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Reduced Order Function Projective Combination Synchronization of Three Josephson Junctions Using Backstepping Technique

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Abstract: In this paper, a new synchronization scheme, combination synchronization, is used to realize reduced order function projective synchronization among three chaotic Josephson junction systems using backstepping technique. In the first case, function projective synchronization of two (2) third order drive systems with a single second order Josephson junction is considered while in the second case, a single third order system is synchronized with two (2) second order system using backstepping. Controllers are designed and simulated to show the efficacy of combination synchronization scheme.

Keywords: *function projective; reduced order synchronization; Josephson junction; combination synchronization.*

Mathematics Subject Classification (2010): 34H10, 93C10.

1 Introduction

Synchronization between two chaotic systems has evolved greatly since its proposition by Pecora and Caroll [1]. Many types of synchronization schemes have been proposed and implemented including complete synchronization (CS) [1], projective synchronization (PS) [2, 3], lag synchronization (LS) [4], modified projective synchronization [5] while techniques such as adaptive control method [6], active control [7], active backstepping [2] and feedback control [8] have been used for design of controllers. Backstepping scheme has been efficient in the design technique for stabilization, tracking and synchronization

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of chaotic systems. According to Tian et al., [9], some of the advantages of the method include: applicability to a variety of chaotic systems irrespective of whether they contain external excitation or not; needs only one controller to realize synchronization of chaotic systems and finally there is no derivative in the controller. Backstepping technique offers faster and better transient error dynamics convergence and synchronization time than the active control technique [2]. The results of chaos synchronization are utilized in biological sciences [10], economics and finance [11], chemical reactions, secure communication [12, 13] and cryptography and data encryption. Recently, synchronization between fractional order and integer order system was reported in [14]

Projective Synchronization (PS) refers to the dynamical behavior in which the responses of two identical systems synchronize up to a constant scaling factor $\alpha \in \Re$ [15]. When $\alpha = 1$ we have complete synchronization and $\alpha = -1$ gives antisynchronization of the systems. Function projective synchronization in which the scaling factor is not a constant value was proposed by Du et al., [16]. A form of projective synchronization referred to as hybrid projective synchronization, in which the different state variables can synchronize up to different scaling factors was implemented by Hu et al., [7, 17]. The hybrid function projective synchronization was extended to different systems with time varying parameters [18], fractional order system [19] and hyperchaotic system [20]. The scaling constant in projective synchronization gives faster communication, hence, the popularity of the scheme.

Until recently, synchronization has been applied to two systems of the same dimension (identical or non-identical), however, natural and practical systems tend to involve systems of different order. As pointed out in [2], there are real situations where systems of different order need to be synchronized e.g the order of the thalamic neurons can be different from the hippocampal neurons, the synchronization between heart and lungs, the synchronization in neuron systems and certain biomechanical systems (such as biological implants), mechanical systems [21]. This motivated the implementation of increased order [22–24] and reduced order synchronization [25, 26].

It was also proposed by Runzi and Yinglan [27] that information signal be transmitted by two different drive systems. For example, we split the transmitted signals into several parts, each part loaded in different systems; or divide time into different intervals, the signals in different intervals loaded in different systems. If this is really so, then the transmitted signals may have stronger anti-attack ability and anti-translated capability than that transmitted by the usual transmission model. Furthermore, in a communication network, there are many users (slave) but one control (master) which connects different users to one another. There is the need to implement a synchronization scheme whereby many users can be connected to and routed through a single master securely. Increased order and reduced order combination synchronization of three different nonlinear systems was implemented using active backstepping design [19]. Synchronization between combination of two drive systems and combination of two response systems in drive-response synchronization model was investigated by Sun et al., [28] in a new scheme referred to as combination-combination synchronization.

To the best of our knowledge, research into reduced order function projective combination synchronization has not been carried out, hence, we set forth in this paper to investigate it. From the aforementioned, we implement a reduced order projective synchronization of (i) two (2) 3-dimensional system and one slave (ii) two (2) one 3-dimensional master system and two (2) 2-dimensional slave system using the backstepping technique. The remainder of the paper is arranged as follows:

2 Reduced-order Function Projective Combination Synchronization of Two Third Order and One Second Order Josephson Junctions

2.1 Design of controller via active backstepping technique

In this section, two third order Josephson junctions in (1) and (2) are taken as the drive systems while one second order non-autonomous Josephson junction (3) is taken as the response system in order to achieve generalized reduced order combination synchronization among the three chaotic Josephson junctions

