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The Structure of the Solution of Delay Differential Equations with One Unstable Positive Equilibrium

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Abstract: This paper studies the equation $\dot{x}(t) = -g(x(t)) + f(x(t-\tau))$ with one trivial equilibrium and only one unstable positive equilibrium. For a class of linear initial values, two sufficient conditions are established to guarantee that the corresponding solutions converge to the trivial equilibrium and the positive equilibrium respectively. All solutions, with the exception of two equilibria, are divided into three classes according to their eventual tendency. The first class solutions are strictly greater than 0 ultimately and converge to it; the second class ones are strictly greater than the positive equilibrium ultimately and converge to it; the third class solutions oscillate about the positive equilibrium up and down and converge to it. Furthermore, the existence of the third class of solutions is determined. Numerical simulations are given to illustrate the main results.

Keywords: *delay differential equations; convergence; oscillatory solution; attractive region; equilibrium.*

Mathematics Subject Classification (2010): 34K05; 34K60; 92B05.

1 Introduction

Delay differential equations are always the research focus of mathematicians dealing with theory of functional differential equations and scientists applying the theory to practical problems. It is not difficult to found a variety of application of delay differential equation in several fields of natural science such as viscoelasticity, mechanics, models for nuclear reactors, distributed networks, heat flow, neural networks, combustion theory, interaction of species, microbiology, learning models, epidemiology, physiology see e.g. [9, 11, 15, 22].

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The introduction of delays makes a much richer range of phenomena possible, however, it also causes sever mathematical complications.

Even with consideration of the simplest-looking equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)), \quad \mu > 0, \tag{1}$$

where $\mu > 0$, just as pointed out by T. Krisztin in paper [17], the dynamics of equation (1) can be very rich. In the monotone feedback case, the properties of equation (1) have been explored comprehensively, including the local and global dynamics, structure of the global attractor, existence and properties of periodic orbit (see [1,7,16,19–21,29]). In the case of a non-monotone feedback function $f(x)$, the dynamics can be very complicated. Though a majority of literatures study the property of equation (1) with nonmonotone feedback (see [2,4,5,8,12]). In general there are still much unknown. One important result comes from paper [26], in which G. Röst and J. Wu showed the existence of the global attractor and gave the bounds of the attractor in the case when $f(x)$ is a general unimodal function, which is the situation for the well-known Nicholson's blowflies equation [10] and the Mackey-Glass equation [25].

Recently C. Huang, Z. Yang, T. Yi and X. Zou [14] investigated the following model

$$\dot{x}(t) = -g(x(t)) + f(x(t-\tau)), \tag{2}$$

where g and f are continuous on $\mathbb{R}^+$ with the values in $\mathbb{R}^+$, and satisfy (F_1) and (F_2).

(F_1) $g(x)$ is strictly increasing on $\mathbb{R}^+$, $\dot{g}(x) > 0$, $g(0) = 0$ and $\lim\limits_{x\to+\infty} g(x) = +\infty$.

(F_2) $f(\xi) > 0$ for all $\xi > 0, f(0) = 0$, and there exists a unique $\xi_0 > 0$, such that $f'(\xi) > 0$ if $0 < \xi < \xi_0$, $f'(\xi_0) = 0 = f'(0)$ and $f'(\xi) < 0$ if $\xi > \xi_0$, furthermore, there also exists a unique $0 < \xi_1 < \xi_0$ such that $f''(\xi) > 0$ if $0 < \xi < \xi_1$, $f''(\xi_1) = 0$ and $f''(\xi) < 0$ if $\xi_1 < \xi < \xi_0$, and $\lim\limits_{\xi\to+\infty} f(\xi) = 0$.

Evidently, the famous Allee-type model with $f(x) = ax^n e^{-x}$ in [23] satisfies conditions (F_1) and (F_2) when $n > 1$. The distinction between the models in [14] and [26] is whether $f'(0) = 0$, it is this property that makes equation (2) have different properties such as multiple equilibria or one unstable positive equilibrium. For equation (2), Huang et.al determined the invariant intervals and the multistability properties of equilibria of equation (2). When the system has only one positive equilibrium, their results imply that the positive equilibrium is unstable, but the equilibria 0 and x_1 have their own local attractive region.

The dynamics of delay differential equations can be affected by many factors. For example, delays can cause the loss of stability and induce oscillations, periodic solutions and the occurrence of Hopf bifurcations [28, 30]. Many papers consider the effect of increasing mortality and harvesting on equation (2) see e.g. [3, 6, 8, 18, 28]. E. Liz and G. Gost [24] obtained some new results for equation (2) with negative Schwarzian derivative. However, the role of initial condition on the property of solutions is not considered. In finite dimensional systems, it is direct to judge the property of orbits by initial value. As we known, systems generated by delay differential equations are infinite dimensional, the previous results can not be applied here. Thus we pay attention to the role of the initial value.

Motivated by the above discussion, we mainly explore the property of the solutions of equation (2) with a class of initial value. Throughout the paper we assume that equation (2) fulfills conditions (F_1'), (F_2) and (F_3).

(F_1') $g(x)$ is strictly increasing on $\mathbb{R}^+$, $\dot{g}(x) > 0$, $\ddot{g}(x) \le 0$, $g(0) = 0$ and $\lim\limits_{x\to+\infty} g(x) = +\infty$.

(F_3) $f(x)$ and $g(x)$ have only one positive intersection point denoted by x_1.

Obviously, Losota's model fulfills (F_1'), (F_2) and (F_3) if $\mu = a(\frac{n-1}{e})^{n-1}$ and $n > 1$. For equation (2) with $\tau = 1$, considering the wide variety of the initial value, we mainly investigate the convergence of the solution with linear initial value $\phi(s) = ks + x_1 + h$ for $-1 \le s \le 0$ and $0 < h \le x_1$. Since $\phi(s)$ is not in the attractive region of 0 or x_1 for some k and h, the results in [14] can not be directly applied to deduce the convergence of the corresponding solutions. Here, we establish two sufficient conditions to ensure that the corresponding solutions converge to 0 and x_1 respectively. Furthermore, we give more detailed description and classification of the solutions of (2). The paper divides all solutions of (2) with the exception of two equilibria into three categories according to their way of convergence. The first class solutions are strictly greater than 0 ultimately and converge to it; the second class ones are strictly greater than x_1 ultimately and converge to it; the third class solutions oscillate about x_1 up and down and converge to it. Moreover, we show the existence of the third class of solutions.

Consider one example of (2) in the form

$$\dot{x}(t) = -\mu x(t) + a_1 x(t-1)^2 e^{-a_2 x(t-1)}, \tag{3}$$

where parameters satisfy $\mu = \frac{a_1}{a_2 e}$ and the two equilibria are 0 and $\frac{1}{a_2}$. We further explore the convergence of the solution with linear initial value $\phi(s) = \frac{1}{a_2}(s+1+h)$ for $-1 \le s \le 0$ and $0 < h < 1$, which is across the attractive region of the two equilibria. When the information about g and f is more specific, the wider range of h can be obtained to guarantee the same convergence.

The rest of the paper is organized as follows. Section 2 mainly presents the basic definitions and introduces some relevant results. Section 3 explores the convergence of the solution with a class of linear initial value. Section 4 divides all the solutions into three classes according to their eventual tendency and shows the existence of the oscillatory solution. In Section 5 an example is given, for a class of linear initial value, more specific relationships are put forward between the location of the line and the eventual tendency of the corresponding solution. In Section 6 numerical simulations are given to illustrate the main results in Sections 4 and 5. In the final section we make a conclusion and present some unsolved issues.

2 Preliminary

Let $C = C([-\tau, 0], \mathbb{R})$ be the Banach space of continuous functions with the norm given by

$$\|\phi\| = \max_{-\tau \le s \le 0} |\phi(s)| \quad \text{for any} \quad \phi \in C.$$

The Banach space C contains the cone as follows,

$$C^+ = \{\phi \in C : \phi(s) \ge 0, -\tau \le s \le 0\}.$$

The usual notations $<, \le$ and $\ll$ can be used to denote the various relations on C generated by the positive cone C^+. In particular, $\phi \le \psi$ holds if $\phi(s) \le \psi(s)$ for $-\tau \le s \le 0$; $\phi < \psi$ holds if $\phi(s) \le \psi(s)$ and $\phi(s) \ne \psi(s)$ for $-\tau \le s \le 0$; $\phi \ll \psi$ holds if $\phi(s) < \psi(s)$ for $-\tau \le s \le 0$. Likewise, there are order relations $>, \ge$ and $\gg$.

Therefore, we can define the order intervals $[\phi,\psi] := \{\xi \in C : \phi \le \xi \le \psi\}$ if $\phi \le \psi$ and $(\phi,\psi) := \{\xi \in C : \phi \ll \xi \ll \psi\}$ if $\phi \ll \psi$.

Solutions of equation (2) are determined by the initial value $x(\theta) = \phi(\theta)$, where$-\tau \le \theta \le 0, \phi \in C$, and we use the universal symbol x_t to denote the state of the system at time t, where $x_t(\theta) = x(t+\theta)$ for $-\tau \le \theta \le 0$. Then $x_0(\theta) = \phi(\theta)$ and $x_t(0) = x(t)$. In order to emphasize the dependence of a solution on the initial value ϕ, we write $x_t(\phi)$ or $x(t,\phi)$. Equation (2) generates a semiflow Φ on C given by

$$\begin{array}{rcl}\Phi : \mathbb{R}^+ \times C & \to & C,\\ (t,\phi) & \mapsto & x_t(\phi) := \Phi_t(\phi).\end{array}$$

We also define the functional $\lambda : C \to \mathbb{R}$ by

$$\lambda(\phi) := -g(\phi(0)) + f(\phi(-\tau)), \forall \phi \in C.$$

So equation (2) can be written as $\dot{x}(t) = \lambda(x_t)$. If $x \in \mathbb{R}$ we denote by x^* the element of C which takes the value x on $[-\tau, 0]$. The set of equilibria for (2) is then given by $E = \{\phi \in C | \phi \equiv x, \lambda(x^*) = 0\}$.

