical equations, automation of logical deduction and automatic proof of theorems, as well as the methods of multi-criterion decision-making. At that period of time a concept and components of intellectualization of a research prototype of the software system EVROLOG-1 with professional artificial intelligence [24] were developed. With Matrosov as its director, the Irkutsk Computing Center of the Siberian Division of Academy of Sciences of USSR became a leader in constructing software for automation of design and research of control systems for complex moving objects [25–27].

2.7 Mathematical modeling of national economy

As early as the end of the 1960s Matrosov extended his interests to the electric power industry and other fields of national economy, as well as the economic, medical, biological and other systems. In particular, the method of VLF was used for the analysis of electric power systems, immunological models, etc. Under Matrosov's supervision a wide range of problems were studied in creating an automation system for solution of problems in modeling and optimization of the fuel and energy complex of USSR. Experimental automation systems were worked out for modeling the development of regional areas [28, 29] including branches of industry, medicine and ecological stability and safe development of these regions. Investigations were carried out on estimation of after-effects of possible technological disasters at oil and chemical facilities. In the 1990s Matrosov arranged the work on creating a social, ecological and economical model of interactions of Russian regional areas allowing for the population migration, production and redistribution of income.

2.8 Problems of global security and stable development

Motivated by the well-known problem of reduction of the strategic offensive weapons, Matrosov initiated investigations of stability of military strategic balance (MSB) of a multi-polar world. In this regard a program system was developed allowing for variation of parameters and characteristics of the weapons and construction of areas of MSB for different scenarios of development of the strategic weapons of Russia and USA. An approach was proposed for the analysis of stable weapon dynamics and stability of MSB based on the method of comparison using VLF for estimating the strategy of defense sufficiency as well as other strategies [30–33].

As early as the 1980s, Matrosov deeply feeled that the popular model of a consumer society, which was universally promoted, is futile. Such a society may cause irreversible changes in the ecosphere and, finally, lead to a global disaster. Under Matrosov's supervision a large body of work was carried out on modifications of the well known model of world dynamics by G. Forrester (USA). Together with Professor A. Onishi (Japan) Matrosov proposed a concept of the international project "Methods and Program Tools for the Analysis of Global Development Stability" (1991).

3 Matrosov's Public Activity

Apart from his intensive scientific activity Matrosov took an active role in public activity. In particular, for several years he participated in the projects realized by the V.I. Vernadsky Foundation and worked at the Commission on Problems of Sustainable Development of The State Duma (Council) of Federal Assembly of The Russian Federation. As the Head of the Center for Modeling of the Sustainable Development of Society at the

Institute of Social and Political Investigations of RAS (1984–2001), Matrosov delivered lectures on stable development of society to the students of Moscow State University [34].

This sketch of Matrosov's accomplishments does not exhaust by any means his versatile and fruitful scientific, professional and public activity. He was a great scientist and a gifted organizer of scientific projects and pursuits. His untethered energy and ability not to spare himself in what he was doing have always impressed those who used to know him closely. His absolute devotion to science has served as an example to others. Matrosov's kindness, collegiality, generosity of a scholar and superb command of the knowledge in his field guaranteed success of his work and work of his collaborators.

Matrosov's organizational, pedagogical and public activities were honored by many state awards and scientific prizes. The abundant scientific legacy of academician Matrosov produced a profound intellectual and humanitarian effect on his readers and had a significant influence on the future generations of researchers.

4 List of Monographs and Books by V.M. Matrosov

[A] Matrosov, V.M. *The Method of Vector Lyapunov Functions: Analysis of Dynamical Properties of Nonlinear Systems.* Fizmatlit, Moscow, 2001. [Russian]

[B] Matrosov, V.M., Anapolsky, L.Yu. and Vassilyev, S.N. *The Method of Comparison in Mathematical System Theory.* Nauka, Novosibirsk, 1980. [Russian]

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Stability Analysis for a Class of Nonlinear Nonstationary Systems via Averaging

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Abstract: A class of nonlinear nonstationary systems of Persidskii type is studied. The right-hand sides of the systems are represented in the form of linear combinations of sector nonlinearities with time-varying coefficients. It is assumed that the coefficients possess mean values. By means of the Lyapunov direct method, it is proved that if the investigated systems are essentially nonlinear, i.e. the right-hand sides of the systems do not contain linear terms with respect to phase variables, then the asymptotic stability of the zero solutions of the corresponding averaged systems implies the local uniform asymptotic stability of the zero solutions for original nonstationary systems. We treat both cases of delay free and time delay systems. Furthermore, it is shown that the proposed approaches can be used as well for the stability analysis of some classes of nonlinear systems with nontrivial linear approximation.

Keywords: *asymptotic stability; Lyapunov function; averaging technique; nonstationary systems; time delay.*

Mathematics Subject Classification (2010): 34D20, 39B72, 34C29.

1 Introduction

A general approach for the stability analysis of nonlinear systems is the Lyapunov direct method (the Lyapunov functions method). By means of this approach, the stability conditions for many types of systems were obtained, see, for example, [9, 11, 17–19, 26] and the references cited therein. However, it should be noted that until now there are no general constructive methods for the finding of Lyapunov functions for nonlinear systems.

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This problem is especially complicated for nonstationary systems [8, 10, 11, 17, 26]. An effective approach for the investigation of dynamical properties of such systems is the averaging technique [10, 11, 13, 17]. This technique allows to reduce stability analysis of time-varying differential equations to the analysis of time-invariant differential equations, possibly resulting in an important simplification. However, it is worth mentioning that the application of the averaging technique is well developed only in the case when original systems are fast time-varying.

In [1, 2], nonlinear nonstationary systems with homogeneous with respect to phase variables right-hand sides have been studied. For such systems, the approach for the Lyapunov functions constructing was proposed. Its application permits to show that if the order of the homogeneity of right-hand sides of the considered time-varying system is greater than one, then the asymptotic stability of the zero solution of the corresponding averaged system implies the same property for the zero solution of the original system. These results have got a further development in [3, 21, 23, 24, 27]. In particular, in [27], a modification of the approach for the Lyapunov functions constructing was suggested. Another techniques for the determination of similar asymptotic stability conditions for time-varying homogeneous systems have been developed in [21, 23]. Recently, these approaches have been extended to nonlinear nonstationary systems with time delay [4–6]. The delay independent asymptotic stability conditions were found on the basis of the stability analysis of corresponding averaged delay free systems.

The principal novelty of the results of the papers [1–6, 21, 23, 24, 27], as compared to the known stability conditions obtained by the application of averaging technique, is that, to guarantee the asymptotic stability for a nonstationary homogeneous system, right-hand sides of the system need not be fast time-varying. It is shown that in the averaging technique, instead of a small parameter providing the fast time-variation of a vector field, the orders of homogeneity can be used.

In the present paper, a class of nonlinear nonstationary systems of Persidskii type [16] is studied. The right-hand sides of the systems are represented in the form of linear combinations of sector nonlinearities with time-varying coefficients. It is assumed that the coefficients possess mean values. By means of the Lyapunov direct method, it is proved that if the investigated systems are essentially nonlinear, i.e. the right-hand sides of the systems do not contain linear terms with respect to phase variables, then the asymptotic stability of the zero solutions of the corresponding averaged systems implies the local uniform asymptotic stability of the zero solutions for original nonstationary systems. We treat both cases of delay free and time delay systems. Furthermore, it is shown that the proposed approaches can be used as well for the stability analysis of some classes of nonlinear systems with nontrivial linear approximation.

2 Statement of the Problem

Consider the ordinary differential equations system

$$\dot{x}_i(t) = \sum_{j=1}^{n} p_{ij}(t) f_j(x_j(t)), \qquad i = 1, \dots, n. \tag{1}$$

Here the functions $f_j(x_j)$ are continuous for $|x_j| < H$, $0 < H \le +\infty$, and belong to a sector-like constrained set defined as follows: $x_j f_j(x_j) > 0$ for $x_j \neq 0$; the coefficients $p_{ij}(t)$ are continuous and bounded for $t \ge 0$. Such systems are widely used in both automatic control [9, 16, 17] and neural networks [15, 16].

We assume that the functions $p_{ij}(t)$ possess mean values $\bar{p}_{ij}$, and the tendencies

$$\frac{1}{T}\int_t^{t+T} p_{ij}(s)ds \to \bar{p}_{ij} \quad \text{as} \quad T \to +\infty, \quad i,j = 1,\dots,n,$$

are uniform with respect to $t \geq 0$. Hence, the coefficients $p_{ij}(t)$ can be represented in the form $p_{ij}(t) = \bar{p}_{ij} + \tilde{p}_{ij}(t)$, with the mean values of the functions $\tilde{p}_{ij}(t)$ equal to zero, $i,j = 1,\dots,n$.

Thus,

$$\dot{x}_i(t) = \sum_{j=1}^{n} \bar{p}_{ij} f_j(x_j(t)), \qquad i = 1,\dots,n, \tag{2}$$

is the averaged system for (1).

It follows from the properties of functions $f_1(x_1),\dots,f_n(x_n)$ that systems (1) and (2) admit the zero solution. We will look for the conditions under which the asymptotic stability of the zero solution of the averaged system implies the same property for the zero solution of original system.

In what follows, we impose some additional restrictions on the right-hand sides in (1).

Assumption 2.1 The matrix $\overline{\mathbf{P}} = \{\bar{p}_{ij}\}_{i,j=1}^{n}$ is diagonally stable [16], i.e. there exist positive numbers $\lambda_1,\dots,\lambda_n$ such that the quadratic form

$$W(\mathbf{x}) = \mathbf{x}^T \left(\overline{\mathbf{P}}^T \mathbf{\Lambda} + \mathbf{\Lambda} \overline{\mathbf{P}} \right) \mathbf{x}$$

is negative definite. Here $\mathbf{x} = (x_1,\dots,x_n)^T$, $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1,\dots,\lambda_n\}$.

Remark 2.1 The problem of matrix diagonal stability is well investigated, see, for example, [16] and references therein.

Remark 2.2 If Assumption 2.1 is fulfilled, then the zero solution of (2) is asymptotically stable, and, for this system, a Lapunov function can be chosen in the form

$$V(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \int_0^{x_i} f_i(s)ds. \tag{3}$$

Remark 2.3 It is well known [28] that if system (1) is linear ($f_j(x_j) = x_j$, $j = 1,\dots,n$), it may be unstable, despite of the asymptotic stability of the corresponding averaged system.

In view of Remark 2.3, hereinafter we consider only the case when the following assumption is fulfilled.

Assumption 2.2 Functions $f_j(x_j)$ can be represented in the form

$$f_j(x_j) = \beta_j x_j^{\mu_j} + g_j(x_j), \quad j = 1,\dots,n,$$

where β_j are positive constants, $\mu_j > 1$ are rational numbers with odd numerators and denominators, and $g_j(x_j)/x_j^{\mu_j} \to 0$ as $x_j \to 0$.

Remark 2.4 Without loss of generality, we assume that $\beta_j = 1$, $j = 1, \ldots, n$, and $\mu_1 \le \ldots \le \mu_n$.

Thus, the investigated equations are essentially nonlinear, and the systems

$$\dot{x}_i(t) = \sum_{j=1}^{n} (\bar{p}_{ij} + \tilde{p}_{ij}(t))\, x_j^{\mu_j}(t), \qquad i = 1, \ldots, n, \tag{4}$$

$$\dot{x}_i(t) = \sum_{j=1}^{n} \bar{p}_{ij} x_j^{\mu_j}(t), \qquad i = 1, \ldots, n, \tag{5}$$

can be considered as systems of the first, in a broad sense, approximation for (1) and (2) respectively.

Let Assumption 2.1 be fulfilled. Then the zero solution of (5) is globally asymptotically stable, and, for this system, the Lyapunov function (3) takes the form

$$V(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \frac{x_i^{\mu_i+1}}{\mu_i + 1}.$$

First, we will show that the zero solution of (4) is locally asymptotically stable. Next, we will determine the stability conditions for a perturbed system, and, on the basis of these conditions, the asymptotic stability of the zero solution of (1) will be proved. Furthermore, along with (1), we will consider the corresponding time-delay system

$$\dot{x}_i(t) = \sum_{j=1}^{n} p_{ij}(t) f_j(x_j(t-\tau)), \quad i = 1, \ldots, n, \quad \tau = \text{const} \ge 0. \tag{6}$$

By the usage of the Lyapunov direct method and the Razumikhin approach [25], for (6), delay independent stability conditions will be found.

3 Sufficient Conditions of Asymptotic Stability

In [3], it was shown that if Assumption 2.1 is fulfilled, and the integrals

$$\int_0^t \tilde{p}_{ij}(s)ds, \quad i, j = 1, \ldots, n, \tag{7}$$

are bounded for $t \in [0, +\infty)$, then the zero solution of (4) is asymptotically stable.

In the present paper, we consider the case when

$$\frac{1}{T} \int_t^{t+T} \tilde{p}_{ij}(s)ds \to 0 \quad \text{as} \quad T \to +\infty, \quad i, j = 1, \ldots, n,$$

uniformly with respect to $t \ge 0$. It is well known [11], that, in this case, integrals (7) may be unbounded.

Theorem 3.1 *Let Assumption 2.1 be fulfilled. Then the zero solution of (4) is uniformly asymptotically stable.*

Proof. By means of the approaches proposed in [1, 2, 27], construct a Lyapunov function for (4) in the form

$$\widetilde{V}(t,\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \frac{x_i^{\mu_i+1}}{\mu_i+1} - \sum_{i,j=1}^{n} \lambda_i L_{ij}(t,\varepsilon)\, x_i^{\mu_i} x_j^{\mu_j}. \tag{8}$$

Here positive numbers $\lambda_1, \dots, \lambda_n$ are chosen in accordance with Assumption 2.1,

$$L_{ij}(t,\varepsilon) = \int_0^t \exp(\varepsilon(s-t))\, \tilde{p}_{ij}(s) ds, \quad i,j = 1,\dots,n,$$

and ε is a positive parameter.

Differentiating $\widetilde{V}(t,\mathbf{x})$ with respect to system (4), we obtain

$$\dot{\widetilde{V}}\big|_{(4)} = \sum_{i,j=1}^{n} \lambda_i \bar{p}_{ij} x_i^{\mu_i} x_j^{\mu_j} + \varepsilon \sum_{i,j=1}^{n} \lambda_i L_{ij}(t,\varepsilon)\, x_i^{\mu_i} x_j^{\mu_j}$$

$$- \sum_{i,j=1}^{n} \lambda_i \mu_i L_{ij}(t,\varepsilon)\, x_i^{\mu_i-1} x_j^{\mu_j} \sum_{k=1}^{n} p_{ik}(t) x_k^{\mu_k}$$

$$- \sum_{i,j=1}^{n} \lambda_i \mu_j L_{ij}(t,\varepsilon)\, x_i^{\mu_i} x_j^{\mu_j-1} \sum_{k=1}^{n} p_{jk}(t) x_k^{\mu_k}.$$

Hence, the estimates

$$a_1 \sum_{i=1}^{n} x_i^{\mu_i+1} - \frac{a_3}{\varepsilon} \sum_{i=1}^{n} x_i^{2\mu_i} \le \widetilde{V}(t,\mathbf{x}) \le a_2 \sum_{i=1}^{n} x_i^{\mu_i+1} + \frac{a_3}{\varepsilon} \sum_{i=1}^{n} x_i^{2\mu_i},$$

$$\dot{\widetilde{V}}\big|_{(4)} \le -a_4 \sum_{i=1}^{n} x_i^{2\mu_i} + a_5 \psi(t,\varepsilon) \sum_{i=1}^{n} x_i^{2\mu_i} + \frac{a_6}{\varepsilon} \sum_{i,j=1}^{n} x_i^{2\mu_i} x_j^{\mu_j-1}$$

are valid for $t \ge 0$, $\mathbf{x} \in \mathbb{R}^n$. Here $a_1, \dots, a_6$ are positive constants independent of chosen value of ε, and

$$\psi(t,\varepsilon) = \max_{i,j=1,\dots,n} \varepsilon \left| L_{ij}(t,\varepsilon) \right|. \tag{9}$$

With the aid of the results of [10], it is easy to verify that $\psi(t,\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly with respect to $t \ge 0$. Therefore, we can find and fix $\varepsilon > 0$ such that $a_5 \psi(t,\varepsilon) < a_4/3$.

Then, for chosen ε and sufficiently small values of $\delta > 0$, the inequalities

$$\frac{a_1}{2} \sum_{i=1}^{n} x_i^{\mu_i+1} \le \widetilde{V}(t,\mathbf{x}) \le 2a_2 \sum_{i=1}^{n} x_i^{\mu_i+1}, \qquad \dot{\widetilde{V}}\big|_{(4)} \le -\frac{a_4}{2} \sum_{i=1}^{n} x_i^{2\mu_i}$$

hold for $t \ge 0$ and $\|\mathbf{x}\| < \delta$ (hereinafter $\|\cdot\|$ denotes the Euclidean norm of a vector). Thus, the Lyapunov function (8) satisfies all the assumptions of the Lyapunov asymptotic stability theorem [9, 26]. This completes the proof. □

Consider now, along with (4), the perturbed system

$$\dot{x}_i(t) = \sum_{j=1}^{n} (\bar{p}_{ij} + \tilde{p}_{ij}(t)) x_j^{\mu_j}(t) + q_i(t,\mathbf{x}(t)), \qquad i = 1,\dots,n. \tag{10}$$

Here functions $q_1(t, \mathbf{x}), \dots, q_n(t, \mathbf{x})$ are defined and continuous in the region $t \geq 0$, $\|\mathbf{x}\| < H$, and, for any $\tilde{H} \in (0, H)$, the estimates

$$|q_i(t, \mathbf{x})| \leq c(\tilde{H}) \sum_{j=1}^{n} |x_j|^{\mu_j}, \quad i = 1, \dots, n,$$

are valid for $t \geq 0$, $\|\mathbf{x}\| < \tilde{H}$, with $c(\tilde{H}) \to 0$ as $\tilde{H} \to 0$. Thus, system (10) admits the solution $\mathbf{x}(t) \equiv \mathbf{0}$, as well.

Theorem 3.2 *Let Assumption 2.1 be fulfilled. Then the zero solution of (10) is uniformly asymptotically stable.*

Proof. Consider the derivative of the Lyapunov function (8) with respect to the perturbed equations. We obtain

$$\dot{\tilde{V}}\Big|_{(10)} \leq -\bar{a}_1 \sum_{i=1}^{n} x_i^{2\mu_i} + \bar{a}_2 \left(\psi(t, \varepsilon) + \frac{c(\tilde{H})}{\varepsilon} \right) \sum_{i=1}^{n} x_i^{2\mu_i} + \frac{\bar{a}_3}{\varepsilon} (1 + c(\tilde{H})) \sum_{i,j=1}^{n} x_i^{2\mu_i} x_j^{\mu_j - 1}$$

for $t \geq 0$, $\|\mathbf{x}\| < \tilde{H}$. Here $\bar{a}_1, \bar{a}_2, \bar{a}_3$ are positive constants independent of chosen values of ε and $\tilde{H}$, and the function $\psi(t, \varepsilon)$ is determined by the formula (9).

In a similar way as in the proof of Theorem 3.1, it is easy to show that if ε and $\tilde{H}$ are sufficiently small, then the estimate

$$\dot{\tilde{V}}\Big|_{(10)} \leq -\frac{\bar{a}_1}{2} \sum_{i=1}^{n} x_i^{2\mu_i}$$

holds for $t \geq 0$ and $\|\mathbf{x}\| < \tilde{H}$. This completes the proof. □

Corollary 3.1 *Let Assumptions 2.1 and 2.2 be fulfilled. Then the zero solution of (1) is uniformly asymptotically stable.*

4 Delay-Independent Stability Conditions

In this section, we will show that the results of Section 3 can be extended to the case of time-delay systems.

Consider the system (6), where $\tau \geq 0$ is a constant delay. Let $PC([-\tau, 0], \mathbb{R}^n)$ be the space of piece-wise continuous functions $\varphi(\theta) : [-\tau, 0] \to \mathbb{R}^n$ with the uniform (supremum) norm $\|\varphi\|_\tau = \sup_{\theta \in [-\tau, 0]} \|\varphi(\theta)\|$, and Ω_H be the set of functions $\varphi(\theta) \in PC([-\tau, 0], \mathbb{R}^n)$ satisfying the inequality $\|\varphi\|_\tau < H$.

By $\mathbf{x}(t, t_0, \varphi)$ we denote a solution of system (6) with the initial conditions $t_0 \geq 0$, $\varphi(\theta) \in \Omega_H$, while $\mathbf{x}_t(t_0, \varphi)$ is the restriction of the solution to the segment $[t - \tau, t]$, i.e. $\mathbf{x}_t(t_0, \varphi) : \theta \to \mathbf{x}(t + \theta, t_0, \varphi)$, $\theta \in [-\tau, 0]$. In some cases, when the initial conditions are not important, or well defined from the context, we write $\mathbf{x}(t)$ and $\mathbf{x}_t$, instead of $\mathbf{x}(t, t_0, \varphi)$ and $\mathbf{x}_t(t_0, \varphi)$, respectively. We will study the impact of delay on the stability of the zero solution of (6).

Consider the averaged system

$$\dot{x}_i(t) = \sum_{j=1}^{n} \bar{p}_{ij} f_j(x_j(t - \tau)), \quad i = 1, \dots, n. \tag{11}$$

Under Assumption 2.1, the zero solution of the corresponding delay free system (2) is asymptotically stable. In [4], it was proved that if no additional restrictions are imposed on the right-hand sides of (11), then an arbitrary small delay may destroy the stability.

In many applications, it is important to have stability conditions under which a system remains stable for any nonnegative value of delay [14, 22]. Such conditions are known as delay-independent ones.

Let Assumption 2.2 be fulfilled. Then the systems

$$\dot{x}_i(t) = \sum_{j=1}^{n} \left(\bar{p}_{ij} + \tilde{p}_{ij}(t)\right) x_j^{\mu_j}(t-\tau), \qquad i = 1, \dots, n, \tag{12}$$

$$\dot{x}_i(t) = \sum_{j=1}^{n} \bar{p}_{ij} x_j^{\mu_j}(t-\tau), \qquad i = 1, \dots, n, \tag{13}$$

are the systems of the first approximation for (6) and (11) respectively.

Delay-independent stability conditions for systems (12) and (13) have been studied in [4]. It was shown that, under Assumption 2.1, the zero solution of (12) is asymptotically stable for any $\tau \geq 0$. Furthermore, if, in addition to Assumption 2.1, the integrals (7) are bounded for $t \in [0, +\infty)$, then the zero solution of (13) is asymptotically stable for any $\tau \geq 0$ as well.

As it was mentioned in Section 3, in the present paper, we consider the case when integrals (7) may be unbounded.

Theorem 4.1 *Let Assumption 2.1 be fulfilled. Then the zero solution of (12) is uniformly asymptotically stable for any* $\tau \geq 0$.

Proof. Choose a Lyapunov function for (12) in the form (8) where positive coefficients $\lambda_1, \dots, \lambda_n$ are determined in accordance with Assumption 2.1.

Consider the derivative of the function with respect to system (12). We obtain

$$\begin{aligned}
\dot{\widetilde{V}}\big|_{(12)} &= \sum_{i,j=1}^{n} \lambda_i \bar{p}_{ij} x_i^{\mu_i}(t) x_j^{\mu_j}(t) + \varepsilon \sum_{i,j=1}^{n} \lambda_i L_{ij}(t,\varepsilon)\ x_i^{\mu_i}(t) x_j^{\mu_j}(t) \\
&- \sum_{i,j=1}^{n} \lambda_i \mu_i L_{ij}(t,\varepsilon)\ x_i^{\mu_i - 1}(t) x_j^{\mu_j}(t) \sum_{k=1}^{n} p_{ik}(t) x_k^{\mu_k}(t-\tau) \\
&- \sum_{i,j=1}^{n} \lambda_i \mu_j L_{ij}(t,\varepsilon)\ x_i^{\mu_i}(t) x_j^{\mu_j - 1}(t) \sum_{k=1}^{n} p_{jk}(t) x_k^{\mu_k}(t-\tau) \\
&+ \sum_{i,j=1}^{n} \lambda_i p_{ij}(t)\ x_i^{\mu_i}(t) \left(x_j^{\mu_j}(t-\tau) - x_j^{\mu_j}(t)\right).
\end{aligned}$$

Hence, if a solution $\mathbf{x}(t)$ of (12) is defined on an interval $[t_0, \hat{t}]$, $0 \leq t_0 < \hat{t}$, then the estimates

$$a_1 \sum_{i=1}^{n} x_i^{\mu_i+1}(t) - \frac{a_3}{\varepsilon} \sum_{i=1}^{n} x_i^{2\mu_i}(t) \leq \widetilde{V}(t, \mathbf{x}(t)) \leq a_2 \sum_{i=1}^{n} x_i^{\mu_i+1}(t) + \frac{a_3}{\varepsilon} \sum_{i=1}^{n} x_i^{2\mu_i}(t),$$

$$\dot{\widetilde{V}}\big|_{(12)} \leq -a_4 \sum_{i=1}^{n} x_i^{2\mu_i}(t) + a_5 \psi(t,\varepsilon) \sum_{i=1}^{n} x_i^{2\mu_i}(t)$$

$$+\frac{a_6}{\varepsilon} \sum_{i,j,k=1}^{n} \left| x_i^{\mu_i}(t) x_j^{\mu_j-1}(t) x_k^{\mu_k}(t-\tau) \right| + a_7 \sum_{i,j=1}^{n} |x_i^{\mu_i}(t)| \left| x_j^{\mu_j}(t-\tau) - x_j^{\mu_j}(t) \right|$$

hold for $t \in [t_0, \hat{t}]$. Here $a_1, \ldots, a_7$ are positive constants independent of the value of ε, and the function $\psi(t,\varepsilon)$ is defined by the formula (9).

Choose and fix $\varepsilon > 0$ for which the inequality $a_5\psi(t,\varepsilon) < a_4/3$ is valid. Let us prove that, for such ε, the Lyapunov function (8) satisfies all the conditions of Theorem 4.2 in [14].

Assume that, for a solution $\mathbf{x}(t)$ of (12), the estimate $\|\mathbf{x}(\xi)\| < \delta$, and the Razumikhin condition $\widetilde{V}(\xi, \mathbf{x}(\xi)) \leq 2\widetilde{V}(t, \mathbf{x}(t))$ are fulfilled for $\xi \in [t-(m+1)\tau, t]$. Here $\delta = \text{const} > 0$, and m is a positive integer such that

$$\frac{(m(\mu_1 - 1) + \mu_1)(\mu_n + 1)}{(\mu_1 + 1)\mu_n} > 1.$$

If the value of δ is sufficiently small, then

$$x_i^{\mu_i+1}(\xi) < 8\frac{a_2}{a_1} \sum_{j=1}^{n} x_j^{\mu_j+1}(t), \qquad i = 1, \ldots, n, \tag{14}$$

for $\xi \in [t - (m+1)\tau, t]$.

With the aid of inequalities (14), it is easy to show that

$$|x_j^{\mu_j}(t-\tau) - x_j^{\mu_j}(t)| = \tau\mu_j\, x_j^{\mu_j-1}(t-\eta_j\tau) \left| \sum_{l=1}^{n} p_{jl} x_l^{\mu_l}(t-\eta_j\tau-\tau) \right|$$

$$\leq b_1 \left(\sum_{l=1}^{n} x_l^{\mu_l+1}(t) \right)^{\frac{\mu_j-1}{\mu_j+1}} \left(\sum_{l=1}^{n} |x_l^{\mu_l}(t)| + \sum_{l=1}^{n} |x_l^{\mu_l}(t-\eta_j\tau-\tau) - x_l^{\mu_l}(t)| \right)$$

$$\leq b_2 \left(\sum_{l=1}^{n} |x_l^{\mu_l}(t)| \right)^{\frac{(\mu_1-1)(\mu_n+1)}{(\mu_1+1)\mu_n}} \left(\sum_{l=1}^{n} |x_l^{\mu_l}(t)| + \sum_{l=1}^{n} |x_l^{\mu_l}(t-\eta_j\tau-\tau) - x_l^{\mu_l}(t)| \right),$$

where $b_1 > 0$, $b_2 > 0$, $0 < \eta_j < 1$, $j = 1, \ldots, n$.

Further, for the functions $|x_l^{\mu_l}(t-\eta_j\tau-\tau) - x_l^{\mu_l}(t)|$, $l = 1, \ldots, n$, the similar estimates can be found.

Successively applying this procedure m times, we obtain

$$\left| x_j^{\mu_j}(t-\tau) - x_j^{\mu_j}(t) \right|$$

$$\leq b_3 \left(\sum_{s=1}^{n} |x_s^{\mu_s}(t)| \right)^{1+\frac{(\mu_1-1)(\mu_n+1)}{(\mu_1+1)\mu_n}} + b_4 \left(\sum_{s=1}^{n} |x_s^{\mu_s}(t)| \right)^{\frac{(m(\mu_1-1)+\mu_1)(\mu_n+1)}{(\mu_1+1)\mu_n}},$$

where b_3 and b_4 are positive constants, $j = 1, \ldots, n$.

Thus, for sufficiently small values of δ, the inequality

$$\dot{\tilde{V}}(t,\mathbf{x}(t)) \le -\frac{a_4}{2}\sum_{i=1}^{n} x_i^{2\mu_i}(t)$$

holds. Hence [14], the zero solution of (12) is uniformly asymptotically stable. This completes the proof. □

Consider now the perturbed system

$$\dot{x}_i(t) = \sum_{j=1}^{n}(\bar{p}_{ij} + \tilde{p}_{ij}(t))x_j^{\mu_j}(t-\tau) + q_i(t,\mathbf{x}(t),\mathbf{x}(t-\tau)), \qquad i = 1,\dots,n. \tag{15}$$

Here functions $q_1(t,\mathbf{x},\mathbf{y}),\dots,q_n(t,\mathbf{x},\mathbf{y})$ are defined and continuous in the region $t \ge 0$, $\|\mathbf{x}\| < H$, $\|\mathbf{y}\| < H$, and, for any $\tilde{H} \in (0,H)$, the estimates

$$|q_i(t,\mathbf{x},\mathbf{y})| \le c(\tilde{H})\sum_{j=1}^{n}\left(|x_j|^{\mu_j} + |y_j|^{\mu_j}\right), \quad i = 1,\dots,n,$$

are valid for $t \ge 0$, $\|\mathbf{x}\| < \tilde{H}$, $\|\mathbf{y}\| < \tilde{H}$, with $c(\tilde{H}) \to 0$ as $\tilde{H} \to 0$.

Theorem 4.2 *Let Assumption 2.1 be fulfilled. Then the zero solution of (15) is uniformly asymptotically stable for any* $\tau \ge 0$.

The proof of the theorem is similar to that of Theorem 4.1.

Corollary 4.1 *Let Assumptions 2.1 and 2.2 be fulfilled. Then the zero solution of (6) is uniformly asymptotically stable for any* $\tau \ge 0$.

5 Stability Conditions for an Automatic Control System

In Sections 3 and 4, it was assumed that the considered systems are essentially nonlinear, i.e. the right-hand sides of the systems do not contain linear terms with respect to phase variables. In this section, we will show that the approaches proposed in the present paper can be used as well for the stability analysis of some classes of nonlinear time-varying systems with nontrivial linear approximations. Right-hand sides of such systems may include linear terms, but linear approximations are critical in the Lyapunov sense [9, 17].

Let the dynamic nonlinear feedback system [17, 26]

$$\begin{cases} \dot{\mathbf{x}}(t) &=& \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,f(\sigma(t)), \\ \dot{\sigma}(t) &=& \mathbf{c}^T\mathbf{x}(t) - f(\sigma(t)), \end{cases} \tag{16}$$

be given. Here $\mathbf{x}(t) \in \mathbb{R}^n$ and $\sigma(t) \in \mathbb{R}$, $\mathbf{A}$ is a constant Hurwitz matrix, $\mathbf{b}$ and $\mathbf{c}$ are constant vectors, $f(\sigma)$ is a sector nonlinearity, which is continuous for $|\sigma| < H$, $0 < H \le +\infty$, and satisfies the condition $\sigma f(\sigma) > 0$ for $\sigma \ne 0$.

Assume that, for system (16), there exists a Lyapunov function of the form

$$V(\mathbf{x},\sigma) = \mathbf{x}^T\mathbf{D}\mathbf{x} + \int_0^{\sigma} f(s)\,ds,$$

where $\mathbf{D}$ is a constant symmetric positive definite matrix, such that the estimate

$$\dot{V}\Big|_{(16)} \leq -b\left(\|\mathbf{x}(t)\|^2 + f^2(\sigma(t))\right), \quad b = \text{const} > 0,$$

holds. The conditions for the existence of the Lyapunov function are well known, see, for instance, [17, 26]. The fulfilment of this assumption implies the asymptotic stability of the zero solution of (16).

Consider now the case when the control law includes a delay and a nonstationary perturbation. Let the system be of the form

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,f(\sigma(t-\tau)), \\ \dot{\sigma}(t) &= \mathbf{c}^T\mathbf{x}(t) - (1+\tilde{p}(t))f(\sigma(t-\tau)). \end{cases} \tag{17}$$

Here $\tau \geq 0$ is a constant delay, while the perturbation $\tilde{p}(t)$ is continuous and bounded for $t \in [0, +\infty)$ function, such that

$$\frac{1}{T}\int_t^{t+T} \tilde{p}(s)ds \to 0 \quad \text{as} \quad T \to +\infty$$

uniformly with respect to $t \geq 0$.

Furthermore, we assume that the nonlinearity $f(\sigma)$ can be represented as follows $f(\sigma) = \beta\sigma^\mu + g(\sigma)$, where $\mu > 1$ is a rational number with odd numerator and denominator, β is a positive constant, and $g(\sigma)/\sigma^\mu \to 0$ as $\sigma \to 0$.

It is worth mentioning that essentially nonlinear control laws were considered in [7, 12, 20]. In particular, in [20], controls of such type were used for solving the problem of angular stabilization of an airplane, whereas, in [12], they were applied for the developing of seismic mitigation devices.