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \frac{1}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3),\\ \dot{x}_3 &= \frac{1}{\beta_L}(x_2 - x_3),\end{aligned} \tag{1}$$

the second drive system is

$$\begin{aligned}\dot{y}_1 &= y_2,\\ \dot{y}_2 &= \frac{1}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3),\\ \dot{x}_3 &= \frac{1}{\beta_L}(y_2 - y_3),\end{aligned} \tag{2}$$

while the response system is given as

$$\begin{aligned}\dot{z}_1 &= z_2 + u_1,\\ \dot{z}_2 &= -\alpha z_2 - \sin z_1 + a + b\sin\omega t + u_2,\end{aligned} \tag{3}$$

where $u_i(t)$, $i = 1, 2$ are the controllers to be designed. We define the error systems as follows

$$\begin{aligned}e_1 &= z_1 - (\alpha_1(t)x_1 + \beta_1(t)y_1 + \alpha_3(t)x_3 + \beta_3(t)y_3),\\ e_2 &= z_2 - (\alpha_2(t)x_2 + \beta_2(t)y_2).\end{aligned} \tag{4}$$

Using the error systems defined in (4) with systems defined in (1), (2) and (3) yields the following error dynamics

$$\begin{aligned}\dot{e}_1 = z_2 + u_1 - \alpha_1(t)x_2 - \beta_1(t)y_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \frac{\beta_3(t)}{\beta_L}(y_2 - y_3)\\ - \dot{\alpha}_1(t)x_1 - \dot{\beta}_1(t)y_1 - \dot{\alpha}_3(t)x_3 - \dot{\beta}_3(t)y_3\end{aligned}$$

$$\begin{aligned}= e_2 + \alpha_2(t)x_2 + \beta_2(t)y_2 - \alpha_1(t)x_2 - \beta_1(t)y_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \frac{\beta_3(t)}{\beta_L}(y_2 - y_3) + u_1\\ - \dot{\alpha}_1(t)x_1 - \dot{\beta}_1(t)y_1 - \dot{\alpha}_3(t)x_3 - \dot{\beta}_3(t)y_3,\end{aligned}$$

$$\begin{aligned}\dot{e}_2 = -\alpha z_2 - \sin z_1 + a + b\sin\omega t + u_2 - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3)\\ - \frac{\beta_2(t)}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3) - \dot{\alpha}_2(t)x_2 - \dot{\beta}_2(t)\end{aligned}$$

$$= -\alpha(e_2 + \alpha_2(t)x_2 + \beta_2(t)y_2) - \dot{\alpha}_2(t)x_2 - \dot{\beta}_2(t) - \sin z_1$$
$$+ a + b\sin\omega t + u_2 - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) - \frac{\beta_2(t)}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3).$$

Thus, the error dynamics of the system can be written as

$$\dot{e}_1 = e_2 + u_1 + A_1, \tag{5}$$

$$\dot{e}_2 = -\alpha e_2 + u_2 + A_2, \tag{6}$$

where

$$A_1 = \alpha_2(t)x_2 + \beta_2(t)y_2 - \alpha_1(t)x_2 - \beta_1(t)y_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \frac{\beta_3(t)}{\beta_L}(y_2 - y_3) - \dot{\alpha}_1(t)x_1$$
$$- \dot{\beta}_1(t)y_1 - \dot{\alpha}_3(t)x_3 - \dot{\beta}_3(t)y_3,$$

$$A_2 = -\alpha(\alpha_2(t)x_2 + \beta_2(t)y_2) - \sin z_1 + a + b\sin\omega t - \dot{\alpha}_2(t)x_2 - \dot{\beta}_2(t) - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2$$
$$- \sin x_1 - x_3) - \frac{\beta_2(t)}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3).$$

Our goal is to find the control functions which will enable the systems (1), (2) and (3) realize generalized reduced order function projective combination synchronization by active backstepping technique. The design procedure includes three steps as shown below:

Step 1. Let $q_1 = e_1$, its time derivative is

$$\dot{q}_1 = \dot{e}_1 = e_2 + u_1 + A_1, \tag{7}$$

where $e_2 = \alpha_1(q_1)$ can be regarded as virtual controller. In order to stabilize q_1-subsystem, we choose the following Lyapunov function $v_1 = \frac{1}{2}q_1^2$. The time derivative of v_1 is

$$\dot{v}_1 = q_1\dot{q}_1 = q_1(\alpha_1(q_1) + u_1 + A_1). \tag{8}$$

Suppose $\alpha_1(q_1) = 0$ and the control function u_1 is chosen as

$$u_1 = -(A_1 + kq_1), \tag{9}$$

then $\dot{v}_1 = -kq_1^2 < 0$, where k is positive constant which represent the feedback gain. Then, $\dot{v}_1$ is negative definite and the subsystem q_1 is asymptotically stable. Since, the virtual controller $\alpha_1(q_1)$ is estimative, the error between e_2 and $\alpha_1(q_1)$ can be denoted by $q_2 = e_2 - \alpha_1(q_1)$. Thus, we have the following (q_1, q_2)-subsystems

$$\begin{aligned} \dot{q}_1 &= q_2 - kq_1, \\ \dot{q}_2 &= -\alpha q_2 + u_2 + A_2. \end{aligned} \tag{10}$$