The positive orbit of ϕ is denoted by $O^+(\phi) = \{\Phi_t(\phi) : t \ge 0\}$. The $\omega(\phi)$ of $\phi \in C^+$ is defined by

$$\omega(\phi) = \cap_{t\ge 0}\overline{\cup_{s\ge t}\Phi_s(\phi)}.$$

i.e, whenever $\psi \in \omega(\phi)$ there exists an infinite sequence t_n such that $\lim\limits_{t_n\to\infty} \Phi_{t_n}(\phi) = \psi$.

The semiflow Φ is said to be monotone provided $\Phi_t(\phi) \le \Phi_t(\psi)$ whenever $\phi \le \psi$ and $t \ge 0$. Φ is called strongly monotone on C^+ if it is monotone and $\Phi_t(\phi) \ll \Phi_t(\psi)$ whenever $\phi < \psi$ and $t > 0$. Φ is said to be eventually strong monotone if it is monotone and whenever $\phi < \psi$ there exists $t_0 > 0$ such that $\Phi_{t_0}(\phi) \ll \Phi_{t_0}(\psi)$. Φ is said to be strongly order-preserving on C^+ if it is monotone and whenever $\phi < \psi$ there exists open subsets $U, V \subset C^+$ and $t_0 > 0$ such that $\phi \in U, \psi \in V$ and $\Phi_{t_0}(U) \le \Phi_{t_0}(V)$. For more knowledge related to functional equations, please refer to [13] and [27].

Proposition 2.1 *[27] If Φ is eventually strongly monotone, then it is strongly order-preserving.*

Here, one main result from Huang et.al [14] about system (2) is as follows.

Theorem 2.1 *[14] For the system (2) fulfilling (F_1'), (F_2) and (F_3) (see Figure 1), $x_0^* = 0^*$ is asymptotically stable and x_1^* is unstable, there exists a heteroclinic orbit $x(t)$, which connects x_0^* and x_1^*. Furthermore, the following results hold:*
(1) $\lim\limits_{t\to\infty} x(t,\phi) = 0$ for $\phi \in [0^*, x_1^*]/\{x_1^*\}$;
(2) $\lim\limits_{t\longrightarrow\infty} x(t,\psi) = x_1$ for $\psi \in [x_1^*, \eta^*]$, where $\eta = \hat{f}^{-1}(f(x_1))$, $\hat{f}$ denotes the restriction of f to the interval $[\xi_0, \infty)$.
(3) The order interval $[0^*, (g^{-1}f(\xi_0))^*]$ is invariant and globally attractive on C^+.

Based on the results of Theorem 2.1, it is clear that $[0^*, \eta_0^*]$ is also invariant and globally attractive on C^+ if $\eta_0 \in [g^{-1}f(\xi_0), \eta]$. If we denote by $\tilde{f}$ the restriction of f to the interval $[0, \xi_0]$, then $\tilde{f}$ is non-decreasing on this interval. The invariance of $[0^*, \xi_0^*]$ and the monotonicity of $\tilde{f}$ guarantee that the semiflow generated by (2) is monotone on $[0^*, \xi_0^*]$ [27].

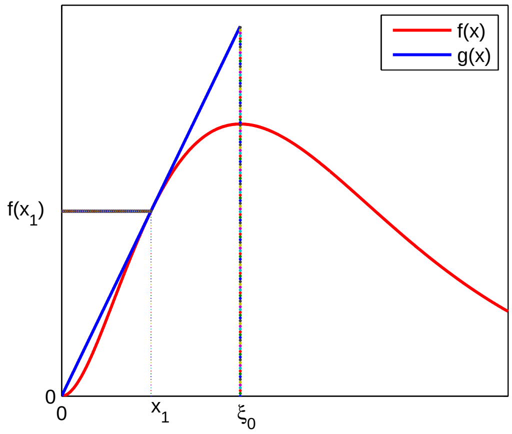

Figure 1: Schematic of equation (2) with one positive equilibrium.

3 The Convergence of the Solution with a Class of Linear Initial Value

This section mainly explores the convergence of the solution of (2) with $\tau = 1$ and linear initial value $\phi(s) = ks + x_1 + h$ for $-1 \le s \le 0$, where $0 < k \le \xi_0$ and $0 < h \le \xi_0 - x_1$. Evidently, ϕ does not completely locate in the attractive region $[0^*, x_1^*]/\{x_1^*\}$ or $[x_1^*, (g^{-1}f(\xi_0))^*]$ for some k and h. Given k, h will determine the convergence of the solution $x(t, \phi)$. Before presenting the principal results, we need to introduce some definitions and explanations. First define a new function $G(x)$, $G(x) = \frac{g(x)}{x}$ if $x > 0$ and $G(0) = g'(0)$. It is easy to check that $G(x)$ is continuous, non-increasing and $G(x) > 0$ by (F_1'). The fact that $\dot{g}(x) > 0$ and $\ddot{g}(x) \le 0$ implies the following definition is meaningful.

$$\delta_1 = \min_{0 \le x \le 2x_1 + \xi_0} g'(x) = g'(2x_1 + \xi_0) > 0 \qquad \text{and} \qquad \delta_2 = \max_{0 \le x \le 2x_1 + \xi_0} g'(x) = g(0) > 0.$$

Therefore, $0 < \delta_1 \le G(x) = \frac{g(x)}{x} \le \delta_2$ for $0 \le x \le 2x_1 + \xi_0$.

From (2) it follows that

$$\dot{x}(t) + x(t)G(x(t)) = f(x(t - \tau)). \tag{4}$$

By multiplying both sides of (4) by $e^{\int_0^t G(x(s))ds}$ and then by integrating from $n\tau$ to t, the solutions of (2) can be obtained for ordinary differential equations on successive intervals of length τ.

$$x(t) = x(n\tau)e^{-\int_{n\tau}^t G(x(s))ds} + \int_{n\tau}^t e^{\int_t^s G(x(\omega))d\omega} f(x(s - \tau))ds \tag{5}$$

with $n \in \mathbb{N}, n\tau \le t \le (n+1)\tau$.

For the initial value $\phi(s) = ks + x_1 + h$, in order to ensure that $\lim_{t\to\infty} x(t, \phi) = x_1$, we give the following hypothesis denoted by (H_0).

Suppose that

$$(H_0) \qquad \xi_0 - x_1 \ge h \ge h_{up} = \frac{-(k - \delta_2 k + \delta_2 x_1) + \sqrt{(k - \delta_2 k + \delta_2 x_1)^2 + 4\delta_2^2 x_1 k}}{2\delta_2}.$$

Theorem 3.1 *Given $\phi(s) = ks + x_1 + h$ for $-1 \le s \le 0$, where $k \le \xi_0$. If h fulfills (H_0), then $\lim_{t\to\infty} x(t, \phi) = x_1$.*

Proof. If $k \le h \le \xi_0 - x_1$, then $\phi(s) \in [x_1^*, \xi_0^*]$, it is clear that $\lim_{t\to\infty} x(t, \phi) = x_1$. If $h \le k$, set $s_1 = -\frac{h}{k}$ and $t_1 = 1 + s_1$. When $0 \le t \le t_1$, from (5) it follows that

$$\begin{aligned}
x(t,\phi) &= e^{-\int_0^t G(x(s,\phi))ds} x(0,\phi) + \int_0^t e^{\int_t^s G(x(\omega,\phi))d\omega} f(x(s-1,\phi))ds \\
&\ge e^{-\int_0^t G(x(s,\phi))ds}(x_1 + h) + f(\phi(-1)) \int_0^t e^{\int_t^s G(x(\omega,\phi))d\omega} ds \\
&= e^{-\int_0^t G(x(s,\phi))ds}(x_1 + h) + f(\phi(-1)) \int_0^t \frac{1}{G(x(s,\phi))} de^{\int_t^s G(x(\omega,\phi))d\omega} \\
&\ge e^{-\int_0^t G(x(s,\phi))ds}(x_1 + h) + \frac{f(\phi(-1))}{\delta_2}(1 - e^{-\int_0^t G(x(s,\phi))ds}) \\
&= (x_1 + h - \frac{f(\phi(-1))}{\delta_2}) e^{-\int_0^t G(x(s,\phi))ds} + \frac{f(\phi(-1))}{\delta_2} \\
&\ge (x_1 + h - \frac{f(\phi(-1))}{\delta_2})(1 - \delta_2 t_1) + \frac{f(\phi(-1))}{\delta_2} \\
&= (1 - \delta_2 t_1)(x_1 + h) + f(\phi(-1)) t_1 \\
&\ge (1 - \delta_2 t_1)(x_1 + h).
\end{aligned}$$

If h satisfies $(1 - \delta_2 t_1)(x_1 + h) \ge x_1$, then $\xi_0 \ge x(t, \phi) \ge x_1$ for $s_1 \le t \le t_1 = s_1 + 1$. By Theorem 2.1, there holds $\lim_{t\to\infty} x(t, \phi) = x_1$.

Therefore, it suffices to show that

$$\begin{aligned}
&(1 - \delta_2 t_1)(x_1 + h) \ge x_1, \qquad (6)\\
i.e.\quad &\frac{\delta_2}{k} h^2 + (1 - \delta_2 + \frac{\delta_2 x_1}{k}) h - \delta_2 x_1 \ge 0.
\end{aligned}$$

It is easy to check that (6) holds if h fulfills (H_0). By the fact that $k \le \xi_0 - x_1$ and Theorem 2.1, there holds $\lim_{t\to\infty} x(t, \phi) = x_1$.

In the following we consider the convergence of the solution of (2) with initial value $\psi(s) = x_1 s + x_1 + h$. First, we introduce some hypotheses as follows.

