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Existence and Multiplicity of Periodic Solutions for a Class of the Second Order Hamiltonian Systems

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Abstract: In this paper, we study the existence and multiplicity of periodic solutions of the following second-order Hamiltonian systems

$$\ddot{x}(t) + V'(t, x(t)) = 0,$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^N$ and $V \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. By using a symmetric mountain pass theorem, we obtain a new criterion to guarantee that second-order Hamiltonian systems has infinitely many periodic solutions. We generalize and improve recent results from the literature. Some examples are also given to illustrate our main theoretical results.

Keywords: *periodic solutions; Hamiltonian systems; mountain pass theorem; symmetric mountain pass theorem.*

Mathematics Subject Classification (2010): 34C25, 58E05, 70H05.

1 Introduction

Consider the second-order Hamiltonian systems

$$\ddot{x}(t) + V'(t, x(t)) = 0, \tag{HS}$$

where $x = (x_1, ..., x_N)$, $V \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $V'(t, x) = \nabla_x V(t, x)$. The existence and multiplicity of periodic solutions for system (HS) have been studied in many papers via critical point theory, see the classical monographs [8] and [10] and the recent papers [5, 6, 12, 13, 15, 18]. In [10], Rabinowitz established the existence of periodic solutions for (HS) under the well known Ambrosetti-Rabinowitz condition:

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(AR) there is a constant $\mu > 2$ such that

$$0 \ < \ \mu V(t,x) \ \leq \ V'(t,x) \ . \ x$$

for all $t \in [0,T]$, $T > 0$, and $x \in \mathbb{R}^N \setminus \{0\}$.

The potential $V(t,x)$ in (HS) is of the following form:

$$V(t,x) \ = \ -\frac{1}{2}L(t)x \ . \ x \ + \ W(t,x),$$

where $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix valued function and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and satisfy:

(W_1) there exist constants $\alpha_0 > 0$ and $d_0 > 0$ such that

$$|W'(t,x)| \ \leq \ d_0 \ (|x|^{\alpha_0} + 1) \ \forall \ t \in [0,T], \ x \in \mathbb{R}^N,$$

He and Wu [6] have obtained some results of the existence of nontrivial $T-$periodic solutions for (HS). See also Fei [5].

Motivated by the ideas of [5–7, 10, 12, 14–18], in this paper we will further study the existence of $T-$periodic solutions for (HS) under some general conditions.

Here and in the following $x \ . \ y$ denotes the inner product of $x, y \in \mathbb{R}^N$ and $|.|$ denotes the associated norm.

Our main results are the two following theorems.

Theorem 1.1 *Assume that V satisfies*

(V_1) $V(t,x) \ = \ -K(t,x) \ + \ W(t,x)$, *where* $K, W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *are* C^1*-maps and are* $T-$*periodic in its first variable with* $T > 0$, *and* $V(t,0) = 0$,

(V_2) $\limsup\limits_{|x| \to 0} \dfrac{V(t,x)}{|x|^2} \ < \ 0$ *uniformly in* $t \in [0,T]$,

(V_3) *there exist constants* $\mu > 2$, $\theta \in [2, \mu)$, $\lambda \in (1,2]$ *and* $b > 0$ *such that*

$$K(t,x) \ \geq \ b\,|x|^\lambda \ , \ K'(t,x) \ . \ x \ \leq \ \theta K(t,x), \ \forall \ (t,x) \in [0,T] \times \mathbb{R}^N,$$

(V_4) *there exist constants* $\sigma \in (1, \ \lambda)$ *and* $C \in \mathbb{R}$ *such that*

$$0 \ \leq \ \mu W(t,x) \ \leq \ W'(t,x) \ . \ x + C\,|x|^\sigma$$

for all $t \in [0,T]$ *and* $x \in \mathbb{R}^N$,

(V_5) *there exist* $\alpha_0(t) > 0$ *and constants* $\alpha_1 > \theta$, $R > 0$ *such that*

$$W(t,x) \ \geq \ \alpha_0(t)\,|x|^{\alpha_1} \ \ \forall (t,x) \in [0,T] \times \mathbb{R}^N, \ |x| \geq R.$$

Then the system (HS) *has a nontrivial* $T-$*periodic solution.*

Moreover, if $V(t,x)$ is symmetric in x, i.e. V satisfies

(V$_6$) $\quad V(t,-x) \;=\; V(t,x), \quad \forall (t,x) \in [0,T] \times \mathbb{R}^N;$

then we obtain the following result by using the symmetric mountain pass theorem.

Theorem 1.2 *Assume that V satisfies $(V_1)-(V_6)$, then the system (HS) has an unbounded sequence of $T-$periodic solutions and, in particular, infinite $T-$periodic solutions.*

Remark 1.1 There are functions K and W which satisfy the hypotheses of Theorem 1.1 and Theorem 2.2, but do not satisfy the corresponding results in [4–7, 10, 12, 14–18].