Step 2. In order to stabilize subsystem (10), a Lyapunov function can be chosen as $v_2 = v_1 + \frac{1}{2}q_2^2$. The time derivative of v_2 is

$$\dot{v}_2 = -q_1^2 + q_2(q_1 - \alpha q_2 + u_2 + A_2). \tag{11}$$

If the control function u_2 is chosen as

$$u_2 = \alpha q_2 - q_1 - A_2 - kq_2, \tag{12}$$

then $\dot{v}_2 = -kq_1^2 - kq_2^2 < 0$, where k is a positive constant which represent the feedback gain. Then, $\dot{v}_2$ is negative definite and the subsystem (q_1, q_2) in (10) is asymptotically stable. This implies that generalized reduced order function projective combination synchronization of the drive systems (1) and (2) and the response system (3) is achieved.

Finally, we have the following subsystems

$$\begin{aligned} \dot{q}_1 &= q_2 - kq_1, \\ \dot{q}_2 &= -q_1 - kq_2. \end{aligned} \tag{13}$$

Now the generalized reduced order function projective combination synchronization is achieved, the following can be obtained.

Let $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then we have Case 1.

Case 1: If the controllers are chosen as

$$\begin{aligned} u_1 &= (\beta_1(t) - \beta_2(t))y_2 + \frac{\beta_3(t)}{\beta_L}(y_2 - y_3) + \dot{\beta}_1(t)y_1 + \dot{\beta}_3(t)y_3 - kq_1, \\ u_2 &= (\alpha - k)q_2 - q_1 + \alpha\beta_2(t)y_2 + \sin z_1 - a - b\sin\omega t + \\ &\quad \frac{\beta_2(t)}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3) + \dot{\beta}_2(t)y_2, \end{aligned} \tag{14}$$

where $q_1 = z_1 - \beta_1(t)y_1 - \beta_3(t)y_3$, $q_2 = z_2 - \beta_2(t)y_2$, then the drive system (2) achieves reduced order modified function projective synchronization with the response system (3). Let $\beta_1(t) = \beta_2(t) = \beta_3(t) = 0$, then we obtain Case 2.

Case 2: If the controllers are chosen as

$$\begin{aligned} u_1 &= (\alpha_1(t) - \alpha_2)x_2 + \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - kq_1 + \dot{\alpha}_1(t)x_1 + \dot{\alpha}_3(t)x_3, \\ u_2 &= (\alpha - k)q_2 - q_1 + \alpha\alpha_2(t)x_2 + \sin z_1 - a - b\sin\omega t + \\ &\quad \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) + \dot{\alpha}_2(t)x_2, \end{aligned} \tag{15}$$

where $q_1 = z_1 - \alpha_1(t)x_1 - \alpha_3(t)x_3$, $q_2 = z_2 - \alpha_2(t)x_2$, then the drive system (1) achieves reduced order modified function projective synchronization with the response system (2).

Suppose $\alpha_1(t) = \alpha_2(t) = \alpha_3(t) = \beta_1(t) = \beta_2(t) = \beta_3(t) = 0$, then we obtain Case 3.

Case 3: If the controllers are chosen as

$$\begin{aligned} u_1 &= -kq_1, \\ u_2 &= (\alpha - k)q_2 - q_1 + \sin z_1 - a - b\sin\omega t, \end{aligned} \tag{16}$$

where $q_1 = z_1$, $q_2 = z_2$, then the equilibrium point $(0, 0, 0)$ of the response system (3) is asymptotically stable.

Suppose $\beta_1(t) = \beta_2(t) = \beta_3(t) = \alpha_1(t) = \alpha_2(t) = \alpha_3(t) = \gamma(t)$, then we obtain Case 4.

Case 4: If the controllers are chosen as

$$\begin{aligned} u_1 &= \frac{\gamma(t)}{\beta_L}(x_2 - x_3 + y_2 - y_3) + \dot{\gamma}(t)(x_1 + y_1 + x_3 + y_3) - kq_1, \\ u_2 &= (\alpha - k)q_2 - q_1 + \alpha\gamma(t)(x_2 + y_2) + \sin z_1 - a - b\sin\omega t + \dot{\gamma}(t)(x_2 + y_2) \\ &\quad - \frac{\gamma(t)}{\beta_C}(g(x_2)x_2 + g(y_2)y_2 + \sin x_1 + \sin y_1 + x_3 + y_3 - 2i), \end{aligned} \tag{17}$$

where $q_1 = z_1 - \gamma(t)(x_1 + y_1 + x_3 + y_3)$, $q_2 = z_2 - \gamma(t)(x_2 + y_2)$, then the drive systems (1) and (2) achieve reduced order function projective combination synchronization with the response system (3).

Let all the scaling functions be $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$, $\beta_1(t)$, $\beta_2(t)$ and $\beta_3(t)$, then we obtain Case 5.