Suppose that

$$(H_1) \qquad 1 \le \frac{\delta_2}{\delta_1} = \alpha < 2$$

and

$$(H_2) \qquad (\alpha - 1)(e^{\delta_2} + \frac{\delta_2}{4} - 1) - (1 - e^{-\frac{\delta_2}{2}})(1 - \frac{\alpha}{2}) < 0.$$

Set

$$\underline{h_1} = \frac{x_1(\sqrt{1 + (2-\alpha)^2(1 - e^{-\frac{\delta_1}{2}})^2} - 1)}{(2-\alpha)(1 - e^{-\frac{\delta_1}{2}})}$$

and

$$\underline{h_2} = \frac{2x_1(\sqrt{\triangle_2} - (\alpha e^{\delta_2} - \alpha - \frac{\delta_2}{2} + 1)}{3\delta_2},$$

where

$$\triangle_2 = (\alpha e^{\delta_2} - \alpha - \frac{\delta_2}{2} + 1)^2 - 3\delta_2((\alpha - 1)(e^{\delta_2} - 1) - \frac{\delta_2}{4}).$$

Set

$$\underline{h_3} = \frac{x_1(\sqrt{\triangle_3} - (\alpha e^{\delta_2} - \alpha - \frac{\delta_2}{2} + 1))}{2((1 - e^{-\frac{\delta_2}{2}})(1 - \frac{\alpha}{2}) - \frac{\alpha}{4}(\delta_2 - 3\delta_1))},$$

where

$$\begin{aligned}\triangle_3 &= (\alpha e^{\delta_2} - \alpha - \frac{\delta_2}{2} + 1)^2 - 4((1 - e^{-\frac{\delta_2}{2}})(1 - \frac{\alpha}{2}) - \frac{\alpha}{4}(\delta_2 - 3\delta_1)) \\ &\quad ((\alpha - 1)(e^{\delta_2} + \frac{\delta_2}{4} - 1) - (1 - e^{-\frac{\delta_2}{2}})(1 - \frac{\alpha}{2})).\end{aligned}$$

Another hypothesis is as follows.

(H_3) $$h \leq h_{down} \triangleq \min\{\underline{h_1}, \underline{h_2}, \underline{h_3}\}.$$

Theorem 3.2 *Given $\psi(s) = x_1 s + x_1 + h$ for $-1 \leq s \leq 0$. If (H_1)–(H_3) hold, then $\lim_{t\to\infty} x(t, \psi) = 0$.*

Proof. Set $s_1 = -\frac{h}{x_1}$ and $s_0 = \frac{s_1 - 1}{2}$, $t_0 = 1 + s_0$ and $t_1 = 1 + s_1$. The aim of the following part is to show $x(t, \psi) \leq x_1$ for $t_0 \leq t \leq 1 + t_0$. Here we divide the proof into four points.

(1) When $0 \leq t \leq t_0$, from (5) it follows that

$$\begin{aligned} x(t,\psi) &= e^{-\int_0^t G(x(s,\psi))ds} x(0,\psi) + \int_0^t e^{\int_t^s G(x(\omega,\psi))d\omega} f(x(s-1,\psi))ds \\ &\leq e^{-\int_0^t G(x(s,\psi))ds}(x_1 + h) + f(\psi(s_0)) \int_0^t e^{\int_t^s G(x(\omega,\psi))d\omega} ds \\ &= e^{-\int_0^t G(x(s,\psi))ds}(x_1 + h) + f(\psi(s_0)) \int_0^t \frac{1}{G(x(s,\psi))} d e^{\int_t^s G(x(\omega,\psi))d\omega} \\ &\leq e^{-\int_0^t G(x(s,\psi))ds}(x_1 + h) + \frac{\delta_2 \psi(s_0)}{\delta_1}(1 - e^{-\int_0^t G(x(s,\psi))ds}) \\ &\leq (x_1 + h)(1 - \frac{\alpha}{2})e^{-\delta_1 t} + \frac{\alpha}{2}(x_1 + h). \end{aligned}$$

Therefore,

$$\begin{aligned} x(t_0,\psi) &\leq (x_1 + h)(1 - \frac{\alpha}{2})e^{-\delta_1 t_0} + \frac{\alpha}{2}(x_1 + h) \\ &\leq (x_1 + h)(1 - \frac{\alpha}{2})(2(e^{-\frac{\delta_1}{2}} - 1)t_0 + 1) + \frac{\alpha}{2}(x_1 + h). \end{aligned} \tag{7}$$

Let

$$(x_1 + h)(1 - \frac{\alpha}{2})(2(e^{-\frac{\delta_1}{2}} - 1)t_0 + 1) + \frac{\alpha}{2}(x_1 + h) \leq x_1, \tag{8}$$

$$i.e. \qquad (1 - \frac{\alpha}{2})(1 - e^{-\frac{\delta_1}{2}})\frac{h^2}{x_1} + h - (1 - \frac{\alpha}{2})(1 - e^{-\frac{\delta_1}{2}})x_1 \leq 0.$$

Since $h \le h_{down} \le \underline{h_1}$, it is easy to check that (8) holds. Therefore, $x(t_0,\psi) \le x_1$.

(2) When $t_1 \le t \le 1$, from (5) it follows that

$$\begin{aligned}
x(t,\psi) &= e^{-\int_0^t G(x(s,\psi))ds}x(0,\psi) + \int_0^t e^{\int_t^s G(x(\omega,\psi))d\omega} f(x(s-1,\psi))ds \\
&\le e^{-\int_0^t G(x(s,\psi))ds}(x_1+h) + f(\psi(s_0)) \int_0^{t_0} e^{\int_t^s G(x(\omega,\psi))d\omega} ds \\
&\quad + f(\psi(s_1)) \int_{t_0}^{t_1} e^{\int_t^s G(x(\omega,\psi))d\omega} ds + f(\psi(0)) \int_{t_1}^{t} e^{\int_t^s G(x(\omega,\psi))d\omega} ds \\
&\le e^{-\int_0^t G(x(s,\psi))ds}(x_1+h) + \frac{\alpha}{2}(x_1+h)(e^{-\int_{t_0}^t G(x(s,\psi))ds} - e^{-\int_0^t G(x(s,\psi))ds}) \\
&\quad + \alpha x_1 (e^{-\int_{t_1}^t G(x(s,\psi))ds} - e^{-\int_{t_0}^t G(x(s,\psi))ds}) + \alpha(x_1+h)(1-e^{-\int_{t_1}^t G(x(s,\psi))ds}) \\
&= e^{-\int_0^t G(x(s,\psi))ds}((1-\frac{\alpha}{2})(x_1+h) + \frac{\alpha}{2}(h-x_1)e^{\int_0^{t_0} G(x(s,\psi))ds} \\
&\quad - \alpha h e^{\int_0^{t_1} G(x(s,\psi))ds}) + \alpha(x_1+h).
\end{aligned}$$

Since

$$\begin{aligned}
& (1-\frac{\alpha}{2})(x_1+h) + \frac{\alpha}{2}(h-x_1)e^{\int_0^{t_0} G(x(s,\psi))ds} - \alpha h e^{\int_0^{t_1} G(x(s,\psi))ds} \\
= \;& x_1(1-\frac{\alpha}{2} - \frac{\alpha}{2}e^{\int_0^{t_0} G(x(s,\psi))ds}) + h(1-\frac{\alpha}{2} - \frac{\alpha}{2}e^{\int_0^{t_1} G(x(s,\psi))ds}) \\
& + \frac{\alpha h}{2}(e^{\int_0^{t_0} G(x(s,\psi))ds} - e^{\int_0^{t_1} G(x(s,\psi))ds}) \\
< \;& 0,
\end{aligned}$$

there holds

$$\begin{aligned}
x(t,\psi) &\le e^{-\delta_2}((1-\frac{\alpha}{2})(x_1+h) + \frac{\alpha}{2}(h-x_1)e^{\int_0^{t_0} G(x(s,\psi))ds} \qquad (9) \\
&\quad - \alpha h e^{\int_0^{t_1} G(x(s,\psi))ds}) + \alpha(x_1+h) \\
&\le e^{-\delta_2}((1-\frac{\alpha}{2})(x_1+h) + \frac{\alpha}{2}(h-x_1)e^{\delta_1 t_0} - \alpha h e^{\delta_1 t_1}) + \alpha(x_1+h) \qquad (10) \\
&\le e^{-\delta_2}((1-\frac{\alpha}{2})(x_1+h) + \frac{\alpha}{2}(h-x_1)(1+\delta_1 t_0) - \alpha h(1+\delta_1 t_1)) + \alpha(x_1+h) \\
&= e^{-\delta_2}(\frac{3\delta_2}{4x_1}h^2 - (\alpha+\frac{\delta_2}{2}-1)h - (\alpha+\frac{\delta_2}{4}-1)x_1) + \alpha(x_1+h).
\end{aligned}$$

Let

$$e^{-\delta_2}(\frac{3\delta_2}{4x_1}h^2 - (\alpha+\frac{\delta_2}{2}-1)h - (\alpha+\frac{\delta_2}{4}-1)x_1) + \alpha(x_1+h) \le x_1, \qquad (11)$$
$$i.e. \qquad \frac{3\delta_2}{4x_1}h^2 + (\alpha e^{\delta_2} - \alpha - \frac{\delta_2}{2} + 1)h + ((\alpha-1)(e^{\delta_2}-1) - \frac{\delta_2}{4})x_1 \le 0.$$

By (H_2), there holds $(\alpha-1)(e^{\delta_2}-1) - \frac{\delta_2}{4} < 0$. Since $h \le h_{down} \le \underline{h_2}$, (11) holds, i.e. $x(t,\psi) \le x_1$ for $t_1 \le t \le 1$.