For example, define a function $K \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ as follows

$$K(t,x) = \begin{cases} |x|^{\frac{5}{4}}\, exp(|x|^{\frac{1}{4}}) + |x|^2\,, \; if \;\; |x| \leq 1, \\ \\ exp(1)\, |x|^{\frac{3}{2}} + |x|^2\,, \; if \;\; |x| > 1. \end{cases}$$

An easy computation shows that K satisfies the condition (V_3) but do not satisfy the corresponding results in [4–7, 10, 12, 14–18]. Define a function $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ as follows

$$W(t,x) = |x|^{\frac{5}{4}}\, exp(|x|^{\frac{1}{4}}).$$

Then we have

$$\begin{aligned} W'(t,x)\ .\ x &= \frac{5}{4}|x|^{\frac{5}{4}}\, exp(|x|^{\frac{1}{4}}) + \frac{1}{4}\, |x|^{\frac{1}{4}}\, |x|^{\frac{5}{4}}\, exp(|x|^{\frac{1}{4}}) \\ &= (\frac{5}{4} + \frac{1}{4}\, |x|^{\frac{1}{4}})\, |x|^{\frac{5}{4}}\, exp(|x|^{\frac{1}{4}}). \end{aligned}$$

So, W does not satisfy (W_1).

Moreover, for any constant $\mu > 2$, we have

$$\mu W(t,x) - W'(t,x)\ .\ x = (\mu - \frac{5}{4} - \frac{1}{4}\, |x|^{\frac{1}{4}})\, |x|^{\frac{5}{4}}\, exp(|x|^{\frac{1}{4}})$$

which yields that

$$0 < \mu W(t,x) - W'(t,x)\ .\ x \leq (\mu - \frac{5}{4})\, |x|^{\frac{5}{4}}\, exp(4\mu - 5)$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ and $0 < |x| < (4\mu - 5)^4$, i.e. the condition (AR) does not hold for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^N \backslash \{0\}$ and

$$\mu W(t,x) - W'(t,x).x \leq 0, \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^N, |x| > (4\mu - 5)^4;$$

then (V_4) holds.

Corollary 1.1 *Assume that V satisfies $(V_1), (V_3) - (V_5)$ and*

(V$'_2$) $\quad W(t,x) = o(|x|^2)$ *as* $|x| \to 0$ *uniformly in* $t \in [0,T]$.

Then the system (HS) has a nontrivial $T-$periodic solution.

Moreover, if V satisfies (V_6) then the system (HS) has an unbounded sequence of $T-$periodic solutions.

2 Proof of the Main Results

Let

$$H_T^1 = \{x : [0,T] \to \mathbb{R}^N,\ x \text{ is absolutely continuous},\ x(0) = x(T),\ \text{and}$$
$$\dot{x} \in L^2([0,T], \mathbb{R}^N)\}$$

Then H_T^1 is a Hilbert space with the norm defined by

$$\|x\| = \left(\int_0^T (|x(t)|^2 + |\dot{x}(t)|^2) dt \right)^{\frac{1}{2}}$$

for $x \in H_T^1$. Consider the functional $\phi : H_T^1 \to \mathbb{R}$ defined by

$$\phi(x) = \int_0^T \left(\frac{1}{2} |\dot{x}(t)|^2 + K(t, x(t)) - W(t, x(t)) \right) dt \ . \tag{1}$$

It is well known that $\phi \in C^1(H_T^1, \mathbb{R})$ and for all $x, y \in H_T^1$

$$\phi'(x)y = \int_0^T (\dot{x}(t).\dot{y}(t) + K'(t, x(t)).y(t) - W'(t, x(t)).y(t))\, dt \ . \tag{2}$$

It is well known that the $T-$periodic solution of system (HS) corresponds to the critical points of ϕ in H_T^1. We will obtain the critical point of ϕ by using the mountain pass theorem and the symmetric mountain pass theorem. We say that ϕ satisfies the Palais-Smale condition if every bounded sequence $\{u_k\}$ in the space H such that $\lim_{k\to\infty} \phi'(u_k) = 0$ contains a convergent subsequence. Therefore we state these theorems.

Theorem 2.1 *[10] Let H be a real Banach space and $\phi \in C^1(H, \mathbb{R})$ satisfying the Palais-Smale condition. If ϕ satisfies the following conditions:*

(i) $\phi(0) = 0$,

(ii) there exist constants $\rho,\ \alpha > 0$ such that $\phi_{/\partial B_\rho(0)} \geq \alpha$,

(iii) there exists $e \in H \backslash \overline{B}_\rho(0)$ such that $\phi(e) \leq 0$.

