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Almost Oscillatory Three-Dimensional Dynamical Systems of First Order Delay Dynamic Equations

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Abstract: In this paper, we investigate oscillation and asymptotic properties for three dimensional systems of first order dynamic equations with delays. Most of our results are new in the discrete case.

Keywords: *time scales; oscillation; three-dimensional dynamical system.*

Mathematics Subject Classification (2010): 39A10.

1 Introduction

In this paper, we investigate three dimensional dynamical systems with delays of the form

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(\tau(t))), \\ y^{\Delta}(t) = b(t)g(z(\tau(t))), \\ z^{\Delta}(t) = \lambda\, c(t)h(x(\tau(t))), \end{cases} \tag{1}$$

on a time scale $\mathbb{T}$, i.e, a closed subset of real numbers, $\tau : \mathbb{T} \to \mathbb{T}$ is a rd-continuous function such that $\tau(t) < t$, $\lim_{t\to\infty} \tau(t) = \infty$, $\lambda = \pm 1$, $a, b : \mathbb{T} \mapsto [0, \infty)$ (not identically zero) and $c : \mathbb{T} \mapsto (0, \infty)$ are rd-continuous functions such that

$$\int_T^{\infty} a(s)\Delta s = \int_T^{\infty} b(s)\Delta s = \infty, \quad T \in \mathbb{T} \tag{2}$$

and $f, g, h : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions satisfying

$$uf(u) > 0, \quad ug(u) > 0, \quad \text{and} \quad uh(u) > 0 \quad \text{for } u \neq 0. \tag{3}$$

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Here, we would like to indicate that none of the functions f, g and h are assumed to be monotone. Sometimes we will assume that functions f, g and h satisfy

$$\frac{f(u)}{\Phi_\alpha(u)} \geq F, \quad \frac{g(u)}{\Phi_\beta(u)} \geq G, \quad \frac{h(u)}{\Phi_\gamma(u)} \geq H \quad \text{for all } u \neq 0, \tag{4}$$

where F, G, H are positive constants and Φ_α, Φ_β and Φ_γ are odd power functions, i.e.

$$\Phi_p(u) = |u|^p \operatorname{sgn} u \quad (p > 0), \qquad p \in \{\alpha, \beta, \gamma\}.$$

This paper is motivated by the papers [1, 2, 6]. In [1], the special case of system (1) has been considered in which $f(u) = u^\alpha$, $g(u) = u^\beta$, $h(u) = u^\gamma$, $\tau(t) = t$, $\lambda = -1$, and α, β, γ are ratios of odd positive integers. In [2], system (1) is considered without delays. The continuous version of a system similar to system (1) without delays in [5] and the discrete version of a system similar to system (1) with delays in [6, 7] have been considered. The results in [8] are the discrete version of these in [1]. It is worth mentioning that our results not only improve results in [6] but also are new in the discrete case.

The main purpose of this paper is to investigate oscillatory and asymptotic behaviour of solutions of system (1). The set up in this paper is as follows: In Section 2, we give preliminary results including some asymptotic behaviour of the solutions of system (1). In Sections 3 and 4, we obtain almost oscillation criteria for solutions of system (1) when $\lambda = -1$ and $\lambda = 1$, respectively.

Here, we consider only unbounded time scales. For an excellent introduction to time scales we refer the interested reader to the books [3, 4].

A proper solution of system (1) is said to be oscillatory if all its components x, y, z are oscillatory. System (1) with $\lambda = 1$ is said to be almost oscillatory if every solution (x, y, z) of system (1) is either oscillatory or

$$\lim_{t\to\infty} | x(t) | = \lim_{t\to\infty} | y(t) | = \lim_{t\to\infty} | z(t) | = \infty. \tag{5}$$

System (1) with $\lambda = -1$ is said to be almost oscillatory if every solution (x, y, z) of system (1) is either oscillatory or

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0. \tag{6}$$

It is necessary to use the following remark in the further sections in order to obtain a contradiction.