Theorem 5.1 *The zero solution of (17) is uniformly asymptotically stable for any value of* $\tau \geq 0$.

Proof. Construct a Lyapunov function for (17) in the form

$$\widetilde{V}(t, \mathbf{x}, \sigma) = \mathbf{x}^T\mathbf{D}\mathbf{x} + \beta\frac{\sigma^{\mu+1}}{\mu+1} + \beta^2\sigma^{2\mu}\int_0^t \exp(\varepsilon(s-t))\,\tilde{p}(s)ds,$$

where ε is a positive parameter. With the aid of this function the subsequent proof is similar to that of Theorem 4.1. □

6 Conclusion

In this paper, for a special class of nonlinear nonstationary systems, new sufficient asymptotic stability conditions of the trivial solution are obtained via the averaging technique. It is proved that, for the considered essentially nonlinear systems, this technique can be applied without requirement of fast time-varying vector field – typical for averaging results.

It is easy to verify that the results obtained for time delay systems remain valid when the systems delays are continuous nonnegative and bounded functions of the time variable. Moreover, these results can be extended to systems with distributed delays as well.

An important direction of future research is application of the developed approaches for the stability analysis of nonlinear nonstationary complex (multiconnected) systems.

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A DTC Neurofuzzy Speed Regulation Concept for a Permanent Magnet Synchronous Machine

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Abstract: Based on Sugeno fuzzy logic system, this paper develops a Neuro-Fuzzy Direct Torque Control (NFDTC) for a Permanent Magnet Synchronous motor (PMSM). The main idea of DTC control is motivated by direct choosing the stators voltage vectors according to the differences between the references of the electromagnetic torque and the stators flux and their reels values calculated and related only on the actual-sizes of the stators. The neurofuzzy regulator is synthesized by using the Sugeno reasoning methods, where the consequences rules are a single order polynomial of inputs defined by three Gaussians fuzzy sets. The parameters of the premises and the conclusions of the fuzzy rules of Sugeno are determined on the base of the input-output data provided by a fuzzy regulator of the Mamdani type, where the linguistic variables of inputs-outputs of the torque, flux and position of the stator flux vectors are of triangular membership functions. The training is based on the extended Kalman filter concept, which allows the determining of the parameters vector of the fuzzy rules so that the output of the Sugeno regulator approaches will be the best possible output of the Mamdani regulator. The simulation results make it possible an effective evaluation of the Kalman extended based filters training algorithms.

Keywords: *DTC; PMSM; Inverter voltage; fuzzy sets; Sugeno methods; extended Mamdani and Kalman filter.*

Mathematics Subject Classification (2010): 03B52, 93C42, 94D05.

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1 Introduction

The use of the PMSM always continues to extend. The technological development made it possible that the permanent magnet synchronous machines are more essential in the field applications of a very high static and dynamic performances demands, especially in the embedded systems fields (aeronautical and aerospace) because of its high power/weight ratio. A noiseless linear process with a constant parameters concept can be controlled accurately by traditional PID regulators; these regulators proved to be sufficient, however the process is subjected to disturbances and its parameters variations are relatively less, especially if the requirements on the precision of adjustment and the dynamic response of the system are not strict. In the contrary case one can have recourse to an auto adaptative solution, which by readjustment of the parameters of the regulators, allows preserving performances fixed in advance in the presence of the disturbances and variation of parameters. Nevertheless, this solution presents the disadvantage of often complex implementation. It is thus possible to solve this problem by using the method of robust commands and neurofuzzy control.

In this paper we apply the neurofuzzy control by the method of Sugeno to the speed regulation of a Permanent Magnet Synchronous Machine. The objective is to synthesize neurofuzzy regulator of Sugeno to three fuzzy sets for each one of: torque, flux and position of the flux vector and whose consequences of the rules are the polynomials of order one. This neurofuzzy regulator is thus deduced by recopying the data inputs outputs provided by a fuzzy regulator of Mamdani to 132 fuzzy rules [1]. The method of copy is based on the approach by extended Kalman filter. In [1], the authors introduce a fuzzy logic controller in conjunction with direct torque control strategy for a permanent magnet synchronous machine. In this controller there are three inputs, which are the error of stator flux, the error of torque and the stator flux angle. The total rule number used is 132 rules. The rules base of the proposed approach contains only 27 rules. Consequently, this approach requires less computing time for its execution compared with the method that is proposed in [1].

2 Mathematical Model of a Permanent Magnet Synchronous Motor

The PMSM model is considered under the following assumptions.

1. The spatial distribution of stator winding is sinusoidal.
2. The saturation is neglected.
3. The damping effect is neglected.

Thus, in the synchronous $d-q$ reference form, the dynamics of PMSM is represented as follows [1,4]

$$\begin{aligned} V_d &= R_s I_d + L_d \frac{dI_d}{dt} - w_r L_q I_q, \\ V_q &= R_s I_q + L_q \frac{dI_q}{dt} - w_r L_d I_d + w_r \varphi_f, \\ T_{em} &= p(\varphi_d I_q - \varphi_q I_d), \end{aligned} \tag{1}$$

with

$$\begin{aligned} \varphi_d &= L_d I_d + \varphi_f, \\ \varphi_q &= L_q I_q, \end{aligned}$$

L_d : direct stator inductance,
L_q : stator inductance in squaring,
φ_f : flux of the permanents magnets.

The total mathematical model is given in the form of space of following state:

$$\begin{aligned} \frac{dI_d}{dt} &= \frac{V_d}{L_d} - \frac{R_s}{L_d} I_d + \frac{L_q}{L_d} w_r I_q p, \\ \frac{dI_q}{dt} &= \frac{V_q}{L_q} - \frac{R_s}{L_q} I_q - p\frac{L_d}{L_q} w_r I_d - \frac{\varphi_f}{L_q} w_r p, \\ \frac{dw_r}{dt} &= p^2 I_q \frac{\varphi_f}{j} + \frac{1}{j}\left[p^2 (L_d - L_q) I_d I_q f_m\right] - p\frac{c_r}{j}, \\ \frac{d\theta_r}{dt} &= p w_r. \end{aligned} \tag{2}$$

3 General Principal of DTC

The direct torque control of the permanent magnet synchronous machine is based on the determination "direct" sequence of order applied to the switches of an inverter of tension. First, we use a fuzzy regulator. Secondly, we replace the latter by a neurofuzzy regulator, whose function is to control the state of the system (the amplitude of stator flux and electromagnetic torque).

3.1 Selection of the voltage vector V_s

The voltage vector V_s is delivered by a three-phase of the voltage source inverter and is given by [5,6]:

$$V_s = \sqrt{\frac{2}{3}}\left(a^0 V_a + a V_b + a^2 V_c\right) \tag{3}$$

with

$$a = \exp(j\frac{2\pi}{3}).$$

By using the logical variables representing the state of the switches, the voltage vector can be written in the form:

$$V_s = \sqrt{\frac{2}{3}} U_0 \left(S_a + a S_b + a^2 S_c\right). \tag{4}$$

As shown in Figure 2 the combinations of the three sizes (S_a, S_b, S_c) allow to generate 8 fixed positions of the vector V_s, two correspond to the null vectors:

$$(S_a, S_b, S_c) = (0,0,0) \quad \text{and} \quad (S_a, S_b, S_c) = (1,1,1). \tag{5}$$

Generally, the space of evolution of stator flux φ_s is delimited in the fixed reference frame

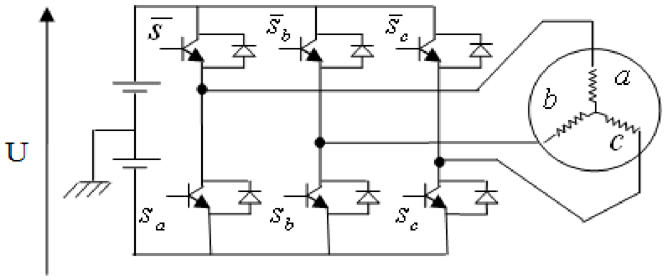

Figure 1: Scheme of the voltage source inverter.

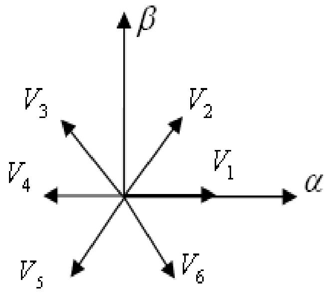

Figure 2: Development of the 8 vectors V_s $(\alpha\beta)$ Stationary reference frame.

V_0	(1 1 1)
V_1	(1 0 0)
V_2	(1 1 0)
V_3	(0 1 0)
V_4	(0 1 1)
V_5	(0 0 1)
V_6	(1 0 1)
V_7	(0 0 0)

Table 1: Development of the 8 possible configurations of the vectors V_s.

(stator) by breaking it up into 6 symmetrical zones compared to the directions of the nonnull voltage vectors. The position of the flux vector in these zones is determined from these components.

When the stator flux vector φ_s is in a numbered zone N, the control of flux and torque can be ensured by selecting one of the nonnull voltage vectors:

N				1	2	3	4	5	6
Tcont	1	flxC	1	V_2	V_3	V_4	V_5	V_6	V_1
			0	V_3	V_4	V_5	V_6	V_1	V_2
Tcont	0	flxC	1	V_0	V_7	V_7	V_0	V_7	V_0
			0	V_7	V_0	V_0	V_7	V_0	V_7

Tcont: Torque Control. flxC: flux Control.

Table 2: Table of commutation for the selection of the voltage vector.

4 Fuzzy Controller

In the hysteresis direct torque control, the errors of torque and flux are directly used to select the switching state of switches of the inverter voltage with any distinction between large or relatively small error. The large or small terms are vague terms containing the concept of fuzzy logic control which allows using a fuzzy controller [1–3]. On the other hand, the torque ripples will be reduced (Figure 3).

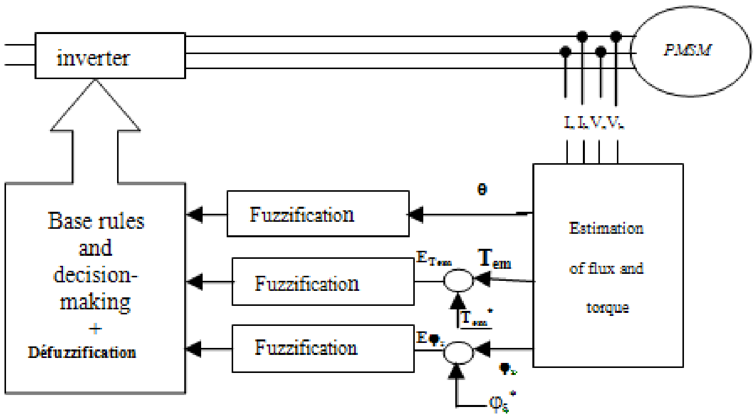

Figure 3: Synoptic scheme of the fuzzy controller of the PMSM.

The studied fuzzy controller has 3 state variables of input and one variable of command in output.

Each variable is represented by fuzzy set. The number of the fuzzy set for each variable is selected to obtain a powerful command with a minimal number of fuzzy rules.

The first fuzzy state variable is the difference between the reference stator flux φ_s^* (in Webers) and the estimated stator flux magnitude φ_s given by:

$$E_{\varphi_s} = \varphi_s^* - |\varphi_s|. \tag{6}$$

The grade of membership distribution is shown in Figure 4(a) which uses a triangular distribution.

The second fuzzy state variable is the difference between the command electromagnetic torque T_{em} and the estimated electromagnetic torque T_{em}^* (error in torque $E_{T_{em}}$) given by:

$$E_{T_{em}} = T_{em}^* - T_{em}. \tag{7}$$

The electromagnetic torque is estimated from the flux and current information which are given in [1]. The grade of membership distribution is shown in Figure 4(b).

The third fuzzy state variable is the angle between stator flux and their reference axis (stator flux angle θ) which is determined by the following relation

$$\theta = \tan^{-1}\left(\frac{\varphi_\beta}{\varphi_\alpha}\right). \tag{8}$$

The universe of discourse of this fuzzy variable is divided into 12 fuzzy sets (θ_1 to θ_{12}). The membership distribution of fuzzy variables is shown in Figure 4(c).

θ_1

$E_{tor}\diagdown E_{\varphi_s}$	P	Z	N
PL	V_1	V_2	V_2
PS	V_1	V_2	V_3
ZE	0	0	0
NS	V_6	0	V_4
NL	V_6	V_5	V_5

θ_2

$E_{tor}\diagdown E_{\varphi_s}$	P	Z	N
PL	V_2	V_2	V_3
PS	V_2	V_3	V_3
ZE	0	0	0
NS	V_6	V_0	V_5
NL	V_6	V_6	V_5

θ_3

$E_{tor}\diagdown E_{\varphi_s}$	P	Z	N
PL	V_2	V_3	V_3
PS	V_2	V_3	V_4
ZE	0	0	0
NS	V_1	0	V_5
NL	V_1	V_6	V_6

θ_4

$E_{tor}\diagdown E_{\varphi_s}$	P	Z	N
PL	V_3	V_3	V_4
PS	V_3	V_4	V_4
ZE	0	0	0
NS	V_1	0	V_6
NL	V_1	V_1	V_6

θ_5

$E_{tor}\diagdown E_{\varphi_s}$	P	Z	N
PL	V_3	V_4	V_4
PS	V_3	V_4	V_5
ZE	0	0	0
NS	V_2	0	V_6
NL	V_2	V_1	V_1

θ_6

$E_{tor}\diagdown E_{\varphi_s}$	P	Z	N
PL	V_5	V_4	V_4
PS	V_4	V_5	V_5
ZE	0	0	0
NS	V_2	0	V_1
NL	V_2	V_2	V_1

θ_7

$E_{tor}\diagdown E_{\varphi_s}$	P	Z	N
PL	V_4	V_5	V_5
PS	V_4	V_5	V_6
ZE	0	0	0
NS	V_3	0	V_1
NL	V_3	V_2	V_2

θ_8

$E_{tor}\diagdown E_{\varphi_s}$	P	Z	N
PL	V_5	V_5	V_6
PS	V_5	V_6	V_6
ZE	0	0	0
NS	V_3	0	V_2
NL	V_3	V_3	V_2

θ_9

$E_{tor}\diagdown E_{\varphi_s}$	P	Z	N
PL	V_5	V_6	V_6
PS	V_5	V_6	V_1
ZE	0	0	0
NS	V_4	0	V_2
NL	V_4	V_3	V_3

θ_{10}

$E_{tor}\diagdown E_{\varphi_s}$	P	Z	N
PL	V_6	V_6	V_1
PS	V_6	V_1	V_1
ZE	0	0	0
NS	V_4	0	V_3
NL	V_4	V_4	V_3

θ_{11}

$E_{tor}\diagdown E_{\varphi_s}$	P	Z	N
PL	V_6	V_1	V_1
PS	V_6	V_1	V_2
ZE	0	0	0
NS	V_5	0	V_3
NL	V_5	V_4	V_4

θ_{12}

$E_{tor}\diagdown E_{\varphi_s}$	P	Z	N
PL	V_1	V_1	V_2
PS	V_1	V_2	V_2
ZE	0	0	0
NS	V_5	0	V_4
NL	V_5	V_5	V_4

Table 3: Set of fuzzy rules for control of PMSM (E_{φ_s}: error of the stator flux, E_{tor}: torque error).

In Figure 5, the output has only one variable of command which is the state of ordering of the switch when the voltage vectors are discrete values.

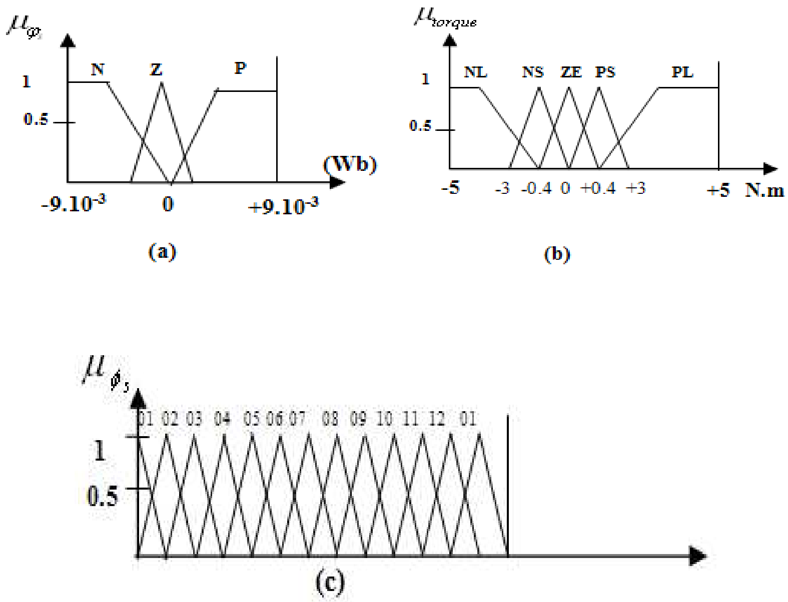

Figure 4: Membership distribution of fuzzy variables for fuzzy controller.

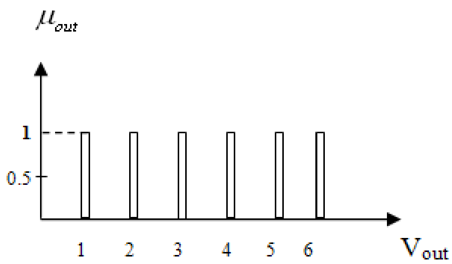

Figure 5: Membership functions variables of fuzzy output.

5 The Sugeno Method

The Sugeno fuzzy logic controller is proposed by Takagi and Sugeno [8], who develops a systematic method of generation of the fuzzy rules starting from a whole of data input-output. In this case, the consequences of the rules are numerical functions, which depend on the current values of the variables of inputs. Being given that each rule has a numerical conclusion, the total output of neurofuzzy controller is obtained by the calculation of a weighted average, and in this manner the time consuming by the procedure of defuzzification is avoided.

Let us designate by e, Δe and δ inputs of the neurofuzzy controller, and by Δu its output. The rules base of the neurofuzzy controller has: $M = m_1 \times m_2 \times m_3$ fuzzy rules of the form:

R_l : If e is F_e and Δe is $F_{\Delta e}$ and δ is F_δ, then

$$\begin{aligned} \Delta u &= f_l[e, \Delta e, \delta] \\ &= p_l e + q_l \Delta e + r_l \delta + z_l \end{aligned} \tag{9}$$

with $l = 1, 2, \ldots, M$, where m_1, m_2 and m_3 are the numbers of fuzzy set associate with e, Δe and δ, respectively. Thus, the output of the neurofuzzy controller is given by the following relation:

$$\Delta u = \frac{\sum_{l=1}^{M} \alpha_l f_l}{\sum_{l=1}^{M} \alpha_l}, \tag{10}$$

where α_l represents the confidence degree or activation of the rule R_l, and is given by:

$$\alpha_l = \mu_{F_e} \mu_{F_{\Delta e}} \mu_{F_\delta}. \tag{11}$$

In our case and for the Sugeno method, the input variables e, Δe and δ are characterized by neurofuzzy set of Gaussian type defined by the relation:

$$\mu(x) = \exp\left[-0.5(v_i(x - c_i))^2\right], \tag{12}$$

where c_i is the average and v_i is the reverse of the variance. Initially, the problem is to determine the parameters: p_l , q_l, r_l and z_l.

6 Determination by Training of the Parameters Sugeno Regulator

The determination of the parameters of neurofuzzy controller of Sugeno constitutes the most difficult phase in the design, taking into account a significant number of parameters to be determined (parameters of the premises and the consequences).

Methods of training, applied specially in neural networks, are more developed for the approximation of an application input output according to a criterion of training. For our case we use an algorithm of training based on Extended Kalman Filter which is usually used to estimate the neural networks parameters. Let us consider a neurofuzzy controller of Sugeno characterized by a vector of parameters θ. Let data set of input-output be $(x(k), d(k))$. Our objective is to find the vectors θ so that the output of neurofuzzy regulator approachs the best possible desired output $d(k)$, i.e. to have $\Delta u\,[x(k), \theta] = d(k)$. Extended Kalman filter approach consists in linearizing the output Δu at any time around the estimated vector $\widehat{\theta}$. This amounts to writing:

$$\begin{aligned} d(k) &= \Delta u\left[x(k); \widehat{\theta}(k-1)\right] + \Psi^T(k)\left[\theta - \widehat{\theta}(k-1)\right], \\ \Psi(k) &= \frac{\partial \Delta u\,[x(k); \theta]}{\partial \theta} / \widehat{\theta}(k-1). \end{aligned} \tag{13}$$

The well-known form of the relation (13) is:

$$\begin{aligned} \widehat{\theta}(k) &= \widehat{\theta}(k-1) + p(k)\Psi(k)e(k), \\ e(k) &= d(k) - \Delta u\left[x(k); \widehat{\theta}(k-1)\right], \end{aligned} \tag{14}$$

where $p(k)$ is the gain of the algorithm of estimate. In the method of the modified gradient, the gain $p(k)$ is selected as a variable. It is given by the following relation [9]:

$$p(k) = \frac{\alpha_1 I}{\alpha_2 \Psi^T(k)\Psi(k)}; \quad \alpha_1 > 0, \alpha_2 > 0. \tag{15}$$

We notice as well that this method requires the calculation of the gradient $\Psi = \frac{\partial \Delta u}{\partial \theta}$, this gradient is calculated by the method of the retropropagation used in the artificial neural network.

For our case, the vector of the parameters is $\theta = [c \;\; v \;\; p \;\; q \;\; r \;\; z]^T$. Consequently, we have:

$$\frac{\partial \Delta u}{\partial \theta} = \left[\frac{\partial \Delta u}{\partial c} \;\; \frac{\partial \Delta u}{\partial v} \;\; \frac{\partial \Delta u}{\partial p} \;\; \frac{\partial \Delta u}{\partial q} \;\; \frac{\partial \Delta u}{\partial r} \;\; \frac{\partial \Delta u}{\partial z}\right], \tag{16}$$

where

$$\frac{\partial \Delta u}{\Delta c_i} = \frac{v_i^2(x_i - c_i)\sum\limits_{k\in I}\alpha_k(f_k - \Delta u)}{\sum\limits_{l=1}^{M}\alpha_l}, \tag{17}$$

$$\frac{\partial \Delta u}{\Delta v_i} = \frac{v_i(x_i - c_i)^2\sum\limits_{k\in I}\alpha_k(f_k - \Delta u)}{\sum\limits_{l=1}^{M}\alpha_l}, \tag{18}$$

$$\frac{\partial \Delta u}{\Delta p_i} = \frac{\alpha_i e}{\sum\limits_{l=1}^{M}\alpha_l}, \quad \frac{\partial \Delta u}{\Delta q_i} = \frac{\alpha_i \Delta e}{\sum\limits_{l=1}^{M}\alpha_l}, \quad \frac{\partial \Delta u}{\Delta r_i} = \frac{\alpha_i \delta}{\sum\limits_{l=1}^{M}\alpha_l}, \tag{19}$$

$$\frac{\partial \Delta u}{\Delta z_i} = \frac{\alpha_i}{\sum\limits_{l=1}^{M}\alpha_l}, \tag{20}$$

with $x_i \in \{e, \Delta e, \delta\}$ and I represents the whole of the indices of the fuzzy rules of which appears the parameter. In our case, the input-output data are obtained by synthesizing a neurofuzzy regulator, while at exploiting the method of Mamdani the linguistic variables of inputs e, Δe, δ and the output variable Δu are described respectively in Figure 4 and Figure 5.

7 Control Algorithm

For the method of Sugeno, the input variables e, Δe, δ are characterized by three fuzzy set Gaussian type:

e: is the input of the electromagnetic torque,

Δe: is the input of the flux,

δ : is the input angles (position) of the stator flux vector.

The fuzzy rules, being used to induce the order for the case of the Sugeno neurofuzzy regulator, are grouped as follows:

if e is NB and Δe is NB and δ is NG, then Δu is f_1;

if e is PB and Δe is PG and δ is PB, then Δu is f_{27}.

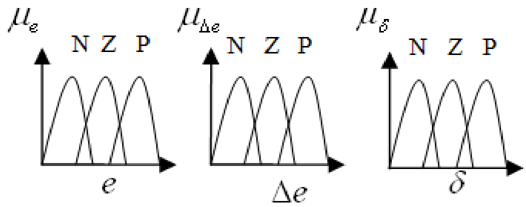

Figure 6: Membership functions of fuzzy input variables.

The training is carried out for the electromagnetic torque control and for speed control. The gains parameters of adaptation are fixed as follows:

$$\alpha_1 = 0.8, \qquad \alpha_2 = 1. \tag{21}$$

Parameters of consequences and premises are gathered in Tables 4 and 5.

8 Simulation Results

In order to test the effectiveness of the training algorithm, we carried out the following sets of control-machines simulation

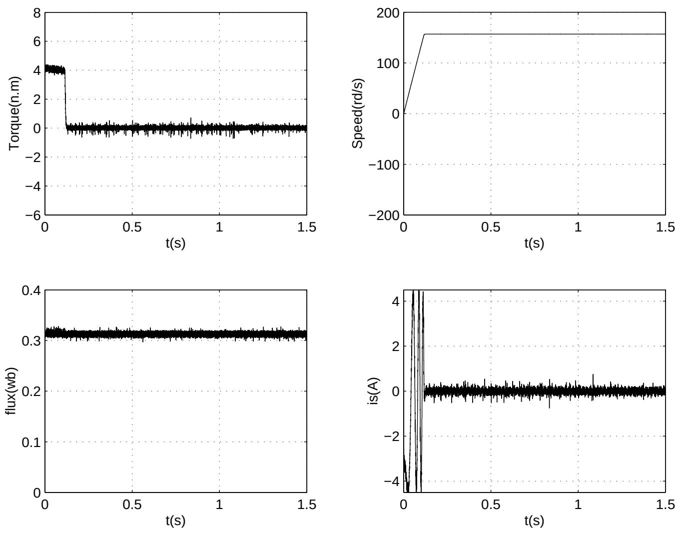

Figure 7: Dynamic behavior of the PMSM controlled by a fuzzy regulator (case of Mamdani).

In Figure 7 we use the following test with Mamdani fuzzy controler:

— No-load start of the process with a reference speed of 157rd/s. We applied load torque of (zero) O N.m. The waveforms obtained in this case show clearly that the revolutions of the machine are followed closely by their references. Both, torque T_{em} and stator current i_s cancel after the transient. And the magnetic flux remains stable by keeping its value with 0.314 Webes.

	p	q	r	z
f_1	-0.001787	$-8.62 * 10^{-5}$	0.0005546	0.0009657
f_2	-0.0001574	$-7.894 * 10^{-6}$	0.0007904	0.0001191
f_3	$-4.157 * 10^{-5}$	$-4.286 * 10^{-6}$	0.002469	$8.91 * 10^{-5}$
f_4	-0.739	-0.05672	0.149	3.921
f_5	0.5889	-0.02495	0.2139	0.3412
f_6	0.01215	-0.009218	0.1277	-0.02831
f_7	-0.4112	0.01933	0.6928	2.881
f_8	-0.215	0.02486	0.268	0.03967
f_9	-0.01169	0.01212	0.1612	0.01257
f_{10}	-0.0001059	$-7.292 * 10^{-6}$	0.0001513	0.000111
f_{11}	$-9.837 * 10^{-6}$	$-1.218 * 10^{-6}$	0.0004298	$3.403 * 10 - 5$
f_{12}	$-6.92 * 10^{-6}$	$-2.245 * 10^{-6}$	0.001392	$5.019 * 10^{-5}$
f_{13}	-0.9648	-0.1822	0.2086	3.159
f_{14}	-0.3653	-0.1021	0.1006	0.9569
f_{15}	0.1277	$9.767 * 10^{-5}$	0.07375	-0.2289
f_{16}	0.1187	-0.03402	-0.03536	0.5158
f_{17}	0.1603	-0.0358	0.003331	0.496
f_{18}	-0.2666	-0.05353	0.03755	-0.0825
f_{19}	$9.041 * 10^{-7}$	$-1.105 * 10^{-8}$	$1.19e - 006$	$4.109 * 10^{-7}$
f_{20}	$1.644 * 10^{-6}$	$-6.729 * 10^{-9}$	$5.699e - 006$	$4.074 * 10^{-7}$
f_{21}	$3.106 * 10^{-7}$	$-7.119e - 009$	$5.613e - 006$	$2.154 * 10^{-7}$
f_{22}	0.406	-0.00212	0.1018	0.1133
f_{23}	0.1034	-0.0006938	0.08979	0.01391
f_{24}	0.02006	$3.082 * 10^{-5}$	0.06282	0.002271
f_{25}	0.2008	-0.000487	-0.05291	0.0424
f_{26}	0.04644	-0.0004289	0.05497	0.001026
f_{27}	0.01833	-0.0001224	0.09209	0.003566

Table 4: The consequences values.

	e		
	NG	Z	PG
c	-2.152	2.403	6.785
v	0.2337	0.2652	0.6746

(a)

	Δe		
	NG	Z	PG
c	-0.1066	-0.01417	0.00829
v	0.00132	0.012	0.01495

(b)

	δ		
	NG	Z	PG
c	-0.2462	14.53	29.76
v	6.608	6.102	6.604

(c)

Table 5: The premises values.

By using the following test with Sugeno neurofuzzy controller in Figure 8:

— We obtain practically the same reponses of Figure 7.

By using the following test with Mamdani fuzzy controller in Figure 9:

— No-load start of the process with a reference speed of 157rd/s. Both, torque T_{em} and stator current i_s cancel after the transient. But from $t = 0.5s$ to $t = 1s$, it applied a nominal load torque of 3.5N.m. The waveforms obtained in this case show clearly by that the revolutions of the machine are followed closely their references.

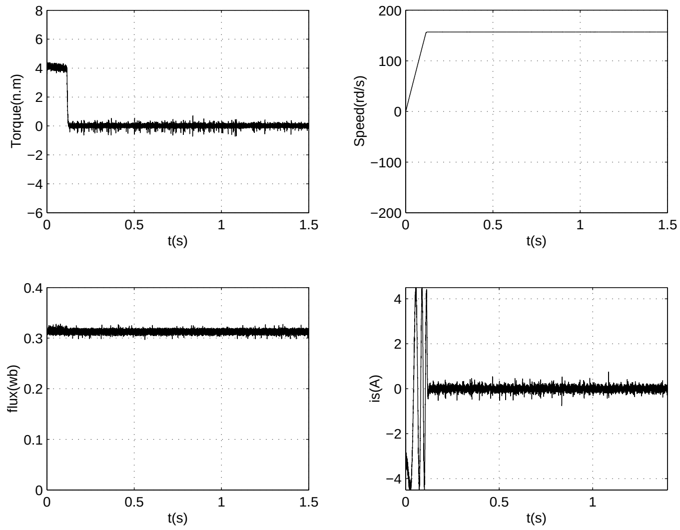

Figure 8: Dynamic behavior of the PMSM controlled by a neurofuzzy regulator (case of Sugeno).

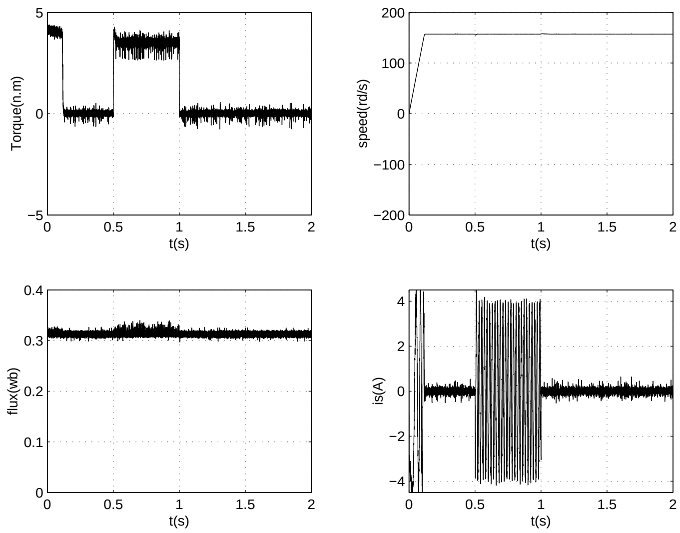

Figure 9: Dynamic behavior of the PMSM controlled by a fuzzy regulator (case of Mamdani).

Each one of torque T_{em}, stator flux and stator current i_s.

By using the following test with Sugeno neurofuzzy controller in Figure 10:

— We obtain practically the same reponses as in Figure 9.

In Figures 11 and 12, we carried out the inversion of direction speed of the PMSM in the two cases (fuzzy and neurofuzzy), with starting the reference of nominal speed of 157rd/s without a load torque at $t = 2s$, it's reversed the reference with -157rd/s.

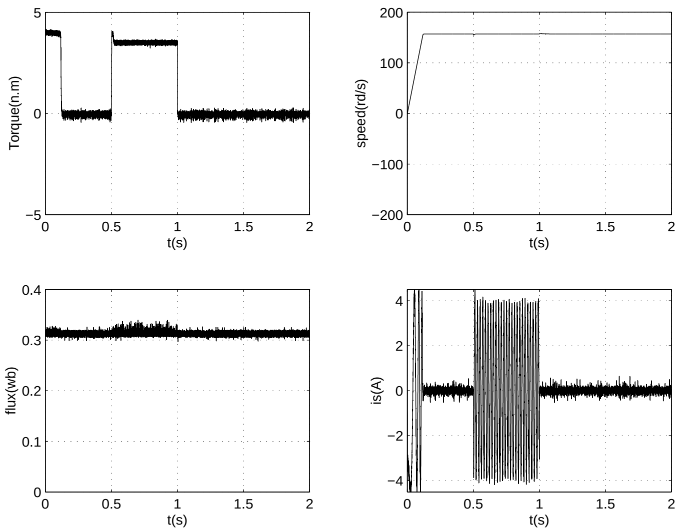

Figure 10: Dynamic behavior of the PMSM controlled by a neurofuzzy regulator (case of Sugeno).