Case 5: If the controllers are chosen as

$$\begin{aligned} u_1 &= -\alpha_2(t)x_2 - \beta_2(t)y_2 + \alpha_1(t)x_2 + \beta_1(t)y_2 + \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) + \frac{\beta_3(t)}{\beta_L}(y_2 - y_3) \\ &\quad + \dot{\alpha}_1(t)x_1 + \dot{\beta}_1(t)y_1 + \dot{\alpha}_3(t)x_3 + \dot{\beta}_3(t)y_3 - kq_1, \\ u_2 &= \alpha(\alpha_2(t)x_2 + \beta_2(t)y_2) + \sin z_1 - a - b\sin\omega t + \dot{\alpha}_2(t)x_2 \\ &\quad + \dot{\beta}_2(t)(y_2) + (\alpha - k)q_2 - q_1 + \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) \\ &\quad + \frac{\beta_2(t)}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3), \end{aligned} \tag{18}$$

$q_1 = z_1 - (\alpha_1(t)x_1 + \beta_1(t)y_1 + \alpha_3(t)x_3 + \beta_3(t)y_3)$, $q_2 = z_2 - (\alpha_2(t)x_2 + \beta_2(t)y_2)$, then the drive systems (1) and (2) achieve reduced order modified function projective combination synchronization with the response system (3).

2.2 Numerical simulation results

The designed controllers are verified in our numerical simulation using the in-built fourth order Runge-Kutta (ode45) routine in Matlab. In the numerical simulation procedure we used the systems parameters within the chaotic region and controllers are chosen in accordance with Case 4. The initial conditions of the drive systems and response system are given as $(x_1, x_2, x_3) = (0, 0, 0)$, $(y_1, y_2, y_3) = (111)$, $(z_1, z_2) = (0, 1)$, $\gamma(t) = 2.0 + 0.01\sin(0.05t)$ and $k = 1$. Corresponding numerical results are as follows: Figure 1 shows the dynamics of the error variables when the controllers are deactivated for $0 \leq t \leq 200$. Figure 2 shows that reduced order combination synchronization among systems (1), (2) and (3) is achieved as indicated by the convergence of the error state variables to zero as soon as the controllers are switch on for $t \geq 80$. Figure 3 shows that the state variables of the drive and the response systems follow the same trajectory when the controllers are activated for $t \geq 80$, this also confirms reduced order combination synchronization among systems (1), (2) and (9). Evidence of reduced order function projective combination synchronization is presented in Figure 4.

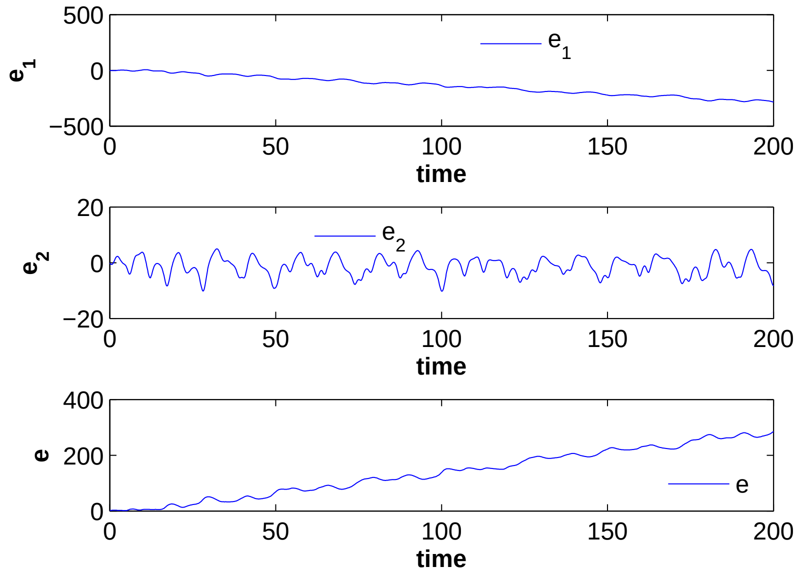

Figure 1: Error dynamics among systems (1), (2) and (3) with control activated at $t \geq 80$.

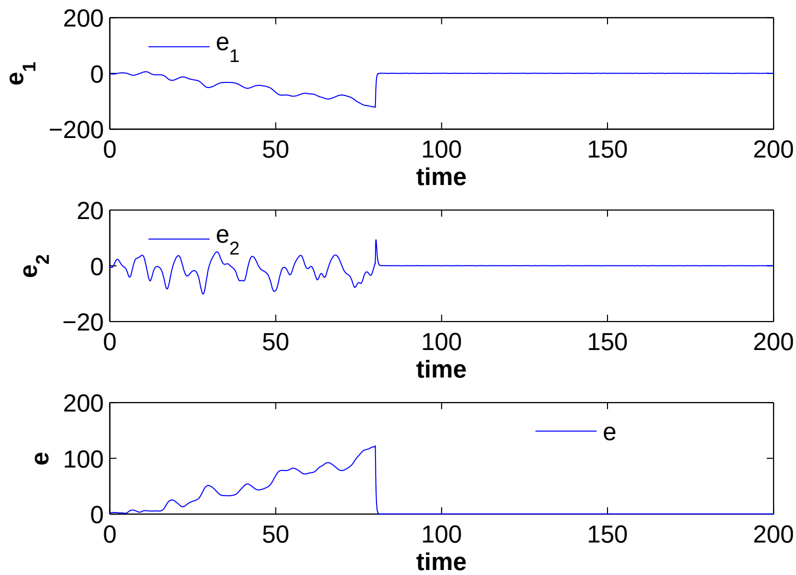

Figure 2: Error dynamics among systems (1), (2) and (9) with control activated activated at $t \geq 80$.