Remark 1.1 (See [1]) Let $a, c \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ such that $\int_T^\infty c(s)\Delta s < \infty$. Then

$$\int_T^\infty a(t) \left(\int_t^\infty c(s)\Delta s \right) \Delta t = \int_T^\infty c(t) \left(\int_T^{\sigma(t)} a(s)\Delta s \right) \Delta t.$$

2 Preliminaries

In this section, we investigate asymptotic behaviour of solutions of system (1) so that we will be able to obtain almost oscillatory systems. The next two results hold regardless if $\lambda = \pm 1$. In the following subsections, we will classify nonoscillatory solutions of system (1) when $\lambda = 1$ and $\lambda = -1$, respectively.

Lemma 2.1 *Assume that condition* (3) *holds. Let* (x, y, z) *be a solution of system* (1) *and let* $x(t)$ *be nonoscillatory for* $t \geq t_0$, $t_0 \in \mathbb{T}$. *Then* (x, y, z) *is nonoscillatory and x,y,z are monotonic for sufficiently large* t.

Proof. Let (x, y, z) be a solution of system (1) such that $x(t)$ is nonoscillatory for $t \geq t_0$. Then we assume that $x(\tau(t)) > 0 \ for \ t \geq t_1 \geq t_0, \ t_1 \in \mathbb{T}$. By the third equation of system (1), we have $z^{\Delta}(t) > 0$ or $z^{\Delta}(t) < 0, \ t \geq t_1 \geq t_0$. This implies that $z(t)$ is monotonic for $t \geq t_1 \geq t_0$ and eventually of one sign for $t \geq t_1$. Let $z(t) > 0, \ z(\tau(t)) > 0 \ for \ t \geq t_2 \geq t_1, \ t_2 \in \mathbb{T}$. Therefore from the second equation of system (1), $y(t)$ is monotonic for $t \geq t_2 \geq t_1$ and eventually of one sign for $t \geq t_2 \geq t_1$. Let $y(\tau(t)) > 0 \ for \ t \geq t_3 \geq t_2$. Similarly, we obtain that x(t) is monotonic for $t \geq t_3 \geq t_2$ from the first equation of system (1). Therefore (x, y, z) is nonoscillatory.

Lemma 2.2 *Assume that conditions* (2) *and* (3) *hold. Let* (x, y, z) *be a nonoscillatory solution of system* (1) *such that* $\lim_{t\to\infty} x(t)$ *is finite, then*

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0.$$

Proof. Assume that (x, y, z) is a nonoscillatory solution of system (1) such that the limit of x is finite. By Lemma 2.1, y is monotonic and hence the limit of y exists. For the sake of contradiction suppose that the limit of y is positive. Therefore, $y(t) > 0$ for large t. Then there exists $t_1 \geq t_0, \ t_1 \in \mathbb{T}$ such that

$$y(\tau(t)) > 0, \quad \tau(t) \geq t_1.$$

From (3), there exist a positive constant K and $t_2 \in \mathbb{T}, \ t_2 \geq t_1$ such that

$$f(y(\tau(t))) > K, \quad \tau(t) \geq t_2.$$

Thus, from the first equation of system (1), we have

$$x^{\Delta}(t) \quad = \quad a(t)f(y(\tau(t))) > a(t)K > 0, \quad \tau(t) \geq t_2.$$

Integrating the above inequality from t_2 to t, we get

$$x(t) > x(t_2) + K \int_{t_2}^{t} a(s)\Delta s.$$

It follows from (2) that

$$\lim_{t\to\infty} x(t) = \infty,$$

but this gives us a contradiction. In the case the limit of y is negative, the proof is similar and hence omitted. Therefore, we get

$$\lim_{t\to\infty} y(t) = 0.$$

Similarly one can show that

$$\lim_{t\to\infty} z(t) = 0$$

by using the second equation of system (1). So this completes the proof.

2.1 Preliminaries when $\lambda = 1$

In this subsection, we will investigate asymptotic behaviour of solutions of system (1) when $\lambda = 1$.