Then ϕ possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} \phi(g(s)),$$

where $B_\rho(0)$ is the open ball in H centered in 0, with radius ρ, $\partial B_\rho(0)$ its boundary and

$$\Gamma = \{g \in C([0,1], H) : g(0) = 0,\ g(1) = e\}.$$

Theorem 2.2 *[10] Let H be a real Banach space, ϕ is even and $\phi \in C^1(H, \mathbb{R})$ satisfyies the Palais-Smale condition. If ϕ satisfies (i) and (ii) of Theorem 2.1 and the following condition:*

(iii') For each finite dimensional subspace $E \subset H$, there is $r = r(E)$ such that $\phi(x) \leq 0$ for $x \in E \backslash B_r(0)$ where $B_r(0)$ is an open ball in H centered in 0, with radius r.

Then ϕ possesses an unbounded sequence of critical values.

In the following, we denote C_i $(i = 1, 2, 3...)$ for different positive constants.

Lemma 2.1 *[7] For all $x \in H_T^1$*

$$\|x\|_\infty \leq C_\infty \|x\| \ . \tag{3}$$

where $\|x\|_\infty = \max\limits_{0\leq t\leq T} |x(t)|$.

2.1 Proof of Theorem 1.1

Let $\gamma_T : H_T^1 \to [0, +\infty)$ be given by

$$\gamma_T(x) = \left(\int_0^T (|\dot{x}(t)|^2 + 2K(t, x(t)))dt\right)^{\frac{1}{2}} \ . \tag{4}$$

By (1) and (4) we have

$$\phi(x) = \frac{1}{2}\ \gamma_T^2(x) - \int_0^T W(t, x(t))dt \ . \tag{5}$$

Moreover, using (V_3) and (2) we obtain

$$\phi'(x)x \leq \int_0^T \left(|\dot{x}(t)|^2 + \theta K(t, x(t))\right) dt - \int_0^T W'(t, x(t)).x(t)dt. \tag{6}$$

It is clear that $\phi(0) = 0$. Firstly, we will show that ϕ satisfies the Palais-Smale condition. Let $(y_j) \subset H_T^1$ be a sequence such that $(\phi(y_j))_{j\in\mathbb{N}}$ is bounded and $\phi'(y_j) \to 0$ as $j \to +\infty$. Then, there exists C_0 such that

$$\phi(y_j) \leq C_0, \quad \|\phi'(y_j)\|_{H_T^1{}^*} \leq C_0, \tag{7}$$

for every $j \in \mathbb{N}$. Without loss of generality, we can assume that $\|y_j\| \neq 0$. Then from (3), (4) and (V_3), we obtain for $j \in \mathbb{N}$

$$\begin{aligned}
\gamma_T^2(y_j) &= \int_0^T \left(|\dot{y}_j(t)|^2 + 2K(t, y_j(t))\right) dt \\
&\geq \int_0^T \left(|\dot{y}_j(t)|^2 + 2b\,|y_j(t)|^\lambda\right) dt \\
&\geq \textstyle\int_0^T |\dot{y}_j(t)|^2\, dt + 2b\,(C_\infty \|y_j\|)^{\lambda-2} \int_0^T |y_j(t)|^2\, dt \\
&\geq \min\left\{1,\ 2b(C_\infty \|y_j\|)^{\lambda-2}\right\}\ \|y_j\|^2 \\
&= \min\left\{\|y_j\|^2, 2bC_\infty^{\lambda-2} \|y_j\|^\lambda\right\}.
\end{aligned} \tag{8}$$

By (4), (6) and (V_4) we have

$$-\frac{\theta}{\mu}\ \gamma_T^2(y_j) \leq \frac{2}{\mu}\ \|\phi'(y_j)\|\ \|y_j\| - \frac{2}{\mu}\int_0^T W'(t, y_j(t))\ .\ y_j(t)\ dt. \tag{9}$$

By Sobolev's embedding theorem, (5), (7), (9) and (V_4) we obtain

$$\left(\frac{\mu-\theta}{\mu}\right)\gamma_T^2(y_j) \leq 2\phi(y_j)+\frac{2}{\mu}\|\phi'(y_j)\|\,\|y_j\|+\frac{2}{\mu}\int_0^T C\,|y_j(t)|^\sigma\,dt$$
$$\leq 2C_0 + C_1\|y_j\| + C_2\|y_j\|^\sigma. \tag{10}$$

Combining (8) with (2.1), we obtain

$$\min\left\{\|y_j\|^2, 2bC_\infty^{\lambda-2}\|y_j\|^\lambda\right\} \leq \frac{\mu}{\mu-\theta}(C_0 + C_1\|y_j\| + C_2\|y_j\|^\sigma). \tag{11}$$