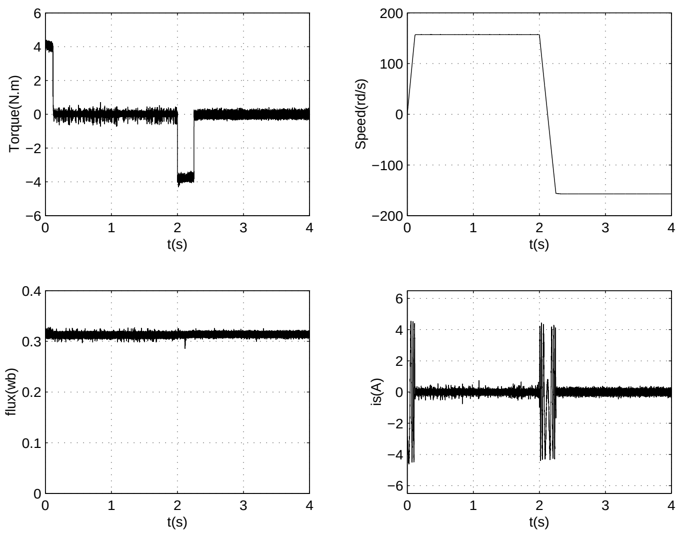

Figure 11: Inversion of direction speed of PMSM controlled by a fuzzy regulator (case of Mamdani).

We notice that the answers on the currents are almost identical too, this shows the effectiveness of the algorithm of training suggested. Learning has been made to keep the same dynamic speed regardless of the dynamics of the electromagnetic torque. Ideally the two results should be identical, but because of the error of learning the results appear different.

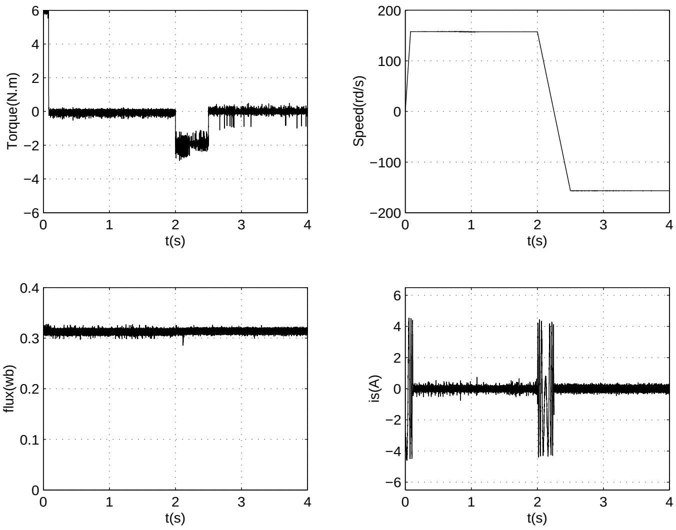

Figure 12: Inversion of direction speed of PMSM controlled by a neurofuzzy regulator (case of Sugeno).

9 Conclusion

In this paper we developed the adjustment of DTC neurofuzzy concept by exploiting the Sugeno methods applied to the PMSM. The DTC strategy is motivated by direct choosing the stators voltage vectors according to the differences between the references of the electromagnetic torque and the stators flux and their reels values calculated and related only on the actual-sizes of the stators. The Sugeno regulator is defined as a polynomial of order one, and the outputs of the regulator depend on its inputs. The Parameters of the premises and the consequences of the neurofuzzy rules of Sugeno are given by re-writing the input-output data obtained by a Mamdani regulator; and the linguistic variables of the inputs, by 3 fuzzy sets. e, Δe and δ are described by 5, 3 and 12 fuzzy sets, respectively. The re-writing concept is obtained by the training while using the extended Kalman filter shows better performance than Mamdani, and got a reduced algorithm tasks. The defuzzification time is less for Sugeno regulator, which is designed only with three membership functions.

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Design of an Optimal Stabilizing Control Law for Discrete-Time Nonlinear Systems Based on Passivity Characteristic

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Abstract: This paper proposes a passivity-based static output feedback law which stabilizes a broad class of nonlinear discrete time systems. This control law is designed in such a way that an arbitrary cost function is also minimized. A general structure with adjustable parameters is considered for the static feedback law. In order to find these parameters for solving the corresponding optimization problem, the genetic optimization algorithm is utilized. An illustrative example shows the effectiveness of the proposed approach.

Keywords: *nonlinear discrete-time systems; passivity-based control; optimal control; genetic optimization algorithm.*

Mathematics Subject Classification (2010): 34D20, 37N35, 70K99, 74H55, 93C10, 93D15.

1 Introduction

The concept of passivity provides a useful tool for the analysis of nonlinear systems [1,2]. The main motivation for studying passivity in the system theory is its connection with stability [3–5]. A very important result in this field is the well known Kalman-Yakubovich-Popov (KYP) Lemma or Positive Real Lemma (PR) which has been specifically developed in the papers ([6, 7]). Also, Byrnes and Isidori [8] have shown that a number of stabilization theorems can be derived from the basic stability property of passive systems.

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The study of dissipative discrete-time systems, first was presented in [9] for linear systems. The motivation of studying dissipative discrete-time systems stems from the fact that passivity properties can simplify the system analysis. In [10, 11] nonlinear discrete-time systems which are affine in the control input, have been studied and some theorems on passivity-based control design were presented. Another approach to passivity in the nonlinear discrete-time case is presented by Monaco and Normand-Cyrot in [12, 13]. They obtained KYP conditions for single-input multiple-output general non-affine-in-input systems. Another problem treated is the action of making a system passive by means of a static state feedback, which is known as feedback passivity (passification). Sufficient conditions to convert MIMO non-passive systems to passive ones have been proposed in the series of papers [14]– [17].

The problem of stabilization is of high importance in the field of control. If a nonlinear discrete-time system is zero-state detectable and passive (with a positive definite and proper storage function) then the origin can be globally stabilized by $u = -\varphi(y)$, where φ is any locally Lipschitz function such that $\varphi(0) = 0$ and $y^T\varphi(y) > 0$ for all $y \neq 0$ ([11]). There is a great freedom in choosing the function $\varphi(y)$. The purpose of this paper is to use this freedom in such a manner that a given cost function be also minimized. Therefore, a general structure is considered for $\varphi(y)$ (which satisfies the above conditions) and adjustable coefficients in the proposed structure are found by a genetic optimization algorithm. This is worth noting that this idea can also be used for passive continuous-time systems.

The remainder of this paper is organized as follows. In the next section, the basic definitions and theorem about passive nonlinear discrete-time systems are presented. Section 3 presents the stabilizing controller design for passive systems in such a way that to minimize an appropriate cost function. A design example is given in Section 4. Finally, conclusions are presented in Section 5.

2 Preliminaries

In this section some basic definitions about the concept of passivity in the discrete-time systems are introduced.

A general class of discrete-time systems can be described by the nonlinear ordinary difference equation in the following discrete-time state space form:

$$\begin{aligned} x(k+1) &= F(x(k), u(k)), \\ y(k) &= H(x(k), u(k)), \end{aligned} \tag{1}$$

where $x \in D \subseteq R^n$ is the state vector, $u \in \mathrm{U} \subseteq R^m$ is the control input, and $y \in R^m$ is the system output. Suppose that F and H are both smooth mappings of the appropriate dimensions. Moreover, assume that $F(0, 0) = 0$ and $H(0, 0) = 0$. In this situation, a positive definite scalar function $V(x(k)) : D \to R$ (where $V(0) = 0$) is addressed as storage function and system (1) is said to be locally passive if there exists a storage function $V(x(k))$ such that:

$$V(F(x, u)) - V(x) \leq y^T u \qquad \forall (x, u) \in \mathrm{D} \times \mathrm{U}, \tag{2}$$

where $\mathrm{D} \times \mathrm{U}$ is a neighborhood of x=0, u=0.

Definition 2.1 [11] The zero dynamics of system (1) is defined by $F^* = F(x, u^*)$, where $(x, u^*) = \{(x, u) :\ \text{s.t. } H(x, u) = 0\}$. A system of the form (1) has a locally passive zero dynamics if there exists a positive definite function $V(x(k)) : D \to R$ such that:

$$V(F(x, u^*)) \leq V(x) \qquad \forall x \in D. \tag{3}$$

Definition 2.2 [11] A system (1) has local relative degree zero at x=0, if

$$\left. \frac{\partial H(x,u)}{\partial u} \right|_{\substack{x=0 \\ u=0}} \tag{4}$$

is nonsingular.

Now, assume that the nonlinear discrete-time system (1) is affine in the control input:

$$\begin{aligned} x(k+1) &= f(x(k)) + g(x(k))u(k), \\ y(k) &= h(x(k)) + J(x(k))u(k). \end{aligned} \tag{5}$$

The system (5) has local relative degree zero if $J(0)$ is nonsingular and it has uniform relative degree zero if $J(x)$ is nonsingular for all $x \in$ D. Additionally, the system (5) is locally zero-state observable if for all $x \in$ D,

$$y(k)|_{u(k)=0} = h(\phi(k, x, 0)) = 0 \quad \forall k \in Z^+ \Rightarrow x = 0, \tag{6}$$

where $\phi(k, x, 0) = f^k(x) = f(f^{k-1}(x)), \forall k > 1$, and $f^0(x) = x$. Also, $f^k(x)$ is the trajectory of the unforced dynamics $x(k+1) = f(x(k))$ from $x(0)$=x. If D $= R^n$, the system is globally zero-state observable. Moreover, system (5) is locally zero-state detectable if for all $x \in$ D, $y(k)|_{u(k)=0} = h(\phi(k, x, 0)) = 0$ and also for all $k \in Z^+$ implies $\lim_{k\to\infty} \phi(k, x, 0) = 0$. Also, if D $= R^n$, the system is globally zero-state detectable.

Another important asset of passive systems is their highly desirable stability properties which may simplify system analysis and controller design procedure. Therefore, transformation of a non-passive system into a passive one is desirable. The use of feedback to transform a non-passive system into a passive one is known as feedback passivation [12].

Definition 2.3 Let $\alpha(x)$ and $\beta(x)$ be smooth functions. Consider a static state feedback control law of the following form:

$$u(x) = \alpha(x) + \beta(x)w(k). \tag{7}$$

A feedback control law of the form (9) is regular if for all $x \in$ D, it follows that $\beta(x)$ is invertible. In order to analyze feedback passivation, the following theorem is taken from [16].

Theorem 2.1 *Consider a system of the form* (5). *Suppose* $h(0) = 0$ *and there exists a storage function V, which is positive definite* C^2 *function (i.e., the storage function and its first and second derivation is continuous). Also,* $V(0) = 0$ *and* $V(f(x) + g(x)u)$ *is quadratic in u. Then, system* (5) *is locally feedback equivalent to a passive system with V as storage function by a regular feedback control law of the form* (7) *if and only if the system has local relative degree zero at* $x = 0$ *and its zero dynamic is locally passive in a neighborhood of* $x = 0$.

Proof. See [11]. □

It is shown in [16] that control law of the form (7) with:

$$\alpha(x) = -J^{-1}(x)h(x) + J^{-1}(x)\bar{h}(x), \tag{8}$$

$$\beta(x) = J^{-1}(x)\bar{J}(x), \tag{9}$$

converts the non-passive nonlinear discrete-system (5) to a new passive dynamic given by:

$$\begin{aligned} x(k+1) &= f^*(x(k)) + g^*(x(k))\bar{h}(x(k)) + g^*(x(k))\bar{J}(x)w(k), \\ y(k) &= \bar{h}(x(k)) + \bar{J}(x)w(k), \end{aligned} \tag{10}$$

where

$$f^*(x) = f(x) - g(x)J^{-1}(x)h(x), \tag{11}$$

$$g^*(x) = g(x)J^{-1}(x), \tag{12}$$

$$\bar{J}(x) = \left(\frac{1}{2}g^{*T}\left.\frac{\partial^2 V}{\partial z^2}\right|_{z=f^*(x)} g^*(x)\right)^{-1}, \tag{13}$$

$$\bar{h}(x) = -\bar{J}(x)\left(\left.\frac{\partial V}{\partial z}\right|_{z=f^*(x)} g^*(x)\right)^{-1}. \tag{14}$$

3 Passivity-Based Optimal Control

Suppose that a system of the form (5) is passive with a positive definite storage function V. Let φ be any smooth mapping such that $\varphi(0) = 0$ and $y^T\varphi(y) > 0$ for all $y \neq 0$. The basic idea of passivity-based control is illustrated in the following theorem [11].

Theorem 3.1 *If system* (5) *is zero-state detectable and passive with storage function V which is proper on R^n, then the smooth output feedback control law* (15) *globally asymptotically stabilizes the equilibrium x=0,*

$$u = -\varphi(y), \quad u, y \in R^m. \tag{15}$$

Proof. See [11]. □

There is a freedom in selection of vector function $\varphi(y)$. In this paper, by use of this freedom we want to design $\varphi(y)$ such that in addition to globally asymptotically stabilizing of the nonlinear system (5) a given cost function be also minimized.

For this purpose, the following general structure for vector function $\varphi(y) = [\varphi_1(y_1), \varphi_2(y_2), ..., \varphi_m(y_m)]^T$ is assumed. It has the structure of a vector function belonging to the first-third quadrant sector, which $y^T\varphi(y) > 0$ for all $y \neq 0$ and also $\varphi(0) = 0$.

$$\varphi_i(y_i) = a_{i0}y_i + a_{i1}y_i^3 + ... + a_{il}y_i^{2l+1} \quad \text{for } i = 1, ..., m, \tag{16}$$

where $a_{i0}, a_{i1}, a_{i2}, ..., a_{il}$ belong to $R^+$and the suitable $l \in z^+ \geq 0$ may be set by the designer. The task is to find these unknown coefficients such that the proposed static output feedback minimizes an appropriate cost function in the form $I(k) = \sum_{\bar{k}=0}^{k} L(x(\bar{k}); u(\bar{k}))$.

In order to obtain the minimum value of the considered cost function, the optimization procedure based on the theory of genetic algorithms is used. The genetic algorithms constitute a class of search and optimization methods, which imitate the principles of natural evolution. A pseudo-code outline of genetic algorithms is shown below. The population of chromosomes at time t is represented by the time-dependent variable $P(t)$, with the initial population of random estimates $P(0)$ [18].

```
procedure GA
begin
 t=0;
 initialize P(t) = P(0);
 evaluate P(t);
 while not finished do
 begin
 t=t+1;
 select P(t) from P (t -1);
 reproduce pairs in P(t) by
 begin
 crossover;
 mutation;
 reinsertion;
 end
 evaluate P(t);
 end
end
```

Therefore, by utilization of the GA optimization process, the best coefficients of the proposed structure (Equation (16)) of output feedback control law may be found in such a way that to minimize the given cost function. In the optimization process, the corresponding cost function is considered as the fitness function of genetic algorithm.

4 Design Example

Consider the following nonlinear discrete-time system:

$$\begin{aligned} x_1(k+1) &= (x_1^2(k) + x_2^2(k) + u(k))\cos(x_2(k)), \\ x_2(k+1) &= (x_1^2(k) + x_2^2(k) + u(k))\sin(x_2(k)), \\ y(k) &= (x_1^2(k) + x_2^2(k)) + \frac{1}{x_1^2(k)+x_2^2(k)-0.25}u(k). \end{aligned} \tag{17}$$

The system (17) is not passive. Considering $V = \frac{1}{2}(x_1^2(k)+x_2^2(k))$ as storage function, the system can be rendered passive by means of a static state feedback control law, due to the fact that $J(x(k)) = \frac{1}{x_1^2(k)+x_2^2(k)-0.25}$ is invertible and the zero dynamics of system (17) is passive [16]. Therefore, the passifying control scheme, i.e. $u = \alpha(x) + \beta(x)w$, proposed by equations (8) and (9) is applied to (17). The passiefied system has the conditions of Theorem 3.1. Consequently, it can be locally asymptotically stabilized by output feedback $w = -\varphi(y)$, where w is the new input of passified system. The goal is finding a proper function $\varphi(y)$ in order to minimize the following cost function of the passified system:

$$I = \frac{1}{2}\sum_{k=0}^{\infty}(w^2(k) + x(k)^T x(k) + y(k)^2).$$

The proposed optimization process has been done for three cases.

Case 1: Only first term of (16) is considered ($\varphi_1(y) = a_0 y$).

Case 2: First two terms of (16) are considered ($\varphi_2(y) = a_0y + a_1y^3$).
Case 3: First three terms of (16) are considered ($\varphi_3(y) = a_0y + a_1y^3 + a_2y^5$).
The nonlinear static functions, resulting from GA optimization procedure are:

$$\begin{aligned} \varphi_1(y) &= 0.02755y, \\ \varphi_2(y) &= 0.0245y + 0.0451y^3, \\ \varphi_3(y) &= 0.021413y + 0.0296y^3 + 0.0258y^5. \end{aligned} \tag{18}$$

The passified dynamic is simulated for the initial conditions x_0=[-1, +1] and the three different control inputs: $w = -\varphi_1(y)$, $w = -\varphi_2(y)$ and $w = -\varphi_3(y)$. Figures 1, 2 and 3, present the responses of output, first and second states of passified dynamic, respectively. The simulation results show that by regulating the adjustable coefficients in (16) a suitable performance may be achieved. Additionally, comparison of results is given in Table 1. As seen, considering more terms of (16) may lead to a better performance.

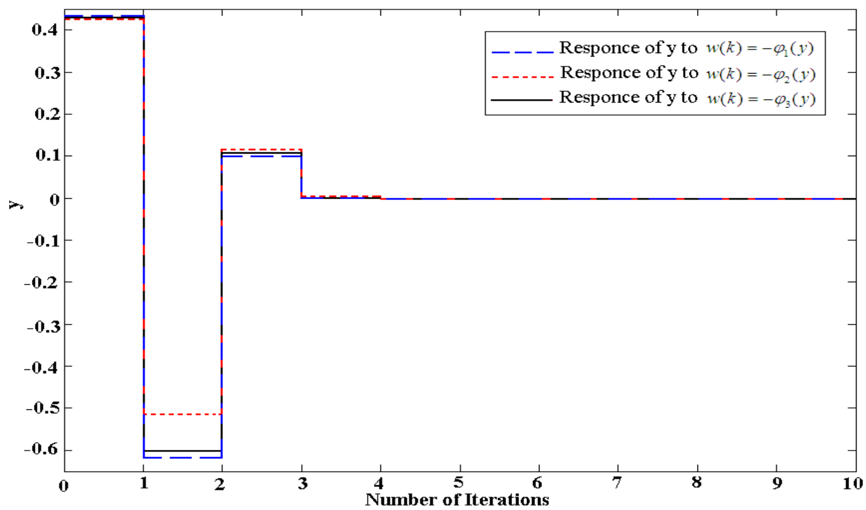

Figure 1: Time response of output $y(k)$.

	$w = -\varphi_1(y)$	$w = -\varphi_2(y)$	$w = -\varphi_3(y)$
$max\,\lvert y\rvert$	0.6178	0.5155	0.6028
I	2.619	2.6097	2.6027

Table 1: The cost functions (I) of control inputs, $w = -\varphi_1(y)$, $w = -\varphi_2(y)$ and $w = -\varphi_3(y)$.

5 Conclusion

In this paper, some properties of nonlinear discrete-time passive systems were studied. Based on the approach of passivity-based control, the output feedback u=-$\varphi(y)$, (where

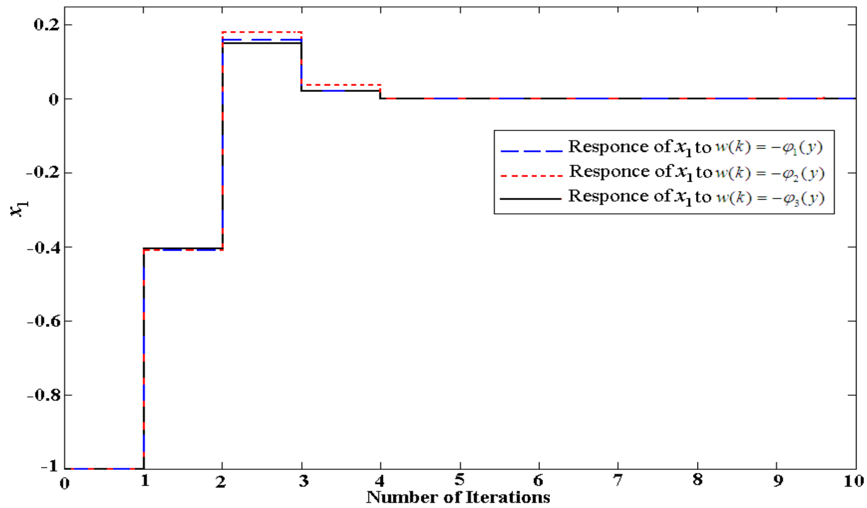

Figure 2: Time response of the first state $x_1(k)$.

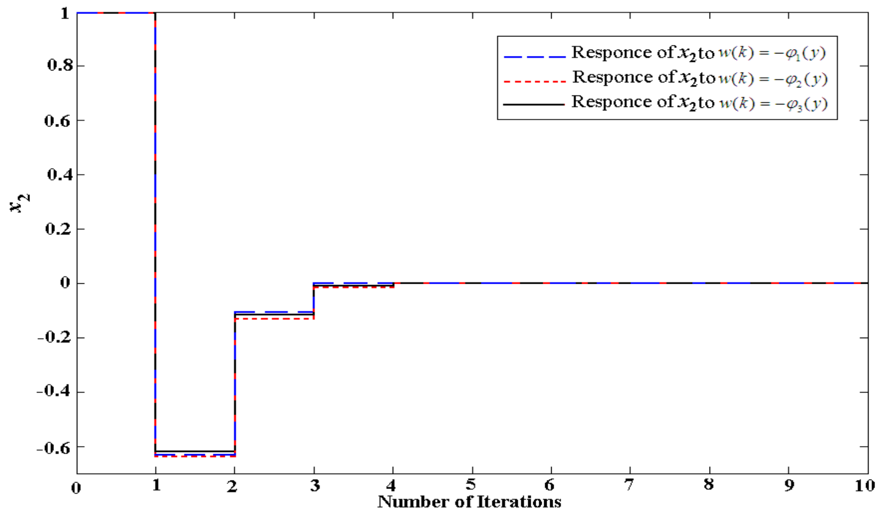

Figure 3: Time response of the second state $x_2(k)$.

$\varphi(y)$ is a smooth function belonging to a first-third quadrant sector) stabilizes the passive system. Having a freedom in choosing $\varphi(y)$, in addition to stabilization, one may consider optimization of the performance of the closed-loop system with respect to an appropriate index. Therefore, an extension (according to a general first-third quadrant sector function) with unknown coefficients was considered for $\varphi(y)$ and these coefficients were found based on the genetic optimization algorithm to minimize an appropriate cost function. Effectiveness of the proposed procedure was illustrated by an example.

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Formal Trigonometric Series, Almost Periodicity and Oscillatory Functions

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Abstract: This paper is the second, in a cycle dedicated to the new approach in constructing new oscillatory functions spaces, taking as primary object the formal trigonometric series and their generalizations, whose terms are of the form $\exp if(t)$, with $f(t)$ functions that belong to various classes. The linear case being considered in the first part of the paper leads to the classical cases of periodicity and almost periodicity, while the generalized case is aimed to obtain more general spaces of oscillatory functions, including those already known, due to V.F. Osipov and Ch. Zhang.

Keywords: *almost periodicity; formal trigonometric series; oscillatory functions spaces.*

Mathematics Subject Classification (2010): 42A75.

1 Introduction

The periodic and, more general, the oscillatory functions/motions appeared in Science and Engineering and other fields of knowledge, have conducted to the development of classical Fourier Analysis of periodic functions and their associated series. While the first traces of this branch of classical analysis can be found in the Mathematics of the XVIII-th century (Euler, for instance), it is the XIX-th century that contains significant results, which stimulated substantially the birth of new theories, contributing vigorously to the new concepts of Modern Analysis (Set Theory, Real variables including Measure and Integral). The Fourier Analysis, as developed until the third decade of the XX-th century, has known a strong impulse due to the emerging of the concept of Almost Periodicity, due to H. Bohr (1923-25), and successfully continued to the present day.

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It is also true that the topics of classical Fourier Analysis have also kept the attention of many leading mathematicians, after the birth of almost periodic functions.

The well known treatises of N.K. Bary (Pergamon, 1964) and A. Zygmund (Cambridge Univ. Press, 2002) contain a wealth of results and information about the periodic functions and their Fourier series, specially obtained before the introduction of the methods of Functional Analysis. More recent publications, due to J.P. Kahane [20], R.E. Edwards [16], G. Folland [19], have brought new ideas and results from this classical, but prolific field.

The concept of *almost periodicity* had several leading contributors to its beginning period. In his famous treatise *Nouvelles Méthodes de la Mécanique Céleste* (1893), H. Poincaré considered the problem of developing a function in a series of sine functions, namely

$$f(t) = \sum_{k=1}^{\infty} f_k \sin \lambda_k t, \quad t \in R, \tag{1}$$

where λ_k are arbitrary real numbers, not necessarily like $\lambda_k = k\omega$, $k \in N$, $\omega > 0$. Poincaré has succeeded to obtain the coefficients f_k, $k \geq 1$, simultaneously introducing the mean value of a function on the whole real line.

Using the complex notations, which became common with the new concept of almost periodicity, formula (1) can be rewritten as

$$f(t) = \sum_{k=1}^{\infty} f_k \exp(i\lambda_k t), \quad t \in R, \tag{2}$$

with $f_k \in \mathcal{C}$ and $\lambda_k \in R$, $k \geq 1$. The coefficients f_k are determined by the formulas

$$f_k = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t) \exp(-i\lambda_k t) dt, \tag{3}$$

in which the Poincaré's mean value (i.e., on an infinite interval) appears:

$$M(g) = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} g(t) dt, \tag{4}$$

$g \in L_{\text{loc}}(R_+, R)$, under the assumption that the limit exists as a finite number.

It is known that most concepts related to almost periodicity, including the Fourier exponents and coefficients (see (3) above) are based on the *mean value* defined in (4).

Other early contributors, preceding the period initiated by H. Bohr, include P. Bohl (1893) and E. Esclangon (1919) who dealt with what was later called *quasiperiodic functions*, a special case of almost periodicity. They have investigated oscillatory functions with a *finite* number of frequencies, the periodic case being concerned with only one basic frequency $(2\pi/\omega)$, ω-period. Some methods encountered to P. Bohl, but particularly to E. Esclangon, have been adapted to the general case of almost periodicity by H. Bohr.

H. Bohr (1887–1951) was the first to create a theory of almost periodicity, in a series of papers (1922–1925) which contained most of the fundamental results of the new theory (Generalized Fourier Analysis). The new theory is marking the beginning of a *second stage* in the study of oscillatory functions, aiming at global behavior of its elements. The theory of almost periodic functions has attracted, in short time, the interest of many mathematicians, including V.V. Stepanov (1925), H. Weyl (1926), A.S.

Besicovitch (1926-1932), S. Bochner (1925-), J. Favard (1926-), J. von Neumann (1934-), B.M. Lewitan (1939-), N.N. Bogoliubov (1930-).

The definition of H. Bohr, for *almost periodic* functions, is showing the fact that these new functions are direct generalizations of the periodic ones:

A continuous function $f : R \to R$ (or $\mathcal{C}$) is called *almost periodic* if the following property holds: to each $\varepsilon > 0$, there corresponds a number $\ell = \ell(\varepsilon) > 0$, such that each interval $(a, a+\ell) \subset R$ contains a number τ with $|f(t+\tau) - f(t)| < \varepsilon$, $t \in R$.

The number τ is called an ε-*almost period* of the functions f and one says that all numbers τ, with the above property, form a *relatively dense set* on R.

This terminology has been present in all the generalizations the almost periodic functions have known so far.

The following two properties of almost periodic functions, in the sense of Bohr, have been readily discovered by Bohr himself, Bochner and Bogoliubov.

A. Approximation property: for each $\varepsilon > 0$, there exists a complex trigonometric polynomial

$$T(t) = T_\varepsilon(t) = \sum_{j=1}^{n} a_j \exp(i\lambda_j t), \quad t \in R, \tag{5}$$

with $\lambda_j \in R,\ a_j \in \mathcal{C}$, such that

$$|f(t) - T_\varepsilon(t)| < \varepsilon, \quad t \in R. \tag{6}$$

Rephrasing the above property, one may say that any almost periodic function (Bohr) can be uniformly approximated on R by trigonometric polynomials of the form (5).

B. Bochner property: the set of translates of an almost periodic function (Bohr), say $\mathcal{F} = \{f(t+h);\ h \in R\}$, is relatively compact in the sense of uniform convergence on R.

Each of properties A and B can be taken as definition for the almost periodic functions in the sense of Bohr. Bogoliubov has given a direct proof of the equivalence between the definition of Bohr and the approximation property, making possible the constructive presentation of the theory.

In what follows, by $AP(R,R)$ or $AP(R,\mathcal{C})$, we will understand the almost periodic set of functions in the sense of Bohr. These sets are actually Banach function spaces, the norm being given by the formula $|f|_{AP} = \sup\{|f(t)|;\ t \in R\}$, which makes sense for each almost periodic function (Bohr), because each function in $AP(R,R)$ or $AP(R,\mathcal{C})$ is bounded on R and uniformly continuous.

The three equivalent properties for the space of almost periodic functions, i.e., the Bohr's definition and A, B, constitute the core of the classical theory and numerous applications to various types of functional equations. See the books by H. Bohr [6], A.S. Besicovitch [5], J. Favard [17], B.M. Levitan [21], C. Corduneanu [9,10], L. Amerio and G. Prouse [2], A.M. Fink [18], S. Zaidman [32], Ch. Zhang [33], W. Maak [23], B.M. Levitan and V.V. Zhikov [22], for most of the evolution of the theory of almost periodic functions, until recently. These references contain a large number of sources in the field, with varied applications in Mathematics and other areas.

Currently, we assist at the beginning of a *third stage* in the development of mathematical concepts and theories to advance the study of various types of vibratory motions, encountered in the description of phenomena examined in Science or Engineering.

We shall touch partially this aspect in the following pages of this paper. It has been realized, by both users and designers of the new tools for investigation of oscillatory phenomena, that periodicity (first stage) and almost periodicity (seconde stage) cannot describe the wide variety of oscillatory or wave-like phenomena that one encounters in science or in the real world.

2 A Remark and Its Consequences

A new approach to build up spaces/classes of oscillatory functions, applicable also to the classical ones (periodic or almost periodic) consists in starting with formal/generic trigonometric series of the form:

$$\sum_{k=1}^{\infty} a_k \exp(i\lambda_k t), \quad t \in R, \tag{7}$$

where $a_k \in \mathcal{C}$, $\lambda_k \in R$, $k \geq 1$, the assumption $\lambda_k \neq \lambda_j$ for $k \neq j$, $k, j \geq 1$, being accepted throughout the paper.

The *main idea* leading to the new approach, in this paper, partially illustrated in our previous paper [11], is to start with formal trigonometric series, of the form (7), as primary material, and identify conditions on the two sequences $\{a_k;\ k \geq 1\} \subset \mathcal{C}$ and $\{\lambda_k;\ k \geq 1\} \subset R$, such that (7) "characterizes" a certain type of oscillatory function, either in the classical category (periodic or almost periodic), or in the new classes of oscillatory functions (e.g., pseudo-almost periodic, to begin with in the third stage of development, or new types, as those investigated by Ch. Zhang [33, 34, 36]).

As we shall see, this new approach works for classes/spaces of classical type, but as well for introducing new spaces of oscillatory (or vibrating?) functions. The answer is not always simple, and to illustrate the situation we will start with the question:

Under what conditions does the series (7) *characterize the space* $AP(R, \mathcal{C})$ *of Bohr almost periodic function?*

Based on the theory of almost periodic functions, the answer has a simple formulation, which is:

Theorem 2.1 *The necessary and sufficient condition, for the series* (7)*, to characterize an almost periodic function of the space* $AP(R, \mathcal{C})$ *is the summability of this series, in the sense of Cesaró-Fejér-Bochner, with respect to the uniform convergence on* R.

Proof. The condition is necessary, because it is well known (see, for instance, Corduneanu [9], [10]) that for a function $f \in AP(R, \mathcal{C})$, whose Fourier series has the form (7), the sequence of trigonometric polynomials

$$\sigma_m(t) = \sum_{k=1}^{n} a_k r_{k,m} \exp(i\lambda_k t), \quad t \in R, \tag{8}$$

$n = n(m)$, with $r_{k,m}$ rationals depending on λ_k and m, but independent of $\{a_k;\ k \geq 1\}$, *converges uniformly* on R to $f(t)$.

The summability condition is also sufficient, because if (7) is summable with respect to the uniform convergence on R, the limit function will belong to $AP(R, \mathcal{C})$.

Let us point out that any linear method of summability, not necessarily the one described by (8), leads to the same conclusion. This ends the proof of Theorem 2.1, which characterizes the formal trigonometric series of the form (7), representing functions in $AP(R, \mathcal{C})$, the first space of almost periodic functions (Bohr).

Remark 2.1 Based on the uniqueness of the Fourier series corresponding to a function from $AP(R, \mathcal{C})$, in case of convergence on R of the series (7), there results that it is the Fourier series of its sum. This case takes place, obviously, when the convergence of (7) is uniform on R, and we can write

$$f(t) = \sum_{k=1}^{\infty} a_k \exp(i\lambda_k t), \quad t \in R. \tag{9}$$

Otherwise, we have to be content with the relationship

$$\lim_{m\to\infty} \sigma_m(t) = f(t), \quad t \in R, \tag{10}$$

uniformly, the $\{\sigma_m(t),\ m \geq 1\}$ being the summability sequence consisting of trigonometric polynomials (e.g., like in (8)). Of course, any trigonometric polynomial $\sum_{k=1}^{n} a_k \exp(i\lambda_k t)$, when regarded as a formal series, is summable, hence Bohr's almost periodic.