Lemma 2.3 *Let conditions* (2) *and* (3) *hold. Assume that* (x, y, z) *is a nonoscillatory solution of system* (1) *with* $\lambda = 1$ *for large* t *and let*

$$\begin{aligned} &\text{Type (a):} \quad sgn\, x(t) = sgn\, y(t) = sgn\, z(t), \\ &\text{Type (c):} \quad sgn\, x(t) = sgn\, y(t) \neq sgn\, z(t). \end{aligned}$$

Then every nonoscillatory solution of system (1) *with* $\lambda = 1$ *is of either Type (a) or Type (c).*

Proof. Let (x, y, z) be a nonoscillatory solution of system (1). Without loss of generality, we assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_0,\ t_0 \in \mathbb{T}$. By Lemma 2.1, both y and z are monotonic. Therefore they are eventually of one sign. First let $z(t) > 0$ and $z(\tau(t)) > 0$ for $t \geq t_0$. Suppose $y(t) < 0$ for $t \geq t_0$. Since y is increasing, $y(\tau(t)) < 0$ for $t \geq t_0$. Since z is increasing, there exist $t_1 \in \mathbb{T}$ and $L > 0$ such that

$$g(z(\tau(t))) > L, \quad \tau(t) \geq t_1. \tag{7}$$

Using (7) and the second equation of system (1) yields

$$y^{\Delta}(t) \quad = \quad b(t)g(z(\tau(t))) > Lb(t), \quad \tau(t) \geq t_1.$$

If we integrate the above inequality from t_1 to t, we obtain

$$y(t) > y(t_1) + L \int_{t_1}^{t} b(s)\Delta s.$$

By (2), $y(t) \to \infty$ as $t \to \infty$, which is a contradiction with the negativity of y. Therefore this case is not possible and so (x, y, z) is of Type (a).

Now let $z(t) < 0$ for $t \geq t_0$. Since z is increasing, $z(\tau(t)) < 0,\ t \geq t_0$. Suppose that $y(t) < 0,\ y(\tau(t)) < 0$ for large t. Then there exist $t_1 \geq t_0$, $t_1 \in \mathbb{T}$ and $v \leq 0$ such that

$$f(y(\tau(t))) \leq v, \quad \tau(t) \geq t_1. \tag{8}$$

We claim that $v = 0$. Assume that $v < 0$ and we will show that this leads to a contradiction. Using (8) and the first equation of system (1) yields

$$x^{\Delta}(t) \quad = \quad a(t)f(y(\tau(t))) \leq va(t), \quad \tau(t) \geq t_1.$$

Integrating the last inequality from t_1 to t, we obtain

$$x(t) \leq x(t_1) + v \int_{t_1}^{t} a(s)\Delta s.$$

By (2), we get $x(t) \to -\infty$ *as* $t \to \infty$, which is a contradiction with the positivity of x. Therefore this case is not possible and so (x, y, z) is of Type (c).

The proof for the case when $x(t) < 0$ for large t is analogous.

Solutions of Type (a) are sometimes called strongly monotone solutions (see, e.g. [5]).

Lemma 2.4 *Let conditions* (2) *and* (3) *hold. Any Type (a) solution* (x, y, z) *of system* (1) *with* $\lambda = 1$ *satisfies*

$$\lim_{t\to\infty} \mid x(t) \mid = \lim_{t\to\infty} \mid y(t) \mid = \infty.$$

Proof. Let (x, y, z) be a Type (a) solution of system (1). Then there exists $t_0 \in \mathbb{T}$ such that $x(\tau(t)) > 0,\ y(\tau(t)) > 0,$ and $z(\tau(t)) > 0$ for $t \geq t_0$. Since y is eventually increasing, there exist $t_1 \geq t_0,\ t_1 \in \mathbb{T}$ and $K > 0$ such that $f(y(\tau(t))) \geq K,\ \tau(t) \geq t_1$. From the first equation of system (1), we have

$$x^{\Delta}(t) = a(t)f(y(\tau(t))) \geq Ka(t), \quad \tau(t) \geq t_1.$$

Integrating the above inequality from t_1 to t yields

$$x(t) \geq x(t_1) + K\int_{t_1}^{t} a(s)\Delta s, \quad \tau(t) \geq t_1.$$

The above inequality together with (2) implies that $\lim_{t\to\infty} x(t) = \infty$. Since z is eventually increasing, there exist $t_2 \geq t_1,\ t_2 \in \mathbb{T}$ and $M > 0$ such that $g(z(\tau(t))) \geq M,\ \tau(t) \geq t_2$. From the second equation of system (1), we have

$$y^{\Delta}(t) = b(t)g(z(\tau(t))) \geq Mb(t), \quad \tau(t) \geq t_2.$$

Integrating the above inequality from t_2 to t gives us

$$y(t) \geq y(t_2) + M\int_{t_2}^{t} b(s)\Delta s, \quad \tau(t) \geq t_2. \tag{9}$$

The above inequality together with (2) implies $\lim_{t\to\infty} y(t) = \infty$. This completes the proof.