(3) When $t_0 \le t \le t_1$, from (5) it follows that

$$\begin{aligned} x(t,\psi) &= e^{-\int_0^t G(x(s,\psi))ds}x(0,\psi) + \int_0^t e^{\int_t^s G(x(\omega,\psi))d\omega} f(x(s-1,\psi))ds \\ &\le e^{-\int_0^t G(x(s,\psi))ds}(x_1+h) + f(\psi(s_0))\int_0^{t_0} e^{\int_t^s G(x(\omega,\psi))d\omega}ds \\ &\quad + f(\psi(s_1))\int_{t_0}^t e^{\int_t^s G(x(\omega,\phi))d\omega}ds \\ &\le e^{-\int_0^t G(x(s,\psi))ds}(x_1+h+\frac{\alpha}{2}(x_1+h)(e^{\int_0^{t_0} G(x(s,\psi))ds}-1) \\ &\quad +\alpha x_1 e^{\int_0^{t_0} G(x(s,\psi))ds}) + \alpha x_1 \\ &= e^{-\int_0^t G(x(s,\psi))ds}((x_1+h)(1-\frac{\alpha}{2}) + \frac{\alpha}{2}(h-x_1)e^{\int_0^{t_0} G(x(s,\psi))ds}) + \alpha x_1. \end{aligned}$$

If $(x_1+h)(1-\frac{\alpha}{2}) + \frac{\alpha}{2}(h-x_1)e^{\int_0^{t_0} G(x(s,\psi))ds} \ge 0$, there holds

$$\begin{aligned} x(t,\psi) &\le e^{-\int_0^{t_0} G(x(s,\psi))ds}((x_1+h)(1-\frac{\alpha}{2}) + \frac{\alpha}{2}(h-x_1)e^{\int_0^{t_0} G(x(s,\psi))ds}) + \alpha x_1 \\ &\le (x_1+h)(1-\frac{\alpha}{2})e^{-\delta_1 t_0} + \frac{\alpha}{2}(h+x_1). \end{aligned} \tag{12}$$

As we have proved that the right-hand part of (12) (i.e.inequaltiy (7)) is less than x_1, which means that $x(t,\psi) \le x_1$ for $t_0 \le t \le t_1$.

If $(x_1+h)(1-\frac{\alpha}{2}) + \frac{\alpha}{2}(h-x_1)e^{\int_0^{t_0} G(x(s,\psi))ds} \le 0$, there holds

$$x(t,\psi) \le e^{-\delta_2}((x_1+h)(1-\frac{\alpha}{2}) + \frac{\alpha}{2}(h-x_1)e^{\int_0^{t_0} G(x(s,\psi))ds}) + \alpha x_1. \tag{13}$$

Subtracting the right-hand part of inequalities (9) from that of (13) gives

$$\alpha h e^{-\delta_2 + \int_0^{t_1} G(x(s,\psi))ds} - \alpha h \le 0.$$

Since the right-hand part of (9) is less than x_1, then $x(t,\psi) \le x_1$ for $t_0 \le t \le t_1$.

(4) When $1 \le t \le 1+t_0$, from (5) it follows that

$$\begin{aligned} x(t,\psi) &= e^{-\int_1^t G(x(s,\psi))ds}x(1,\psi) + \int_1^t e^{\int_t^s G(x(\omega,\psi))d\omega} f(x(s-1,\psi))ds \\ &\le e^{-\int_1^t G(x(s,\psi))ds}x(1,\psi) + f(\psi(0))\int_1^t e^{\int_t^s G(x(\omega,\psi))d\omega}ds \\ &\le e^{-\int_1^t G(x(s,\psi))ds}x(1,\psi) + \alpha(x_1+h)(1-e^{-\int_1^t G(x(s,\psi))ds}) \\ &\le (x(1,\psi)-\alpha(x_1+h))e^{-\delta_2 t_0} + \alpha(x_1+h). \end{aligned}$$

Since (10) implies that

$$x(1,\psi) \le e^{-\delta_2}((1-\frac{\alpha}{2})(x_1+h) + \frac{\alpha}{2}(h-x_1)e^{\delta_1 t_0} - \alpha h e^{\delta_1 t_1}) + \alpha(x_1+h),$$

there holds

$$\begin{aligned} x(t,\psi) &\le e^{-\delta_2}((1-\frac{\alpha}{2})(x_1+h)e^{-\delta_2 t_0}+\frac{\alpha}{2}(h-x_1)e^{\delta_1 t_0-\delta_2 t_0}-\alpha h e^{\delta_1 t_1-\delta_2 t_0})+\alpha(x_1+h) \\ &\le e^{-\delta_2}((1-\frac{\alpha}{2})(x_1+h)(2(e^{-\frac{\delta_2}{2}}-1)t_0+1)+\frac{\alpha}{2}(h-x_1)(1+\delta_1 t_0-\delta_2 t_0) \\ &\quad -\alpha h(1+\delta_1 t_1-\delta_2 t_0))+\alpha(x_1+h) \qquad (b=1-e^{-\frac{\delta_2}{2}}) \\ &= e^{-\delta_2}((b(1-\frac{\alpha}{2})-\frac{\alpha}{4}(\delta_2-3\delta_1))\frac{h^2}{x_1}-(\alpha+\frac{\delta_2}{2}-1)h \\ &\quad +((\alpha-1)(\frac{\delta_2}{4}-1)-b(1-\frac{\alpha}{2}))x_1)+\alpha(x_1+h). \end{aligned}$$

Letting the right-hand part of the above inequality be less than x_1, by equivalent transformation, we have

$$\begin{aligned} &(b(1-\frac{\alpha}{2})-\frac{\alpha}{4}(\delta_2-3\delta_1))\frac{h^2}{x_1}+(\alpha e^{\delta_2}-\alpha-\frac{\delta_2}{2}+1)h \\ &+((\alpha-1)(e^{\delta_2}+\frac{\delta_2}{4}-1)-b(1-\frac{\alpha}{2}))x_1 \le 0. \end{aligned} \tag{14}$$

Based on (H_2) and the fact that $h \le h_{down} \le \underline{h_3}$, (14) holds, i.e. $x(t,\psi) \le x_1$ for $1 \le t \le 1+t_0$.

As a conclusion, $x(t,\psi) \le x_1$ for $t_0 \le t \le 1+t_0$ if (H_1)–(H_3) hold. By Theorem 2.1, there holds $\lim\limits_{t\to\infty} x(t,\psi)=0$.

4 The Classification of Solutions and the Existence of Oscillatory Solution

This section is devoted to divide all solutions of (2) into three categories according to their eventual tendency and show the existence of oscillatory solution. First the definition of oscillatory solutions is formulated as follows.

Definition 4.1 [9, 11, 12] The solution $x(t,\phi)$ of (2) with initial value $\phi \in C^+$ is said to be oscillatory about $\bar{x}$, if there exists a sequence $\{\xi_n\} \to \infty$ as $n \to \infty$ such that $x(\xi_n,\phi)=\bar{x}$ and $x(t,\phi)-\bar{x}$ simultaneously has positive and negative values in (ξ_n,ξ_{n+1}) for $n=1,2,3,\cdots$. Otherwise, $x(t,\phi)$ is said to be non-oscillatory about $\bar{x}$.

For the systems of delay differential equations, there are various ways to define oscillation. For instance, in [9, 11] the real function x is said to be oscillatory about zero if x has arbitrarily large zeros. Here the definition is stricter than those mentioned above. Consider $x(t)=\sin t+2$, which is oscillatory about 1 according to the concept in [9, 11]. However, it is non-oscillatory about 1 according to Definition 4.1.

Theorem 4.1 *If $x(t,\phi)$ is oscillatory about x_1, then $\lim\limits_{t\to\infty} x(t,\phi)=x_1$.*

Proof. First we assert that the semiflow generated by (2) is eventually strongly monotone on $[0^*,\xi_0^*]$, then by Proposition 2.1, it is strongly order-preserving. For any $\phi,\psi \in [0^*,\xi_0^*]$, if $\phi < \psi$, there exists a $t_0 \in [0,\tau]$ such that $x(t_0,\phi) < x(t_0,\psi)$. Otherwise, $x(t,\phi)=x(t,\psi)$ for $0 \le t \le \tau$.

From (5) it follows immediately that

$$\begin{aligned} x(t,\phi)-x(t,\psi) &= e^{-\int_0^t G(x(s,\phi))ds}x(0,\phi)-e^{-\int_0^t G(x(s,\psi))ds}x(0,\psi) \\ &\quad +\int_0^t e^{\int_t^s G(x(\omega,\phi))d\omega}f(x(s-\tau,\phi))ds \\ &\quad -\int_0^t e^{\int_t^s G(x(\omega,\psi))d\omega}f(x(s-\tau,\psi))ds, \\ i.e. \qquad 0 &= \int_0^t e^{\int_t^s G(x(\omega,\phi))d\omega}(f(x(s-\tau,\phi))-f(x(s-\tau,\psi)))ds \end{aligned}$$

with $0 \le t \le \tau$. By the fact that $x(t,\phi) \le x(t,\psi) \le \xi_0$ and $f(x)$ is strictly increasing on $[0,\xi_0]$, there holds $x(s-\tau,\phi)=x(s-\tau,\psi)$ for $0 \le s \le \tau$, i.e. $\phi=\psi$, which contradicts the assumption.

Replacing $n\tau$ in (5) by t_0, we have

$$x(t)=e^{-\int_{t_0}^t G(x(s))ds}x(t_0)+\int_{t_0}^t e^{\int_t^s G(x(\omega))d\omega}f(x(s-\tau))ds$$

with $t_0 \le t \le t_0+\tau$. By the fact that $x(t,\phi) \le x(t,\psi) \le \xi_0$, $f(x)$ is strictly increasing on $[0,\xi_0]$ and $G(x)$ is non-increasing, there holds $f(x(t,\phi)) \le f(x(t,\psi))$ and $G(x(t,\phi)) \ge G(x(t,\psi))$. Furthermore,

$$\begin{aligned} x(t,\phi)-x(t,\psi) &\le e^{-\int_{t_0}^t G(x(s,\phi))ds}x(t_0,\phi)-e^{-\int_{t_0}^t G(x(s,\psi))ds}x(t_0,\psi) \\ &< 0 \qquad \text{whenever} \qquad t_0 \le t \le t_0+\tau, \end{aligned}$$

i.e. for $\phi<\psi$, there exists a $t_1=t_0+\tau$ such that $x_{t_1}(\phi) \ll x_{t_1}(\psi)$, then the semiflow generated by (2) is eventually strongly monotone. Therefore, it is strongly order-preserving on $[0^*,\xi_0^*]$.