In order to establish Theorem 2.1, we needed to rely on Bohr's properties of almost periodic functions.

What if we start with the new definition for $AP(R, \mathcal{C})$, a fact made possible by Theorem 2.1?

It turns out that the most basic properties can be routinely derived from the new definition. We shall list a few of them, leaving the task of proof to the reader.

a) An almost periodic function in Bohr's sense is bounded on R.

b) An almost periodic function in Bohr's sense is uniformly continuous on R.

c) If $f \in AP(R, \mathcal{C})$ and $c \in \mathcal{C}$, then $cf \in AP(R, \mathcal{C})$, as well as $\bar{f}$.

d) If $f, g \in AP(R, \mathcal{C})$, then $f + g \in AP(R, \mathcal{C})$ also fg.

e) If $f \in AP(R, \mathcal{C})$ and $h \in R$, then $f(t + h) = f_h(t)$ and $f(ht) = f^h(t)$ both belong to $AP(R, \mathcal{C})$.

More basic properties of Bohr's almost periodic functions can be "rediscovered" if we introduce a topology/convergence in the set of all formal trigonometric series (7). We shall not proceed on this way, preferring instead on relying on every fact in the existing theory of almost periodicity, as soon as essential connections are established.

Let us give one more example to illustrate the fact that, starting from trigonometric series, one can proceed successfully to the construction of various spaces of almost periodic functions. On behalf of Theorem 2.1 (and even of the new definition of AP-space), the approximation is assured by the summability assumption. This means that, for any $f \in AP(R, \mathcal{C})$, one can construct a sequence of trigonometric polynomials, say $\{f_n; n \geq 1\} \subset AP(R, \mathcal{C})$, such that $\lim f_n(t) = f(t)$, uniformly on R, as $n \to \infty$.

Starting from A, the space $AP(R, \mathcal{C})$ has been constructed by Bogoliubov in 1930's. This direct approach is discussed in detail in the book by Corduneanu [9].

The new approach, starting from trigonometric series as background material, instead of trigonometric polynomials, is not meant to be a substitute for other existing approaches. It has been shown in Corduneanu [9] or Shubin [27] that various applications make sense in this approach, and properties can be emphasized that were unknown before, in case of the spaces we have denoted by $AP_r(R,\mathcal{C})$, $1 \le r \le 2$, obtained by the procedure of *completion* of the linear space of trigonometric polynomials. Briefly, the space $AP_r(R,\mathcal{C})$ is defined as consisting of all series (7), satisfying the convergence condition

$$\sum_{k=1}^{\infty} |a_k|^r < \infty, \quad r \in [1,2], \ r \text{ fixed.} \tag{11}$$

This space is a linear space over $\mathcal{C}$, the norm being given by

$$\left| \sum_{k=1}^{\infty} a_k \exp(i\lambda_k t) \right|_r = \left(\sum_{k=1}^{\infty} |a_k|^r \right)^{1/r}, \tag{12}$$

the right hand side of (12) being known as Minkowski's norm.

The case $r = 1$ leads to the space of almost periodic functions with absolutely convergent series of Fourier coefficients. We have called this space Poincaré's space of almost periodic functions, and it is well known that it can be organized as a Banach algebra (see, for instance, Corduneanu [10]). It is denoted by $AP_1(R,\mathcal{C})$.

The other extreme, $r = 2$, leads to the Besicovitch space $B^2 = AP_2$ of almost periodic functions, the largest in which the Parseval's formula holds true:

$$\sum_{k=1}^{\infty} |a_k|^2 = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt, \tag{13}$$

where $f(t)$ is the function associated to the series (7), in the manner we shall describe in subsequent lines. What appears in the right hand side of (13), according to (12) where $r = 2$, is actually the square of the *seminorm* of the function space $B^2(R,\mathcal{C})$

$$|f|_{B^2}^2 = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt, \tag{14}$$

valid for all series/functions satisfying (11), for $r = 2$. As one sees from (14), the Poincaré's mean value on R is deeply involved in dealing with generalizations of Bohr's almost periodic functions.

The scale of spaces, of almost periodic functions, extended from the Poincaré's space $AP_1(R,\mathcal{C})$, to the Besicovitch space $B^2(R,\mathcal{C}) = AP_2(R,\mathcal{C})$, has been introduced and investigated in some detail in the recent paper by Corduneanu [11].

Applications of these spaces of almost periodic functions have been recently given in the papers by Corduneanu [11], Mahdavi [24] and Corduneanu and Li [14], concerning some classes (linear and nonlinear) of functional differential equations of the form

$$\dot{x}(t) = (Ax)(t) + (Fx)(t), \quad t \in R, \tag{15}$$

where A is a linear operator acting on an $AP_r(R,\mathcal{C})$ space, while $F : AP_r(R,\mathcal{C}) \to AP_r(R,\mathcal{C})$ is, generally, nonlinear. It is useful to notice that the operator A could involve convolution type terms, the convolution product being defined by the formula

$$(K * x)(t) \sim \sum_{j=1}^{\infty} \widetilde{x}_j \exp(i\lambda_j t), \quad t \in R, \tag{16}$$

with x represented by the series

$$x(t) \sim \sum_{k=1}^{\infty} x_k \exp(i\lambda_k t), \quad t \in R, \tag{17}$$

and

$$\widetilde{x}_k = x_k \int_R K(s) \exp(-i\lambda_k s) ds, \quad k \geq 1. \tag{18}$$

The sign $\sim$ will be used to mark the relationship between trigonometric series and its associated function, as in (16) and (17). In order for (18) to make sense, it will be assumed that $K \in L^1(R, \mathcal{C})$.

It can be easily checked that

$$|K * x|_r \leq |K|_{L^1} \cdot |x|_r, \quad r \in [1, 2], \tag{19}$$

for each $x \in AP_r(R, \mathcal{C})$. The inequality (19) is a replica of a similar one, namely

$$|f * g|_{L^p} \leq |f|_{L^1} \cdot |g|_{L^p}, \quad p \geq 1,$$

which is often used in convolution problems. Actually, the convolution product, in this generalized form, has been used in the above referenced papers by Corduneanu, Mahdavi and Li.

3 The Besicovitch Space $B^2(R, \mathcal{C})$

It is known that the space B^2 has properties that have been used in several applications, and presents various features making it more accessible to connections with other topics. Such a situation is not encountered when dealing with the Besicovitch space $B = B^1(R, \mathcal{C})$, even though this space is known as the largest for which the Fourier series can be associated to its elements. We will consider the space $B(R, \mathcal{C})$ in a subsequent section of this paper.

The construction of the space $B^2(R, \mathcal{C})$, starting from our approach (point of view), is rather simple. We know from the classical theory that, for each $f \in B^2(R, \mathcal{C})$, the Parseval formula

$$\sum_{k=1}^{\infty} |a_k|^2 = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt, \tag{20}$$

where

$$f \sim \sum_{k=1}^{\infty} a_k \exp(i\lambda_k t), \tag{21}$$

represents the connection between the function f and its Fourier series. Also, we know that for each sequence $\{a_k;\ k \geq 1\} \in \ell^2(N, \mathcal{C})$ =the complex Hilbert space, there exists $f \in B^2(R, \mathcal{C})$ such that (21) holds true.

Our basic assumption for constructing $B^2(R, \mathcal{C})$, starting from the set of trigonometric series of the form (7), will be

$$\sum_{k=1}^{\infty} |a_k|^2 < \infty, \tag{22}$$

which is the same as $\{a_k;\ k \in N\} \in \ell^2(N, \mathcal{C})$.

Consider now a series like (21), and see what we can get if one searches its convergence in the norm derived from Poincaré's mean value on the real axis.

Why do we appeal to this type of convergence?

I think because it has proven to be a very important tool in Fourier Analysis (second stage), and hope to be also successful in the future. The procedure to be followed to define the space $B^2(R, \mathcal{C})$ and emphasize some of its properties has the origin in the theory of orthogonal functions. In this field of investigation, closely related to Fourier Analysis, there are numerous monographs and treatises. We send the reader to the classical references Alexits [1] and Sansone [26].

In order to apply this procedure to the case of almost periodic functions, the following elementary result is useful:

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \exp(i\lambda t) dt = \begin{cases} 1, & \text{for } \lambda = 0, \\ 0, & \text{for } \lambda \neq 0. \end{cases} \tag{23}$$

Equation (23) is an orthogonality relation, which clearly appears when one considers a sequence of complex exponentials $\{\exp(i\lambda_k t);\ k \geq 1\}$, with $\lambda_k \neq \lambda_j$ for $k \neq j$, and derive from (23) the relation

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \exp[i(\lambda_k - \lambda_j)t] dt = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \tag{24}$$

Let us return to the assumption (22), and notice that

$$\begin{aligned} \left| \sum_{k=n+1}^{n+p} a_k \exp(i\lambda_k t) \right|^2 &= \left\langle \sum_{k=n+1}^{n+p} a_k \exp(i\lambda_k t), \sum_{k=n+1}^{n+p} \bar{a}_k \exp(-i\lambda_k t) \right\rangle \\ &= \sum_{k=n+1}^{n+p} |a_k|^2 + \sum_{\substack{k,j=n+1 \\ k \neq j}}^{n+p} a_k \bar{a}_j \exp[(i(\lambda_k - \lambda_j)t]. \end{aligned}$$

If one takes (24) into account, and takes the Poincaré's mean value of both sides in the last equation, one obtains the relation

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \left| \sum_{k=n+1}^{n+p} a_k \exp(i\lambda_k t) \right|^2 dt = \sum_{k=n+1}^{n+p} |a_k|^2. \tag{25}$$

Now, taking into account our assumption (22), we see from (25) that the series converges on R, with respect to the seminorm $f \to |f|_{B^2}$, defined by

$$|f|_{B^2}^2 = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt, \tag{26}$$

the right hand side of (26) being finite. Indeed, in the way we have obtained (25), one also obtains

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \left| \sum_{k=1}^{n} a_k \exp(i\lambda_k t) \right|^2 dt = \sum_{k=1}^{n} |a_k|^2, \tag{27}$$

and letting $n \to \infty$, there results on behalf of (22) (the seminorm is continuous!) the formula (26), or

$$\sum_{k=1}^{\infty} |a_k|^2 = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt, \tag{$*$}$$

which is nothing else but Parseval's formula. See also formula (20).

A legitimate question is now whether the convergence, in the sense of the norm derived from Poincaré's mean value, defines a function belonging to $L^2_{\text{loc}}(R, \mathcal{C})$, such that (26) may have a meaning?

The answer to this question is positive and we shall dwell in getting it. If one denotes by $A > 0$ the sum of the series $\sum_{k=1}^{\infty} |a_k|^2$ in (22), then (27) allows us to write the inequality, valid when $n \geq 1$,

$$\int_{-\ell}^{\ell} \left| \sum_{k=1}^{n} a_k \exp(i\lambda_k t) \right|^2 dt < 2\ell(A + \varepsilon), \tag{28}$$

for $\ell \geq \ell(\varepsilon)$. Let us fix now ℓ as mentioned above, and read (28) as follows: the series in (21), under assumption (22), converges on the interval $[-\ell, \ell]$, in the space $L^2([-\ell, \ell], \mathcal{C})$. We assign now to $\ell \geq \ell(\varepsilon)$ a sequence of values $\{\ell_m;\ m \geq 1\}$, such that $\ell_m \nearrow \infty$ as $m \to \infty$. Since on each interval $[-\ell_m, \ell_m]$ the series in (21) is L^2-convergent, there results that we deal with convergence in $L^2_{\text{loc}}(R, \mathcal{C})$. The limit function, we have denoted by $f(t)$, satisfies the equation

$$f(t) = \sum_{k=1}^{\infty} a_k \exp(i\lambda_k t), \quad a.e. \ \ t \in R, \tag{29}$$

the a.e. convergence being the consequence of the fact $f(t) \in L^2_{\text{loc}}(R, \mathcal{C})$. Therefore, we have the right to substitute (29) to (21), and we can now associate to each series, which satisfies (22), a function $f(t) \in L^2_{\text{loc}}(R, \mathcal{C})$. This function is exactly the sum of the series (21), which generates it in the way shown above when proving the convergence in $L^2_{\text{loc}}(R, \mathcal{C})$.

At this point in the discussion, it is very important to look more in detail at the correspondence from series to functions, as established above. The following remark is necessary. Namely, since the right hand side in (26) remains unchanged, when the integrand $|f(t)|^2$ is changed into $|f(t) - f_0(t)|^2$, with $f_0(t)$ such that

$$|f_0|_{B^2} = \left\{ \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f_0(t)|^2 dt \right\}^{1/2} = 0, \tag{30}$$

it means that the correspondence from series to function is not one to one (as it happens in $AP(R, \mathcal{C})$). More precisely, to each series in (21), one associates a set of functions $f \in L^2_{\text{loc}}$, for which formula ($*$) is verified. This set of functions is nothing else but the translation of the null space of Poincaré's functional, i.e., the space $\mathcal{N}_0$ of those functions for which (30) is satisfied. Let us notice that one of these functions is $f_0(t) = \exp(-|t|)$, $t \in R$.

Let us denote by $\mathcal{B}$ the set of all trigonometric series like (21), such that (22) holds true for each series. We shall denote by $\widetilde{\mathcal{B}}$ the space of all functions $f \in L^2_{\text{loc}}(R, \mathcal{C})$, corresponding to series from $\mathcal{B}$, by means of the procedure described above, that lead to the Parseval's formula ($*$). See also the relation given by formula (29).

Before introducing the Besicovitch space of almost periodic functions, $B^2 = B^2(R, \mathcal{C})$, let us point out the fact that formula $(*)$ in this section is the vehicle that helps us to deal with either manner of constructing the space B^2. We shall prove, first, the following.

Lemma 3.1 *The set $\mathcal{B}$, organized as a linear seminormed space, is complete. Hence, it is isometric and isomorphic to a B-space (see Yosida* [31]*).*

Proof. Since the elements of $\mathcal{B}$ are series like (7), and the coefficients verify condition (22), it is to be expected that the Hilbert space $\ell^2(N, \mathcal{C})$ will play an important role in investigating properties of $\mathcal{B}$. Indeed, let us consider a Cauchy sequence $\{x_k;\ k \geq 1\} \subset \mathcal{B}$. This means that, for any $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$, with the property

$$|x_n - x_m|_{\mathcal{B}} < \varepsilon \ \text{ for } n, m \in N(\varepsilon). \tag{31}$$

Since each $x_k \in \mathcal{B}$ can be regarded as an element in the Hilbert space $\ell^2(N, \mathcal{C})$, i.e., its representation in $\mathcal{B}$ is

$$x_k \sim \sum_{j=1}^{\infty} a_k^j \exp(i\lambda_j t), \tag{32}$$

with $\{a_k^j;\ j \geq 1\} \subset \ell^2(N, \mathcal{C})$, (31) takes the form

$$\sum_{j=1}^{\infty} |a_n^j - a_m^j|^2 < \varepsilon^2, \quad \text{for } n, m \geq N(\varepsilon). \tag{33}$$

Starting from (33), by a routine procedure (see for detailed discussion, for instance, V. Trénoguine [30]) one obtains the existence of an element/series in $\mathcal{B}$, say x, such that $x \sim \sum_{j=1}^{\infty} a^j \exp(i\lambda_j t)$. The coefficients a^j, $j \geq 1$, are limits for subsequences of the sequences $\{a_k^j;\ k \geq 1\}$, $j \in N$.

Remark 3.1 According to our notation, it appears that the set of λ_k's is common to all series involved in the representation of the elements x_k, $k \geq 1$. This is not a restriction, because the union of all such exponents to all x_k's $k \geq 1$, is a countable set. Therefore, one may have to add some terms, in the representations, whose coefficients are zero. In such a way, we can use the same complex exponentials for each $x_k \in \mathcal{B}$, $k \geq 1$.

Remark 3.2 In case we have two series in $\mathcal{B}$, say $\sum_{j=1}^{\infty} a_j \exp(i\lambda_j t)$ and $\sum_{j=1}^{\infty} b_j \exp(i\lambda_j t)$, the equation $(*)$ allows us to write

$$\sum_{j=1}^{\infty} |a_j - b_j|^2 = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t) - g(t)|^2 dt, \tag{34}$$

from which we derive

$$a_j = b_j,\ j \geq 1, \ \text{ iff } \ f - g \in \mathcal{N}_0, \tag{35}$$

where $\mathcal{N}_0$ =the null space, has been defined above in this sections. In other words, two functions $f, g \in \widetilde{\mathcal{B}}$ generate the same series in $\mathcal{B}$, in case, and only in case $f - g \subset \mathcal{N}_0$.

Remark 3.3 From the relationship

$$\sum_{j=1}^{\infty} |a_n^j - a_m^j|^2 = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |x_n(t) - x_m(t)|^2 dt,$$

which results from Parseval equation, written for the difference $x_n(t) - x_m(t)$, in accordance with the representation (32), one derives the conclusion that the linear space $\widetilde{\mathcal{B}} \subset L^2_{\text{loc}}(R, \mathcal{C})$ is also complete in the topology induced by the seminorm $|\cdot|_{B^2}$, as defined by (26).

To summarize the above discussion, we shall state the following.

Theorem 3.1 *The Banach space $\mathcal{B}$ of series like* (7)*, under assumption* (22)*, with the norm*

$$\left| \sum_{k=1}^{\infty} a_k \exp(i\lambda_k t) \right|_{B^2} = \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2}, \tag{36}$$

is completely determined, as described above. First of its realizations is the model also described above, starting with the set $\mathcal{B}$, and endowing it until the Banach space $B^2 = B^2(R, \mathcal{C})$ is constructed. A second realization (*isomorphism plus isometry*)*, also described above, consists in the model starting with the set $\widetilde{\mathcal{B}} \subset L^2_{\text{loc}}(R, \mathcal{C})$, which is isomorphic and isometric to $\mathcal{B}$, modulo $\mathcal{N}_0$ – the null space in $\widetilde{\mathcal{B}}$. The integral norm on the factor space $\widetilde{\mathcal{B}}/\mathcal{N}_0 = B^2$ is given by the formula* (26)*.*

The **proof**, to be complete, also requires to prove that $\mathcal{N}_0$ is a closed subspace of $\widetilde{\mathcal{B}}$, in the topology of the seminorm (26) on $\widetilde{\mathcal{B}}$.

Let $f_n \to f$ in $\widetilde{\mathcal{B}}$, as $n \to \infty$, and assume $\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f_n(t)|^2 dt = 0$, $n \geq 1$. Let us show that $\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt = 0$. This follows from the Minkowski's inequality

$$\left\{ (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt \right\}^{1/2} \leq \left\{ (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t) - f_n(t)|^2 dt \right\}^{1/2} + \left\{ (2\ell)^{-1} \int_{-\ell}^{\ell} |f_n(t)|^2 dt \right\}^{1/2},$$

which implies, as $\ell \to \infty$, $|f(t)|_{B^2} \leq |f(t) - f_n(t)|_{B^2}$. Now, letting $n \to \infty$, one obtains $|f(t)|_{B^2} = 0$, which means $f \in \mathcal{N}_0$.

For definitions and details concerning the factor space, see Yosida [31] and Swartz [29].

Finally, let us notice that Remark 3.3 to Lemma 3.1 proves the completeness of $\widetilde{\mathcal{B}}$, with respect to the seminorm (26), which is needed in obtaining the completeness, and hence the Banach type space for $B^2(R, \mathcal{C})$ – as a quotient or factor space.

With these considerations, related to the construction of the Besicovitch space $B^2(R, \mathcal{C})$, we end the proof of Theorem 3.1.

We have dealt with $B^2(R, \mathcal{C})$ in Corduneanu [10], when the notation $AP_2(R, \mathcal{C})$ has been used to stress its connection with the spaces $AP_r(R, \mathcal{C})$, $r \in [1, 2)$. But these spaces, all of them subsets of $B^2 = AP_2$, have different topologies, stronger than the topology of B^2. Moreover, the approximation property has been taken as definition, instead of starting with trigonometric series. Properties similar to A and B have been emphasized for the $AP_r(R, \mathcal{C})$-spaces.

In concluding this section, we shall recall the fact that in the book by Corduneanu [10], the construction of the space $\mathcal{B}^2(R,\mathcal{C}) = AP_2(R,\mathcal{C})$ is based on the approximation property applied in the Macinkiewicz' space $\mathcal{M}_2(R,\mathcal{C})$, taking the closure of the set of trigonometric polynomials.

4 The Besicovitch Space $B(R,\mathcal{C})$

In Besicovitch [4], one finds the construction of the spaces B^p for $p > 1$, the case $p = 1$ conducing to a more difficult treatment, with definitions for the upper and lower mean values. The difference with respect to the case $p > 1$ comes from the fact that Hölder inequality, in case $p = 1$ leads to the conjugate index $q = \infty$, while for L^∞ we don't have an integral norm. But, this tool is systematically used in building the theory of B^p-spaces when $p > 1$. It is known that the seminorm which plays the main role in constructing the spaces $B^p(R,\mathcal{C})$, $p < \infty$, is given by

$$|f|^p_{B^p} = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^p dt. \tag{37}$$

In the preceding section we have obtained and dealt with (37), in the case $p = 2$. But our approach was based on taking the trigonometric series as departing object and the condition (22) imposed on these series that characterize the B^2-functions.

The seminorm (37), for $p = 1$, which is the Poincaré's mean value of $|f(t)|$, will be also of great use in our approach to construct the space B.

Instead of starting from a condition similar to (22), which apparently does not exist, even though for $AP_r(R,\mathcal{C})$ spaces, $r \in (1,2]$, it has been helpful, we shall start from the space $AP(R,\mathcal{C})$ of Bohr, which has been characterized in our approach by Theorem 2.1.

In the space $AP(R,\mathcal{C})$, due to the summability of its associated series, the approximation property is valid. This means that the set of trigonometric polynomials, a fraction of the set $\mathcal{ST}$ of trigonometric series like (7) is everywhere dense in $AP(R,\mathcal{C})$, with respect to the uniform convergence on R. As it is well known (see, for instance, Lewitan [21] or Corduneanu [10]), once the approximation property is established, one can easily derive the existence of the mean value for each $f \in AP(R,\mathcal{C})$, starting from the obvious fact that the mean value exists for each trigonometric polynomial (equal to the term without complex exponential, if any, otherwise $= 0$).

The main properties of the mean value $M\{f\}$, $f \in AP(R,\mathcal{C})$, are

(a) $M\{\bar{f}\} = \overline{M\{f\}}$;

(b) $M\{\alpha f + \beta g\} = \alpha M\{f\} + \beta M\{g\}$, $\alpha, \beta \in \mathcal{C}$, $f, g \in AP(R,\mathcal{C})$;

(c) $f(t) \geq 0$ on R implies $M\{f\} \geq 0$, $f \in A(P,R)$ and $M\{f\} = 0$ implies $f \equiv 0$;

(d) $|M\{f\}| \leq M\{|f|\}$, $f \in AP(R,\mathcal{C})$.

Let us notice that the map $f \to M\{|f|\}$, from $AP(R,R)$ into R is a norm. Indeed, for $f, g \in AP(R,\mathcal{C})$, one has $|f+g| \leq |f|+|g|$, which leads to $M\{|f+g|\} \leq M\{|f|\}+M\{|g|\}$. Property (c) is a consequence of the uniqueness.

Lemma 4.1 *In the topology induced by the mean value norm, the space $AP(R,\mathcal{C})$ is always incomplete (denoted by $AP_M(R,\mathcal{C})$).*

Proof. The proof will be conducted on the principle of *reductio ad absurdum.* Hence, let us assume that the set of elements in $AP(R,\mathcal{C})$, with the norm $M\{|f|\}$, is complete. Therefore, it is a Banach space. Then the identity map, which is one-to-one, is a linear operator acting from the Banach space $AP(R,\mathcal{C})$, in its associate $AP_M(R,\mathcal{C})$, endowed with the mean-value norm $M\{|f|\}$. According to the Banach theorem on the continuity of the inverse operator, we derive that the identity map (which coincides with its inverse) is continuous from $AP_M(R,\mathcal{C})$ onto $AP(R,\mathcal{C})$. This fact implies the existence of a constant $C > 0$, such that

$$\sup\{|f(t)|;\ t \in R\} \leq CM\{|f|\}, \quad f \in AP(R,\mathcal{C}). \tag{38}$$

By an example, we shall prove now that (38) is impossible, and therefore our assumption that $AP_M(R,\mathcal{C})$ is complete is *false.*

Let us consider the sequence of periodic functions, defined by $f_n(t+1) = f_n(t)$, $t \in R$, $n \geq 2$, and for $t \in [0,1)$ by

$$f_n(t) = \begin{cases} 1-nt, & 0 \leq t < n^{-1}, \\ 0, & n^{-1} \leq t < 1-n^{-1}, \\ 1-n+nt, & 1-n^{-1} \leq t \leq 1. \end{cases} \tag{39}$$

Since periodic functions are almost periodic (Bohr), i.e. in $AP(R,R) \subset AP(R,\mathcal{C})$, we obtain $M\{f_n\} = n^{-1}$, $n \geq 2$, while $\sup f_n = 1$, $n \geq 2$. Hence, one should have $1 \leq C/n$, $n \geq 2$, which is obviously impossible. This ends the proof of Lemma 4.1.

Further, on our way to construct the space $B = B(R,\mathcal{C})$, we shall complete the space $AP_M(R,\mathcal{C})$, following the usual procedure (see, for instance, Trénoguine [30], or Yosida [31]).

Let us denote by $\mathcal{B}$ the linear complete space which is the (unique, up to isomorphism) completion of the space $AP_M(R,\mathcal{C})$. One has $AP_M(R,\mathcal{C}) \subset \mathcal{B}(R,\mathcal{C})$, more precisely $AP_M(R,\mathcal{C})$ can be identified with a set which is everywhere dense in $\mathcal{B}(R,\mathcal{C})$.

Applying the Hahn-Banach theorem on extension of functionals, from subspaces to a larger space, we can infer that the seminorm $M\{|f|\}$, which is defined on $AP_M(R,\mathcal{C})$, admits an extension to $\mathcal{B}(R,\mathcal{C})$, with preservation of its basic properties. If one denotes by $\widetilde{M}\{|f|\}$ the extension of M from $AP_M(R,\mathcal{C})$ to $\mathcal{B}(R,\mathcal{C})$, then $\widetilde{M}\{|f|\} = M\{|f|\}$ for each $f \in AP_M(R,\mathcal{C})$ and $\widetilde{M}$ satisfies on $\mathcal{B}$ the properties (a), (b), (c), (d), excepting the part of (c) which makes $AP_M(R,\mathcal{C})$ a normed space (not a seminormed one!).

A natural question arises at this point in our discussion. Namely, how do we know that the new elements in the completed space are functions locally integrable on$R$, so that $\widetilde{M}\{|f|\}$ makes sense.

The answer to this question results from the following considerations (also encountered when constructing $B^2(R,\mathcal{C})$, in the preceding section). If one considers a Cauchy sequence in $AP_M(R,\mathcal{C})$, say $\{f_k;\ k \geq 1\} \subset AP_M$, from $\lim_{\ell\to\infty}(2\ell)^{-1}\int_{-\ell}^{\ell}|f_n(t)-f_m(t)|dt < \varepsilon$, for $n,m \geq N(\varepsilon)$, one derives the inequality

$$\int_{-\ell}^{\ell}|f_n(t)-f_m(t)|dt < (2\ell+1)\varepsilon, \tag{40}$$

for $n,m \geq N(\varepsilon)$ and $\ell \geq L(\varepsilon)$. As proceeded in the preceding section, one obtains that $F(t) = \lim f_m(t)$, as $m \to \infty$, in $L^1_{\text{loc}}(R,\mathcal{C})$. Hence, we are assured that in order to complete the normed space $AP_M(R,\mathcal{C})$, it is sufficient to add functions which are in $L^1_{\text{loc}}(R,\mathcal{C})$. Of course, this situation takes place when the Cauchy sequence $\{f_k;\ k \geq 1\}$ does not have its limit in $AP_M(R,\mathcal{C})$.

So far, we have constructed a complete seminormed space, not a Banach space yet, denoted by $\mathcal{B}(R,\mathcal{C})$, the seminorm being the mean-value functional $f \to \widetilde{M}\{|f|\}$.

The last step to achieve the construction of the Besicovitch space $B(R,\mathcal{C})$, as a Banach space, is to take the factor space $\mathcal{B}/\mathcal{N}_0$, where $\mathcal{N}_0$ stands for the null space of the functional

$$\widetilde{M}\{|f|\} = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|dt. \tag{41}$$

For the construction of the factor space $\mathcal{B}/\mathcal{N}_0$, in order to obtain by means of this procedure a normed complete space (Banach), we need to show that $\mathcal{N}_0$ is a closed subspace of $\mathcal{B}$. Indeed, assume that $\{f_k;\ k \geq 1\} \subset \mathcal{B}$ is such that $\widetilde{M}(|f_k|) = 0$, $k \geq 1$, and

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f_k(t) - f(t)|dt = 0. \tag{42}$$

We need to prove that $f \in \mathcal{N}_0$, i.e., $\widetilde{M}(|f|) = 0$. Taking into account the relationship $|f(t)| \leq |f(t) - f_k(t)| + |f_k(t)|$, we obtain

$$\widetilde{M}(|f|) \leq \widetilde{M}(|f - f_k|) + \widetilde{M}(|f_k|), \quad k \geq 1, \tag{43}$$

and since $\widetilde{M}(|f_k|) = 0,\ k \geq 1$, while $\widetilde{M}(|f - f_k|) \to 0$ as $k \to \infty$, there results $\widetilde{M}(|f|) = 0$. This means $f \in \mathcal{N}_0$, and this is what we wanted to prove. Summarizing the discussion about the construction of the space $B = B(R,\mathcal{C})$, carried out above, we can formulate the following

Theorem 4.1 *The Besicovitch space $B = B(R,\mathcal{C})$ is constructed by the following procedure:*

1) *One starts with the Bohr space of almost periodic functions $AP(R,\mathcal{C})$ (see Theorem 2.1 above), which generates the incomplete normed space $AP_M(R,\mathcal{C})$, according to Lemma 4.1.*

2) *The (unique) completion of $AP_M(R,\mathcal{C})$, denoted by $\mathcal{B} = \mathcal{B}(R,\mathcal{C})$, is a seminormed complete space, with the seminorm $f \to \widetilde{M}(|f|)$ =the extended mean value/norm in $AP_M(R,\mathcal{C})$, defined by (41).*

3) *The Banach space $B = B(R,\mathcal{C})$ is the factor space $\mathcal{B}/\mathcal{N}_0$, with $\mathcal{N}_0$ the null space of the seminorm $\widetilde{M}\{|f|\}$, $f \in B$.*

The **proof** of Theorem 4.1 has been completed above, in this section, while the construction procedure is motivated by the known results on *completion* of seminormed spaces, as well as on the construction of the *factor space.* For details in this regard, see Yosida [31] and Swartz [29].

In concluding this section, we shall briefly discuss some properties of the space B, including its relationships with other spaces of almost periodic functions.

From the construction of the space $B(R,\mathcal{C})$ described above, there results several properties that we shall consider below.

First, let us notice the fact that the approximation property is valid, in the norm of the space $B(R,\mathcal{C})$. This means that for $f \in B$ and each $\varepsilon > 0$, one can determine a trigonometric polynomial of the form (5), such that

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t) - T_\varepsilon(t)|dt < \varepsilon, \tag{44}$$

is satisfied.

Second, the mean value of any function $g \in B(R,\mathcal{C})$ exists, being given by (4).

The proof of this statement can be found in Besicovitch [4] or Corduneanu [11].

Third, the mean value $f \to \widetilde{M}\{f\}$ satisfies conditions (a), (b), (d) mentioned above in this section, while in (c) only the first statement remains true.

Indeed, $\widetilde{M}(|f|) = 0$ does not imply $f = 0$, but only $f \in \mathcal{N}_0$. One has to take into account that $|\widetilde{M}(t)| \leq \widetilde{M}(|f|)$, which is an obvious property. The property also shows that $f \to \widetilde{M}(f)$ is a continuous functional on B (or $\mathcal{B}$).

Fourth, once established the existence of the mean value $\widetilde{M}(f)$, for each $f \in B(R,\mathcal{C})$, one can find the Fourier series associated to $f \in B(R,\mathcal{C})$, which represents the trigonometric series of the form (7), characterizing not only f (as an individual function), but the equivalence class which contains f, i.e., any other g for which

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t) - g(t)|dt = 0.$$

Fifth, besides the spaces $AP(R,\mathcal{C})$ and $B^2(R,\mathcal{C})$, $B(R,\mathcal{C})$ is also containing the Stepanov's space of almost periodic functions, $S = S(R,\mathcal{C})$, which is defined as the set of all $f \in L^1_{\text{loc}}(R,\mathcal{C})$, such that

$$\sup\left\{\int_t^{t+1} |f(s)|ds;\ t \in R\right\} = |f|_S < \infty. \tag{45}$$

Since for large $\ell > 0$ we can write for $f \in S$

$$\ell^{-1}\int_0^{\ell} |f(s)|ds \leq \ell^{-1}\left(\int_0^1 |f(s)|ds + \int_1^2 |f(s)|ds + \dots + \int_{[\ell]}^{[\ell]+1} |f(s)|ds\right) \leq \ell^{-1}([\ell]+1)|f|_S,$$

one obtains, as $\ell \to \infty$, the inequality

$$|f|_B \leq |f|_S, \quad f \in S(R,\mathcal{C}), \tag{46}$$

which tells us that $S \subset B$.

We took into account that one has

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(s)|ds = \lim_{\ell\to\infty} \ell^{-1} \int_0^{\ell} |f(s)|ds = \lim_{\ell\to\infty} \ell^{-1} \int_{-\ell}^{0} |f(s)|ds,$$

which can be found in most books on almost periodic functions (for instance, Corduneanu [11]).