It follows from (11) that $\|y_j\|$ is bounded in H_T^1. In a similar way as in Proposition 4.3 in [8], we can prove that (y_j) has a convergent subsequence in H_T^1. Hence, ϕ satisfies the Palais-Smale condition. Now, let us show that ϕ satisfies assumption (ii) of Theorem 2.1. By (V_2), there exist constants $\alpha_0, \rho_0 > 0$ such that

$$V(t,x) \leq -\alpha_0|x|^2 \tag{12}$$

for all $|x| \leq \rho_0$ and $t \in [0,T]$. Choose $\rho = \frac{\rho_0}{C_\infty}$ and let $S = \left\{x \in H_T^1,\ \|x\| = \rho\right\}$. By 3, we have $\|x\|_\infty \leq \rho_0$, for all $x \in S$, which together with (12) implies

$$\begin{array}{rcl}\phi(x) & = & \frac{1}{2}\int_0^T|\dot{x}(t)|^2\,dt - \int_0^T V(t,x(t))\,dt\\ & \geq & \frac{1}{2}\int_0^T|\dot{x}(t)|^2\,dt + \alpha_0\int_0^T|x(t)|^2\,dt\\ & \geq & \min\left\{\frac{1}{2},\alpha_0\right\}\rho^2 := \alpha.\end{array}$$

for every $x \in S$.

It remains to prove that ϕ satisfies assumption (iii) of Theorem 2.1. By (V_3) we have

$$K(t,x) \leq C_3|x|^\theta + C_4 \quad \forall\ (t,x) \in [0,T]\times\mathbb{R}^N, \tag{13}$$

where $C_3 = \sup\limits_{t\in[0,T],|x|=1} K(t,x)$ and $C_4 = \sup\limits_{t\in[0,T],|x|\leq 1} K(t,x)$. By (1) and (13) we have, for every $s \in \mathbb{R}\setminus\{0\}$ and $x \in H_T^1\setminus\{0\}$,

$$\phi(sx) \leq \frac{s^2}{2}\int_0^T|\dot{x}(t)|^2\,dt + C_3 s^\theta\int_0^T|x(t)|^\theta\,dt + C_5 - \int_0^T W(t,sx(t))\,dt. \tag{14}$$

Take some $Q \in H_T^1$ such that $\|Q\| = 1$. Then there exists a subset Ω of positive measure of $[0,T]$ such that $Q(t) \neq 0$ for $t \in \Omega$. Take $s > 1$ such that $s|Q(t)| \geq R$ for $t \in \Omega$. Then by (V_4), (V_5) and (14)

$$\phi(sQ) \leq C_6 s^\theta - s^{\alpha_1}\int_\Omega \alpha_0(t)\,|Q(t)|^{\alpha_1}\,dt. \tag{15}$$

Since $\alpha_0(t) > 0$ and $\alpha_1 > \theta$, (15) implies that $\phi(sQ) < 0$ for some $s > 1$ such that $s|Q(t)| \geq R$ for $t \in \Omega$ and $s\|Q\| > \rho$. By Theorem 1.1, ϕ possesses a critical value $c \geq \alpha > 0$ given by

$$c = \inf_{g\in\Gamma}\max_{s\in[0,1]}\phi(g(s)),$$

where

$$\Gamma = \{g \in C([0,1],H) : g(0) = 0,\ g(1) = e\}.$$

Hence, there is $x \in H_T^1$ such that $\phi(x) = c, \phi'(x) = 0$. The proof of Theorem 1.1 is complete.

2.2 Proof of Theorem 1.2

(V_6) implies that ϕ is even. By Theorem 2.1 and the proof of Theorem 1.1, it suffices to prove that ϕ satisfies (iii') of Theorem 2.2.

Let $E \subset H_T^1$ be a finite dimensional subspace. From the proof of Theorem 1.1 we know that for any $Q \in E \subset H_T^1$ such that $\|Q\| = 1$, there is $s_Q > 1$ such that $\phi(sQ) < 0$, for every $|s| \geq s_Q > 1$. Since $E \subset H_T^1$ is a finite dimensional subspace, we can choose $r = r(E) > 0$ such that

$$\phi(x) < 0, \ \forall \ x \in E \setminus B_r(0).$$

Hence, by Theorem 2.1, ϕ possesses an unbounded sequence of critical values $(c_n)_{n\in\mathbb{N}}$ with $c_n \to +\infty$. The proof of Theorem 1.2 is complete.

2.3 Proof of Corollary 1.1.

It follows from (V_3) and (V_2')

$$\limsup_{|x|\to 0} \frac{V(t,x)}{|x|^2} \leq \limsup_{|x|\to 0} \left(\frac{W(t,x)}{|x|^2} - b\,|x|^{\lambda-2} \right) < 0$$

uniformly in $t \in [0,T]$, which implies the conditions (V_2). An easy application of Theorem 2.1 and Theorem 2.2 will show that Corollary 1.1 holds.

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