If $x(t,\phi)$ is oscillatory about x_1 with $0^* \le \phi < \xi_0^*$, by Theorem 3.7 in [27], we have $\omega(\phi) <$ or $= \omega(\xi_0^*)=\{x_1^*\}$. If the former holds, the compactness of $\overline{O^+(\phi)}$ suggests that $\omega(\phi)$ is nonempty, compact, invariant and connected, so $0^* \in \omega(\phi)$. Obviously, $0^* \le \omega(\phi)$, Corollary 2.4 in [27] implies that $\omega(\phi)=\{0^*\}$, which contradicts the oscillation of $x(t,\phi)$. Thus $\omega(\phi)=\{x_1^*\}$, and $x_{t_k}(\phi) \to x_1^*$ if and only if $x_{t_k}(\xi_0^*) \to x_1^*$. The fact that $x(t,\xi_0^*) \to x_1$ implies $x(t,\phi) \to x_1$.

Based on the global attractivity of $[0^*,\xi_0^*]$, the solution $x(t,\phi)$ with $\phi \in C^+$ oscillating about x_1 will eventually tend to x_1.

Proposition 4.1 *Given any $\phi \in C^+ \setminus \{0^*,x_1^*\}$, only one of the following results holds:*

(1) $x(t,\phi)$ enters $(0,x_1)$ ultimately , thus $\lim\limits_{t\to\infty} x(t,\phi)=0$.

(2) $x(t,\phi)$ enters $(x_1,\xi_0]$ ultimately, thus $\lim\limits_{t\to\infty} x(t,\phi)=x_1$.

(3) $x(t,\phi)$ oscillates about x_1, thus $\lim\limits_{t\to\infty} x(t,\phi)=x_1$.

Proof. Assume, by contradiction, that there exists $\phi \in C^+ \setminus \{0^*,x_1^*\}$, a T and a sequence $\{\xi_n\} \to \infty$ as $n \to \infty$ such that one of the following two cases holds.
(a) $x(\xi_n,\phi)=x_1$ for $n=1,2,3,\cdots$, and $x(t,\phi) \ge x_1$ for $t>T$.
(b) $x(\xi_n,\phi)=0$ for $n=1,2,3,\cdots$, and $0 \le x(t,\phi) < x_1$ for $t>T$.

Assume that case (a) holds. Choose a sufficiently large $\xi_n > T + 2\tau$ and denote it by ξ_{n_0+2} such that $x(\xi_{n_0+2}, \phi) = x_1$. Note that $x(t, \phi)$ eventually enters $[x_1, \xi_0]$ and the derivative of $x(t, \phi)$ is continuous, then $\dot{x}(\xi_n, \phi) = 0$ for $n = 1, 2, 3, \cdots$.
Therefore,

$$\begin{aligned} 0 = \dot{x}(\xi_{n_0+2}, \phi) &= -g(x(\xi_{n_0+2}, \phi)) + f(x(\xi_{n_0+2} - \tau, \phi)), \\ \text{i.e.} \quad g(x_1) &= f(x(\xi_{n_0+2} - \tau, \phi)), \end{aligned}$$

which implies that $x(\xi_{n_0+2} - \tau, \phi) = x_1$. Here, denote $\xi_{n_0+2} - \tau$ by ξ_{n_0+1} and $\xi_{n_0+2} - 2\tau$ by ξ_{n_0} for brevity. Then they satisfy the following coditions.
(a1) $x(\xi_{n_0+i}, \phi) = x_1$ where $i = 0, 1, 2$.
(a2) $\dot{x}(\xi_{n_0+i}, \phi) = 0$ where $i = 0, 1, 2$.

Let ξ_{n_0+1} be an initial point of integration in (5), then

$$x(t, \phi) = e^{-\int_{\xi_{n_0+1}}^{t} G(x(s,\phi))ds} x(\xi_{n_0+1}, \phi) + \int_{\xi_{n_0+1}}^{t} e^{\int_t^s G(x(\omega,\phi))d\omega} f(x(s - \tau, \phi))ds \qquad (15)$$

with $\xi_{n_0+1} \le t \le \xi_{n_0+1} + \tau = \xi_{n_0+2}$.
Replacing t by ξ_{n_0+2} in (15) gives

$$x(\xi_{n_0+2}, \phi) = x_1 e^{-\int_{\xi_{n_0+1}}^{\xi_{n_0+2}} G(x(s,\phi))ds} + \int_{\xi_{n_0+1}}^{\xi_{n_0+2}} e^{\int_{\xi_{n_0+2}}^{s} G(x(\omega,\phi))d\omega} f(x(s - \tau, \phi))ds, \qquad (16)$$

i.e.

$$\begin{aligned} x_1(1 - e^{-\int_{\xi_{n_0+1}}^{\xi_{n_0+2}} G(x(s,\phi))ds}) &= \int_{\xi_{n_0+1}}^{\xi_{n_0+2}} e^{\int_{\xi_{n_0+2}}^{s} G(x(\omega,\phi))d\omega} f(x(s - \tau, \phi))ds \\ &= \int_{\xi_{n_0+1}}^{\xi_{n_0+2}} \frac{f(x(s - \tau, \phi))}{G(x(s, \phi))} G(x(s, \phi)) e^{\int_{\xi_{n_0+2}}^{s} G(x(\omega,\phi))d\omega} ds. \end{aligned}$$

Note that

$$\int_{\xi_{n_0+1}}^{\xi_{n_0+2}} G(x(s, \phi)) e^{\int_{\xi_{n_0+2}}^{s} G(x(\omega,\phi))d\omega} ds = 1 - e^{-\int_{\xi_{n_0+1}}^{\xi_{n_0+2}} G(x(s,\phi))ds},$$

by equivalent transformation, (16) becomes

$$0 = \int_{\xi_{n_0+1}}^{\xi_{n_0+2}} \left(\frac{f(x(s - \tau, \phi))}{x_1 G(x(s, \phi))} - 1\right) G(x(s, \phi)) e^{\int_{\xi_{n_0+2}}^{s} G(x(\omega,\phi))d\omega} ds. \qquad (17)$$

By the fact that $\xi_0 \ge x(t, \phi) \ge x_1$ for $t > T$, $f(x)$ increases on $[x_1, \xi_0]$ and $G(x)$ is non-increasing, there holds

$$\frac{f(x(s - \tau, \phi))}{x_1 G(x(s, \phi))} \ge \frac{f(x_1)}{x_1 G(x(s, \phi))} = \frac{G(x_1)}{G(x(s, \phi))} \ge 1.$$

Equality in (17) holds if and only if $x(s - \tau, \phi) = x_1$ and $G(x(s, \phi)) = G(x_1)$ for $\xi_{n_0+1} \le s \le \xi_{n_0+2}$. Induction implies $\phi = x_1$, which contradicts the assumption. Similarly, case (b) does not hold. So far the proof is completed.

In the following part, attention will be paid to show the existence of the oscillatory solution. Here consider the initial value $\phi(s) = ks + b$ for $-\tau \le s \le 0$, where $0 < k \le \min\{\frac{x_1}{\tau}, \frac{\xi_0 - x_1}{\tau}\}$ and $x_1 \le b \le \xi_0$. Given k, the parameter b will determine the eventual tendency of the solution $x(t, \phi)$. In order to stress the dependence of the eventual tendency of $x(t, \phi)$ on the parameter b, we abbreviate $\phi(s)$ to ϕ^b.

Proposition 4.2 *Given $\phi \in C^+$, if $\lim_{t\to\infty} x(t,\phi) = 0$, then there exists a $\delta > 0$ such that $\lim_{t\to\infty} x(t,\psi) = 0$ for any $\psi \in O(\phi,\delta)$.*

Proof. If $\lim_{t\to\infty} x(t,\phi) = 0$, then there exists a $T_0 > 0$ such that $x(t,\phi) < x_1$ for $t \in [T_0, T_0 + 2\tau]$. Set $l = \max_{T_0 \le t \le T_0+2\tau} x(t,\phi)$, $\epsilon = (x_1 - l)/3$ and $T = T_0 + 2\tau$, by the continuous dependence of solutions on the initial value [13, 15, 27], there exists a $\delta(\epsilon, T) > 0$ such that $|x(t,\phi) - x(t,\psi)| < \epsilon$ for $0 \le t \le T$ and any $\psi \in O(\phi,\delta)$. This means that $x(t,\psi) < x_1$ for $T_0 \le t \le T_0 + 2\tau$. Therefore $\lim_{t\to\infty} x(t,\psi) = 0$ by Theorem 2.1.

Remark 4.1 From the above proposition it is easy to get the following conclusion. If $b = b_0$, i.e. the initial value $\phi(s) = ks + b_0$ for $-\tau \le s \le 0$, and $\lim_{t\to\infty} x(t,\phi^{b_0}) = 0$, then there exists a $\delta > 0$ such that $\lim_{t\to\infty} x(t,\phi^b) = 0$ for any $b \in O(b_0,\delta) \cap [x_1, \xi_0]$.

Remark 4.2 The above proposition can not be generalized to $\lim_{t\to\infty} x(t,\phi) = x_1$, i.e. if $\lim_{t\to\infty} x(t,\phi) = x_1$, it does not provide that there exists a $\delta > 0$, such that $\lim_{t\to\infty} x(t,\psi) = x_1$ for any $\psi \in O(\phi,\delta)$. This case can be confirmed in the following part. The following proposition is a special case.