Lemma 2.5 *Let* (2) *and* (3) *hold. Assume that* (x, y, z) *is a Type (c) solution of system* (1) *with* $\lambda = 1$*. Then*

$$\lim_{t\to\infty} z(t) = 0.$$

Proof. Assume that (x, y, z) is a Type (c) solution of system (1). Without loss of generality, assume that $x(\tau(t)) > 0$ for $t \geq t_0$, $t_0 \in \mathbb{T}$. Then $y(t) > 0,\ z(t) < 0,\ t \geq t_0$. Since z is increasing, $\lim_{t\to\infty} z(t) \leq 0$. Suppose that $\lim_{t\to\infty} z(t) < 0$. Then there exist $t_1 \geq t_0$, $t_1 \in \mathbb{T}$ and $S < 0$ such that $g(z(\tau(t))) \leq S$, $\tau(t) \geq t_1$. Integrating the second equation of system (1) from t_1 to t, we have

$$y(t) \leq y(t_1) + S\int_{t_1}^{t} b(s)\Delta s, \quad \tau(t) \geq t_1$$

and therefore (2) implies that $\lim_{t\to\infty} y(t) = -\infty$. But this contradicts the fact that $y(t) > 0$ for $t \geq t_0$. Therefore, $\lim_{t\to\infty} z(t) = 0$. This completes the proof.

2.2 Preliminaries when $\lambda = -1$

In this subsection, we will investigate the asymptotic behaviour of solutions of system (1) when $\lambda = -1$.

Lemma 2.6 *Let conditions* (2) *and* (3) *hold. Then any nonoscillatory solution* (x, y, z) *of system* (1) *with* $\lambda = -1$ *is one of the following types:*

$$\begin{aligned} &\text{Type (a):} \quad sgn\, x(t) = sgn\, y(t) = sgn\, z(t) \quad for \quad large\, t; \\ &\text{Type (b):} \quad sgn\, x(t) = sgn\, z(t) \neq sgn\, y(t) \quad for \quad large\, t. \end{aligned}$$

Proof. Let (x, y, z) be a nonoscillatory solution of system (1). Without loss of generality, we assume that $x(t) > 0$, $x(\tau(t)) > 0$ for $t \geq t_0$. By Lemma 2.1, both y and z are monotonic and they are eventually of one sign. We now show that z cannot be negative. Suppose that $z(t) < 0$ for $t \geq t_0$ to obtain a contradiction. Then there exists $t_1 \geq t_0$, $t_1 \in \mathbb{T}$ such that $z(\tau(t)) < 0$ for $\tau(t) \geq t_1$. Then there exist $t_2 \in \mathbb{T}$, $t_2 \geq t_1$ and a constant $d \leq 0$ such that

$$g(z(\tau(t))) \leq d, \quad \tau(t) \geq t_2. \tag{10}$$

We claim that $d = 0$. Assume that $d < 0$ and we will show that this leads to a contradiction. If we use (10) together with the second equation of system (1), we obtain

$$y^{\Delta}(t) \quad = \quad b(t)g(z(\tau(t))) \leq db(t), \quad \tau(t) \geq t_2.$$

Integrating the above inequality from t_2 to t, we get

$$y(t) \leq y(t_2) + d \int_{t_2}^{t} b(s)\Delta s.$$

In view of (2), $y(t) \to -\infty$ *as* $t \to \infty$. Therefore, there exist $t_3 \in \mathbb{T}$, $t_3 \geq t_2$ and a negative constant v such that

$$y(\tau(t)) < v, \quad \tau(t) \geq t_3. \tag{11}$$

From (3) and (11), there exist $K < 0$ and $t_4 \in \mathbb{T}$, $t_4 \geq t_3$ such that

$$f(y(\tau(t))) \leq K, \quad \tau(t) \geq t_4. \tag{12}$$

Using (12) together with the first equation of system (1), we obtain

$$x^{\Delta}(t) \quad = \quad a(t)f(y(\tau(t))) \leq Ka(t), \quad \tau(t) \geq t_4.$$

If we integrate the last inequality from t_4 to t, we get

$$x(t) < x(t_4) + K \int_{t_4}^{t} a(s)\Delta s.$$

By (2), we have $x(t) \to -\infty$ as $t \to \infty$, but this contradicts the fact that $x(t) > 0$ for all $t \geq t_0$. This implies that $z(t) > 0$ for all $t \geq t_0$.