As far as the inclusion $B^2 \subset B$ is concerned, it follows from the inequality

$$(2\ell)^{-1}\int_{-\ell}^{\ell} |f(s)|ds \leq \left[(2\ell)^{-1}\int_{-\ell}^{\ell} |f(s)|^2 ds\right]^{1/2},$$

valid for $\ell > 0$ and each $f \in B^2(R,\mathcal{C}) \subset L^2_{\text{loc}}(R,\mathcal{C})$, on behalf of Cauchy's integral inequality (special case of Hölder's inequality).

Sixth, because the approximation property by trigonometric polynomials is assured for functions in $B(R,\mathcal{C})$, there results that the property B, mentioned in Introduction, is valid. As it is known, the Bochner's property (i.e., relative compactness) of the family of translates if f, $\mathcal{F} = \{f(t+h);\ h \in R\}$, implies Bohr's property. Of course, all these properties are meant in the sense of the norm of the space $B(R,\mathcal{C})$. More precisely, for $f \in B(R,\mathcal{C})$, to any $\varepsilon > 0$ there corresponds $\ell = \ell(\varepsilon)$, such that each interval $(a, a+\ell) \ni \tau$, such that $|f(t+\tau) - f(t)|_B < \varepsilon$, $t \in R$.

Seventh, the space $B(R,\mathcal{C})$ has been already involved in work pertaining to the third stage of the development of the theory of oscillatory functions. See the book by Ch. Zhang [33], which contains the theory of pseudo-almost periodic functions. When defining the space $PAP(R,\mathcal{C})$ of these functions, the B-norm is involved, together with that of BC-space (the supremum norm, on R). One has the inclusion $PAP(R,\mathcal{C}) \subset BC(R,\mathcal{C})$, but the pseudo-almost periodicity appears as perturbation of the classical case of Bohr. An example of the use of space $B(R,\mathcal{C})$ in proving existence of almost periodic solutions to certain functional equations is given in Corduneanu [9]. The solutions are in $B^2(R,\mathcal{C})$.

Eighth, the interest for oscillatory functions/solutions comes from their significance in the physical problems, and their frequent use. In the paper of Staffans [28], an example of a function belonging to the Weyl's space (see Besicovitch [4]) is provided, which does not present the oscillatory character. It is understood that the space $B(R,\mathcal{C})$ may contain functions whose behaviour may not be classified as oscillatory.

We shall make a final remark about the manner of introducing the space $B(R,\mathcal{C})$. Namely, if we start again from the set of trigonometric series, of the form (7), the Cauchy's type convergence condition

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \left| \sum_{k=n+1}^{n+p} a_k \exp(i\lambda_k t) \right| dt < \varepsilon, \tag{47}$$

for $n \geq N(\varepsilon)$, $p \geq 1$, is, very likely, leading to the space $B(R,\mathcal{C})$ after the operations used already (completion, factor space). We have used this approach in constructing the space $B^2(R,\mathcal{C})$. In that case, we have been essentially helped by condition (22) imposed on the coefficients of the candidate series. It is obvious that (47) is the condition guaranteeing the convergence of the series (7) in the space $\mathcal{B}$ or B (after factorization). The approach we have used in this section relies substantially on the facts known in the classical theory.

5 Some Preliminaries for Oscillatory Functions Spaces

Both classes of oscillatory functions, amply investigated during the last two centuries, are representable by means of series of the form (7). It does not mean that the series are convergent in the usual sense, but the procedure that can be associated to them, in various ways, allow the construction of corresponding functions (e.g., by summability methods or by convergence in certain nonclassical norms, usually inducing a weaker type of convergence than the sup norm). They are useful, because they permit the construction of the function, in a manner that leads to results that can be used in applications.

We have in mind the Fourier Analysis in the classical framework, but also its extension to various classes of almost periodic functions, starting with the functions in $AP(R,\mathcal{C})$, or $AP(R,R)$.

Let us point out that the problem of convergence of Fourier series, which constitute a special form of series (7), has been in the attention of famous mathematicians for a long time. An example constructed by Kolmogorov (see the treatises by Bary and Zygmund, quoted in Introduction) shows that there exists Fourier series, in the classical sense, nowhere convergent on the interval $[-\pi, \pi]$. It is also worth mentioning the fact that the attention paid to the convergence of series of the form (7) is directed to their convergence on the finite interval $[-\pi, \pi]$, even though each term of the series is defined on the whole R. This feature is not, generally, agreeing with the needs of applications, when large interval of time can be involved, such as it happens in Celestial Mechanics or in other types (could be man made) of evolutionary systems.

Some of the latest example of oscillatory systems/functions led to the investigation of series of a much more general form than (7), namely

$$\sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)] \tag{48}$$

with $\{a_k;\ k \geq 1\} \subset \mathcal{C}$, and $\lambda_k(t),\ k \geq 1$, some real functions defined on R, and such that certain orthogonality conditions are verified.

We shall use again the Poincaré's mean value on R, and write these conditions in the form

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \exp[i(\lambda_k(t) - \lambda_j(t))]dt = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{49}$$

where $k, j \geq 1$, and $\lambda_k(t) \neq \lambda_j(t)$ for $k \neq j$, with $\lambda_k(t) \in L^1_{\text{loc}}(R, R),\ k \geq 1$, while $\{a_k;\ k \geq 1\} < \mathcal{C}$ satisfy (22).

The following assertion shows how a certain type of convergence, applied to the series (48), can help to associate a function or set of functions to it.

Lemma 5.1 *Consider the series* (48)*, under the above stated conditions for the functions* $\lambda_k(t)$*,* $k \geq 1$*, and* $\{a_k;\ k \geq 1\} \subset \ell^2(N, \mathcal{C})$*. Then the series* (48) *converges on* R*, with respect to the* B^2*-seminorm, i.e.,*

$$f \to \left[\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt\right]^{1/2}, \tag{50}$$

which implies convergence in $L^2_{\text{loc}}(R, R)$.

The **proof** of Lemma 5.1 is completely similar to that given in the section of this paper dedicated to the construction of the space $B^2(R, \mathcal{C})$, where $\lambda_k(t) = \lambda_k t,\ t \in R$, $\lambda_k \in R$, $k \geq 1$. As shown there, one can write

$$f(t) = \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)], \quad t \in R, \tag{51}$$

the convergence (on R) being that of the space $L^2([-\ell, \ell], R)$, for each $\ell > 0$.

An important aspect in the development of the approach of constructing classes/ spaces of oscillatory functions, starting from series of the form (51), under condition (22) for the coefficients, is the finding/construction of sets consisting of function $\lambda(t) : R \to R$, from which we can recruit sequences satisfying the conditions stipulated in Lemma 5.1.

We owe to Ch. Zhang [34], [35], [36] the finding of such a set of functions (polynomials), which allowed him to construct spaces of oscillatory functions, called *strong limit power* functions. These functions are obtained by the uniform approximation procedure from a set of polynomials, forming a group, under usual addition. These polynomials, actually "generalized polynomials", are defined as follows:

$$\lambda(t) = \begin{cases} \sum_{j=1}^{m} c_j t^{\alpha_j}, & t \geq 0, \\ -\sum_{j=1}^{m} c_j (-t)^{\alpha_j}, & t < -0, \end{cases} \tag{52}$$

where $c_j \in \mathcal{C},\ j \geq 1$ and $\alpha_1 > \alpha_2 > ... > \alpha_m > 0$ are arbitrary positive numbers. Then, one considers generalized polynomials of the form

$$P(t) = \sum_{k=1}^{n} a_k \exp[i\lambda_k(t)], \quad t \in R, \tag{53}$$

with each $\lambda_k(t)$ as described in (52). It is obvious that each $\lambda(t)$ in (52) is an odd function (like $\sin t$), a property which plays an important role in existence of the mean value on R.

Then, the orthogonality conditions (49) are satisfied, and one can proceed to the construction of the space $SLP(R, \mathcal{C})$ – strong limit power – as follows: $f \in SLP(R, \mathcal{C})$ if for every $\varepsilon > 0$, there exists a generalized polynomial of the form(53), such that

$$|f(t) - P_\varepsilon(t)| < \varepsilon, \quad t \in R. \tag{54}$$

From (54) we read that sup-norm is the one for $SLP(R, \mathcal{C})$.

The SLP space defined above is a Banach space, and each $f \in SLP(R, \mathcal{C})$ can be related to a generalized Fourier series, such that

$$f(t) \sim \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)], \quad t \in R, \tag{55}$$

which satisfies the Parseval equality

$$\sum_{k=1}^{\infty} |a_k|^2 = M\{|f|^2\} \tag{56}$$

with the coefficients

$$a_k = M\{f(t)e^{-i\lambda_k(t)}\}. \tag{57}$$

Many properties of $AP(R, \mathcal{C})$ can be adapted to the $SLP(R, \mathcal{C})$ space. We can say that the space $SLP(R, \mathcal{C})$ is a "copy" of the Bohr space, with considerable extension of the class of functions involved.

The mean value functional $M\{f\}$ is the Poincaré's mean value on R, and possesses other properties that appear in the case of the space $AP(R, \mathcal{C})$. See also the papers by Ch. Zhang and C. Meng [37], [38].

We can now proceed to construct a space of almost periodic functions, relying on Lemma 5.1, and using the same procedure as in case of the space $B^2(R, \mathcal{C})$. In this way, we shall obtain a larger space than $SLP(R, \mathcal{C})$, because we shall use the seminorm that appears in (50). This space will be richer than the space $SLP(R, \mathcal{C})$, possessing less properties, but still pertaining to the oscillatory type.

We will denote this space, to be constructed, by $B^2_\lambda(R, \mathcal{C})$, where the index λ designates the fact that only polynomials of the form (53) will be used as exponents for the complex exponentials involved.

The space $B^2_\lambda(R, \mathcal{C})$ will be a space of oscillatory functions, and as $SLP(R, \mathcal{C})$, will be part of the *third* period in the development of classical Fourier Analysis.

6 Construction of the Space $B^2_\lambda(R, \mathcal{C})$

The space $B^2_\lambda(R, \mathcal{C})$ will be constructed in the manner used in case of the Besicovitch space $B^2(R, \mathcal{C})$. The first step is to start with formal generalized series of the form

$$\sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)], \quad t \in R, \tag{58}$$

instead of the trigonometric series (7). The function $\lambda_k(t)$, $k \geq 1$, are generalized polynomials as those defined by the formula (52) and used in constructing the SLP-space of Ch. Zhang [35], [36]. By applying Lemma 5.1, we shall associate a function f in $L^2_{\text{loc}}(R, \mathcal{C})$, such that

$$f(t) = \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)], \quad \text{a.e. on } R, \tag{59}$$

and following step by step the construction of the space $B^2(R, \mathcal{C})$ in a previous section, we find the equation

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \left| \sum_{k=n}^{n+p} a_k \exp[i\lambda_k(t)] \right|^2 dt = \sum_{k=n}^{n+p} |a_k|^2, \; n \geq 1, \; p \geq 1, \tag{60}$$

which, on behalf of (22), assures the convergence of the series (59) in $L^2_{\text{loc}}(R, \mathcal{C})$. Hence, we can write the formula

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt = \sum_{k=1}^{\infty} |a_k|^2, \tag{61}$$

which is the same as (56).

Formula (61) is the Parseval equation for the function $f \in \widetilde{B}^2_\lambda(R, \mathcal{C})$, which is defined as the set of functions respresentable in the form (59), with $\{a_k;\ k \geq 1\} \in \ell^2(N, \mathcal{C})$, and convergence in $L^2_{\text{loc}}(R, \mathcal{C})$. The connection between $f \in \widetilde{B}^2_\lambda(R, \mathcal{C})$ and the coefficients a_k is given by (57), formulas easy to obtain from (59) and the above procedure.

The set of functions, we have denoted by $\widetilde{B}^2_\lambda(R, \mathcal{C})$, is naturally organized as a seminormed linear space, with the seminorm in the left hand side of (61), taken at power 1/2, i.e.,

$$f \to \left\{ \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt \right\}^{1/2}. \tag{62}$$

In order to prove the *completeness* of this seminormed space, one needs to proceed again like in the case of construction of the Besicovitch space $B^2(R, \mathcal{C})$. The key condition is again the assumption (22) on the coefficients of complex exponentials, and the validity of Parseval's type formula (61). In other words, everything reduces to the structure of the space $\ell^2(N,\mathcal{C})$. See Remark 3.3 to Theorem 2.1.

The last step in constructing the space $B^2_\lambda(R,\mathcal{C})$ consists in taking the factor space of $\widetilde{B}^2_\lambda(R,\mathcal{C})$, modulo the subspace of zero-seminorm elements in this space.

If the subspace above, say $\mathcal{N}_{0\lambda}$ is closed in the topology induced by the seminorm (62), then the factor space is a Banach space. Apparently, this is the case, but it is to be seen if the argument used in case of Besicovitch space $B^2(R,\mathcal{C})$ is valid in this situation. Otherwise, the final result is a seminormed complete space, which is widely accepted in Functional Analysis (see, for instance, Yosida [31] or Swartz [29]).

In other words, the last step may not be necessary in the construction of $B^2_\lambda(R,\mathcal{C})$, the space $\widetilde{B}^2_\lambda(R,\mathcal{C})$ constituting the complete seminormed space, which can be useful in various applications.

A few final remarks, related to the content of this paper, may be in order to conclude it.

First, this paper (a continuation of Corduneanu [9]), pursues the idea of constructing spaces of oscillatory functions, generalizing those encountered in the study of periodic functions (classical Fourier Analysis), of almost periodic functions and, lately, of new spaces of oscillatory functions, taking as starting point the set (say $\mathcal{TS}$) of formal trigonometric series (in complex form). By imposing various conditions to the formal series, one obtains old or new classes/spaces of oscillatory functions, with properties that allow their use in applications (particularly, in Engineering, whose impulse has been felt in mathematical research). See references to Zhang [34].

Second, this approach in constructing new spaces of oscillatory functions led to various classes of almost periodic functions, as the $AP_r(R,\mathcal{C})$, $r\in[1,2]$, allowing to obtain a scale of almost periodic function spaces, with a good potential of applications to the theory of functional equations and the introduction of new concepts, like the generalization of the convolution product (see Corduneanu [8], for instance).

Third, the series characterizing various classes, generally, are not convergent in the classical sense (i.e., uniformly or in Lebesque's spaces), and in order to have a better tool for investigation, it would be desirable to "descend" from these rather abstract functions, to more affordable ones, necessary in numerical analysis and in many applications. For instance, to each series in $AP_r(R,\mathcal{C})$ or in $AP_r(R,R)$, with $r\in(1,2)$, one can attach the series (in $AP_1(R,\mathcal{C})$), $\sum_{k=1}^{\infty}|a_k|^r\exp(i\lambda_k t)$, i.e., an absolutely convergent series. Can we take some advantage from the investigation of the operator $T_r: AP_r\to AP_1$,

$$\sum_{k=1}^{\infty}a_k\exp(i\lambda_k t)\to\sum_{k=1}^{\infty}|a_k|^r\exp(i\lambda_k t)?$$

We have also formulated this problem in Corduneanu [9].

Fourth, the approach based on dealing with formal series in order to obtain new classes of oscillatory functions, appears to be adequate in advancing the study of more and more intricate functions occurring in applied fields. The work of Ch. Zhang [34–36] is highly illustrative in this regard. One has to note also the contribution of V.F. Osipov [25], who presented a special case of the oscillatory functions of Fresnel type (for instance, the type of oscillations corresponding to the $\sin t^2$), and who dedicated a whole volume to this kind of problems.

Fifth, the method of formal series must be used, in particular, for finding oscillatory solutions of various classes of functional equations. In order to be applicable to partial differential equations, a theory of oscillatory functions, with values in Hilbert or Banach spaces, appears necessary. We will finish soon a paper, dedicated to the existence of such solutions, in which hyperbolic equations are tested – these representing the natural type to possess such solutions (but not only).

Sixth, one problem of great importance in constructing new spaces of oscillatory functions is finding adequate systems $\{\lambda_k(t);\ k \geq 1\}$, satisfying the orthogonality condition (49).

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Existence and Uniqueness of a Nontrivial Solution for Second Order Nonlinear m-Point Eigenvalue Problems on Time Scales

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Abstract: In this paper, by introducing a new operator, improving and generating a p-Laplace operator for some $p \geq 2$, we study the existence and uniqueness of a nontrivial solution for nonlinear m-point eigenvalue problems on time scales. We obtain several sufficient conditions of the existence and uniqueness of nontrivial solution of the eigenvalue problems when λ is in some interval. Our approach is based on the Leray - Schauder nonlinear alternative.

Keywords: *nontrivial solutions; eigenvalue problems; fixed point theorems; time scales.*

Mathematics Subject Classification (2010): 34B15, 39A10.

1 Introduction

In this paper, we are concerned with the existence and uniqueness of a nontrivial solution for the following second order m-point eigenvalue problems on time scales:

$$(\varphi(h(t)u^{\triangle}(t)))^{\nabla} + \lambda f(t, u(t), u^{\triangle}(t)) = 0, \qquad t \in [0, T], \tag{1}$$

$$\alpha u(\rho(0)) - \beta u^{\triangle}(\rho(0)) = C_0(\sum_{i=1}^{m-2} \alpha_i u^{\triangle}(\xi_i)), \qquad u^{\triangle}(T) = 0, \tag{2}$$

where $\varphi : R \to R$ is an increasing homeomorphism and homomorphism such that $\varphi(0) = 0$, $\lambda > 0$ is a parameter, $\xi_i \in [0, T]$ with $0 < \xi_1 < ... < \xi_{m-2} < T$, $\alpha > 0$ and $\beta \geq 0$.

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A projection $\varphi : R \to R$ is called an increasing homeomorphism and homomorphism, if the following conditions are satisfied:
1) If $x \le y$, then $\varphi(x) \le \varphi(y)$, for all $x, y \in R$.
2) φ is a continuous bijection and its inverse mapping is also continuous.
3) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in R$.

Moreover, throughout the paper the following conditions hold for α_i, f, h, C_0 and φ^{-1}:
(A_1) $\alpha_i \in [0,\infty), i = 1, 2, ..., m-2$ and $f \in C_{ld}([0,T] \times R \times R)$.
(A_2) $h \in C([\rho(0), T], (0,\infty))$ and h is increasing on $[\rho(0), T]$.
(A_3) $C_0(v)$ is a continuous function on R and satisfies the condition that there exists $A > 0$ such that $|C_0(v)| \le A|v|$, for all $v \in R$.
(A_4) For all $x, y \in R$, $|C_0(x) - C_0(y)| \le C_0(|x-y|)$.
(A_5) For all $x, y \ge 0$, $\varphi^{-1}(x+y) \le \varphi^{-1}(x) + \varphi^{-1}(y)$.

A time scale T is a nonempty closed subset of R. We make the assumption that $0 \in \mathrm{T}_k$ and $T \in \mathrm{T}^k$. By an interval $[0,T]$, we always mean the intersection of the real interval $[0,T]$ with T_k^k; that is $[0,T] \cap \mathrm{T}_k^k$. Some basic definitions and theorems on time scales can be found in the books [4, 5].

Recently, for $\phi_p(u) = |u|^{p-2}u, p > 1$, p-Laplacian problems with two-point, three-point and multi-point boundary value conditions for ordinary differential equations and finite difference equations have been studied extensively, see [8, 11, 13, 15]. For the existence problems of positive solutions of boundary value problems on time scales, some authors have obtained many results; for details, see [2, 7, 9, 10, 12, 14, 16] and the references therein. However, for the increasing homeomorphism and homomorphism operator, the research has proceeded very slowly. Especially for the existence of countably many positive solutions for dynamic equations on time scales still remain unknown.

In this paper we define a new operator φ which is an increasing homeomorphism and homomorphism with $\varphi(0) = 0$. For existence result we need that the assumption (A_5) is provided by this operator. Since the condition (A_5) is not satisfied for $\phi_p(u), 1 < p < 2$, our paper generalizes p-Laplacian operator ϕ_p for $p \ge 2$.

In [9], He considered the existence of positive solutions of the p-Laplacian dynamic equations on time scales:

$$\begin{aligned} &(\phi_p(u^{\triangle}(t)))^{\nabla} + a(t)f(u(t)) = 0, \qquad t \in (0,T),\\ &\alpha u(0) - B_0(u^{\triangle}(\eta) = 0, \qquad u^{\triangle}(T) = 0, \end{aligned}$$

or

$$\alpha u^{\triangle}(0) = 0, \qquad u(T) - B_1(u^{\triangle}(\eta)) = 0,$$

where $\eta \in (0, \rho(T))$. He obtained the existence of at least double and triple positive solutions of this problem by using a new double fixed-point theorem and triple fixed-point theorem, respectively.

In [15], Yao studied the existence of positive solutions for the following semipositone second-order boundary value problem:

$$\begin{aligned} &u''(t) = \lambda q(t) f(t, u(t), u'(t)), \qquad t \in (0,1),\\ &\alpha u(0) - \beta u'(0) = d, \qquad u(1) = 0, \end{aligned}$$

where $d > 0$, $\alpha \ge 0$, $\beta \ge 0$, $\alpha + \beta > 0$ and $q(t)f(t, u(t), u'(t)) \ge 0$ on a suitable subset of $[0,1] \times [0,\infty) \times (-\infty,\infty)$. His proofs are based on the Leray-Schauder fixed-point theorem and the localization method.

In [12], Lianga and Zhanga show the sufficient conditions for the existence of countably many positive solutions by using the fixed-point index theory and a new fixed-point theorem in cones for the following boundary value problem on time scales:

$$(\varphi(u^{\triangle}(t)))^{\nabla} + a(t)f(u(t)) = 0, \qquad t \in [0,T]_{\mathrm{T}},$$
$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \qquad u^{\triangle}(T) = 0,$$

where $\varphi : R \to R$ is an increasing homeomorphism and positive homomorphism and $\varphi(0) = 0$, $\xi_i \in [0,T]_{\mathrm{T}}$ with $0 < \xi_1 < ... < \xi_{m-2} < T$ and $0 < \sum_{i=1}^{m-2} \alpha_i < 1$, $a(t) : [0,T]_{\mathrm{T}} \to [0,\infty)$ and has countably many singularities in $[0,T]_{\mathrm{T}}$.

This paper is organized as follows. In Section 2, we present some lemmas that will be used to prove our main results and we will establish two new theorems of existence and uniqueness of nontrivial solutions of (1.1)–(1.2). In Section 3, we will give some examples to illustrate the main results in this paper.

2 Main Results

To prove the main results in this paper, we will employ some several lemmas. The following lemma is based on the linear BVP

$$(\varphi(h(t)u^{\triangle}(t)))^{\nabla} + \lambda y(t) = 0, \qquad t \in [0,T], \tag{3}$$
$$\alpha u(\rho(0)) - \beta u^{\triangle}(\rho(0)) = C_0(\sum_{i=1}^{m-2} \alpha_i u^{\triangle}(\xi_i)), \qquad u^{\triangle}(T) = 0. \tag{4}$$

Lemma 2.1 *If $y \in C_{ld}([0,T], R)$, then the problem $(2.3)-(2.4)$ has a unique solution*

$$u(t) = \int_{\rho(0)}^{t} \frac{1}{h(s)}\varphi^{-1}(\int_s^T \lambda y(r)\nabla r)\Delta s + \frac{\beta}{\alpha}\frac{1}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^T \lambda y(r)\nabla r)$$
$$+\frac{1}{\alpha}C_0(\sum_{i=1}^{m-2}\alpha_i \frac{1}{h(\xi_i)}\varphi^{-1}(\int_{\xi_i}^T \lambda y(r)\nabla r)).$$

Let Y denote the Banach space $C^1_{ld}[0,T]$ with the norm
$\|u\|_1 = \|u\| + \|u^{\triangle}\| = max_{t\in[0,T]}|u(t)| + max_{t\in[0,T]}|u^{\triangle}(t)|$.

Lemma 2.2 [6] *Let X be a real Banach space and Ω be a bounded open subset of X, $0 \in \Omega$, $F : \overline{\Omega} \to X$ be a completely continuous operator.Then either there exist $x \in \partial\Omega$, $\mu > 1$ such that $F(x) = \mu x$ or there exists a fixed point $x^* \in \overline{\Omega}$.*

The main results of this paper are the following.

Theorem 2.1 *Suppose that $(A_1), (A_2), (A_3), (A_5)$ hold, $f(t,0,0) \not\equiv 0$, $t \in [0,T]$ and there exist nonnegative functions $p, q, a \in L^1[0,T]$ such that*

$$|f(t,u,v)| \le p(t)\varphi(|u|) + q(t)\varphi(|v|) + a(t),\ forall(t,u,v) \in [0,T] \times R^2,$$

and there exists $t_0 \in [0,T]$ such that $p(t_0) \neq 0$ or $q(t_0) \neq 0$. Then there exists a constant $\lambda^* > 0$ such that for any $0 < \lambda \le \lambda^*$, the problem $(1.1)-(1.2)$ has at least one nontrivial solution $u^* \in Y$.

Proof. By Lemma 2.1, the problem (1.1) – (1.2) has a solution $u = u(t)$ if and only if u solves the operator equation

$$u(t) = Fu(t) = \int_{\rho(0)}^{t} \frac{1}{h(s)} \varphi^{-1}(\int_{s}^{T} \lambda y(r)\nabla r)\Delta s + \frac{\beta}{\alpha} \frac{1}{h(\rho(0))} \varphi^{-1}(\int_{\rho(0)}^{T} \lambda y(r)\nabla r)$$
$$+ \frac{1}{\alpha} C_0 (\sum_{i=1}^{m-2} \alpha_i \frac{1}{h(\xi_i)} \varphi^{-1}(\int_{\xi_i}^{T} \lambda y(r)\nabla r))$$

in Y. So we only need to seek a fixed point of F in Y. It follows that this operator $F : Y \to Y$ is a completely continuous operator from the references [1, 3, 14].

From the condition (A_3), there exists $A > 0$ such that

$$|C_0 (\sum_{i=1}^{m-2} \alpha_i \frac{1}{h(\xi_i)} \varphi^{-1}(\int_{\xi_i}^{T} \lambda f(r, u, u^{\triangle})\nabla r))| \leq A |\sum_{i=1}^{m-2} \alpha_i \frac{1}{h(\xi_i)} \varphi^{-1}(\int_{\xi_i}^{T} \lambda f(r, u, u^{\triangle})\nabla r)|.$$

Let $M^* = \frac{1}{\alpha}(\alpha(T - \rho(0)) + \beta + K + \alpha)M$ and $N^* = \frac{1}{\alpha}(\alpha(T - \rho(0)) + \beta + K + \alpha)N$. where $M = \varphi^{-1}(\int_{\rho(0)}^{T} (p(r) + q(r))\nabla r), N = \varphi^{-1}(\int_{\rho(0)}^{T} a(r)\nabla r)$ and $K = A \sum_{i=1}^{m-2} \alpha_i$.

Since $|f(t, 0, 0)| \leq a(t)$ for all $t \in [0, T]$, we know that $N > 0$, from $p(t_0) \neq 0$ or $q(t_0) \neq 0$, we readily obtain $M > 0$. Moreover, $M^*, N^* > 0$ since $\alpha, M, N > 0$.

Let $r = \frac{N^*}{M^*}$ and $\Omega = \{u \in C^1_{ld}([0, T]) : \|u\|_1 < r\}$. Suppose $u \in \partial\Omega$, $\mu > 1$ such that $Fu = \mu u$. Then

$$\mu r = \mu\|u\|_1 = \|Fu\|_1 = \|Fu\| + \|(Fu)^{\triangle}\|.$$

For all $t \in [0, T]$, we have

$$|Fu(t)| \leq \int_{\rho(0)}^{T} \frac{1}{h(s)} \varphi^{-1}(\int_{s}^{T} \lambda|f(r, u, u^{\triangle})|\nabla r)\Delta s + \frac{\beta}{\alpha} \frac{1}{h(\rho(0))} \varphi^{-1}(\int_{\rho(0)}^{T} \lambda|f(r, u, u^{\triangle})|\nabla r)$$
$$+ \frac{A}{\alpha} \sum_{i=1}^{m-2} \alpha_i \frac{1}{h(\xi_i)} \varphi^{-1}(\int_{\xi_i}^{T} \lambda|f(r, u, u^{\triangle})|\nabla r)$$
$$\leq \int_{\rho(0)}^{T} \frac{1}{h(\rho(0))} \varphi^{-1}(\int_{s}^{T} \lambda[p(r)\varphi(|u|) + q(r)\varphi(|u^{\triangle}|) + a(r)]\nabla r)\Delta s$$
$$+ \frac{\beta}{\alpha} \frac{1}{h(\rho(0))} \varphi^{-1}(\int_{\rho(0)}^{T} \lambda[p(r)\varphi(|u|) + q(r)\varphi(|u^{\triangle}|) + a(r)]\nabla r)$$
$$+ \frac{A}{\alpha} \sum_{i=1}^{m-2} \alpha_i \frac{1}{h(\rho(0)))} \varphi^{-1}(\int_{\xi_i}^{T} \lambda[p(r)\varphi(|u|) + q(r)\varphi(|u^{\triangle}|) + a(r)]\nabla r)$$
$$\leq \int_{\rho(0)}^{T} \frac{1}{h(\rho(0))} \varphi^{-1}(\lambda[\int_{\rho(0)}^{T} \varphi(\|u\|_1)(p(r) + q(r))\nabla r + \int_{\rho(0)}^{T} a(r)\nabla r])\Delta s$$
$$+ \frac{\beta}{\alpha} \frac{1}{h(\rho(0))} \varphi^{-1}(\lambda[\int_{\rho(0)}^{T} \varphi(\|u\|_1)(p(r) + q(r))\nabla r + \int_{\rho(0)}^{T} a(r)\nabla r])$$

$$+\frac{A}{\alpha}\frac{1}{h(\rho(0))}\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\lambda[\int_{\rho(0)}^{T}\varphi(\|u\|_1)(p(r)+q(r))\nabla r+\int_{\rho(0)}^{T}a(r)\nabla r])$$

$$\leq \frac{1}{h(\rho(0))}\int_{\rho(0)}^{T}\varphi^{-1}(\lambda)[\|u\|_1 M+N]\Delta s$$

$$+\frac{\beta}{\alpha}\frac{1}{h(\rho(0))}\varphi^{-1}(\lambda)[\|u\|_1 M+N]+\frac{1}{\alpha}\frac{\varphi^{-1}(\lambda)}{h(\rho(0)}[\|u\|_1 M+N]A\sum_{i=1}^{m-2}\alpha_i.$$

Then

$$\|Fu\|\leq \frac{\varphi^{-1}(\lambda}{h(\rho(0))}\|u\|_1\frac{1}{\alpha}(\alpha(T-\rho(0))+\beta+K)M+\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\frac{1}{\alpha}(\alpha(T-\rho(0))+\beta+K)N.$$

For all $t\in[0,T]$, we have

$$|(Fu)^{\triangle}(t)|\leq \frac{1}{h(t)}\varphi^{-1}(\int_{t}^{T}\lambda|f(r,u,u^{\triangle})|\nabla r)$$

$$\leq \frac{1}{h(\rho(0))}\varphi^{-1}(\int_{t}^{T}\lambda[p(r)\varphi(|u|)+q(r)\varphi(|u^{\triangle}|)+a(r)]\nabla r)$$

$$\leq \frac{1}{h(\rho(0))}\varphi^{-1}(\lambda[\varphi(\|u\|_1)\int_{\rho(0)}^{T}(p(r)+q(r))\nabla r+\int_{\rho(0)}^{T}a(r)\nabla r])$$

$$\leq \frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u\|_1\varphi^{-1}(\int_{\rho(0)}^{T}(p(r)+q(r))\nabla r)+\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^{T}a(r)\nabla r)$$

$$=\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u\|_1 M+\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}N.$$

Then $\|(Fu)^{\triangle}\|\leq \frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u\|_1 M+\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}N$. Thus, we get

$$\|Fu\|_1\leq \frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u\|_1 M^*+\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}N^*.$$

Choose $\lambda^*=\varphi(\frac{h(\rho(0))}{2M^*})$. Then when $0<\lambda\leq\lambda^*$, we have

$$\mu r=\mu\|u\|_1=\|Fu\|_1\leq \frac{1}{2M^*h(\rho(0))}M^*h(\rho(0))\|u\|_1+\frac{N^*}{2M^*}.$$

Consequently, $\mu r\leq \frac{1}{2}r+\frac{1}{2}r=r$.

This contradicts $\mu>1$, by Lemma 2.2, F has a fixed point $u^*\in\overline{\Omega}$, since $f(t,0,0)\not\equiv 0$, then when $0<\lambda\leq\lambda^*$, the problem $(1.1)-(1.2)$ has a nontrivial solution $u^*\in Y$. This completes the proof.

Theorem 2.2 *Suppose that* $(A_1),(A_2),(A_3),(A_4)$ *and* (A_5) *hold, and* $f:[0,T]\times R^2\to(-\infty,0]$ *or* $f:[0,T]\times R^2\to[0,\infty)$ *is ld-continuous,* $f(t,0,0)\not\equiv 0$, $t\in[0,T]$ *and there exist nonnegative functions* $p_1,q_1\in L^1[0,T]$ *such that*

$$|f(t,u_1,v_1)-f(t,u_2,v_2)|\leq p_1(t)\varphi(|u_1-u_2|)+q_1(t)\varphi(|v_1-v_2|)$$

and there exists $t_0 \in [0,T]$ *such that* $p_1(t_0) \neq 0$ *or* $q_1(t_0) \neq 0$. *Then there exists a constant* $\lambda^* > 0$ *such that for any* $0 < \lambda \leq \lambda^*$, *the problem* (1.1) − (1.2) *has a unique nontrivial solution* $u^* \in Y$.

Proof. If $u_2 = v_2 = 0$, then we have

$$|f(t,u_1,v_1)| \leq p_1(t)\varphi(|u_1|) + q_1(t)\varphi(|v_1|) + |f(t,0,0)|.$$

From Theorem 2.1, we know that the problem (1.1)–(1.2) has a nontrivial solution $u^* \in Y$.