Proposition 4.3 *If $b = \xi_0$, i.e, the initial value $\phi(s) = ks + \xi_0$ for $-\tau \le s \le 0$, then there exists a $\delta > 0$ such that $\lim_{t\to\infty} x(t,\phi^b) = x_1$ for any $b \in [\xi_0 - \delta, \xi_0]$.*

Proof. Note that $\phi^{\xi_0} \in [x_1^*, \xi_0^*]$, the argument of Theorem 4.1 implies that there exists a T_1 such that $x(t,\phi) > x_1$ for $t \ge 0$. Let $T_2 = T_1 + 2\tau$, $l = \min_{0\le t\le 2\tau} x(t,\phi^{\xi_0})$ and $\epsilon = (l - x_1)/3$, by the continuous dependence of solutions on the initial value [13,15,27], there exists a $\delta(\epsilon, T_2) > 0$, when $b \in [\xi_0 - \delta, \xi_0]$, $0 \le x(t,\phi^{\xi_0}) - x(t,\phi^b) < \epsilon$ for $0 \le t \le T_2$. This means that $x(t,\phi^b) > x_1$ for $T_1 \le t \le T_2$, so $\lim_{t\to\infty} x(t,\phi^b) = x_1$ by Theorem 2.1.

Theorem 4.2 *There exists an initial value ϕ such that $x(t,\phi)$ oscillates about x_1.*

Proof. Consider the linear initial value $\phi(s) = ks + b$ for $-\tau \le s \le 0$. We restrict b to $[x_1, \xi_0]$. Then there must exist a $b_0 \in (x_1, \xi_0)$ such that $x(t,\phi^b)$ oscillates about x_1. Otherwise, given $b \in [x_1, \xi_0]$, by Theorem 4.1, Propositions 4.2 and 4.3, there exists a δ such that $\lim_{t\to\infty} x(t,\phi^{b'}) = \lim_{t\to\infty} x(t,\phi^b) = 0$ or x_1 for any $b' \in O(b,\delta)$. This contradicts the finiteness of b, which is restricted to $[x_1, \xi_0]$. Thus such a b_0 exists, i.e. the oscillatory solution exists.

In the following section, denote

$$B := \{ \quad b \quad | \quad x_1 \le b \le \xi_0, \quad \lim_{t\to\infty} x(t,\phi^b) = x_1\}, \quad \beta = \inf B,$$

$$A := \{ \quad b \quad | \quad x_1 \le b \le \xi_0, \quad \lim_{t\to\infty} x(t,\phi^b) = 0\}, \quad \alpha = \sup A.$$

Proposition 4.4 *The solution $x(t,\phi^\alpha)$ oscillates about x_1, $\lim_{t\to\infty} x(t,\phi) = 0$ for $\phi \in [0^*, \phi^\alpha)$ and $\lim_{t\to\infty} x(t,\phi) = x_1$ for $\phi \in [\phi^\alpha, \xi_0^*]$.*

Proof. If $x(t,\phi^\alpha)$ does not oscillate about x_1, then it will eventually enter the domain $(x_1,\xi_0]$ or $(0,x_1)$. If it enters $(0,x_1)$, by Proposition 4.2, there exists a δ such that $\lim_{t\to\infty} x(t,\phi^b)=0$ for $b\in O(\alpha,\delta)$, which contradicts the definition of α. Similarly, it will not eventually enter the domain $(x_1,\xi_0]$. Therefore $x(t,\phi^\alpha)$ oscillates about x_1. The second part is clear by the monotonicity of the semiflow generated by (2).

In the same way, we can immediately get the following result.

Corollary 4.1 *The solution $x(t,\phi^\beta)$ oscillates about x_1 and $\alpha=\beta$.*

Remark 4.3 For system (2) with $\tau=1$ and the initial value $\phi(s)=x_1(s+1+h)$ in Section 3, according to Theorem 4.2, there exists a h_0 such that $x(t,\phi)$ oscillates about x_1 and then converges to it if $h=h_0$, $\lim_{t\to\infty} x(t,\phi)=0$ if $0\le h<h_0$ and $\lim_{t\to\infty} x(t,\phi)=x_1$ if $h_0\le h\le \xi_0-x_1$.

5 Example

This section mainly investigates model (3)

$$\dot{x}(t)=-\mu x(t)+a_1x(t-1)^2e^{-a_2x(t-1)},$$

where parameters satisfy $\mu=\frac{a_1}{a_2e}$ and the two equilibria are 0 and $\frac{1}{a_2}$. Their attractive regions are $[0^*,(\frac{1}{a_2})^*]\setminus\{(\frac{1}{a_2})^*\}$ and $[(\frac{1}{a_2})^*,(\hat{f}^{-1}(f(\frac{1}{a_2})))^*]$ respectively [14]. Let us set the linear initial value $\phi(s)=\frac{1}{a_2}(s+1+h)$ for $-1\le s\le 0$ and $0<h<1$. Obviously, ϕ does not completely locate in any attractive region. The parameter function h will determine the convergence of the solution $x(t,\phi)$.

The following two theorems describe the relationship between the eventual tendency of the solution $x(t,\phi)$ and the parameter μ (i.e. a_1 and a_2).

Theorem 5.1 *Set $h_1(\mu)=\frac{\mu}{\mu+1}$ for $0<\mu<\infty$ and $\phi(s)=\frac{1}{a_2}(s+1+h)$ for $-1\le s\le 0$, if $h_1\le h\le 1$, then $\lim_{t\to\infty} x(t,\phi)=\frac{1}{a_2}$.*

The proof of this theorem is given in Appendix A. For system (3) and the initial value with slope $\frac{1}{a_2}$, if μ increases, the ratio of the intercept to $\frac{1}{a_2}$ needs to be increased appropriately so that the corresponding solution converges to $\frac{1}{a_2}$. If μ decreases, appropriate reduction in the ratio can still guarantee that the corresponding solution converges to $\frac{1}{a_2}$.

According to Theorem 3.1, $h_{up}=\frac{-1+\sqrt{1+4\mu^2}}{2\mu}\frac{1}{a_2}$, i.e. $\lim_{t\to\infty} x(t,\phi)=\frac{1}{a_2}$ if $\phi(s)=\frac{1}{a_2}(s+1)+h$ for $h_{up}\le h\le\frac{1}{a_2}$. Note that $h_{up}\ge\frac{1}{a_2}h_1$, it implies that Theorem 5.1 gives wider range of linear initial value, the corresponding solutions of which converge to the positive equilibrium of system (3).

Theorem 5.2 *Set*

$$h_2(\mu)=\begin{cases}\dfrac{\mu}{3(\mu+1)}, & 0<\mu\le 1,\\[2mm] \dfrac{1}{6\mu}, & 1<\mu<\infty,\end{cases}$$

and $\psi(s)=\frac{1}{a_2}(s+1+h)$ for $-1\le s\le 0$, if $0\le h\le h_2(\mu)$, then $\lim_{t\to\infty} x(t,\psi)=0$.

The proof of this theorem is given in Appendix A. Note that in the case $0 < \mu \leq 1$, for system (3) and the initial value with slope $\frac{1}{a_2}$, the ratio of the intercept to $\frac{1}{a_2}$ needs to be decreased appropriately so that the corresponding solution converges to 0 if μ decreases. Appropriate increase in the ratio still can guarantee that the corresponding solution converges to 0 if μ increases.

According to Theorem 3.2, $\underline{h_2} = \frac{\frac{\mu}{2} - e^{\mu} + \sqrt{\triangle_2}}{1.5 a_2 \mu}$ where $\triangle_2 = e^{2\mu} - \mu e^{\mu} + \mu^2$. Note that $h_{down} \leq \underline{h_2} \leq \frac{1}{a_2} h_2$, it implies that Theorem 5.2 gives wider range of linear initial value, the corresponding solutions of which converge to the trivial equilibrium of system (3).

Remark 5.1 The above two parameter functions indeed guarantee that the corresponding solution belongs to the first class and the second class mentioned in Proposition 4.1. However, they are just sufficient conditions. For system (3) with the initial $\phi(s) = \frac{1}{a_2}(s + 1 + h)$, according to Theorem 4.2, there exists a h_0 such that $x(t, \phi)$ oscillates about $\frac{1}{a_2}$ and then converges to it if $h = h_0$, $\lim\limits_{t\to\infty} x(t, \phi) = 0$ if $0 \leq h < h_0$ and $\lim\limits_{t\to\infty} x(t, \phi) = \frac{1}{a_2}$ if $h_0 \leq h \leq \frac{2}{a_2}$.

6 Simulations

In this section, numerical simulations are given to illustrate some results in Sections 4 and 5.

Consider the model from Section 5

$$\dot{x}(t) = -\mu x(t) + a_1 x(t-1)^2 e^{-a_2 x(t-1)}$$

and the initial value $\phi(s) = \frac{1}{a_2}(s + 1 + h(\mu))$ for $-1 \leq s \leq 0$.

Simulation 1: Let $h(\mu) = h_1(\mu) = \frac{\mu}{\mu+1}$ for $0 < \mu < \infty$.

Case A: Fix $a_1 = e$.

(1) Choose $a_2 = 10$, then $\mu = \frac{1}{10}$ and $\phi_1(s) = \frac{1}{10}(s + \frac{12}{11})$. From Theorem 5.1 it follows $\lim\limits_{t\to\infty} x(t, \phi_1) = \frac{1}{10}$ (see Figure 2).

(2) Choose $a_2 = 4$, then $\mu = \frac{1}{4}$ and $\phi_2(s) = \frac{1}{4}(s + \frac{6}{5})$. From Theorem 5.1 it follows $\lim\limits_{t\to\infty} x(t, \phi_2) = \frac{1}{4}$. However, if set $\phi_3(s) = \frac{1}{4}(s + \frac{12}{11})$, simulation implies $\lim\limits_{t\to\infty} x(t, \phi_3) = 0$ (see Figure 4).

(3) Choose $a_2 = 1$, then $\mu = 1$ and $\phi_4(s) = s + \frac{3}{2}$. From Theorem 5.1 it follows $\lim\limits_{t\to\infty} x(t, \phi_4) = 1$. However, if set $\phi_5(s) = s + \frac{6}{5}$, simulation implies $\lim\limits_{t\to\infty} x(t, \phi_5) = 0$ (see Figure 6).

Case B: Fix $a_2 = 5$.