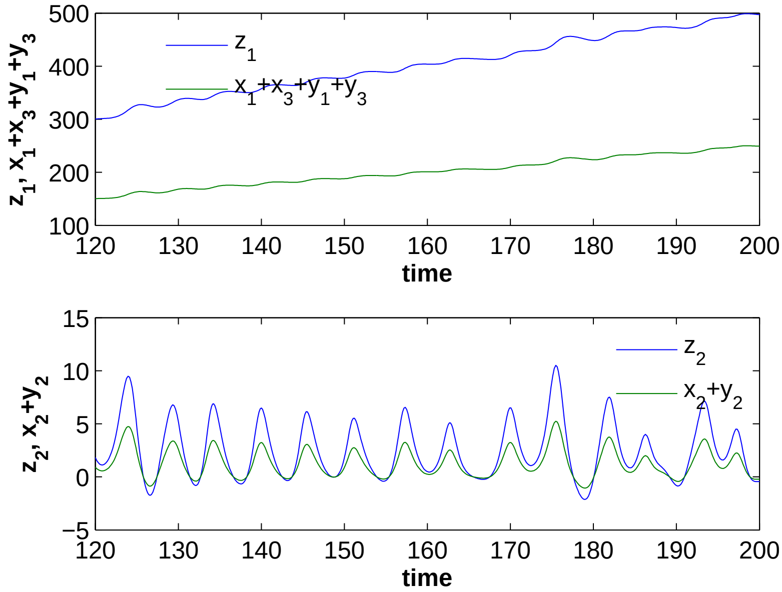

Figure 3: Dynamics of state variables with control applied.

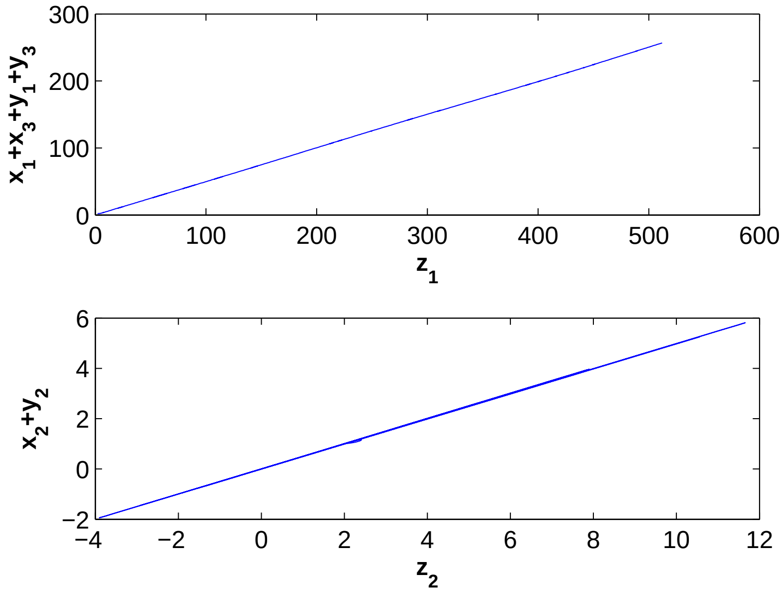

Figure 4: Evidence of projective synchronization.

3 Generalized Reduced-order Function Projective Combination Synchronization of One Third Order and Two Second Order Josephson Junctions

3.1 Design of controller via active backstepping technique

In this section, one third order Josephson junction in (1) is taken as the drive system while two second order non-autonomous Josephson junctions in (19) and (20) are taken as the response systems in order to achieve reduced order function projective combination synchronization among the three chaotic Josephson junctions

$$\begin{aligned} \dot{y}_1 &= y_2 + u_1, \\ \dot{y}_2 &= -\alpha y_2 - \sin y_1 + a + b\sin\omega t + u_2, \end{aligned} \tag{19}$$

$$\begin{aligned} \dot{z}_1 &= z_2 + u_3, \\ \dot{z}_2 &= -\alpha z_2 - \sin z_1 + a + b\sin\omega t + u_4, \end{aligned} \tag{20}$$

where u_1, u_2, u_2 and u_4 are the controllers to be designed. We define the error systems as follows

$$\begin{aligned} e_1 &= z_1 + y_1 - (\alpha_1(t)x_1 + \alpha_3(t)x_3), \\ e_2 &= z_2 + y_2 - \alpha_2 x_2. \end{aligned} \tag{21}$$