One can show the proof similarly for the case when $x(t) < 0$ eventually for $t \geq t_0$.

Lemma 2.7 *Let conditions* (2) *and* (3) *hold. Assume* (x, y, z) *is a Type (b) solution of system* (1) *with* $\lambda = -1$. *Then*

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0.$$

Proof. Assume (x, y, z) is a Type (b) solution of system (1) such that $x(t) > 0$, $y(t) < 0$, $z(t) > 0$, $z(\tau(t)) > 0$ for $t \geq t_0$, $t_0 \in \mathbb{T}$. Since $y(t)$ is increasing, we have $\lim_{t\to\infty} y(t) \leq 0$. Assume $\lim_{t\to\infty} y(t) \neq 0$. Then there exist $t_1 \geq t_0$ and a constant $L < 0$ such that $y(\tau(t)) \leq L$ for $\tau(t) \geq t_1$. From (3), there exists $K < 0$ such that

$$f(y(\tau(t))) \leq K, \quad \tau(t) \geq t_1. \tag{13}$$

Integrating the first equation of system (1) from t_1 to t and using (13), we have

$$x(t) \leq x(t_1) + K \int_{t_1}^{t} a(s)\Delta s, \quad \tau(t) \geq t_1$$

and so (2) implies $\lim_{t\to\infty} x(t) = -\infty$. This contradicts the positivity of x and therefore $\lim_{t\to\infty} y(t) = 0$. In a similar way, we can show that $\lim_{t\to\infty} z(t) = 0$.

In the next two sections, we will obtain almost oscillation criteria for system (1).

3 Almost Oscillatory System (1) When $\lambda = -1$

The next two results in this section are new in the discrete case and can be found in [[2], Theorem 4.1, Theorem 4.2 and Theorem 4.3.] without delays.

Theorem 3.1 *Let conditions* (2) *and* (3) *hold. Assume*

$$\int_{T}^{\infty} c(s)\Delta s = \infty, \quad T \in \mathbb{T}. \tag{14}$$

Then system (1) *with* $\lambda = -1$ *is almost oscillatory.*

Proof. Assume (x, y, z) is a nonoscillatory solution of system (1). By Lemma 2.6, nonoscillatory solutions are of either Type (a) or Type (b). Assume (x, y, z) is a Type (a) solution. Without loss of generality, assume that there exists $t_0 \in \mathbb{T}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $y(t) > 0$, $y(\tau(t)) > 0$, and $z(t) > 0$ for $t \geq t_0$. Since $x(t)$ is eventually increasing, there exist $L > 0$ and $t_1 \geq t_0$ such that $x(\tau(t)) > L$ for $\tau(t) \geq t_1$. From (3), there exist $K > 0$ and $t_2 \in \mathbb{T}$, $t_2 \geq t_1$ such that

$$h(x(\tau(t))) \geq K, \quad \tau(t) \geq t_2. \tag{15}$$

Integrating the third equation of system (1) from t_2 to t and using (15), we have

$$z(t) \geq z(t_2) + K \int_{t_2}^{t} c(s)\Delta s, \quad \tau(t) \geq t_2,$$

and so this implies that $\lim_{t\to\infty} z(t) = \infty$, which is a contradiction with the boundedness of z. Therefore, (x, y, z) can not be a Type (a) solution. Therefore all nonoscillatory solutions are of Type (b). Without loss of generality, assume that there exists $t_0 \in \mathbb{T}$ such that $x(t) > 0,\ y(t) < 0,\ y(\tau(t)) < 0,\ z(t) > 0,\ t \geq t_0$. By Lemma 2.7, we have $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0$. So it is enough to show that $\lim_{t\to\infty} x(t) = 0$. Since x is eventually decreasing, there exists $t_1 \geq t_0$ such that $\lim_{t\to\infty} x(t) = M \geq 0,\ t \geq t_1$. Therefore there exists $t_2 \geq t_1$ such that $x(\tau(t)) \geq M,\ \tau(t) \geq t_2$. By (3), there exist $K > 0$ and $t_3 \geq t_2$ such that

$$h(x(\tau(t))) \geq K, \quad \tau(t) \geq t_3. \tag{16}$$

Integrating the third equation of system (1) from t_3 to t and using (16), we get

$$z(t) \leq z(t_3) - K \int_{t_3}^{t} c(s)\Delta s, \quad \tau(t) \geq t_3.$$

and as $t \to \infty$, we get a contradiction with the boundedness of z. So $\lim_{t\to\infty} x(t) = 0$. This completes the proof.