(1) Choose $a_1 = e$, then $\mu = \frac{1}{5}$ and $\psi_1(s) = \frac{1}{5}(s + \frac{7}{6})$. From Theorem 5.1 it follows $\lim\limits_{t\to\infty} x(t, \psi_1) = \frac{1}{5}$ (see Figure 3).

(2) Choose $a_1 = 3e$, then $\mu = \frac{3}{5}$ and $\psi_2(s) = \frac{1}{5}(s + \frac{11}{8})$. From Theorem 5.1 it follows $\lim\limits_{t\to\infty} x(t, \psi_2) = \frac{1}{5}$. However, if set $\psi_3(s) = \frac{1}{5}(s + \frac{7}{6})$, simulation implies $\lim\limits_{t\to\infty} x(t, \psi_3) = 0$ (see Figure 5).

(3) Choose $a_1 = 20e$, then $\mu = 4$ and $\psi_4(s) = \frac{1}{5}(s + \frac{9}{5})$. From Theorem 5.1 it follows $\lim\limits_{t\to\infty} x(t, \psi_4) = \frac{1}{5}$. However, if set $\psi_5(s) = \frac{1}{5}(s + \frac{11}{8})$, simulation implies $\lim\limits_{t\to\infty} x(t, \psi_5) = 0$ (see Figure 7).

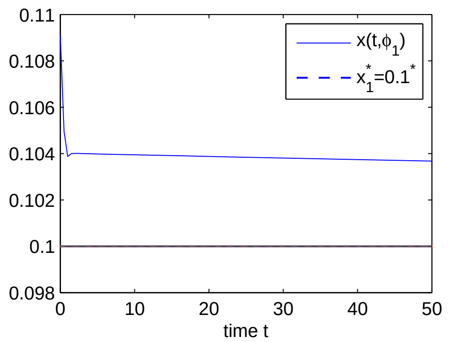

Figure 2: The numerical solution of $\dot{x}(t) = -\frac{x(t)}{10} + ex(t-1)^2 e^{-10x(t-1)}$ with the initial value ϕ_1.

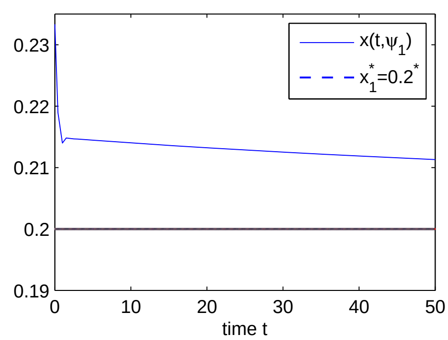

Figure 3: The numerical solution of $\dot{x}(t) = -\frac{x(t)}{5} + ex(t-1)^2 e^{-5x(t-1)}$ with the initial value ψ_1.

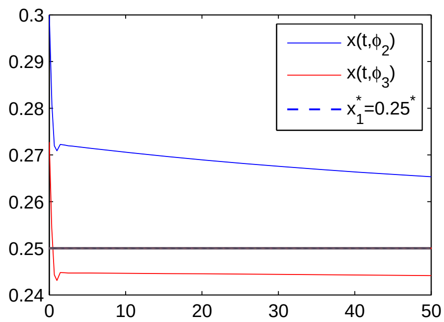

Figure 4: The numerical solutions of $\dot{x}(t) = -\frac{x(t)}{4} + ex(t-1)^2 e^{-4x(t-1)}$ with the initial value ϕ_2 and ϕ_3.

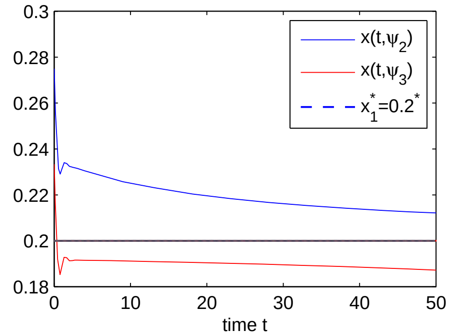

Figure 5: The numerical solutions of $\dot{x}(t) = -\frac{3x(t)}{5} + 3ex(t-1)^2 e^{-5x(t-1)}$ with the initial value ψ_2 and ψ_3.

Remark 6.1 For model (3) with $\tau = 1$ and the linear initial value with slope $\frac{1}{a_2}$, if μ increases, the ratio of the intercept to $\frac{1}{a_2}$ needs to be increased appropriately to ensure the same convergence of the corresponding solution. Otherwise, it probably converges to 0. If μ decreases, appropriate reduction in the ratio can still guarantee that the corresponding solution converges to $\frac{1}{a_2}$.

Simulation 2: Let $h(\mu) = h_2(\mu) = \frac{\mu}{3(\mu+1)}$ for $0 < \mu \le 1$.

Case A: Fix $a_1 = e$.

(1) Choose $a_2 = 1$, then $\mu = 1$ and $\phi_1(s) = s + \frac{7}{6}$. From Theorem 5.2 it follows $\lim\limits_{t\to\infty} x(t, \phi_1) = 0$ (see Figure 8).

(2) Choose $a_2 = 4$, then $\mu = \frac{1}{4}$ and $\phi_2(s) = \frac{1}{4}(s + \frac{16}{15})$. From Theorem 5.2 it follows $\lim\limits_{t\to\infty} x(t, \phi_2) = 0$. However, if set $\phi_3(s) = \frac{1}{4}(s + \frac{7}{6})$, simulation implies $\lim\limits_{t\to\infty} x(t, \phi_3) = \frac{1}{4}$ (see Figure 10).

(3) Choose $a_2 = 10$, then $\mu = \frac{1}{10}$ and $\phi_4(s) = \frac{1}{10}(s + \frac{34}{33})$. From Theorem 5.2 it follows $\lim\limits_{t\to\infty} x(t, \phi_4) = 0$. However, if set $\phi_5(s) = \frac{1}{10}(s + \frac{16}{15})$, simulation implies $\lim\limits_{t\to\infty} x(t, \phi_5) = \frac{1}{10}$

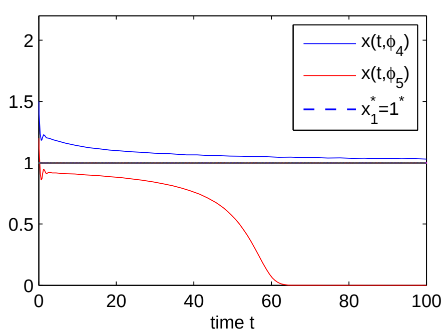

Figure 6: The numerical solutions of $\dot{x}(t) = -x(t)+ex(t-1)^2e^{-x(t-1)}$ with the initial value ϕ_4 and ϕ_5.

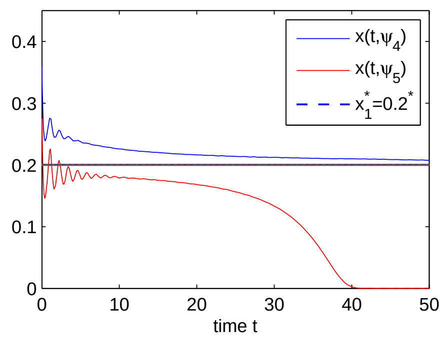

Figure 7: The numerical solutions of $\dot{x}(t) = -4x(t)+20ex(t-1)^2e^{-5x(t-1)}$ with the initial value ψ_4 and ψ_5.

(see Figure 12).

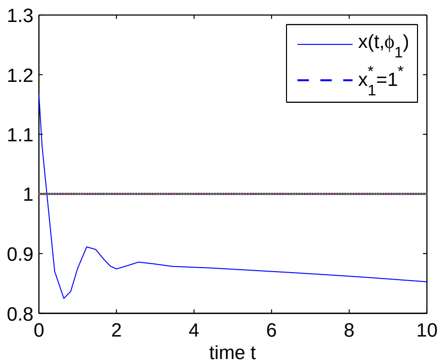

Figure 8: The numerical solution of $\dot{x}(t) = -x(t)+ex(t-1)^2e^{-x(t-1)}$ with the initial value ϕ_1.

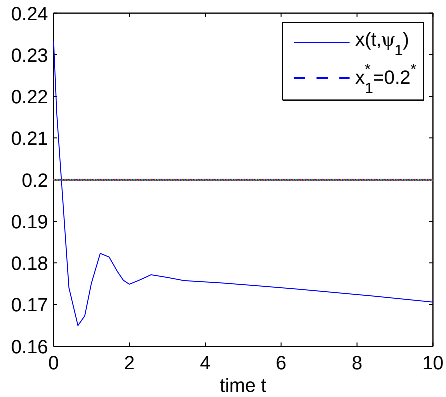

Figure 9: The numerical solution of $\dot{x}(t) = -x(t) + 5ex(t-1)^2e^{-5x(t-1)}$ with the initial value ψ_1.

Case B: Fix $a_2 = 5$.
(1) Choose $a_1 = 5e$, then $\mu = 1$ and $\psi_1(s) = \frac{1}{5}(s + \frac{7}{6})$. From Theorem 5.2 it follows $\lim_{t\to\infty} x(t, \psi_1) = 0$ (see Figure 9).
(2) Choose $a_1 = 2e$, then $\mu = \frac{2}{5}$ and $\psi_2(s) = \frac{1}{5}(s + \frac{23}{21})$. From Theorem 5.2 it follows $\lim_{t\to\infty} x(t, \psi_2) = 0$. However, if set $\psi_3(s) = \frac{1}{5}(s + \frac{7}{6})$, simulation implies $\lim_{t\to\infty} x(t, \psi_3) = \frac{1}{5}$ (see Figure 11).
(3) Choose $a_1 = e$, then $\mu = \frac{1}{5}$ and $\psi_4(s) = \frac{1}{5}(s + \frac{19}{18})$. From Theorem 5.2 it follows $\lim_{t\to\infty} x(t, \psi_4) = 0$. However, if set $\psi_5(s) = \frac{1}{5}(s + \frac{23}{21})$, simulation implies $\lim_{t\to\infty} x(t, \psi_5) = \frac{1}{5}$ (see Figure 13).