Using the error systems defined in (21) with systems defined in (1), (19) and (20) yields the following error dynamics

$$\begin{aligned} \dot{e}_1 &= z_2 + y_2 + u_3 - \alpha_1(t)x_2 + u_1 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \dot{\alpha}_1(t)x_1 - \dot{\alpha}_3(t)x_3 \\ &= e_2 + (\alpha_2(t) - \alpha_1(t))x_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) + u_3 + u_1 - \dot{\alpha}_1(t)x_1 - \dot{\alpha}_3(t)x_3, \end{aligned}$$

$$\begin{aligned} \dot{e}_2 = {} & -\alpha z_2 - \sin z_1 + a + b\sin\omega t + u_4 - \alpha y_2 - \sin y_1 + a + b\sin\omega t \\ & + u_2 - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) - \dot{\alpha}_2(t)x_2 \\ = {} & -\alpha(e_2 + \alpha_2(t)x_2) - \sin z_1 + a + b\sin\omega t) - \dot{\alpha}_2(t)x_2 \\ & + u - \sin y_1 + a + b\sin\omega t) + u_2 - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3). \end{aligned}$$

Thus, the error dynamics of the system can be written as:

$$\dot{e}_1 = e_2 + U_1 + B_1, \tag{22}$$

$$\dot{e}_2 = -\alpha e_2 + U_2 + B_2, \tag{23}$$

where

$$B_1 = (\alpha_2(t) - \alpha_1(t))x_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \dot{\alpha}_1(t)x_1 - \dot{\alpha}_3(t)x_3,$$

$$B_2 = -\alpha\alpha_2(t)x_2) - \sin z_1 + 2a + 2b\sin\omega t - \sin y_1 - \dot{\alpha}_2(t)x_2 - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3),$$

$$U_1 = u_1 + u_3, \quad U_2 = u_2 + u_4.$$

Our goal is to find the control functions which will enable the systems (7), (19) and (20) realize reduced order function projective combination synchronization by active backstepping technique. The design procedure includes three steps as shown below:

Step 1. Let $q_1 = e_1$, its time derivative is

$$\dot{q}_1 = \dot{e}_1 = e_2 + U_1 + B_1, \tag{24}$$

where $e_2 = \alpha_1(q_1)$ can be regarded as virtual controller. In order to stabilize q_1-subsystem, we choose the following Lyapunov function $v_1 = \frac{1}{2}q_1^2$ and its time derivative of v_1 is

$$\dot{v}_1 = q_1\dot{q}_1 = q_1(\alpha_1(q_1) + U_1 + B_1). \tag{25}$$

Suppose $\alpha_1(q_1) = 0$ and the control function U_1 is chosen as

$$U_1 = -(B_1 + kq_1), \tag{26}$$

then $\dot{v}_1 = -kq_1^2 < 0$, where k is positive constant which represents the feedback gain. Then, $\dot{v}_1$ is negative definite and the subsystem q_1 is asymptotically stable. Since the virtual controller $\alpha_1(q_1)$ is estimative, the error between e_2 and $\alpha_1(q_1)$ can be denoted by $q_2 = e_2 - \alpha_1(q_1)$. Thus, we have the following (q_1, q_2)-subsystems

$$\begin{aligned} \dot{q}_1 &= q_2 - kq_1, \\ \dot{q}_2 &= -\alpha q_2 + U_2 + B_2. \end{aligned} \tag{27}$$

Step 2. In order to stabilize system (27), a Lyapunov function can be chosen as $v_2 = v_1 + \frac{1}{2}q_2^2$. The time derivative of v_2 is

$$\dot{v}_2 = -q_1^2 + q_2(q_1 - \alpha q_2 + U_2 + B_2). \tag{28}$$

If the control function u_2 is chosen as

$$U_2 = -B_2 - kq_2 + \alpha q_2 - q_1, \tag{29}$$

then $\dot{v}_2 = -kq_1^2 - kq_2^2 < 0$, where k is positive constant which represents the feedback gain. Then, $\dot{v}_2$ is negative definite and the subsystem (q_1, q_2) in (27) is asymptotically stable. This implies that the drive system (1) and the response systems (19) and (20) achieve reduced order function projective combination synchronization. Finally, we have the following subsystems

$$\begin{aligned} \dot{q}_1 &= q_2 - kq_1, \\ \dot{q}_2 &= -q_1 - kq_2. \end{aligned} \tag{30}$$

Here we limit our results to only two major Corollaries.

Let $\alpha_1 = \alpha_2 = \alpha_3$, $u_1 = u_3$ and $u_2 = u_4$, then we have Case 6.