Example 3.1 Let $\mathbb{T} = \mathbb{Z}$. Then we consider the following system

$$\begin{cases} \Delta x_n = a_n f(y_{n-l}), \\ \Delta y_n = b_n g(z_{n-l}), \\ \Delta z_n = \lambda c_n h(y_{n-l}), \end{cases} \tag{17}$$

where l is a given positive integer and $\lambda = -1$. Here $a_n, b_n : \mathbb{N}_{n_0} \to \mathbb{R}_+ \cup \{0\},\ c_n : \mathbb{N}_{n_0} \to \mathbb{R}_+$ such that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \infty, \tag{18}$$

where $n_0 \in \mathbb{N} = \{1, 2, \ldots\}$, R_+ is the set of positive real numbers. Also $f, g, h : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying (3). If

$$\sum_{n=1}^{\infty} c_n = \infty, \tag{19}$$

then system (17) with $\lambda = -1$ is almost oscillatory by Theorem 3.1.

For the next two theorems, we assume that

$$\int_T^{\infty} c(s)\Delta s < \infty, \quad T \in \mathbb{T}. \tag{20}$$

Theorem 3.2 *Let $\lambda = -1$ in system (1). Assume condition (3) holds and there exist positive constants F, G, α, β such that*

$$\frac{f(u)}{\Phi_\alpha(u)} \geq F, \quad \frac{g(u)}{\Phi_\beta(u)} \geq G \quad \textit{for small } u \neq 0. \tag{21}$$

If

$$\int_T^{\infty} b(s) \left(\int_{\tau(s)}^{\infty} c(v)\Delta v \right)^{\beta} \Delta s = \infty, \quad T \in \mathbb{T}, \tag{22}$$

or

$$\int_T^\infty a(t)\left(\int_{\tau(t)}^\infty b(s)\left(\int_{\tau(s)}^\infty c(v)\Delta v\right)^\beta \Delta s\right)^\alpha \Delta s = \infty, \quad T \in \mathbb{T}, \tag{23}$$

then every nonoscillatory solution of system (1) *that fulfils Type (b) satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume that (x, y, z) is a nonoscillatory solution of system (1) of Type (b). Without loss of generality assume that $x(t) > 0,\ y(t) < 0,\ y(\tau(t)) < 0$ and $z(t) > 0$ for $t \geq t_0$. From the first equation of system (1), x is nonincreasing, and therefore x has a nonnegative limit. Assume that $\lim_{t\to\infty} x(t) > 0$. Then there exists $t_1 \geq t_0$ such that $x(\tau(t)) \geq 0,\ \tau(t) \geq t_1$. By (3), there exist $t_2 \geq t_1$ and $K > 0$ such that

$$h(x(\tau(t))) \geq K, \quad \tau(t) \geq t_2. \tag{24}$$

Integrating the third equation of system (1) from $\tau(t)$ to ∞ and using (24), we obtain

$$z(\tau(t)) \geq K \int_{\tau(t)}^\infty c(s)\Delta s, \quad \tau(t) \geq t_2,$$

where we use Lemma 2.7. By (21) there exist $t_3 \geq t_2,\ t_3 \in \mathbb{T}$ and $G > 0$ such that

$$g(z(\tau(t))) \geq GK^\beta \left(\int_{\tau(t)}^\infty c(s)\Delta s\right)^\beta, \quad \tau(t) \geq t_3. \tag{25}$$

Integrating the second equation of system (1) from t_3 to t and using (25), we obtain

$$\begin{aligned} y(t) &= y(t_3) + \int_{t_3}^t b(s)g(z(\tau(s)))\Delta s \\ &\geq y(t_3) + GK^\beta \int_{t_3}^t b(s)\left(\int_{\tau(s)}^\infty c(v)\Delta v\right)^\beta \Delta s, \quad \tau(t) \geq t_3. \end{aligned}$$