Remark 6.2 For model (3) with $\tau = 1$ and the linear initial value with slope $\frac{1}{a_2}$, if μ decreases, the ratio of the intercept to $\frac{1}{a_2}$ needs to be decreased appropriately to ensure

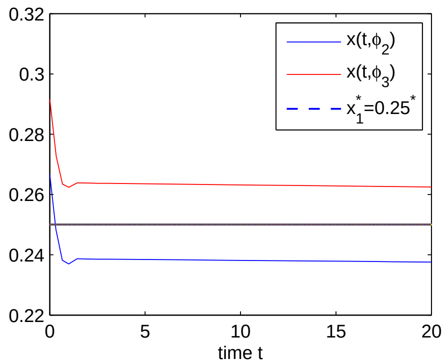

Figure 10: The numerical solutions of $\dot{x}(t) = -\frac{x(t)}{4} + ex(t-1)^2 e^{-4x(t-1)}$ with the initial value ϕ_2 and ϕ_3.

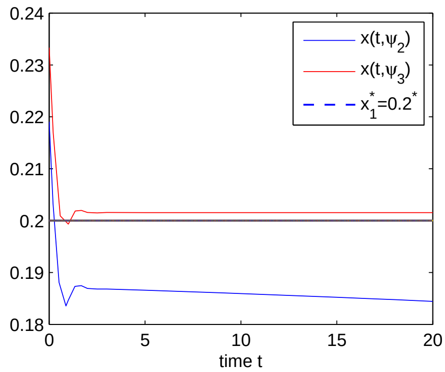

Figure 11: The numerical solutions of $\dot{x}(t) = -\frac{2x(t)}{5} + 2ex(t-1)^2 e^{-5x(t-1)}$ with the initial value ψ_2 and ψ_3.

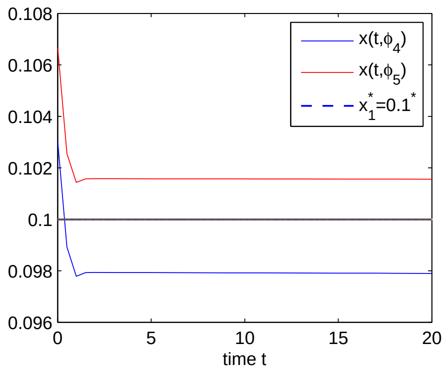

Figure 12: The numerical solutions of $\dot{x}(t) = -\frac{x(t)}{10} + ex(t-1)^2 e^{-10x(t-1)}$ with the initial value ϕ_4 and ϕ_5.

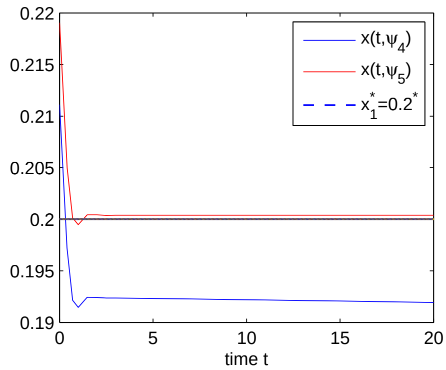

Figure 13: The numerical solutions of $\dot{x}(t) = -\frac{x(t)}{5} + ex(t-1)^2 e^{-5x(t-1)}$ with the initial value ψ_4 and ψ_5.

the same convergence of the corresponding solution. Otherwise, it probably converges to $\frac{1}{a_2}$. If μ increases, appropriate increase in the ratio can still guarantee that the corresponding solution converges to 0.

Simulation 3: Set $a_1 = e$, $a_2 = 1$ and $\tau = 1$, then $\mu = 1$. The model is:

$$\dot{x}(t) = -x(t) + ex(t-1)^2 e^{-x(t-1)}. \tag{18}$$

For the initial value $\phi(s) = s + b$, by Proposition 4.2, there must exist a special b_0 such that the solution $x(t, \phi^{b_0})$ oscillates about 1. By making use of the dichotomy, the range of b_0 is given as follows.

Step 1: Set $\phi_1(s) = s + \frac{4}{3}$ and $\phi_2(s) = s + \frac{7}{6}$, by Theorem 5.1 and 5.2, we have $\lim\limits_{t\to\infty} x(t, \phi_1) = 1$ and $\lim\limits_{t\to\infty} x(t, \phi_2) = 0$ (see Figure 14).

Step 2: Set $\phi_3(s) = s + \frac{5}{4}$ (i.e. $\frac{1}{2}(\phi_1 + \phi_2)$) and $\phi_4(s) = s + \frac{31}{24}$ (i.e. $\frac{1}{2}(\phi_1 + \phi_3)$), simulation implies that $\lim\limits_{t\to\infty} x(t, \phi_3) = 0$ and $\lim\limits_{t\to\infty} x(t, \phi_4) = 1$ (see Figure 15).

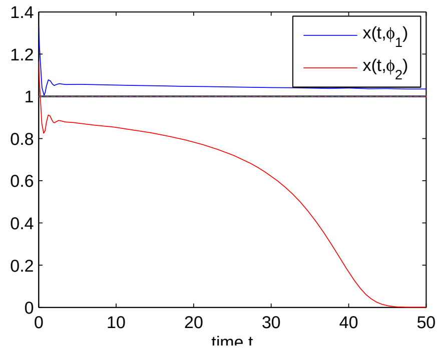

Figure 14: The numerical solutions of (18) with the initial value ϕ_1 and ϕ_2.

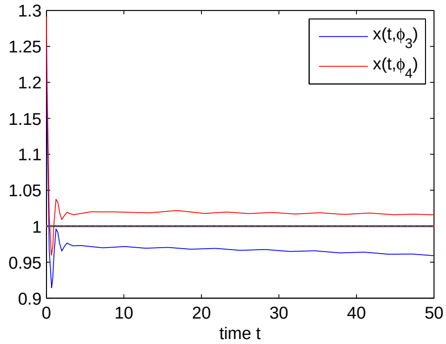

Figure 15: The numerical solutions of (18) with the initial value ϕ_3 and ϕ_4.

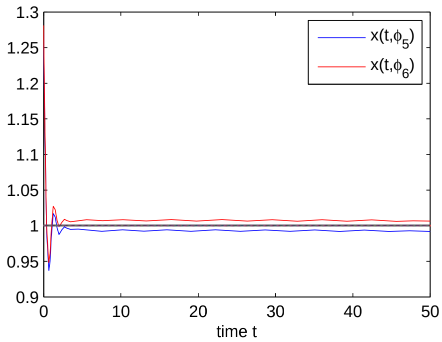

Figure 16: The numerical solutions of (18) with the initial value ϕ_5 and ϕ_6.

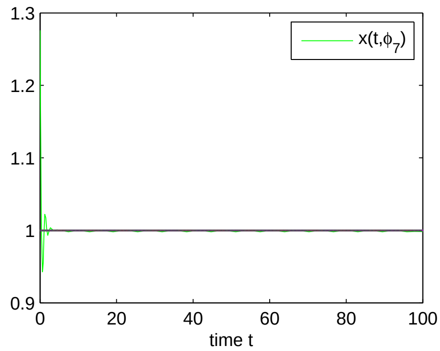

Figure 17: The numerical solutions of (18)with the initial value ϕ_7.

Step 3: Set $\phi_5(s) = s + \frac{61}{48}$ (i.e. $\frac{1}{2}(\phi_3 + \phi_4)$) and $\phi_6(s) = s + \frac{123}{96}$ (i.e. $\frac{1}{2}(\phi_4 + \phi_5)$), simulation implies that $\lim\limits_{t\to\infty} x(t, \phi_5) = 0$ and $\lim\limits_{t\to\infty} x(t, \phi_6) = 1$ (see Figure 16).

Step 4: Set $\phi_7(s) = s + \frac{245}{192}$ (i.e.$\frac{1}{2}(\phi_5 + \phi_6)$), the convergence of $x(t, \phi_7)$ is not evident (see Figure 17). Therefore the special b_0 that makes $x(t, \phi^{b_0})$ oscillate about 1 must locate in $[\frac{61}{48}, \frac{123}{96}]$.

7 Conclusions and Discussions

For equation (2) with $\tau = 1$, when the unique positive equilibrium is not globally asymptotic stable, the initial value plays an important role in practical problems. In order to ensure that the solution converges to the trivial or positive equilibrium, i.e. population size or density disappears or approximates a positive steady state, we need to fully consider the effects of the initial value. Since the form of initial value is so abundant, the paper conducts a preliminary study of the convergence of the solution with the initial value $\phi(s)$, which means that population size or density increases linearly in the initial stage. Theorem 3.1 implies that $\lim\limits_{t\to\infty} x(t, \phi) = x_1$ if (H_0) is satisfied. Theorem 3.2 implies that $\lim_{t\to\infty} x(t,\phi) = 0$ if (H_1)–(H_3) hold. By the monotonicity of the flow generated by (2), we show the existence of the oscillatory solution, and prove that the solution oscillating about x_1 must converge to it. Furthermore, we give more detailed descriptions and classifications of all solutions of (2). When an example (3) is given, wider range of h compared to that of the general case is established to guarantee that the corresponding solution converges to 0 and x_1 respectively.

For equation (2) with a class of linear initial value $\phi(s)$, by the argument of Theorem 4.2, there exists a unique h_0 such that $x(t,\phi)$ oscillates about x_1 if $h = h_0$, $x(t,\phi)$ converges to 0 if $0 \le h < h_0$ and $x(t,\phi)$ converges to x_1 if $h_0 \le h < \xi_0 - x_1$. However, which h_0 should be chosen needs to be further explored.

Here we mainly investigate the convergence of the solution with the initial value that is linear and across the attractive region of 0 and x_1. However, in real-world problems, the initial value is various. When $\phi(s)$ is in other form and not in the attractive region of 0 and x_1 such as $\phi(s) = k\sin s + x_1 + h$, new method needs to be explored to found the condition which guarantees that the corresponding solution converges to 0 or x_1.

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