Now, we shall use the Banach fixed theorem to show the uniqueness of nontrivial solution of the problem (1.1) − (1.2). For $|Fu_1(t) - Fu_2(t)|$, we have

$$\begin{aligned}
&|\int_{\rho(0)}^t \frac{1}{h(s)}\varphi^{-1}(\int_s^T \lambda f(r,u_1,u_1^{\triangle})\nabla r)\Delta s - \int_{\rho(0)}^t \frac{1}{h(s)}\varphi^{-1}(\int_s^T \lambda f(r,u_2,u_2^{\triangle})\nabla r)\Delta s \\
&+\frac{\beta}{\alpha}\frac{1}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^T \lambda f(r,u_1,u_1^{\triangle})\nabla r) - \frac{\beta}{\alpha}\frac{1}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^T \lambda f(r,u_2,u_2^{\triangle})\nabla r) \\
&+\frac{1}{\alpha}C_0(\sum_{i=1}^{m-2}\alpha_i\frac{1}{h(\xi_i)}\varphi^{-1}(\int_{\xi_i}^T \lambda f(r,u_1,u_1^{\triangle})\nabla r)) \\
&-\frac{1}{\alpha}C_0(\sum_{i=1}^{m-2}\alpha_i\frac{1}{h(\xi_i)}\varphi^{-1}(\int_{\xi_i}^T \lambda f(r,u_2,u_2^{\triangle})\nabla r))| \\
&\leq \int_{\rho(0)}^t \frac{1}{h(s)}|\varphi^{-1}(\int_s^T \lambda f(r,u_1,u_1^{\triangle})\nabla r) - \varphi^{-1}(\int_s^T \lambda f(r,u_2,u_2^{\triangle})\nabla r)|\Delta s \\
&+\frac{\beta}{\alpha}\frac{1}{h(\rho(0))}|\varphi^{-1}(\int_{\rho(0)}^T \lambda f(r,u_1,u_1^{\triangle})\nabla r) - \varphi^{-1}(\int_{\rho(0)}^T \lambda f(r,u_2,u_2^{\triangle})\nabla r)| \\
&+\frac{1}{\alpha}C_0(|\sum_{i=1}^{m-2}\alpha_i\frac{1}{h(\xi_i)}[\varphi^{-1}(\int_{\xi_i}^T \lambda f(r,u_1,u_1^{\triangle})\nabla r) - \varphi^{-1}(\int_{\xi_i}^T \lambda f(r,u_2,u_2^{\triangle})\nabla r)]|) \\
&\leq \int_{\rho(0)}^T \frac{1}{h(s)}|\varphi^{-1}(\int_s^T \lambda f(r,u_1,u_1^{\triangle})\nabla r) - \varphi^{-1}(\int_s^T \lambda f(r,u_2,u_2^{\triangle})\nabla r)|\Delta s \\
&+\frac{\beta}{\alpha}\frac{1}{h(\rho(0))}|\varphi^{-1}(\int_{\rho(0)}^T \lambda f(r,u_1,u_1^{\triangle})\nabla r) - \varphi^{-1}(\int_{\rho(0)}^T \lambda f(r,u_2,u_2^{\triangle})\nabla r)| \\
&+\frac{A}{\alpha}\sum_{i=1}^{m-2}\alpha_i\frac{1}{h(\xi_i)}|\varphi^{-1}(\int_{\xi_i}^T \lambda f(r,u_1,u_1^{\triangle})\nabla r) - \varphi^{-1}(\int_{\xi_i}^T \lambda f(r,u_2,u_2^{\triangle})\nabla r)| \\
&\leq \int_{\rho(0)}^T \frac{1}{h(\rho(0))}\varphi^{-1}(\lambda)\varphi^{-1}(\int_{\rho(0)}^T |f(r,u_1,u_1^{\triangle}) - f(r,u_2,u_2^{\triangle})|\nabla r)\Delta s \\
&+\frac{\beta}{\alpha}\frac{1}{h(\rho(0))}\varphi^{-1}(\lambda)\varphi^{-1}(\int_{\rho(0)}^T |f(r,u_1,u_1^{\triangle}) - f(r,u_2,u_2^{\triangle})|\nabla r) \\
&+\frac{A}{\alpha}\sum_{i=1}^{m-2}\alpha_i\frac{1}{h(\xi_i)}\varphi^{-1}(\lambda)\varphi^{-1}(\int_{\xi_i}^T |f(r,u_1,u_1^{\triangle}) - f(r,u_2,u_2^{\triangle})|\nabla r)
\end{aligned}$$

$$\leq \frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\int_{\rho(0)}^{T}\varphi^{-1}(\int_{\rho(0)}^{T}(p_1(r)\varphi(|u_1-u_2|)+q_1(r)\varphi(|u_1^{\triangle}-u_2^{\triangle}|))\nabla r)\Delta s$$

$$+\frac{\beta}{\alpha}\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^{T}(p_1(r)\varphi(|u_1-u_2|)+q_1(r)\varphi(|u_1^{\triangle}-u_2^{\triangle}|))\nabla r)$$

$$+\frac{A}{\alpha}\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\int_{\rho(0))}^{T}(p_1(r)\varphi(|u_1-u_2|)+q_1(r)\varphi(|u_1^{\triangle}-u_2^{\triangle}|))\nabla r)$$

$$\leq \frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\int_{\rho(0)}^{T}\varphi^{-1}(\int_{\rho(0)}^{T}\varphi(\|u_1-u_2\|_1)(p_1(r)+q_1(r))\nabla r)\Delta s$$

$$+\frac{\beta}{\alpha}\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^{T}\varphi(\|u_1-u_2\|_1)(p_1(r)+q_1(r))\nabla r)$$

$$+\frac{A}{\alpha}\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\int_{\rho(0)}^{T}\varphi(\|u_1-u_2\|_1)(p_1(r)+q_1(r))\nabla r)$$

$$=\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u_1-u_2\|_1(T-\rho(0))M_1+\frac{\beta}{\alpha}\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u_1-u_2\|_1M_1$$

$$+\frac{1}{\alpha}\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u_1-u_2\|_1M_1A\sum_{i=1}^{m-2}\alpha_i.$$

Then

$$\|Fu_1-Fu_2\|\leq \frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u_1-u_2\|_1\frac{1}{\alpha}(\alpha(T-\rho(0))+\beta+K)M_1,$$

where $M_1=\varphi^{-1}(\int_{\rho(0)}^{T}(p_1(r)+q_1(r))\nabla r)$, A is a constant such that

$$C_0(|\textstyle\sum_{i=1}^{m-2}\alpha_i\frac{1}{h(\xi_i)}[\varphi^{-1}(\int_{\xi_i}^{T}\lambda f(r,u_1,u_1^{\triangle})\nabla r)-\varphi^{-1}(\int_{\xi_i}^{T}\lambda f(r,u_2,u_2^{\triangle})\nabla r)]|)$$
$$\leq A|\sum_{i=1}^{m-2}\alpha_i\frac{1}{h(\xi_i)}[\varphi^{-1}(\int_{\xi_i}^{T}\lambda f(r,u_1,u_1^{\triangle})\nabla r)-\varphi^{-1}(\int_{\xi_i}^{T}\lambda f(r,u_2,u_2^{\triangle})\nabla r)]|$$

and $K=A\sum_{i=1}^{m-2}\alpha_i$. For all $t\in[0,T]$, we have

$$|((Fu_1)^{\triangle}-(Fu_2)^{\triangle})(t)|\leq \frac{1}{h(t)}\varphi^{-1}(\int_{t}^{T}\lambda|f(r,u_1,u_1^{\triangle})-f(r,u_2,u_2^{\triangle})|\nabla r)$$
$$\leq \frac{1}{h(t)}\varphi^{-1}(\int_{\rho(0)}^{T}\lambda[p_1(r)\varphi(|u_1-u_2|)+q_1(r)\varphi(|u_1^{\triangle}-u_2^{\triangle}|)]\nabla r)$$
$$\leq \frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^{T}\varphi(\|u_1-u_2\|_1)(p_1(r)+q_1(r))\nabla r).$$

Then

$$\|(Fu_1)^{\triangle}-(Fu_2)^{\triangle}\|\leq \frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u_1-u_2\|_1M_1.$$

So, we get

$$\|Fu_1-Fu_2\|_1\leq \frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u_1-u_2\|_1M_1^{*},$$

where $M_1^* = \dfrac{1}{\alpha}(\alpha(T-\rho(0)) + \beta + K + \alpha)M_1$.

Choose $\lambda^* = \varphi(\dfrac{h(\rho(0))}{2M_1^*})$. Then when $0 < \lambda \leq \lambda^*$, we have

$$\|Fu_1 - Fu_2\|_1 \leq \frac{1}{2}\|u_1 - u_2\|_1.$$

Thus the problem (1.1)–(1.2) has a unique solution for $0 < \lambda \leq \lambda^*$.

Corollary 2.1 *Suppose that* $(A_1), (A_2), (A_3), (A_5)$ *hold,* $f : [0,T] \times R^2 \to R$ *is ld-continuous,* $f(t,0,0) \not\equiv 0, t \in [0,T]$ *and*

$$0 \leq l = \limsup_{|u|+|v| \to +\infty} \max_{t\in[0,T]} \frac{|f(t,u,v)|}{\varphi(|u|) + \varphi(|v|)} < +\infty. \tag{5}$$

Then there exists a constant $\lambda^* > 0$ *such that for any* $0 < \lambda \leq \lambda^*$*, the problem* $(1.1)-(1.2)$ *has at least one nontrivial solution* $u^* \in Y$*.*

Proof. Let $\varepsilon > 0$. By (2.5), there exists $H > 0$ such that

$$|f(t,u,v)| \leq (l+\varepsilon)(\varphi(|u|) + \varphi(|v|)), \quad |u| + |v| \geq H, t \in [0,T].$$

Let $K = \max_{t\in[0,T],|u|+|v|\leq H} |f(t,u,v)|$. Then for $(t,u,v) \in [0,T] \times R^2$, we have

$$|f(t,u,v)| \leq (l+\varepsilon)\varphi(|u|) + (l+\varepsilon)\varphi(|v|) + K.$$

From Theorem 2.1, we know that the problem (1.1)–(1.2) has at least one nontrivial solution $u^* \in Y$.

Corollary 2.2 *Suppose that* $(A_1), (A_2), (A_3), (A_5)$ *hold,* $f : [0,T] \times R^2 \to R$ *is ld-continuous,* $f(t,0,0) \not\equiv 0, t \in [0,T]$ *and*

$$0 \leq l = \limsup_{|u|+|v| \to +\infty} \max_{t\in[0,T]} \frac{|f(t,u,v)|}{\varphi(|u|)} < +\infty,$$

or

$$0 \leq l = \limsup_{|u|+|v| \to +\infty} \max_{t\in[0,T]} \frac{|f(t,u,v)|}{\varphi(|v|)} < +\infty.$$

Then there exists a constant $\lambda^* > 0$ *such that for any* $0 < \lambda \leq \lambda^*$*, the problem* $(1.1)-(1.2)$ *has at least one nontrivial solution* $u^* \in Y$*.*

Corollary 2.3 *Suppose that* $(A_1), (A_2), (A_3), (A_4)$ *and* (A_5) *hold,* $f : [0,T] \times R^2 \to [0,\infty)$ *is ld-continuous,* $f(t,0,0) \not\equiv 0,\ t \in [0,T]$ *,* $C_0(v)$ *satisfies the condition that there exists* $B > 0$ *such that* $Bv \leq C_0(v)$*, for all* $v \geq 0$ *and there exist nonnegative functions* $p_1, q_1 \in L^1[0,T]$ *such that*

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq p_1(t)\varphi(|u_1-u_2|) + q_1(t)\varphi(|v_1-v_2|)$$

and there exists $t_0 \in [0,T]$ *such that* $p_1(t_0) \neq 0$ *or* $q_1(t_0) \neq 0$*. Then there exists a constant* $\lambda^* > 0$ *such that for any* $0 < \lambda \leq \lambda^*$*, the problem* $(1.1)-(1.2)$ *has a positive unique solution* $u^* \in Y$*.*

3 Examples

In this section, we will give some examples to illustrate our main results.

Example 3.1 Let $T = [0,1] \cup \{2\} \cup [3,5]$. We consider the following second order eigenvalue problem

$$(\varphi((t+4)u^{\triangle}(t)))^{\nabla} + \lambda(te^{-t}u^2 - (u^{\triangle})^2 t^2 \sin t + \cos t) = 0, \quad t \in [0,4], \tag{6}$$

$$u(0) = \frac{1}{2}|u^{\triangle}(1) + u^{\triangle}(2)|, \quad u^{\triangle}(4) = 0, \tag{7}$$

where $h(t) = t+4, \alpha = 1, \beta = 0, T = 4, \xi_1 = 1, \xi_2 = 2, \alpha_1 = \alpha_2 = \dfrac{1}{2}$,

$$\varphi(u) = \begin{cases} -u^2, & u < 0, \\ u^2, & u \geq 0, \end{cases}$$

and $C_0(x) = |x|$. Then we can take $A = 1$ so that $|C_0(x)| \leq A|x|$ for all $x \in R$. Thus $K = A(\alpha_1 + \alpha_2) = 1$.

Noticing, for all $t \in [0,4]$, f satisfies

$$|f(t,u,v)| = |te^{-t}u^2 - v^2t^2 \sin t + \cos t| \leq t|u|^2 + t^2|v|^2 + 1.$$

Then $|f(t,u,v)| \leq t\varphi(|u|) + t^2\varphi(|v|) + 1$. It is easy to see by calculating that

$$M = \varphi^{-1}\left(\int_0^4 (r^2 + r)\nabla r\right) = \sqrt{\frac{74}{3}}, \; M^* = \frac{1}{\alpha}(\alpha(T - \rho(0)) + \beta + K + \alpha)M = 6\sqrt{\frac{74}{3}}.$$

So, we have $\lambda^* = \varphi(\dfrac{h(0)}{2M^*}) \approx 0.0045$. Then by Theorem 2.1, we know that the problem (3.6)–(3.7) has nontrivial solution $u^* \in Y$ for any $\lambda \in (0, \lambda^*]$.

Example 3.2 Let $T = \{0\} \cup \{\dfrac{1}{n} : n \in N\} \cup [2,4]$. We consider the following second order eigenvalue problem

$$(\varphi(e^t u^{\triangle}(t)))^{\nabla} + \lambda(t^2 \sin u + t) = 0, \qquad t \in [0,3], \tag{8}$$

$$2u(0) - u^{\triangle}(0) = \frac{1}{3}|u^{\triangle}(\frac{1}{5}) + u^{\triangle}(\frac{12}{5}) + u^{\triangle}(\frac{14}{5})|, \quad u^{\triangle}(3) = 0, \tag{9}$$

where $\varphi(u) = u$, $h(t) = e^t, \alpha = 2, \beta = 1, T = 3, \xi_1 = \dfrac{1}{5}, \xi_2 = \dfrac{12}{5}, \xi_3 = \dfrac{14}{5}, \alpha_1 = \alpha_2 = \alpha_3 = \dfrac{1}{3}$ and $C_0(x) = |x|$. Then we can take $A = 1$ so that $|C_0(x)| \leq A|x|$ for all $x \in R$. Thus, $K = A(\alpha_1 + \alpha_2 + \alpha_3) = 1$.

Noticing, for all $t \in [0,3]$, f satisfies

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| = |t^2 \sin u_1 + t - t^2 \sin u_2 - t| \leq t^2|u_1 - u_2|.$$

It is easy to see by calculating that

$$M_1 = \int_0^3 r^2 \nabla r = \frac{\pi^2 + 38}{6} \text{ and } M_1^* = 5(\frac{\pi^2 + 38}{6}).$$

Thus, we have $\lambda^* = \dfrac{h(0)}{2M_1^*} \approx 0.0125$. Then by Theorem 2.2, we know that the problem (3.8) – (3.9) has a unique solution $u^* \in Y$ for any $\lambda \in (0, \lambda^*]$.

In the following example we will take the p-Laplacian operator $\phi_4(u)$ such that $\phi_p(u) = |u|^{p-2}u$ for $p > 1$ and $(\phi_p)^{-1} = \phi_q$ with $\frac{1}{p} + \frac{1}{q} = 1$ which is the special case of φ.

Example 3.3 Let $T = [0,1] \cup [2,7]$. We consider the following second order eigenvalue problem

$$(\phi_4((t+2)u^{\triangle}(t)))^{\nabla} + \lambda(\arctan(u^2 + (u^{\triangle})^2) + t^2 \sinh t) = 0,\ t \in [0,4], \tag{10}$$

$$2u(0) = \frac{1}{3}u^{\triangle}(1) + \frac{2}{3}u^{\triangle}(3), \quad u^{\triangle}(4) = 0, \tag{11}$$

where $h(t) = t+2, \alpha = 2, \beta = 0, T = 4, \xi_1 = 1, \xi_2 = 3, \alpha_1 = \frac{1}{3}, \alpha_2 = \frac{2}{3}$ and $C_0(x) = x$. Then we can take $A = 1$ so that $|C_0(x)| \leq A|x|$ for all $x \in R$. Thus $K = A(\alpha_1 + \alpha_2) = 1$. It is clear that

$$\limsup_{|u|+|v|\to+\infty} \max_{t\in[0,4]} \frac{|\arctan(u^2+v^2) + t^2 \sinh t|}{\phi_4(|u|) + \phi_4(|v|)} = 0.$$

Choosing $\epsilon = \frac{1}{2}$, we get

$$M = \phi_q(\int_0^4 (\frac{1}{2} + \frac{1}{2})\nabla r) = \sqrt[3]{4} \text{ and } M^* = \frac{1}{\alpha}(\alpha(T - \rho(0)) + \beta + K + \alpha)M = \frac{11}{2}\sqrt[3]{4}.$$

So, we have $\lambda^* = \phi_4(\frac{h(0)}{2M^*}) \approx 0.0015$. Then by Corollary 2.1, we know that the problem (3.10)–(3.11) has nontrivial solution $u^* \in Y$ for any $\lambda \in (0, \lambda^*]$.

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Infinitely Many Solutions for a Discrete Fourth Order Boundary Value Problem

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Abstract: By using variational methods and critical point theory, the authors obtain criteria for the existence of infinitely many solutions to the fourth order discrete boundary value problem

$$\begin{cases} \Delta^4 u(t-2) - \alpha\Delta^2 u(t-1) + \beta u(t) = \lambda f(t, u(t)), \quad t \in [1, T]_{\mathbb{Z}}, \\ u(0) = \Delta u(-1) = \Delta^2 u(T) = 0, \ \Delta^3 u(T-1) - \alpha\Delta u(T) = \mu g(u(T+1)), \end{cases}$$

where $T \geq 2$ is an integer, $[1, T]_{\mathbb{Z}} = \{1, 2, \ldots, T\}$, $\alpha, \beta, \lambda, \mu \in \mathbb{R}$ are parameters, $f \in C([1, T]_{\mathbb{Z}} \times \mathbb{R}, \mathbb{R})$, and $g \in C(\mathbb{R}, \mathbb{R})$. Several consequences of their main theorems are also presented. One example is included to show the applicability of the results.

Keywords: *discrete boundary value problem; infinitely many solutions; fourth order; variational methods.*

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1 Introduction

Throughout this paper, for any integers a and b with $a \leq b$, let $[a, b]_{\mathbb{Z}}$ denote the discrete interval $\{a, a+1, \ldots, b\}$. Here, we are concerned with the existence of solutions of the four-parameter fourth order discrete boundary value problem (BVP)

$$\begin{cases} \Delta^4 u(t-2) - \alpha \Delta^2 u(t-1) + \beta u(t) = \lambda f(t, u(t)), & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = \Delta u(-1) = \Delta^2 u(T) = 0, \ \Delta^3 u(T-1) - \alpha \Delta u(T) = \mu g(u(T+1)), \end{cases} \tag{1.1}$$

where $T \geq 2$ is an integer, Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, $\Delta^k u(t) = \Delta^{k-1}(\Delta u(t))$ for $k = 2, 3, 4$, $\alpha, \beta, \lambda, \mu$ are four parameters with $\alpha, \beta \in \mathbb{R}$, $\lambda \in (0, \infty)$, $\mu \in [0, \infty)$, $f \in C([1, T]_{\mathbb{Z}} \times \mathbb{R}, \mathbb{R})$, and $g \in C(\mathbb{R}, \mathbb{R})$. By a *solution* of (1.1), we mean a function $u \in C([-1, T+2]_{\mathbb{Z}}, \mathbb{R})$ satisfying (1.1). We assume throughout, and without further mention, that the following condition holds:

(H1) α and β satisfy

$$1 + \alpha_-(T+1)^2 + \beta_- T^2 (T+1)^2 > 0,$$

where $\alpha_- = \min\{\alpha, 0\}$ and $\beta_- = \min\{\beta, 0\}$.

Difference equations appear in numerous settings and forms, both in mathematics and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology, and other fields ([1,19]). In recent years, many researchers have paid a lot of attention to fourth order BVPs for difference equations with various boundary conditions. The reader may refer to [2,6,7,11,13,14,16–18,20,22,26,28] and the included references for some recent work.

We point out, depending on the values of the parameters α, β, λ, and μ, that BVP (1.1) covers many problems as special cases. For instance, if $\alpha = \beta = 0$ and $\mu = 1$, BVP (1.1) becomes

$$\begin{cases} \Delta^4 u(t-2) = \lambda f(t, u(t)), & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = \Delta u(-1) = \Delta^2 u(T) = 0, \ \Delta^3 u(T-1) = g(u(T+1)). \end{cases} \tag{1.2}$$

The continuous version of BVP (1.2), i.e., the problem

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u), & t \in (0, 1), \\ u(0) = u'(0) = u''(1) = 0, \ u'''(1) = g(u(1)), \end{cases}$$

has recently been investigated in [24] where results for the existence of three solutions are obtained. Notice that BVPs for fourth order differential equations have been extensively studied in the literature. For a small sample of recent work, see [9,12,14,15,23–25].

The existence of three solutions of BVP (1.1) has been studied in [11]. In this paper, we continue our study on BVP (1.1). We apply variational methods and critical point theorem to establish some criteria for the existence of infinitely many solutions of BVP (1.1). We also present several consequences of our main theorems. Our analysis is mainly based on a recent theorem on critical points that appeared in [3,21]; see Lemma 4.1 below. This lemma and its variations have been frequently used to obtain multiplicity results for nonlinear problems of a variational nature; see, for example, [3–5,8,10,21] and the references therein. Our proofs are partly motivated by these papers.

The rest of this paper is organized as follows. Section 2 contains some preliminary lemmas, Section 3 contains the main results of this paper and one illustrative example, and the proofs of the main results are presented in Section 4.

2 Preliminary Lemmas

We define a real vector space

$$X = \left\{u : [-1, T+2]_{\mathbb{Z}} \to \mathbb{R} \ : \ u(-1) = u(0) = 0,\ \Delta^2 u(T) = 0\right\}. \tag{2.1}$$

For any $u \in X$, we let

$$||u||_X = \left(\sum_{t=1}^{T+1} \left(|\Delta^2 u(t-2)|^2 + \alpha |\Delta u(t-1)|^2 \right) + \beta \sum_{t=1}^{T} |u(t)|^2 \right)^{1/2}.$$

Let

$$\rho = (T+1)^{3/2} \left(1 + \alpha_-(T+1)^2 + \beta_- T^2 (T+1)^2\right)^{-1/2}. \tag{2.2}$$

Clearly, $\rho > 0$ by condition (H1).

The following result is taken from [11, Lemma 2.1].

Lemma 2.1 *For any $u \in X$, we have*

$$\sum_{t=1}^{T+1} \left(|\Delta^2 u(t-2)|^2 + \alpha |\Delta u(t-1)|^2 \right) + \beta \sum_{t=1}^{T} |u(t)|^2 \geq 0$$

and

$$|u(t)| \leq \rho ||u||_X \quad \text{for } t \in [1, T+1]_{\mathbb{Z}}. \tag{2.3}$$

Hence, $||\cdot||_X$ is a norm on X with which X becomes a $T+1$ dimensional separable and reflexive Banach space.

For any $u \in X$, let the functionals Φ and Ψ be defined by

$$\Phi(u) = \frac{1}{2} ||u||_X^2 \tag{2.4}$$

and

$$\Psi(u) = \sum_{t=1}^{T} F(t, u(t)) - \frac{\mu}{\lambda} G(u(T+1)), \tag{2.5}$$

where

$$F(t, x) = \int_0^x f(t, s) ds, \quad (t, x) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}, \tag{2.6}$$

and

$$G(x) = \int_0^x g(s) ds, \quad x \in \mathbb{R}. \tag{2.7}$$

Then, Φ and Ψ are well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at $u \in X$ are the functionals $\Phi'(u)$ and $\Psi'(u)$ given by

$$\Phi'(u)(v) = \sum_{t=1}^{T+1} \left(\Delta^2 u(t-2) \Delta^2 v(t-2) + \alpha \Delta u(t-1) \Delta v(t-1) \right) + \beta \sum_{t=1}^{T} u(t) v(t)$$

and

$$\Psi'(u)(v) = \sum_{t=1}^{T} f(t, u(t)) v(t) - \frac{\mu}{\lambda} g(u(T+1)) v(T+1)$$

for any $v \in X$.

Lemma 2.2 below follows from [11, Lemma 2.3].

Lemma 2.2 *The function $u \in X$ is a critical point of the functional $\Phi - \lambda\Psi$ if and only if u is a solution of BVP* (1.1).

3 Main Results

In this section, we present our main results. In what follows, let X, ρ, F, and G be defined by (2.1), (2.2), (2.6), and (2.7), respectively. For convenience, we use the following notation:

$$A = \liminf_{\xi\to\infty} \frac{\sum_{t=1}^{T} \max_{|x|\le\xi} F(t,x)}{\xi^2}, \quad B = \limsup_{\xi\to\infty} \frac{\sum_{t=1}^{T} F(t,\xi)}{\xi^2}, \tag{3.1}$$

$$C = \liminf_{\xi\to 0^+} \frac{\sum_{t=1}^{T} \max_{|x|\le\xi} F(t,x)}{\xi^2}, \quad D = \limsup_{\xi\to 0^+} \frac{\sum_{t=1}^{T} F(t,\xi)}{\xi^2}, \tag{3.2}$$

$$\lambda_1 = \frac{2+\alpha+\beta T}{2B}, \quad \lambda_2 = \frac{1}{2\rho^2 A}, \tag{3.3}$$

$$\lambda_3 = \frac{2+\alpha+\beta T}{2D}, \quad \lambda_4 = \frac{1}{2\rho^2 C}.$$

In the following, we assume that

(H2) $A, \ B, \ C, \ D \ge 0$.

We also use the convention that $1/a = \infty$ when $a = 0$.

We now state our main results in the paper.

Theorem 3.1 *Assume that*

$$A < \frac{B}{\rho^2(2+\alpha+\beta T)}. \tag{3.4}$$

Then, for each $\lambda \in (\lambda_1, \lambda_2)$, for each function $g \in C(\mathbb{R}, \mathbb{R})$ with

$$g(x) \le 0 \text{ on } \mathbb{R} \quad \text{and} \quad G_\infty = \liminf_{\xi\to\infty} \frac{G(\xi)}{\xi^2} > -\infty, \tag{3.5}$$

and for each $\mu \in [0, \overline{\mu}_1)$ with

$$\overline{\mu}_1 = \frac{1 - 2\rho^2\lambda A}{-2\rho^2 G_\infty}, \tag{3.6}$$

BVP (1.1) *has a sequence of solutions that is unbounded in X.*

Theorem 3.2 *Assume that*

$$C < \frac{D}{\rho^2(2+\alpha+\beta T)}. \tag{3.7}$$

Then, for each $\lambda \in (\lambda_3, \lambda_4)$, for each function $g \in C(\mathbb{R}, \mathbb{R})$ satisfying (3.4)*, and for each $\mu \in [0, \overline{\mu}_2)$ with*

$$\overline{\mu}_2 = \frac{1 - 2\rho^2\lambda C}{-2\rho^2 G_\infty},$$

BVP (1.1) *has a sequence of solutions converging uniformly to zero in X.*

Remark 3.1 For Theorems 3.1 and 3.2, we make the following comments.

(a) It is easy to verify that condition (H) implies $2+\alpha+\beta T>0$. Thus, $\lambda_1\ge 0$ and $\lambda_3\ge 0$.

(b) By the assumptions (3.4) and (3.7), we see that $\lambda_1<\lambda_2$ and $\lambda_3<\lambda_4$. This assures that the intervals (λ_1,λ_2) and (λ_3,λ_4) are nonempty.

(c) The interval $[0,\overline{\mu}_1)$ is well defined since $\overline{\mu}_1>0$ under the condition that $\lambda<\lambda_2$.

(d) The interval $[0,\overline{\mu}_2)$ is well defined since $\overline{\mu}_2>0$ under the condition that $\lambda<\lambda_4$.

The following results are direct consequences of Theorems 3.1 and 3.2.

Corollary 3.1 *Assume that* (3.4) *holds. Then, for each $\lambda\in(\lambda_1,\lambda_2)$, the BVP*

$$\begin{cases}\Delta^4u(t-2)-\alpha\Delta^2u(t-1)+\beta u(t)=\lambda f(t,u(t)), \quad t\in[1,T]_{\mathbb{Z}},\\ u(0)=\Delta u(-1)=\Delta^2u(T)=0,\ \Delta^3u(T-1)-\alpha\Delta u(T)=0,\end{cases}\tag{3.8}$$

has a sequence of solutions which is unbounded in X.

Corollary 3.2 *Assume that* (3.7) *holds. Then, for each $\lambda\in(\lambda_3,\lambda_4)$, BVP* (3.8) *has a sequence of solutions converging uniformly to zero in X.*

Corollary 3.3 *Assume that $A=0$ and $B=\infty$. Then, for each $\lambda\in(0,\infty)$, for each function $g\in C(\mathbb{R},\mathbb{R})$ with*

$$g(x)\le 0 \quad \text{on } \mathbb{R} \quad \text{and} \quad G_\infty=\liminf_{\xi\to\infty}\frac{G(\xi)}{\xi^2}=0,\tag{3.9}$$

and for each $\mu\in[0,\infty)$, BVP (1.1) *has a sequence of solutions which is unbounded in X.*

Corollary 3.4 *Assume that $C=0$ and $D=\infty$. Then, for each $\lambda\in(0,\infty)$, for each function $g\in C(\mathbb{R},\mathbb{R})$ satisfying* (3.9)*, and for each $\mu\in[0,\infty)$, BVP* (1.1) *has a sequence of solutions converging uniformly to zero in X.*

Corollary 3.5 *Assume that $A<\frac{B}{2(T+1)^3}$. Then, for each $\lambda\in\left(\frac{1}{B},\frac{1}{2A(T+1)^3}\right)$ and each function $g\in C(\mathbb{R},\mathbb{R})$ satisfying* (3.9)*, BVP* (1.2) *has a sequence of solutions which is unbounded in X.*

Corollary 3.6 *Assume that $C<\frac{D}{2(T+1)^3}$. Then, for each $\lambda\in\left(\frac{1}{D},\frac{1}{2C(T+1)^3}\right)$ and each function $g\in C(\mathbb{R},\mathbb{R})$ satisfying* (3.9)*, BVP* (1.2) *has a sequence of solutions converging uniformly to zero in X.*

We conclude this section with the following example where the construction of the nonlinear function $f(t,x)$ is partly motivated by [10, Example 3.1].

Example 3.1 Let $T \geq 2$ be an integer, $\{a_n\}$ and $\{b_n\}$ be sequences defined by $b_1 = 2$, $b_{n+1} = b_n^6$, and $a_n = b_n^4$ for $n \in \mathbb{N}$. Let $f : [0,T]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ be a positive continuous function defined by

$$f(t,x) = t^2 \begin{cases} b_1^3\sqrt{1-(1-x)^2}+1, & x \in [0,b_1], \\ (a_n - b_n^3)\sqrt{1-(a_n-1-x)^2}+1, & x \in \cup_{n=1}^{\infty}[a_n-2,a_n], \\ (b_{n+1}^3 - a_n)\sqrt{1-(b_{n+1}-1-x)^2}+1, & x \in \cup_{n=1}^{\infty}[b_{n+1}-2,b_{n+1}], \\ 1, & \text{otherwise}. \end{cases}$$

Let α, $\beta \in \mathbb{R}$ satisfy (H). We claim that for each $\lambda \in (0,\infty)$ and $\mu \in [0,\infty)$, the BVP

$$\begin{cases} \Delta^4 u(t-2) - \alpha\Delta^2 u(t-1) + \beta u(t) = \lambda f(t,u(t)), \quad t \in [1,T]_{\mathbb{Z}}, \\ u(0) = \Delta u(-1) = \Delta^2 u(T) = 0,\ \Delta^3 u(T-1) - \alpha\Delta u(T) = -\mu(u(T+1))^{2/3}, \end{cases} \tag{3.10}$$

has a sequence of solutions which is unbounded in X.