If we assume (22), then we have $\lim_{t\to\infty} y(t) = \infty$, but this contradicts the fact that $\lim_{t\to\infty} y(t) = 0$. So $\lim_{t\to\infty} x(t) = 0$. Assume (23). Integrating the second equation of system (1) from $\tau(t)$ to ∞ and using the fact that $\lim_{t\to\infty} y(t) = 0$ and (25), we obtain

$$-y(\tau(t)) \geq GK^\beta \int_{\tau(t)}^\infty b(s)\left(\int_{\tau(s)}^\infty c(v)\Delta v\right)^\beta \Delta s, \quad \tau(t) \geq t_3,$$

By (21), there exists $F > 0$ such that

$$\begin{aligned} f(y(\tau(t))) &\leq Fy^\alpha(\tau(t)) \\ &\leq -FG^\alpha K^{\alpha\beta}\left[\int_{\tau(t)}^\infty b(s)\left(\int_{\tau(s)}^\infty c(v)\Delta v\right)^\beta \Delta s\right]^\alpha, \quad \tau(t) \geq t_3. \end{aligned}$$

Integrating the first equation of system (1) from t_3 to t yields

$$\begin{aligned} x(t)-x(t_3) &= \int_{t_3}^{t} a(s)f(y(\tau(s)))\Delta s \\ &\le -FG^{\alpha}K^{\alpha\beta}\int_{t_3}^{t} a(s)\left[\int_{\tau(s)}^{\infty} b(v)\left(\int_{\tau(v)}^{\infty} c(\eta)\Delta\eta\right)^{\beta}\Delta v\right]^{\alpha}\Delta s, \quad \tau(t)\ge t_3. \end{aligned}$$

This implies that $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction by (23). This completes the proof.

Example 3.2 Let $\mathbb{T}=\mathbb{Z}$. Then we consider system (17) with $\lambda=-1$. Assume there exist positive constants F, G, α, β such that (21) holds. If

$$\sum_{i=1}^{\infty} b_i\left(\sum_{r=i-l}^{\infty} c_r\right)^{\beta} = \infty$$

or

$$\sum_{i=1}^{\infty} a_i\left(\sum_{s=i-l}^{\infty} b_s\left(\sum_{r=s-l}^{\infty} c_r\right)^{\beta}\right)^{\alpha} = \infty$$

holds, then every nonoscillatory solution of system (17) that fulfils Type (b) satisfies $\lim_{t\to\infty} x(t) = 0$ by Theorem 3.2.

Theorem 3.3 *Assume conditions* (2), (3) *and* (4) *hold. Let* $\alpha\beta\gamma<1$. *If*

$$\int_{t_3}^{\infty} c(t)\left(\int_{t_2}^{\tau(t)} a(s)\left(\int_{t_1}^{\tau(s)} b(v)\Delta v\right)^{\alpha}\Delta s\right)^{\gamma}\Delta t = \infty, \quad t_1, t_2, t_3 \in \mathbb{T}, \tag{26}$$

then every nonoscillatory solution of system (1) *with* $\lambda=-1$ *is of Type (b). In addition, if* (22) *holds, then system* (1) *is almost oscillatory.*

Proof. Suppose that (x,y,z) is a nonoscillatory solution of system (1) with $\lambda=-1$. Then by Lemma 2.6, (x,y,z) is of either Type (a) or Type (b). Suppose that (x,y,z) is a Type (a) solution. Without loss of generality, assume $x(t)>0,\ x(\tau(t))>0,\ y(t)>0,\ y(\tau(t))>0,\ z(t)>0$ for $t\ge t_0,\ t_0\in\mathbb{T}$. Integrating the second equation of system (1) from $t_1\ge t_0,\ t_1\in\mathbb{T}$ to $\tau(t)$ and using the positivity of y yield

$$y(\tau(t)) \ge y(\tau(t)) - y(t_1) = \int_{t_1}^{\tau(t)} b(s)g(z(\tau(s)))\Delta s, \quad \tau(t)\ge t_1.$$