Case 6.: If the controllers are chosen as

$$\begin{aligned} u_1 = u_3 = \frac{1}{2}(\frac{\alpha_1(t)}{\beta_L}(x_2 - x_3) + \dot{\alpha}_1(t)(x_1 + x_3) - kq_1), \\ u_2 = u_4 = \frac{1}{2}(\alpha - k)q_2 - q_1 + (\alpha\alpha_1(t) + \dot{\alpha}_1(t))x_2 + \sin z_1 \\ + \sin y_1 - 2a - 2b\sin\omega t - \frac{\alpha_1(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3), \end{aligned}$$

where $e_1 = z_1 - \alpha_1(x_1 + x_3)$, $e_2 = z_2 - \alpha_1 x_2$, then the drive system (1) achieves reduced order function projective combination synchronization with the response systems (19) and (20).

Let all the scaling functions be $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$ with $u_1 = u_3$ and $u_2 = u_4$, then we have Case 7.

Case 7: If the controllers are chosen as

$$u_1 = u_3 = \frac{1}{2}((\alpha_1(t) - \alpha_2(t))x_2 + (\frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) + \dot{\alpha}_1(t)x_1 + \dot{\alpha}_3(t)x_3 - kq_1),$$

$$\begin{aligned} u_2 = u_4 = \frac{1}{2}(\alpha - k)q_2 - q_1 + (\alpha\alpha_2(t) + \dot{\alpha}_2(t)x_2)x_2 + \sin z_1 + \sin y_1 \\ - 2a - 2b\sin\omega t - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3), \end{aligned}$$

where $e_1 = z_1 + y_1 - (\alpha_1(t)x_1 + \alpha_3(t))$, $e_2 = z_2 + y_2 - \alpha_2(t)x_2$, then reduced order modified function projective combination synchronization is achieved between the drive system (1) and the response systems (19) and (20).

3.2 Numerical simulation results

The designed controllers are verified in our numerical simulation using the in-built fourth order Runge-Kutta (ode45) routine in Matlab. In the numerical simulation procedure we used the systems parameters within the chaotic region and controllers are chosen in accordance with Case 6. The initial conditions of the drive systems and response system are given as $(x_1, x_2, x_3) = (0, 0, 0)$, $(y_1, y_2, y_3) = (111)$, $(z_1, z_2) = (0, 1)$, $\gamma(t) = 2.0 + 0.01\sin(0.05t)$ and $k = 1$. Corresponding numerical results are as follows: Figure 5 shows the dynamics of the error variables when the controllers were deactivated. Figure 6 shows that reduced order function projective combination synchronization among systems (1), (19) and (20) is achieved as indicated by the convergence of the error state variables to zero as soon as the controllers are switch on for $t \geq 80$. Figure 7 shows that the state variables of the drive and response systems follow the same trajectory when the controllers are activated for $t \geq 80$, this also confirms reduced order function projective combination synchronization among systems (1), (19) and (20). Figure 8 presents evidence of reduced order function projective synchronization among the systems.

4 Conclusion

Reduced order function projective combination synchronization of three chaotic systems consisting of: (i) two third order chaotic Josephson junctions as drives and one second order chaotic Josephson junction as response system; (ii) one third order chaotic Josephson junction as the drive and two second order chaotic Josephson junctions as the slaves via active backstepping technique has been achieved. We showed from the theoretical analysis that various controllers which is suitable for different type of synchronization scheme can be obtained from the general results. Furthermore, reduced order function projective combination synchronization has more potential application to secure communication systems and biological systems.

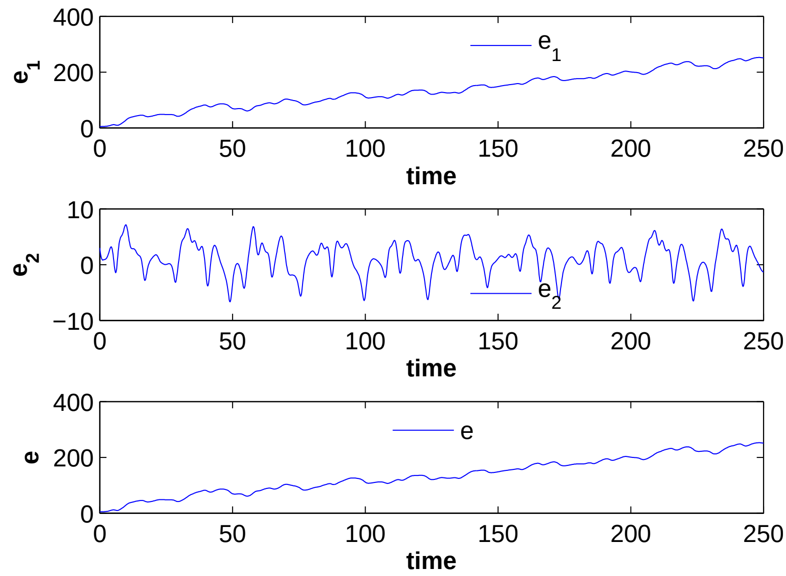

Figure 5: Error dynamics without control activated.