In fact, with $g(x) = -x^{2/3}$, it is clear that BVP (3.10) is a special case of BVP (1.1) and that (3.9) holds. Let $F(t,x)$ be defined by (2.6). Then, for $t \in [1,T]_{\mathbb{Z}}$, simple computations yield

$$\begin{aligned} F(t,a_n) &= t^2\Bigg(\int_0^{a_n} 1ds + b_1^3\int_0^2 \sqrt{1-(1-s)^2}\ ds \\ &\quad + \sum_{i=1}^{n}\int_{a_i-2}^{a_i}(a_i-b_i^3)\sqrt{1-(a_i-1-s)^2}\ ds \\ &\quad + \sum_{i=1}^{n-1}\int_{b_{i+1}-2}^{b_{i+1}}(b_i^3-a_i)\sqrt{1-(b_{i+1}-1-s)^2}\ ds\Bigg) \\ &= t^2\left(\frac{\pi}{2}a_n + a_n\right) \end{aligned}$$

and

$$\begin{aligned} F(t,b_n) &= t^2\Bigg(\int_0^{b_n} 1ds + b_1^3\int_0^2 \sqrt{1-(1-s)^2}\ ds \\ &\quad + \sum_{i=1}^{n-1}\int_{a_i-2}^{a_i}(a_i-b_i^3)\sqrt{1-(a_i-1-s)^2}\ ds \\ &\quad + \sum_{i=1}^{n-1}\int_{b_{i+1}-2}^{b_{i+1}}(b_i^3-a_i)\sqrt{1-(b_{i+1}-1-s)^2}\ ds\Bigg) \\ &= t^2\left(\frac{\pi}{2}b_n^3 + b_n\right). \end{aligned}$$

Thus,

$$\lim_{n\to\infty}\frac{F(t,a_n)}{a_n^2} = 0 \quad \text{and} \quad \lim_{n\to\infty}\frac{F(t,b_n)}{b_n^2} = \infty \quad \text{for } t \in [1,T]_{\mathbb{Z}}.$$

Then, for A and B defined in (3.1), it is easy to see that

$$A = \liminf_{\xi\to\infty}\frac{F(t,\xi)\sum_{t=1}^{T}t^2}{\xi^2} = 0 \quad \text{and} \quad B = \limsup_{\xi\to\infty}\frac{F(t,\xi)\sum_{t=1}^{T}t^2}{\xi^2} = \infty. \tag{3.11}$$

Thus, all the conditions of Corollary 3.3 are satisfied. The claim then follows directly from Corollary 3.3.

4 Proofs of the Main Results

The proofs of our theorems are based on the following lemma obtained in [3, Theorem 2.1]. This result is a supplement of the variational principle of Ricceri [21, Theorem 2.5].

Lemma 4.1 *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty,r)} \frac{\left(\sup_{v \in \Phi^{-1}(-\infty,r)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}, \tag{4.1}$$

and

$$\gamma := \liminf_{r\to\infty} \varphi(r), \qquad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then:

(a) For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional $I_\lambda := \Phi - \lambda\Psi$ to $\Phi^{-1}(-\infty, r)$ admits a global minimum that is a critical point (local minimum) of I_λ in X.

(b) If $\gamma < \infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either

(b_1) I_λ possesses a global minimum, or

(b_2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that

$$\lim_{n\to\infty} \Phi(u_n) = \infty.$$

(c) If $\delta < \infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either

(c_1) there is a global minimum of Φ which is a local minimum of I_λ, or

(c_2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ which converges weakly to a global minimum of Φ.

The proof of Theorem 3.1 relies on Lemma 4.1 (b).

Proof of Theorem 3.1. Let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by (2.4) and (2.5), respectively. Then, it is clear that Φ and Ψ satisfy all the regularity assumptions given in Lemma 4.1.

By the definition of A in (3.1), there exists a sequence $\{\xi_n\}$ of positive numbers such that $\lim_{n\to\infty} \xi_n = \infty$ and

$$A = \lim_{n\to\infty} \frac{\sum_{t=1}^{T} \max_{|x|\le\xi_n} F(t,x)}{\xi_n^2}. \tag{4.2}$$

Let $r_n = \frac{\xi_n^2}{2\rho^2}$. Then, for any $u \in X$ with $\Phi(u) < r_n$, from (2.3), we have

$$\max_{t\in[1,T+1]_{\mathbb{Z}}} |u(t)| \le \rho||u||_X < \rho(2r_n)^{1/2} = \xi_n. \tag{4.3}$$

Note that $0 \in \Phi^{-1}(-\infty, r_n)$ and $\Psi(0) = 0$. Then, by (4.1) and (3.5),

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v)\right) - \Psi(u)}{r_n - \Phi(u)} \\ &\le \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v)}{r_n} \\ &\le \frac{\sum_{t=1}^{T} \max_{|x|\le\xi_n} F(t,x) - \frac{\mu}{\lambda}\min_{|s|\le\xi_n} G(s)}{r_n} \\ &= 2\rho^2 \frac{\sum_{t=1}^{T} \max_{|x|\le\xi_n} F(t,x) - \frac{\mu}{\lambda} G(\xi_n)}{\xi_n^2}.\end{aligned}$$

Thus, from (3.5) and (4.2), we see that, for γ defined in Lemma 4.1,

$$\gamma \le \liminf_{n\to\infty} \varphi(r_n) \le 2\rho^2 \left(A - \frac{\mu}{\lambda} G_\infty\right) < \infty. \tag{4.4}$$

We claim that

$$\text{if } \lambda \in (\lambda_1, \lambda_2) \text{ and } \mu \in [0, \overline{\mu}_1), \text{ then } \lambda \in (0, 1/\gamma). \tag{4.5}$$

In fact, it is clear that $\lambda > 0$. Now, when $\lambda \in (\lambda_1, \lambda_2)$ and $\mu \in [0, \overline{\mu}_1)$, from (3.6) and (4.4), we have

$$\gamma \le 2\rho^2 \left(A - \frac{\overline{\mu}_1}{\lambda} G_\infty\right) = 2\rho^2 \left(A + \frac{1 - 2\rho^2\lambda A}{2\rho^2\lambda}\right) = \frac{1}{\lambda},$$

and so, $\lambda < 1/\gamma$. Thus, (4.5) holds.

Let $\lambda \in (\lambda_1, \lambda_2)$ and $\mu \in [0, \overline{\mu}_1)$ be fixed. Then, in view of (4.4) and (4.5), by Lemma 4.1 (b), it follows that one of the following alternatives holds

(b_1) either $I_\lambda := \Phi - \lambda\Psi$ has a global minimum, or

(b_2) there exists a sequence $\{u_n\}$ of critical points of I_λ such that $\lim_{n\to\infty} ||u_n||_X = \infty$.

In what follows, we show that alternative (b_1) does not hold. By the definition of B in (3.1), there exists a sequence $\{\eta_n\}$ of positive numbers such that $\lim_{n\to\infty} \eta_n = \infty$ and

$$B = \lim_{n\to\infty} \frac{\sum_{t=1}^{T} F(t, \eta_n)}{\eta_n^2}. \tag{4.6}$$

For each $n \in \mathbb{N}$, define a function $w_n : [-1, T+2]_{\mathbb{Z}} \to \mathbb{R}$ by

$$w_n(t) = \begin{cases} 0, & t = -1, 0, \\ \eta_n, & t \in [1, T+2]_{\mathbb{Z}}. \end{cases} \tag{4.7}$$

Then, $w_n \subseteq X$. Moreover, from (2.4) and (2.5), it is easy to see that

$$\Phi(w_n) = \frac{1}{2}(2+\alpha+\beta T)\eta_n^2$$

and

$$\Psi(w_n) = \sum_{t=1}^{T} F(t,\eta_n) - \frac{\mu}{\lambda}G(\eta_n).$$

Note that $G(\eta_n) \le 0$ by (3.5). Then, we have

$$\begin{aligned} I_\lambda(w_n) &= \Phi(w_n) - \lambda\Psi(w_n) \\ &= \frac{1}{2}(2+\alpha+\beta T)\eta_n^2 - \lambda\sum_{t=1}^{T} F(t,\eta_n) + \mu G(\eta_n) \\ &\le \frac{1}{2}(2+\alpha+\beta T)\eta_n^2 - \lambda\sum_{t=1}^{T} F(t,\eta_n). \end{aligned} \tag{4.8}$$

Now, we consider two cases.

Case 1: $B < \infty$. From the fact that $\lambda > \lambda_1$ and the definition of λ_1 in (3.3), we have $B - \frac{2+\alpha+\beta T}{2\lambda} > 0$. Let

$$\epsilon \in \left(0, B - \frac{2+\alpha+\beta T}{2\lambda}\right). \tag{4.9}$$

From (4.6), there exists $N_1 \in \mathbb{N}$ such that

$$\sum_{t=1}^{T} F(t,\eta_n) > (B-\epsilon)\eta_n^2 \quad \text{for } n \ge N_1.$$

This, together with (4.8), implies that

$$I_\lambda(w_n) \le \left(\frac{1}{2}(2+\alpha+\beta T)\eta_n^2 - \lambda(B-\epsilon)\right)\eta_n^2.$$

Thus, from (4.9) and the fact that $\lim_{n\to\infty}\eta_n = \infty$, we have $\lim_{n\to\infty} I_\lambda(w_n) = -\infty$.

Case 2: $B = \infty$. Choose

$$M > \frac{2+\alpha+\beta T}{2\lambda}. \tag{4.10}$$

Then, (4.6) implies that there exists $N_2 \in \mathbb{N}$ such that

$$\sum_{t=1}^{T} F(t,\eta_n) > M\eta_n^2 \quad \text{for } n \ge N_2.$$

Thus, from (4.8),

$$I_\lambda(w_n) \le \left(\frac{1}{2}(2+\alpha+\beta T)\eta_n^2 - \lambda M\right)\eta_n^2.$$

Then, from (4.10) and the fact that $\lim_{n\to\infty}\eta_n = \infty$, we have $\lim_{n\to\infty} I_\lambda(w_n) = -\infty$.

Combining the above two cases, we see that the functional I_λ is always unbounded from below. Hence, the alternative (b_1) does not hold. Therefore, there exists a sequence

$\{u_n\}$ of critical points of I_λ such that $\lim_{n\to\infty} ||u_n||_X = \infty$. Applying Lemma 2.2 completes the proof of the theorem. □

Using Lemma 4.1 (c) and arguing as in the proof of Theorem 3.1, we can prove Theorem 3.2. For the completeness, we give the proof below.

Proof of Theorem 3.2. Let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by (2.4) and (2.5), respectively. Then, as before, Φ and Ψ satisfy all the regularity assumptions given in Lemma 4.1.

By the definition of C in (3.2), there exists a sequence $\{\xi_n\}$ of positive numbers such that $\lim_{n\to\infty} \xi_n = 0$ and

$$C = \lim_{n\to\infty} \frac{\sum_{t=1}^{T} \max_{|x|\le\xi_n} F(t,x)}{\xi_n^2}.$$

By the fact that $\inf_X \Phi = 0$ and the definition δ, we have $\delta = \liminf_{r\to 0^+} \varphi(r)$. Then, as in showing (4.4) and (4.5) in the proof of Theorem 3.1, we can prove that $\delta < \infty$ and that if $\lambda \in (\lambda_3, \lambda_4)$ and $\mu \in [0, \overline{\mu}_2)$, then $\lambda \in (0, 1/\delta)$. Let $\lambda \in (\lambda_3, \lambda_4)$ and $\mu \in [0, \overline{\mu}_2)$ be fixed. Then, by Lemma 4.1 (c), we see that one of the following alternatives holds

(c_1) either there is a global minimum of Φ which is a local minimum of $I_\lambda = \Phi - \lambda\Psi$, or

(c_2) there exists a sequence $\{u_n\}$ of critical points of I_λ which converges weakly to a global minimum of Φ.

In the following, we show that alternative (c_1) does not hold. By the definition of C in (3.2), there exists a sequence $\{\eta_n\}$ of positive numbers such that $\lim_{n\to\infty} \eta_n = 0$ and

$$C = \lim_{n\to\infty} \frac{\sum_{t=1}^{T} F(t,\eta_n)}{\eta_n^2}. \tag{4.11}$$

For each $n \in \mathbb{N}$, let $w_n : [-1, T+2]_{\mathbb{Z}} \to \mathbb{R}$ be defined by (4.7) with the above η_n. Then, as in the cases 1 and 2 of the proof of Theorem 3.1, we can obtain that, for n large enough, if $C < \infty$, then

$$I_\lambda(w_n) \le \left(\frac{1}{2}(2+\alpha+\beta T)\eta_n^2 - \lambda(C-\epsilon)\right)\eta_n^2,$$

where

$$\epsilon \in \left(0, C - \frac{2+\alpha+\beta T}{2\lambda}\right),$$

and if $C = \infty$, then

$$I_\lambda(w_n) \le \left(\frac{1}{2}(2+\alpha+\beta T)\eta_n^2 - \lambda M\right)\eta_n^2,$$

where M satisfies (4.10). Therefore, we always have $I_\lambda(w_n) < 0$ for large n. Then, since $\lim_{n\to\infty} I_\lambda(w_n) = I_\lambda(0) = 0$, we see that 0 is not a local minimum of I_λ. This, together with the fact that 0 is the only global minimum of Φ, shows that alternative (c_1) does not hold. Therefore, there exists a sequence $\{u_n\}$ of critical points of I_λ which converges weakly (and thus also strongly) to 0. An application of Lemma 2.2 completes the proof of the theorem. □

Finally, we point out that Corollaries 3.1, 3.3, and 3.5 follow directly from Theorem 3.1, and Corollaries 3.2, 3.4, and 3.6 are obviously consequences of Theorem 3.2.

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Design of Robust PID Controller for Power System Stabilization Using Bacterial Foraging Algorithm

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Abstract: In this paper, a novel bacterial foraging algorithm (BFA) based approach for robust and optimal design of PID controller connected to power system stabilizer (PSS) is proposed for damping low frequency power oscillations of a single machine infinite bus bar (SMIB) power system. This paper attempts to optimize three parameters (Kp, Ki, Kd) of PID-PSS based on foraging behaviour of Escherichia coli bacteria in human intestine. The problem of robustly selecting the parameters of the power system stabilizer is converted to an optimization problem which is solved by bacterial foraging algorithm with a carefully selected objective function. The eigenvalue analysis and the simulation results obtained for internal and external disturbances for a wide range of operating conditions show the effectiveness and robustness of the proposed BFAPSS. Further, the time domain simulation results when compared with those obtained using conventional PSS and Particle Swarm Optimization (PSO) based PSS show the superiority of the proposed design.

Keywords: *Bacterial Foraging Algorithm; Power system stabilizer; Power system Stability; PID controller.*

Mathematics Subject Classification (2010): 34D20, 34H15, 93D21.

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1 Introduction

Power systems are highly non-linear and exhibit low frequency oscillations due to poor damping caused by the high-gain, fast-acting automatic voltage regulator (AVR) employed in the excitation system. The power system utilities employ power system stabilizers (PSSs) to introduce supplementary stabilizing signals into the excitation system to increase the damping of the low frequency oscillations. Among various types of PSSs, the fixed-structure lag-lead type is preferred by the utilities due to its operational simplicity and ease of tuning PSS parameters. However, the robustness of these PSS under changing conditions is a major concern.

The concept of PSSs and their tuning procedures were well explained in literature. A well-tuned lag-lead type PSS can effectively improve dynamic stability. Many approaches have been proposed to tune PSSs, such as the sensitivity approach [4], pole placement technique [2], and the damping torque approach [1]. Global optimization technique like genetic algorithm (GA) [5], Particle Swarm Optimization (PSO) [12], tabu search [6] and simulated annealing (SA) [7] are attracting the attention in the field of PSS parameter optimization in recent times. But when the system has a highly epistatic objective function (i.e., where the parameters being optimized are highly correlated) and number of parameters to be optimized are large, GA has been reported to exhibit degraded efficiency [8]. Bacterial foraging algorithm has been proposed and introduced as a new evolutionary technique in [9]. Passino et al. pointed out that the foraging algorithms can be integrated in the framework of evolutionary algorithms. To overcome the drawbacks of conventional methods for PSS design, a new optimization scheme known as bacterial foraging (BF) is used for the PSS parameter design. This algorithm (BFA) appeared as a promising one for handling the optimization problems [13]. It is a computational intelligence based technique that is not largely affected by the size and nonlinearity of the problem and can converge to the optimal solution in many cases where many analytical methods fail to converge. Considering the strength of this algorithm, it is employed in the present work for the optimal tuning the parameters of the PSS.

In this paper a new/improved BFA-based optimal determination of PID-PSS parameters is presented which overcomes the shortcomings of previous works. In order to design a robust PSS which guarantees stability of system in a wide range of operating conditions, the objective function is defined such that the resultant time response is restricted to lie within specific bounds as well as limiting the amount of overshooting of power system response when subjected to disturbances. The performance of the BFAPSS is compared with those obtained with other techniques such as conventional and Particle Swarm Optimization (PSO) by plotting the time response curves for step disturbance. Further, the robustness of the controller so designed is established by choosing any one set of parameters for a particular operating condition and testing its performance with its fixed structure for other operating conditions too.

2 Power System Model Studied

The system considered in this paper is a synchronous machine connected to an infinite bus through a transmission line, as shown in Figure 1.The linear incremental model of a synchronous machine connected to a large system is shown in Figure 2.

The state equation under a particular loading condition can be written as [1].

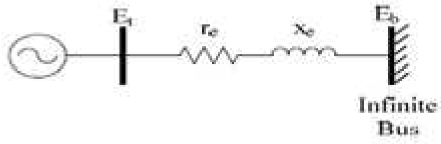

Figure 1: Single Machine connected to Infinite Bus System.

$$dx(t)/dt = Ax(t) + Bu(t), \quad (1)$$

$$y(t) = Cx(t), \quad (2)$$

where $x(t)$ is the state vector, $u(t)$ is the control input and $y(t)$ is the output and A, B, C are the matrices of appropriate dimensions. The following physical variables are chosen as the state and output for the power system under consideration.

$$x(t) = [\Delta(t) \quad \Delta\omega(t) \quad \Delta Eq'(t) \quad \Delta Efd(t)]^T, \quad (3)$$

$$y(t) = [0 \quad 1 \quad 0 \quad 0]. \quad (4)$$

The system matrices as taken from [1] are given below

$$A = \begin{vmatrix} 0 & 314 & 0 & 0 \\ -K1/M & -D/M & 0 & 0 \\ -K4/M & 0 & -1/K3T'do & 0 \\ KeK5/Ta & 0 & -K6Ke/Te & -1/Te \end{vmatrix},$$

$$B = [0 \quad 0 \quad 0 \quad Ke/Te], \quad (5)$$

$$C = [0 \quad 1 \quad 0 \quad 0]. \quad (6)$$

The parameters K1-K6 in system matrix A are functions of real power output P and reactive power output Q of the generator [11, 12]. Thus it is observed that the elements of the A matrix change as the operating point of the generator changes. When the system is perturbed it is possible that it becomes unstable or operates with sustained oscillations. It is therefore necessary to design a PSS which will guarantee stability of the system and suppress these unwanted oscillations. Further, it is necessary to change the PSS parameters according to the drift in the operating conditions.

The main objective of this work is to design the power system stabilizer using Bacterial Foraging Algorithm such that the controller structure so designed rejects the internal and external disturbances and is immune to machine parameters variations.

3 Particle Swarm Optimization

Particle Swarm Optimization (PSO) is a population based stochastic optimization technique developed by Eberhart and Kennedy [12]. It shares many similarities with evolutionary computation techniques such as Genetic Algorithms (GA). The system is initialized with a population of random particles where each particle is a candidate solution.

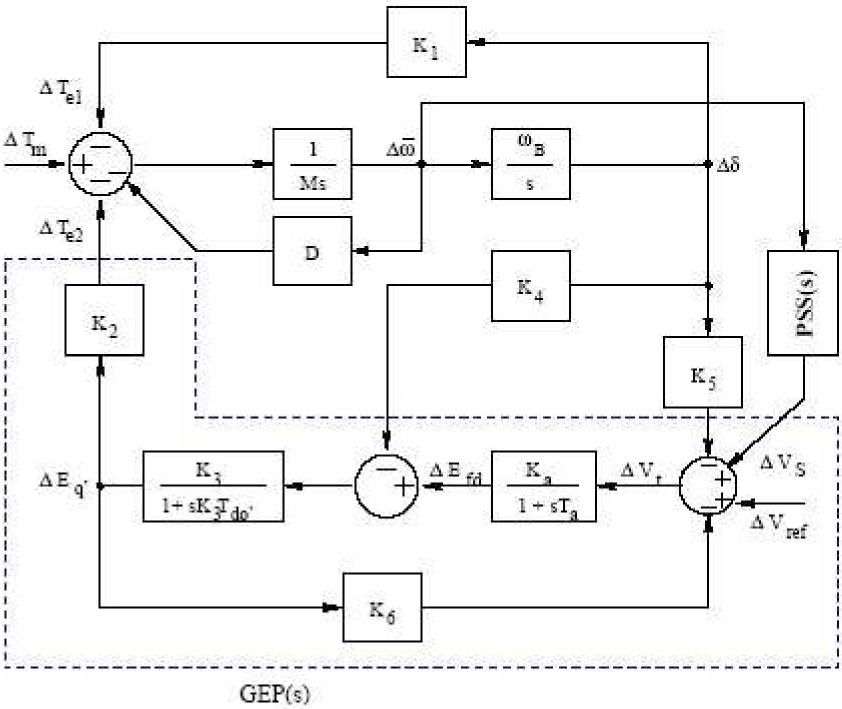

Figure 2: Linearized model of a synchronous machine with an exciter and stabilizer.

The particles fly through the problem space by following the current optimum particles and searches for optima by updating their positions. However, unlike GA, PSO has no evolution operators such as crossover and mutation. The advantages of PSO over GA are the ease of programming and fast convergence [8, 9]. In the PSO algorithm, each particle updates its velocity and position by the following relationships:

$$Vi^{k+1} = wVi^k + c1rand1(pbesti - Si^k) + c2rand2(gbesti - Si^k), \quad (7)$$

$$Si^{k+1} = S^k + Vi^{k+1}, \quad (8)$$

where $c1$ and $c2$ are cognition and social parameters respectively, $rand1()$ and $rand2()$ are constant numbers in the range of [0,1], w is the inertia weight. Vi represents the velocity of the i^{th} particle and Si is its position, pbesti and gbesti are local best and global best positions respectively. The velocity of particle in equation (7) depends on its previous velocity, its own thinking and social psychological adaptation of the population. The PSO algorithm starts with random initialization of population and velocity. The search for the optimum solution is continued unless one of the stopping criteria is reached. The stopping criteria are: either the maximum iterations are reached, or there is no further improvement in the optimal solution. The values of parameters for PSO used in this study are as follows: No. of particles 20; No. of swarms 3 (Kp, Ki, Kd); No. iteration=

500; Maximum particle velocity (upper-lower bound) / No. iteration = 0.05; $c1, c2 = 2, 2$; $wmax = 0.9, wmin = 0.4$.

To compute the optimum parameter values of PID-PSS shown in Figure 4, a 0.1 step change in reference mechanical torque (ΔTm) is assumed and the performance index in equation (9) is minimized using Particle Swarm Optimization. The settling time (ts) and peak overshoot ($\Delta \omega p$) are evaluated for each iteration.

4 Bacterial Foraging Algorithm

Bacterial foraging algorithm is inspired by an activity called "chemotaxis" exhibited by bacterial foraging behaviors. Motile bacteria such as E. coli and salmonella propel themselves by rotation of the flagella. To move forward, the flagella rotates counterclockwise and the organism "swims" or "runs" while a clockwise rotation of the flagellum causes the bacterium to randomly "tumble" itself in a new direction and swim again. Alternation between "swim" and "tumble" enables the bacterium to search for nutrients in random directions. Swimming is more frequent as the bacterium approaches a nutrient gradient. Tumbling, hence direction changes, is more frequent as the bacterium moves away from some food to search for more. Basically, bacterial chemotaxis is a complex combination of swimming and tumbling that keeps bacteria in places of higher concentrations of nutrients. The foraging strategy of Escherichia coli bacteria present in human intestine can be explained by three processes, namely chemotaxis, reproduction, and elimination-dispersal [9].

In Chemotaxis, a unit walk with random direction represents a "tumble" and a unit walk with the same direction in the last step indicates a "run". $C(i)$ is called the run length unit parameter, is the chemo tactic step size during each run or tumble. With the activity of run or tumble at each step of the chemotaxis process, a step fitness will be evaluated. In the reproduction step, all bacteria are stored in reverse order according to the health status. Here only the first half of the population survives, and a surviving bacterium splits into two identical ones, which are then placed in the same locations. Thus, the population of bacteria keeps constant. It is possible that in the local environment, the life of a population of bacteria changes either gradually by consumption of nutrients or suddenly due to some other influence. Events can kill or disperse all the bacteria in a region. They have the effect of possibly destroying the chemotactic progress, but in contrast, they also assist it, since dispersal may place bacteria near good food sources. Elimination and dispersal helps in reducing the behavior of stagnation (i.e., being trapped in a premature solution point or local optima). The flow chart of the iterative algorithm is shown in Figure 3.

The bacteria with large run length unit $C(i)$ have the exploring ability and stay for a while in several domains containing local optima. It can also escape from the local optima to enter the domain with global optima. On the other hand, a bacterium with small run length unit $C(i)$ is attracted into the domain with local optima and exploits this local minimum for its whole life cycle. It is therefore necessary to choose the value of $C(i)$ with larger value for faster convergence. In this algorithm, cost function value is taken as objective function and the bacterium having minimum cost function (J) is retained for the next generation. For swarming, the distances of all the bacteria in a new chemotactic stage are evaluated from the global optimum bacterium till that point. To speed up the convergence, a simple heuristic rule to update one of the coefficients (C) of BFA algorithm is formulated.

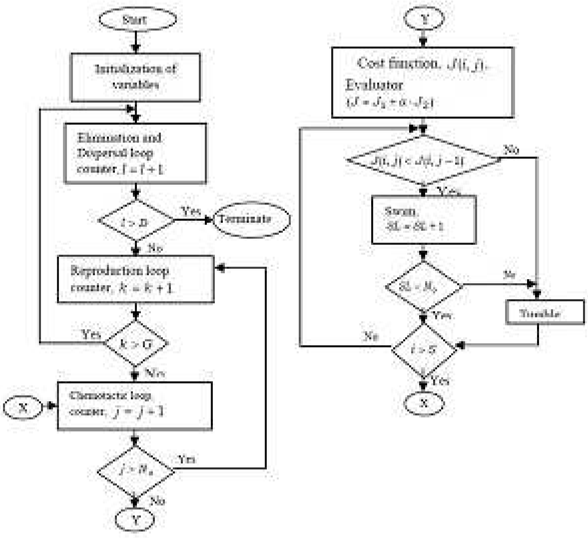

Figure 3: Flowchart of Bacterial Foraging Algorithm.

5 BFA based Tuning of PID-PSS

PID (proportional integral derivative) control is one of the earlier control strategies. Its early implementation was in pneumatic devices, followed by vacuum and solid state analog electronics, before arriving at today's digital implementation of microprocessors. It has a simple control structure which was understood by plant operators and which they found relatively easy to tune. Since many control systems using PID control have proved satisfactory, it still has a wide range of applications in industrial control. It has been found possible to set satisfactory controller parameters from less plant information than a complete mathematical model. In the proposed design approach, the PID control structure shown in Figure 4 is used as the power system stabilizer as opposed to the traditional lead-lag controller. In Figure 4, the speed deviation is the input to the

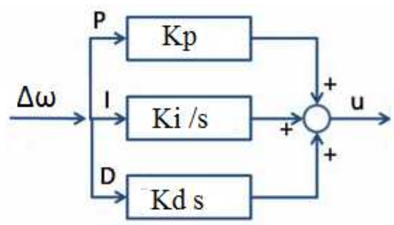

Figure 4: The PID power system stabilizer.

controller and u is the supplementary stabilizing signal. The PID parameters Kp, Ki, and Kd are tuned using the BFA technique discussed in Section 4. To compute the optimum parameter values, a 0.1 step change in reference mechanical torque (ΔTm) is assumed and the performance index

$$F = \frac{1}{(1 + \Delta\omega p)(1 + ts)} \tag{9}$$

is minimized using bacterial foraging algorithm. The settling time (ts) and peak overshoot ($\Delta\omega p$) are evaluated for each iteration. The PID parameters selected using the above objective function are used to form the augmented A matrix as given below:

$$\begin{pmatrix} 0 & 314 & 0 & 0 & 0 \\ -K1/M & 0 & -K2/M & 0 & 0 \\ -K4/T'd0 & 0 & -1/K3T'do & 1/T'do & 0 \\ MKe(-K5 + MKi/314 - KdK1)MTe & KeKp/Te & -MK6Ke - K2KeKd/Te & 1/Te & Ke/Te \\ MKi/314 - KdK1/MTw & Kp/Tw & -K2Kd/MTw & 0 & -1/Tw \end{pmatrix}.$$

The following machine parameters are chosen for study $xd = 1.6; x'd = 0.32; xq = 1.55; vt0 = 1.05;$ $\omega = 100 rad/s; T'd0 = 6.0s D = 0.0; M = 10.0; re = 0; xe = 0.4; Ke = 50.0; Te = 0.05s; T = 5s$. The parameters for BFA used in this study are as follows: $Nc = 5, Nre = 4, Ned = 10, Ns = 4, datt = 0.01, hrep = 0.01, watt = 0.4, wrep = 0.42, w = 0.8, c1 = 2.0$ and $c2 = 2.0$.

6 Tuning Results and Discussion

Simulation tests were made using a computational program that represents the single machine connected to infinite bus bar system. The machine with PID-PSS is represented as 5th order state space model with saturation neglected.

The different operating conditions [2] considered are given in Table 1. The simulation study for the operating conditions mentioned using Bacterial Foraging Algorithm (BFA) is carried out for a step disturbance of 0.1 mechanical torque (ΔTm). Simulation study is also carried out for the mentioned operating conditions for the PSS designed using conventional and Particle Swarm Optimization (PSO). The conventional PSS parameters are calculated using frequency response method. The PSS parameters obtained by the application of conventional, PSO and BFA along with the corresponding eigenvalues are shown in Table 2. From Table 2, it is observed that the real parts of closed loop eigenvalues obtained using BFAPSS are shifted to the left half of the s-plane which provides more damping. The time response specifications obtained from the transient response curves are shown in Table 3.

From Figures 5–7 and Table 3, it is observed that the performance of the PSS designed using BFA is far superior compared to the PSS designed using conventional as well as Particle Swarm Optimization (PSO). Figure 8 illustrates the convergence of the objective function with Particle Swarm Optimization (PSO) and BFA. From the convergence characteristics it is clear that BFA offers superior performance than (PSO). Figure 9 shows the speed deviation for different operating conditions with BFA PSS when the system is subjected to 0.1 p.u step disturbance in the reference input voltage ($\Delta Vref$).

In power system the operating condition changes very fast. The controller designed for one operating condition may not give satisfactory performance to other operating conditions. Therefore, it becomes necessary that the controller parameters need to be tuned according to the changes in the operating condition which is very difficult to accomplish online even using very fast computer. Therefore it is necessary to design a PSS which is robust in behaviour. From Table 2 the eigen values obtained for the power system with BFAPSS do not change appreciably which suggests the robustness of the PSS. It is therefore possible to choose the PSS parameters obtained by BFA at any one operating condition which can be chosen and retained for other operating conditions also.

Operating Conditions			
Operating points	P1	P2	P3
Real Power(P)	1.2	0.9	0.7
Reactive Power(Q)	0.2	0.3	0.2

Table 1: Operating conditions of the machine.

Operating Conditions	CPS	PSOPSS	BFAPSS
P=1.2,Q=0.2	Kpss=9.2734 T1=0.3806 T2=0.1	Kp=12.74 Ki=12.84 Kd=14.35	Kp=31.58 Ki=6.3202 Kd=32.32
Eigenvalues	-21.2515 +4.9661i -21.2515 -4.9661i - 0.7438 + 6.6601i - 0.7438 - 6.6601i -5.6514	-14.914 +20.104i -14.914 -20.104i -11.569 -4.7024 -0.30501	-15.01+29.1567i -15.01-29.1567i - 13.4750 -6.8227 -0.3272
P=0.9,Q=0.3	Kpss=7.6451 T1=0.48746 T2=0.1	Kp=10.66 Ki=4.556 Kd=14.82	Kp=48.98 Ki=9.0988 Kd=25.17
Eigenvalues	-21.3386 +4.1240i -21.3386 -4.1240i -0.6869 +6.5345i -0.6869 -6.5345i -5.3633	-14.938 +20.454i -14.938 -20.454i -11.699 -4.8983 -0.32239	-14.5016+26.0508i -14.5016-26.0508i - 13.8296 -6.1569 -0.3241
P=0.7,Q=0.2	Kpss=5.571 T1=0.6776 T2=0.1	Kp=40.99 Ki=7.650 Kd=6.306	Kp=38.05 Ki=8.5241 Kd=36.97
Eigenvalues	-21.3488 + 3.3392i -21.3488 - 3.3392i - 0.6133 +6.2708i -0.6869 -6.5345i -5.1956	-13.69 +12.755i -13.69 -12.755i -10.956 -1.6008 -0.26435	-15.1515+ 31.322i -15.1515- 31.322i - 13.3897 -7.0191 -0.3264

Table 2: Eigenvalue analysis.

For the study of robustness, the PSS parameters designed using BFA for light load condition are chosen. With these PSS parameters fixed at all operating conditions the dynamic response of the system for 0.1p.u mechanical disturbance (ΔTm) for light, normal and heavy operating conditions are obtained and plotted as shown in Figure 10 and Figure 11. From Figure 10 and Figure 11, it is evident that the oscillations due to

Operating Point	PSOPSS		BFAPSS	
	Settling time	Peakoveshoot	Settling time	Peakoveshoot
P=1.2;Q=0.2	2.77	$0.66X10^{-}4$	3.4	$0.57X10^{-}4$
P=0.9;Q=0.3	3.13	$0.69X10^{-}4$	1.9	$0.63X10^{-}4$
P=0.7;Q=0.2	1.34	$1.45X10^{-}4$	2.4	$0.83X10^{-}4$

Table 3: Settling time max. overshoot comparison.

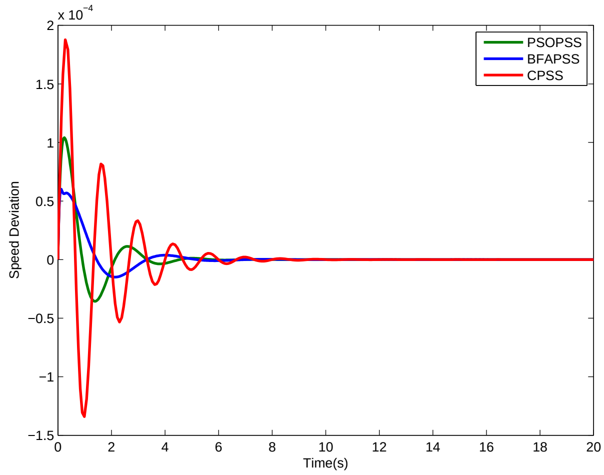

Figure 5: Speed deviation for operating condition $(P = 1.2, Q = 0.2)$.