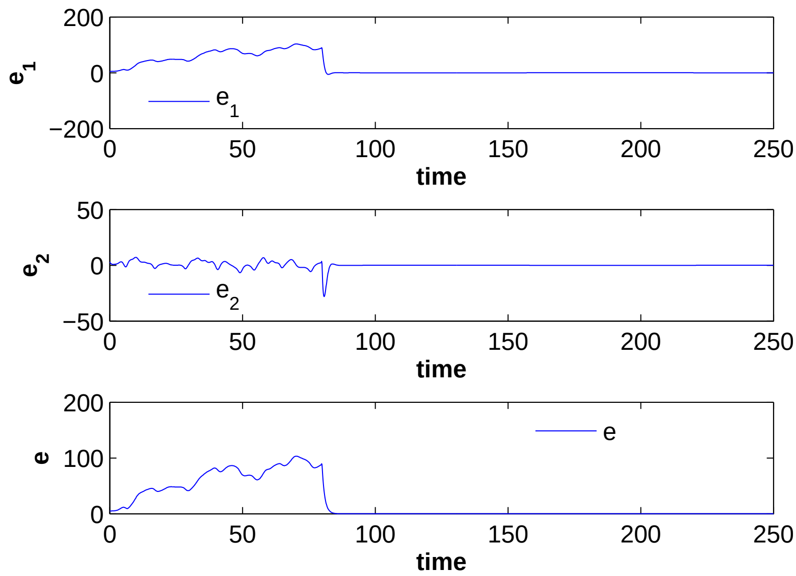

Figure 6: Error dynamics with control activated.

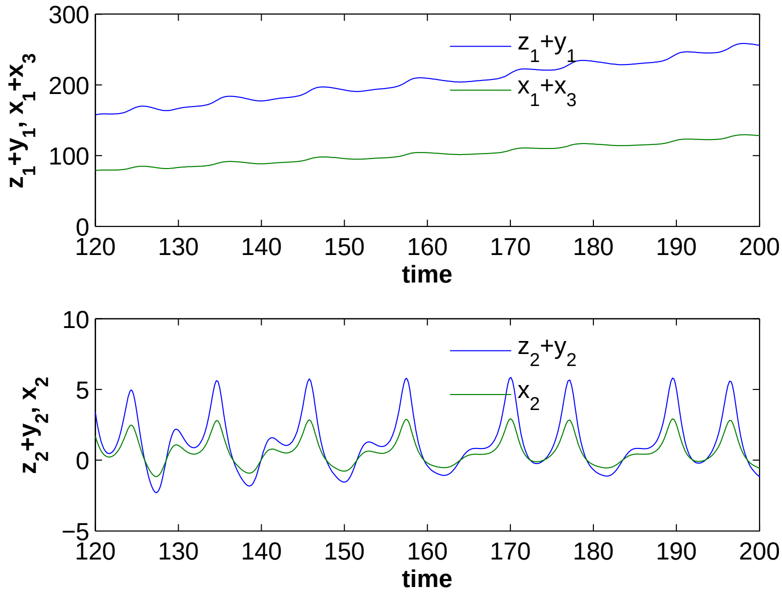

Figure 7: Dynamics of state variables with control activated.

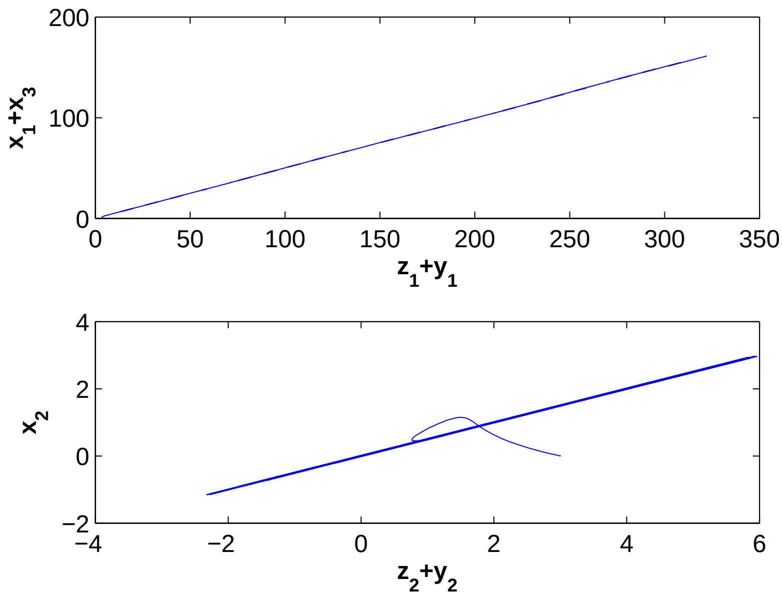

Figure 8: Evidence of projective synchronization.

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