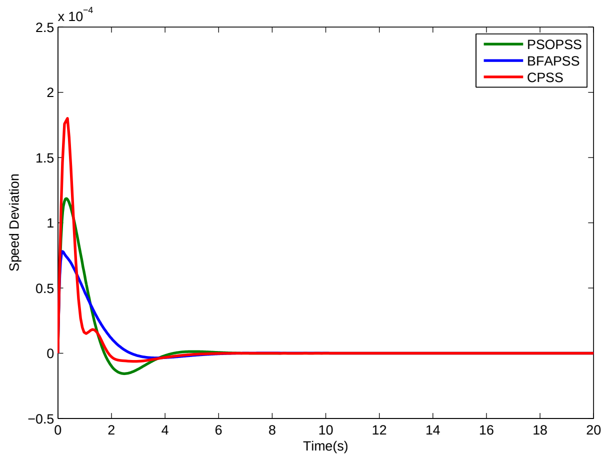

Figure 6: Speed deviation for operating condition $(P = 0.9, Q = 0.3)$.

disturbances are completely suppressed and the system rejects external disturbances at

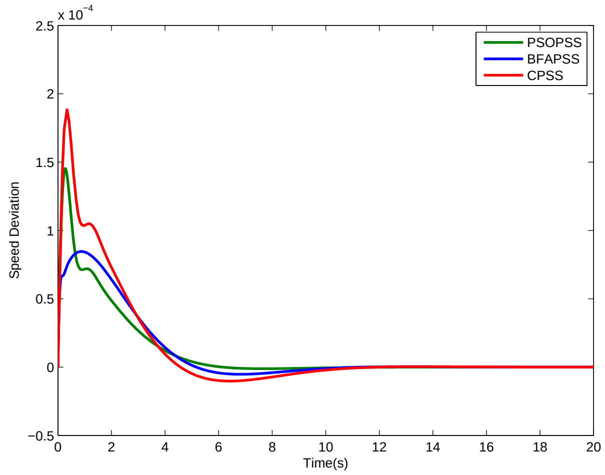

Figure 7: Speed deviation for operating condition $(P = 0.7, Q = 0.2)$.

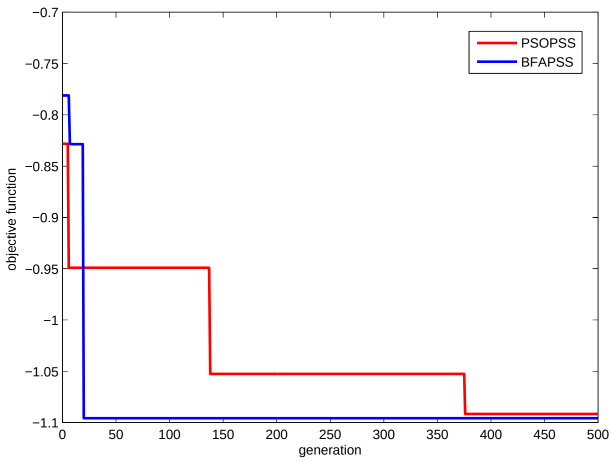

Figure 8: Convergence comparison of PSO and BFA.

all operating conditions. In addition, the system performance with the proposed PSS is much better than that of PSOPSS and the oscillations are damped out much faster. This illustrates the potential and superiority of the proposed design approach to obtain an optimal set of PSS parameters.

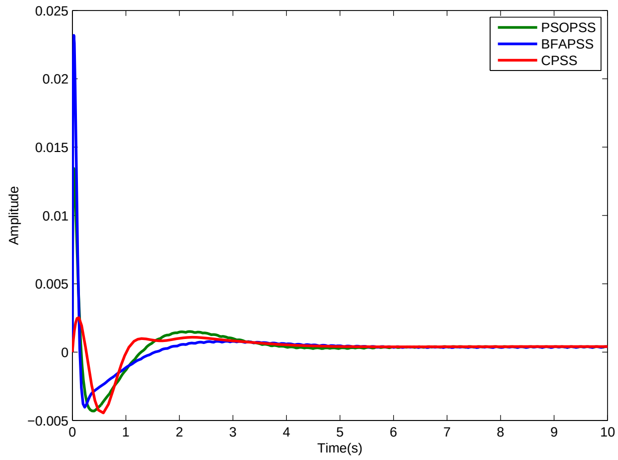

Figure 9: Speed deviation for different operating conditions for a 0.1 p.u step change in reference input voltage ($\Delta Vref$)

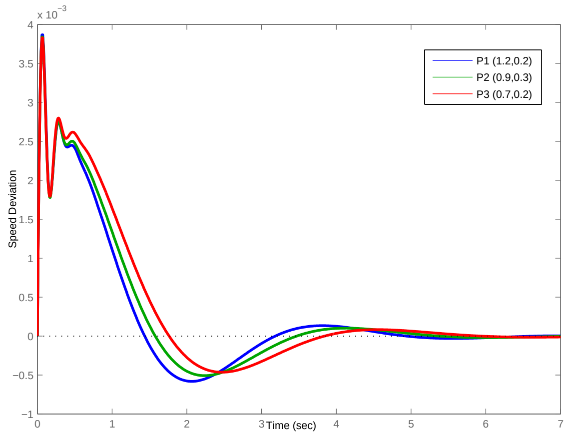

Figure 10: Speed deviation for different operating conditions using BFA PSS with $Kp = 38.0553$, $Ki = 8.5241$, $Kd = 36.9748$.

7 Conclusion

In this study, optimal design of robust power system stabilizer (PSS) for single machine system using Bacterial Foraging Algorithm is proposed. Eigenvalue analysis under different operating conditions reveals that undamped and lightly damped oscillation modes are shifted to a specific stable zone in the s-plane. These results show the potential of Bacterial Foraging Algorithm for optimal design of PSS parameters. Further, from the

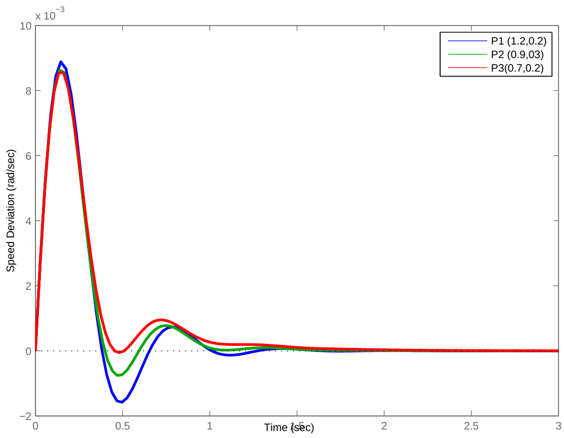

Figure 11: Speed deviation for different operating conditions using PSOPSS with $Kp = 40.997, Ki = 7.6503, Kd = 6.30605$.

simulation results it is observed that, when a system is subjected to internal and external disturbances by retaining the same structure and parameter of the controller which was obtained for any one operating condition works effectively over a wide range of loading conditions which is very difficult to accomplish on line. This shows the robustness of the controller designed using BFA. Furthermore, the simulation results also show that the proposed method in this paper gives much improved performance when compared to the performance of conventional and Particle Swarm Optimization (PSO) based design of controller for PSS. Further, the convergence of the objective function in the proposed method is much faster when compared with Particle Swarm Optimization (PSO).

In this paper, the linear incremental model of single machine connected to infinite bus has been considered for the design of PID controller even though the actual system is highly non-linear one and therefore it becomes necessary to validate the results obtained here by laboratory test which has been taken for future work. The Bacterial Foraging Algorithm (BFA) remains to be tried out for designing controllers in the capacitive area and also for multi-machine complex power system.

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Synchronization Between a Fractional Order Chaotic System and an Integer Order Chaotic System

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Abstract: This paper deals with synchronization between a fractional order Coullet chaotic system and an integer order Rabinovich-Fabrikant chaotic system by using tracking control and stability theory of fractional order system. An effective controller is designed to synchronize these two systems. Numerical simulations have been done by using Mathematica and Matlab both. Numerical solutions via Grünwald-Letnikov method have been used in Matlab. Numerical results show that method is effective and feasible.

Keywords: *synchronization; fractional order derivatives; fractional order coullet system, integer order Rabinovich-Fabrikant chaotic system, tracking control method, Grünwald-Letnikov method.*

Mathematics Subject Classification (2010): 37B25, 37D45, 37N30, 37N35, 70K99.

1 Introduction

Synchronization is the dynamical process by which two or more oscillators adjust their rhythms due to a weak interaction [38]. This problem has received the great attention in the literature due to its importance in engineering and physical sciences, as well as in the challenging biological and social entities [38,39,44]. Chaotic synchronization did not attract much attention until Pecora and Carroll [34] introduced a method to synchronize two identical chaotic systems with different initial conditions in 1990 and they demonstrated that chaotic synchronization could be achieved by driving or replacing one of the variables of a chaotic system with a variable of another similar chaotic device. From then

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on, enormous studies have been done by researchers on the synchronization of dynamical systems. In the last two decades considerable research has been done in non-linear dynamical systems and their various properties. One of the most important properties is synchronization. Synchronization techniques have been improved in recent years and many different methods are applied theoretically as well as experimentally to synchronize the chaotic-systems including adaptive control [7,10,27], backstepping design [46–48], active control [5,23,52], nonlinear control [6,33] and observer based control method [11,50]. Using these methods, numerous synchronization problems of well-known chaotic systems such as Lorenz, Chen, Lü and Rössler system have been worked on by many researchers. In sequel to the study of chaotic systems, chaotic dynamics of fractional order systems has also been studied popularly. Since many real objects are generally fractional, so fractional calculus opens wide ways to describe a real object more accurately than the classical integer methods. So the fractional order methods become global and allow greater degree of flexibility in the study of dynamical models. Due to advantage over integer methods they have a lot of important applications in the various fields such as control theory [35, 42], viscoelastic [3], diffusion [9, 25], bioengineering [29], dielectric polarization [45], electrode-electrolyte polarization [24], electromagnetic waves [22], medicine [19] etc. Chaotic dynamics of fractional order systems is becoming an important field of investigation in nonlinear dynamics. Although the fractional calculus is more than three century old subject, yet in past few years it has increased rapidly. Analysis of fractional order dynamical systems has been studied by authors in [31,32,40]. Geometric and physical interpretation of fractional integration and fractional differentiation has also been studied by Podlubny [41]. In the continuation of study of chaos in fractional order dynamical systems, one of the important property synchronization of dynamical systems of fractional order has also got much attention. We can see many works on chaos in fractional order systems: in Chen's system [26], Volta system [37], Rössler system [12], Chua system [21], Duffing oscillators [18], cellular network [1], Lorenz system [20] etc. And synchronization between identical as well as non-identical fractional order systems has been presented in [4, 13, 14, 16, 17, 28, 49, 51, 53, 54].

The aim of this study is to investigate the synchronization behavior between an integer order system and a fractional order chaotic system. Synchronization between different orders has its own importance since it plays an important role in security of communication as well as it can also generate hybrid chaotic transient signals before final states. So it is necessary to synchronize two different order systems. Here we have used tracking control method to synchronize Rabinovich-Fabrikant integer order system and Coullet chaotic system of fractional order. Numerical simulations have been done by using both Matlab and Mathematica. For fractional order system we have used Grünwald-Letnikov method [40].

2 Preliminaries

In this section we mention some fundamental properties and definitions of fractional order derivatives.

2.1 Fractional derivatives and its approximations

Fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator ${}_aD_t^{\alpha}$, where a and t are the limits of the operation and α is the fractional order which can be a complex number, $R(\alpha)$ denotes the real part of α. The continuous integro-differential operator is defined as:

$$ {}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & R(\alpha) > 0, \\ 1, & R(\alpha) = 0, \\ \int_a^t d\tau^{-\alpha}, & R(\alpha) < 0. \end{cases} $$

The three definitions used for general fractional differintegral are Grünwald-Letnikov(GL) definition, the Riemann-Liouville (RL) and Caputo's definition [40]. The GL definition is given as:

$$ {}_aD_t^\alpha f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} f(t - jh), $$

where $[\cdot]$ denotes the integer part. And the RL definition is given by

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, n-1 < \alpha < n, $$

where $\Gamma(\cdot)$ is the gamma function. The Caputo fractional derivative is

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, n-1 < \alpha < n. $$

For numerical calculations of fractional-order derivatives we have used Grünwald-Letnikov method which is derived from Grünwald-Letnikov definition. It is also called Power Series Expansion method [8, 15, 36].

2.2 Methodology for synchronization between fractional order and integer order chaotic system

In this section we put a glimpse of methodology for synchronization between fractional order and integer order chaotic system via tracking control. Consider the following n-dimensional fractional order chaotic system as drive (master) system

$$ \frac{d^{q_\alpha} x}{dt^{q_\alpha}} = f(x), \tag{2.1} $$

where $x \in \mathbb{R}^n$, fractional order $q_\alpha = (q_{\alpha_1}, q_{\alpha_2}, ...q_{\alpha_n})^T$; $(0 < q_{d_i} < 1)$ may be unequal. $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a differentiable function. Now, consider the following n-dimensional chaotic system of integer order as :

$$ \frac{dy}{dt} = g(y), $$

where $y \in \mathbb{R}^n$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a differentiable function and construct the following integer order response system:

$$ \frac{dy}{dt} = g(y) + u(y, x), \tag{2.2} $$

where $u(y, x)$ is the controller to be designed via tracking control method.

Our goal in this paper is to design controller $u(y, x)$ such that

$$ \lim_{t \to \infty} \|e\| = \lim_{t \to \infty} \|y - x\| = 0, $$

where $\| \cdot \|$ is the Euclidean norm (here x in response system (2.2) belongs to chaotic system (2.1)), then the systems (2.1) and (2.2) will be synchronized.

2.3 Stability of Fractional order systems

An autonomous system $D^q X = AX, X(0) = 0$, with $0 < q \leq 1$, $X \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is asymptotically stable iff $|\arg\lambda| > q\pi/2$ is satisfied for all eigenvalues (λ)of matrix A. Also this system is stable iff $|\arg\lambda|\geq q\pi/2$ is satisfied for all eigenvalues of a matrix A and those critical eigenvalues which satisfy $|\arg\lambda| > q\pi/2$ have geometric multiplicity one [30].

3 System Description

3.1 Rabinovich-Fabrikant chaotic system of integer order

The Rabinovich-Fabrikant chaotic system is a set of three coupled ordinary differential equations exhibiting chaotic behavior for certain values of parameters. They are named after Mikhail Rabinovich and Anatoly Fabrikant, who described them in 1979 [43]. The equations of system are:

$$\left.\begin{aligned} \dot{y_1} &= y_2(y_3 - 1 + y_1^2)+\gamma y_1, \\ \dot{y_2} &= y_1(3y_3 + 1 - y_1^2)+\gamma y_2, \\ \dot{y_3} &= -2y_3(y_1 y_2+\alpha), \end{aligned}\right\} \tag{3.1}$$

where α and γ are constant parameters that control the evolution of the system. For some values of α and γ, the system is chaotic but for other it tends to a stable periodic orbit. Figures given below show the chaotic behavior of Rabinovich-Fabrikant system with different values of parameters.

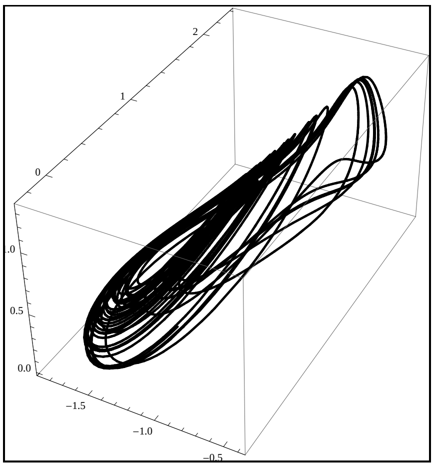

Figure 1: Chaotic behavior of the system with $\alpha = 0.87$ and $\gamma = 1.1$.

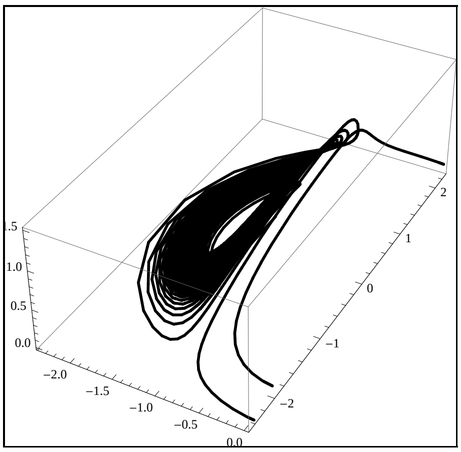

Figure 2: Chaotic behavior having tendency of stable periodic orbit with $\alpha = 0.14$ and $\gamma = 0.1$.

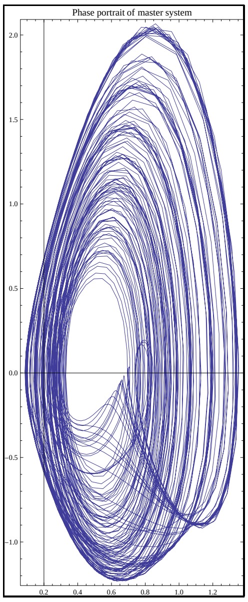

Figure 3: Phase portrait shows chaotic behavior of the system.

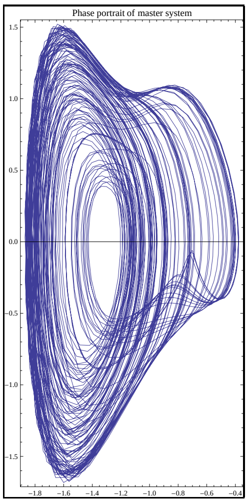

Figure 4: Phase portrait shows chaotic behavior of the system.

3.2 Coullet chaotic system of fractional order

The Coullet chaotic system consists of three fractional order differential equations with orders q_1, q_2, and q_3, respectively [2]

$$\left.\begin{aligned} \frac{d^{q_1}x_1}{dt^{q_1}} &= x_2, \\ \frac{d^{q_2}x_2}{dt^{q_2}} &= x_3, \\ \frac{d^{q_3}x_3}{dt^{q_3}} &= ax_1 + bx_2 + cx_3 + dx_1^3, \end{aligned}\right\} \tag{3.2}$$

where $a = 0.8$, $b = -1.1$, $c = -0.45$, and $d = -1$. We can vary values of q_1, q_2, and q_3 accordingly. Figures given below show chaotic behavior of the system with different values of q_1, q_2, and q_3, respectively.

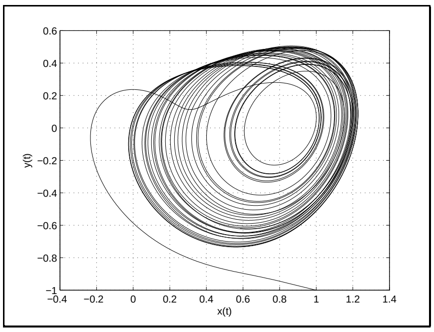

Figure 5: Chaotic attractor of the Coullet system in xy-plane with $q_1 = 0.90$, $q_2 = 0.97$, and $q_3 = 0.95$.

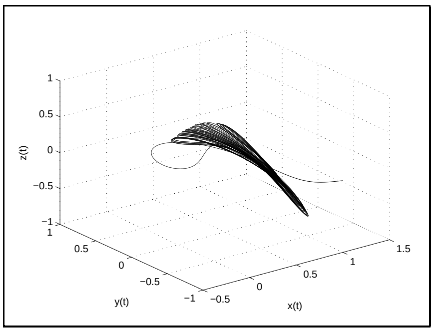

Figure 6: 3D chaotic attractor of the Coullet system with $q_1 = 0.90$, $q_2 = 0.97$, and $q_3 = 0.95$.

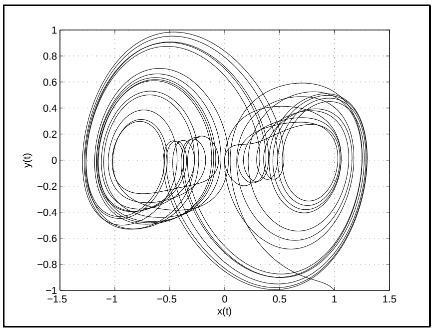

Figure 7: Chaotic attractor of the Coullet system in xy-plane with $q_1 = q_2 = q_3 = 0.98$.

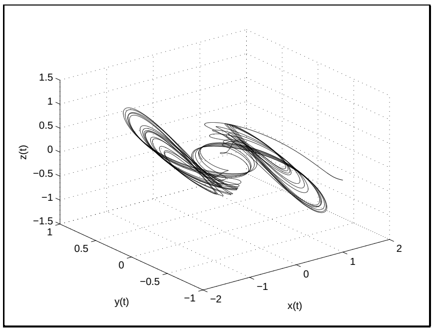

Figure 8: 3D chaotic attractor of the Coullet system with $q_1 = q_2 = q_3 = 0.98$.

3.3 Synchronization between Coullet chaotic system of fractional order and Rabinovich-Fabrikant chaotic system of integer order via Tracking control

In this section, we synchronize a fractional order derivative and an integer order derivative via tracking control. Consider fractional order Coullet system as a drive (master) system:

$$\left.\begin{aligned} \frac{d^{q_1}x_1}{dt^{q_1}} &= x_2, \\ \frac{d^{q_2}x_2}{dt^{q_2}} &= x_3, \\ \frac{d^{q_3}x_3}{dt^{q_3}} &= ax_1 + bx_2 + cx_3 + dx_1^3, \end{aligned}\right\} \tag{3.3}$$

where $a = 0.8$, $b = -1.1$, $c = -0.45$, and $d = -1$. Here we have taken $q_1 = q_2 = q_3 = 0.98$ and $q_1 = 0.90$, $q_2 = 0.97$, and $q_3 = 0.95$. One can take any other values of q_1, q_2 and q_3 $(0 < q \leq 1)$.

Now integer order Rabinovich-Fabrikant chaotic system is:

$$\left.\begin{aligned} \dot{y_1} &= y_2(y_3 - 1 + y_1^2) + \gamma y_1, \\ \dot{y_2} &= y_1(3y_3 + 1 - y_1^2) + \gamma y_2, \\ \dot{y_3} &= -2y_3(y_1 y_2 + \alpha), \end{aligned}\right\} \tag{3.4}$$

where α and γ are constant parameters. For $\alpha = 0.87$ and $\gamma = 1.1$ system is chaotic but for $\alpha = 0.14$ and $\gamma = 0.1$ (see Figures 1 and 2) it tends to a stable periodic orbit. Now construct Rabinovich-Fabrikant chaotic system as response system. The response system is:

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \end{pmatrix} = \begin{pmatrix} y_2(y_3 - 1 + y_1^2) + \gamma y_1 \\ y_1(3y_3 + 1 - y_1^2) + \gamma y_2 \\ -2y_3(y_1 y_2 + \alpha) \end{pmatrix} + \theta(x) + \tau(y, x), \tag{3.5}$$

where $\theta(x)$ is compensation controller and $\tau(y, x)$ is feedback controller.

According to methodology, we can obtain compensation controller for response system (3.5) as follows:

$$\theta(x) = \frac{dx}{dt} - g(x) = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} - \begin{pmatrix} x_2(x_3 - 1 + x_1^2) + \gamma x_1 \\ x_1(3x_3 + 1 - x_1^2) + \gamma x_2 \\ -2x_3(x_1x_2 + \alpha) \end{pmatrix}. \tag{3.6}$$

So from equation (3.5) and (3.6),

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \end{pmatrix} = \begin{pmatrix} y_2(y_3 - 1 + y_1^2) + \gamma y_1 \\ y_1(3y_3 + 1 - y_1^2) + \gamma y_2 \\ -2y_3(y_1y_2 + \alpha) \end{pmatrix} + \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix}$$

$$- \begin{pmatrix} x_2(x_3 - 1 + x_1^2) + \gamma x_1 \\ x_1(3x_3 + 1 - x_1^2) + \gamma x_2 \\ -2x_3(x_1x_2 + \alpha) \end{pmatrix} + \tau(y, x). \tag{3.7}$$

Let error $e_i = y_i - x_i$; $i = 1, 2, 3$. Then error system can be obtained from (3.7) described by

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \end{pmatrix} - \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \begin{pmatrix} y_2(y_3 - 1 + y_1^2) + \gamma y_1 \\ y_1(3y_3 + 1 - y_1^2) + \gamma y_2 \\ -2y_3(y_1y_2 + \alpha) \end{pmatrix}$$

$$- \begin{pmatrix} x_2(x_3 - 1 + x_1^2) + \gamma x_1 \\ x_1(3x_3 + 1 - x_1^2) + \gamma x_2 \\ -2x_3(x_1x_2 + \alpha) \end{pmatrix} + \tau(y, x).$$

This implies

$$\begin{pmatrix} \frac{de_1}{dt} \\ \frac{de_2}{dt} \\ \frac{de_3}{dt} \end{pmatrix} = G_1(x, e) + G_2(x, e), \tag{3.8}$$

where $G_1(x, e) = g(x_i + e_i) - g(x_i)$ and $G_2(x, e) = \tau(x_i + e_i, x_i)$; $i = 1, 2, 3$. Now the vector function $G_1(x, e)$ is

$$\begin{pmatrix} 2x_1x_2e_1 + 2\gamma x_1e_1 + 2x_1e_1e_2 + x_2e_1^2 + e_2x_1^2 + e_2x_3 + e_3x_2 + e_2e_3 - e_2 + e_2e_1^2 + \gamma e_1^2 \\ 3x_1e_3 + 3x_3e_1 - 3e_1x_1^2 - 3x_1e_1^2 + 3e_1e_3 + e_1 - e_1^3 + 2\gamma x_2e_2 + \gamma e_1^2 \\ -2\alpha e_3 - 2x_1x_2e_3 - 2x_1x_3e_2 - 2x_1e_2e_3 - 2e_1x_2x_3 - 2x_2e_1e_3 - 2x_3e_1e_2 - 2e_1e_2e_3 \end{pmatrix}.$$

Hence, we can choose that

$$\overline{e_1} = e_1, \overline{e_2} = (e_2, e_3)^T, A_1 = (0), A_2 = \begin{pmatrix} 0 & 0 \\ 0 & -2\alpha \end{pmatrix},$$

$F_1(x, \overline{e_1}, \overline{e_2})$

$$= \left(\; 2x_1x_2e_1 + 2\gamma x_1e_1 + 2x_1e_1e_2 + x_2e_1^2 + e_2x_1^2 + e_2x_3 + e_3x_2 + e_2e_3 - e_2 + e_2e_1^2 + \gamma e_1^2 \; \right),$$

$$F_{21}(x, \overline{e_1}, \overline{e_2}) = \begin{pmatrix} 3x_3e_1 - 3e_1x_1^2 - 3x_1e_1^2 + 3e_1e_3 + e_1 - e_1^3 \\ -2e_1x_2x_3 - 2x_2e_1e_3 - 2x_3e_1e_2 - 2e_1e_2e_3 \end{pmatrix},$$

$$F_{22}(x, \overline{e_1}, \overline{e_2}) = \begin{pmatrix} 3x_1e_3 - 2\gamma e_2x_2 + \gamma e_2^2 \\ -2e_3x_1x_2 \end{pmatrix}.$$

So, the vector function

$$G_1(x, e) = \begin{pmatrix} A_1\overline{e_1} + F_1(x, \overline{e_1}, \overline{e_2}) \\ A_2\overline{e_2} + F_{21}(x, \overline{e_1}, \overline{e_2}) + F_{22}(x, \overline{e_1}, \overline{e_2}) \end{pmatrix}. \tag{3.9}$$

Obviously, $\lim_{e_1 \to 0} F_{21}(x, \overline{e_1}, \overline{e_2}) = 0$. According to tracking control method we can define feedback controller as

$$G_2(x, e) = \begin{pmatrix} \Omega_1(x, \overline{e_1}, \overline{e_2}) \\ \Omega_2(x, \overline{e_1}, \overline{e_2}) \end{pmatrix} = \begin{pmatrix} \Lambda_1\overline{e_1} - F_1(x, \overline{e_1}, \overline{e_2}) \\ \Lambda_2\overline{e_2} - F_{22}(x, \overline{e_1}, \overline{e_2}) \end{pmatrix}. \tag{3.10}$$

So, from equations (3.9) and (3.10) response system (3.8) can be rewritten as

$$\begin{cases} \frac{d\overline{e_1}}{dt} = (A_1 + \Lambda_1)\overline{e_1}, \\ \frac{d\overline{e_2}}{dt} = (A_2 + \Lambda_2)\overline{e_2} + F_{21}(x, \overline{e_1}, \overline{e_2}), \end{cases} \tag{3.11}$$

so, we choose now suitable $A_1 + \Lambda_1 \in \mathbb{R}^1$ and $A_2 + \Lambda_2 \in \mathbb{R}^{2\times 2}$, which satisfy $|\arg\lambda| > \pi/2$ (here $q = 1$). As equation (3.11) is asymptotically stable with equilibrium point $e_1 = 0$ and $\overline{e_2} = 0$. Obviously, $\lim_{t\to\infty} \|e_1\| = 0$ and $\lim_{e_1\to 0} F_{21}(x, \overline{e_1}, \overline{e_2}) = 0$, then the synchronization between response system and master system can be achieved.

4 Numerical Simulations

Parameters of the integer order Rabinovich-Fabrikant chaotic system are $\alpha = 0.87$ and $\gamma = 1.1$ and for fractional order Coullet system $a = 0.8$, $b = -1.1$, $c = -0.45$, and $d = -1$. The fractional order is taken to be $q = q_1 = q_2 = q_3 = 0.98$ and $q_1 = 0.97, q_2 = 0.95$ and $q_3 = 0.90$ for which the systems are chaotic. In equation (3.11) we have chosen $\Lambda_1 = (-1)$ and $\Lambda_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, which leads to stability conditions as eigenvalue of $A_1 + \Lambda_1$ is $\lambda_1 = -1$ and eigenvalues of $A_2 + \Lambda_2$ are $\lambda_2 = -1, \lambda_3 = -1.74$ when $\alpha = 0.87$ and $\lambda_2 = -1, \lambda_3 = -0.28$ when $\alpha = 0.14$. The initial conditions for master and slave systems $[x_1(0), x_2(0), x_3(0)] = [0.1, 0.4, 0.3]$ and $[y_1(0), y_2(0), y_3(0)] = [-1, 0, 0.5]$ respectively and for $[e_1(0), e_2(0), e_3(0)] = [-1.1, -0.4, 0.2]$ diagrams of convergence of errors given below are the witness of achieving synchronization between master and slave system.

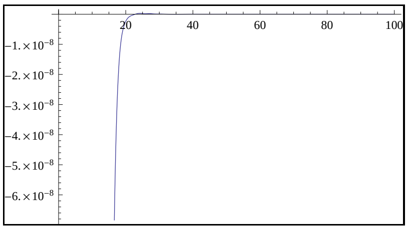

Figure 9: Convergence error of e_1, $t = [0, 100]$ with $\alpha = 0.87$.

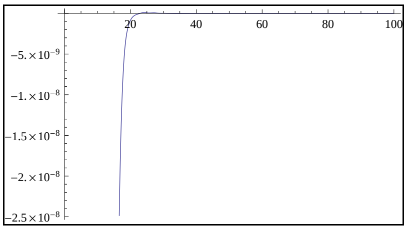

Figure 10: Convergence error of e_2, $t = [0, 100]$ with $\alpha = 0.87$.

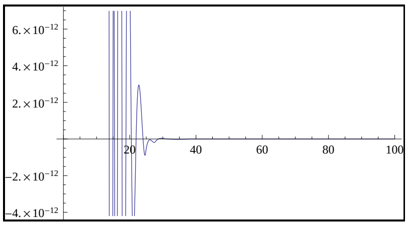

Figure 11: Convergence error of e_3, $t = [0, 100]$ with $\alpha = 0.87$.

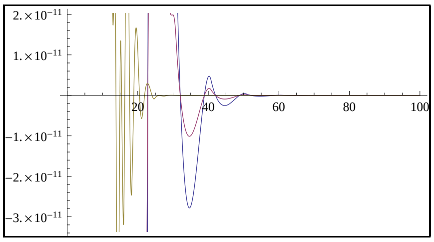

Figure 12: Combined Convergence error of e_1, e_2, and e_3, $t = [0, 100]$ with $\alpha = 0.87$.

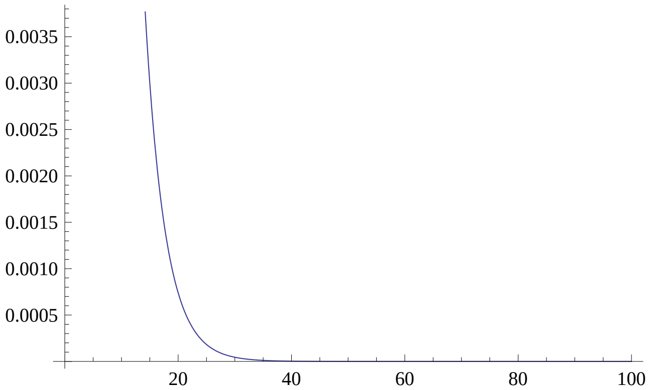

Figure 13: Graph between $e = \sqrt{e_1^2 + e_2^2 + e_3^2}$ and $t = [0, 100]$ with $\alpha = 0.87$ shows synchronization between drive and response system.

5 Conclusion

In this paper, we have investigated synchronization behavior between an integer order Rabinovich-Fabrikant chaotic system and fractional order Coullet chaotic system via tracking control method and stability of fractional order systems. The results are validated by numerical simulations using Mathematica and Matlab both. Synchronization between two different orders has more advantage over synchronization between same order systems. Synchronization between two different order chaotic systems is more beneficial to enhance security of communication.

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