By (3) and (4), there exist $G>0$ and $t_2\ge t_1,\ t_2\in\mathbb{T}$ such that

$$g(z(\tau(t))) \ge Gz^{\beta}(\tau(t)),\ \tau(t)\ge t_2.$$

Therefore, we obtain

$$\begin{aligned} y(\tau(t)) &\ge G\int_{t_1}^{\tau(t)} b(s)z^{\beta}(\tau(s))\Delta s \ge G\int_{t_1}^{\tau(t)} b(s)z^{\beta}(s)\Delta s \\ &\ge Gz^{\beta}(t)\int_{t_1}^{\tau(t)} b(s)\Delta s, \quad \tau(t)\ge t_2 \end{aligned}$$

or

$$y^{\alpha}(\tau(t)) \geq G^{\alpha} z^{\alpha\beta}(t) \left(\int_{t_1}^{\tau(t)} b(s)\Delta s \right)^{\alpha}, \quad \tau(t) \geq t_2. \tag{27}$$

By (3), (4) and (27), there exist $F > 0$ and $t_3 \geq t_2$, $t_3 \in \mathbb{T}$ such that

$$f(y(\tau(t))) \geq F y^{\alpha}(\tau(t)) \geq F G^{\alpha} z^{\alpha\beta}(t) \left(\int_{t_1}^{\tau(t)} b(s)\Delta s \right)^{\alpha}, \quad \tau(t) \geq t_3. \tag{28}$$

Integrating the first equation of system (1) from $t_3 \geq t_2$ to $\tau(t)$ and using (28)

$$\begin{aligned} x(\tau(t)) &\geq x(\tau(t)) - x(t_3) \\ &= \int_{t_3}^{\tau(t)} a(s) f(y(\tau(s)))\Delta s \\ &\geq FG^{\alpha} \int_{t_3}^{\tau(t)} a(s) z^{\alpha\beta}(s) \left(\int_{t_1}^{\tau(s)} b(v)\Delta v \right)^{\alpha} \Delta s \\ &\geq FG^{\alpha} z^{\alpha\beta}(t) \int_{t_3}^{\tau(t)} a(s) \left(\int_{t_1}^{\tau(s)} b(v)\Delta v \right)^{\alpha} \Delta s, \quad \tau(t) \geq t_3. \end{aligned}$$

or

$$x^{\gamma}(\tau(t)) > F^{\gamma} G^{\alpha\gamma} z^{\alpha\beta\gamma}(t) \left[\int_{t_3}^{\tau(t)} a(s) \left(\int_{t_1}^{\tau(s)} b(v)\Delta v \right)^{\alpha} \Delta s \right]^{\gamma}, \quad \tau(t) \geq t_3.$$

By (3) and (4), there exist $H > 0$ and $t_4 \geq t_3$ such that

$$h(x(\tau(t))) \geq H x^{\alpha}(\tau(t)), \quad \tau(t) \geq t_4.$$

From the third equation of system (1), we have

$$\begin{aligned} -z^{\Delta}(t) &= c(t) h(x(\tau(t))) \\ &\geq H c(t) x^{\gamma}(\tau(t)) \\ &> F^{\alpha} H G^{\alpha\gamma} c(t) z^{\alpha\beta\gamma}(t) \left[\int_{t_3}^{\tau(t)} a(s) \left(\int_{t_1}^{\tau(s)} b(v)\Delta v \right)^{\alpha} \Delta s \right]^{\gamma}, \quad \tau(t) \geq t_4. \end{aligned}$$

Dividing both sides of the above inequality by $z^{\alpha\beta\gamma}(t)$, we have

$$\frac{-z^{\Delta}(t)}{z^{\alpha\beta\gamma}(t)} > F^{\alpha} H G^{\alpha\gamma} c(t) \left[\int_{t_3}^{\tau(t)} a(s) \left(\int_{t_1}^{\tau(s)} b(v)\Delta v \right)^{\alpha} \Delta s \right]^{\gamma}, \quad \tau(t) \geq t_4.$$

Integrating the above inequality from t_4 to t yields

$$\int_{t_4}^{t} \frac{-z^{\Delta}(t)}{z^{\alpha\beta\gamma}(t)} \Delta t > F^{\alpha} H G^{\alpha\gamma} \int_{t_4}^{t} c(p) \left[\int_{t_3}^{\tau(t)} a(s) \left(\int_{t_1}^{\tau(s)} b(v)\Delta v \right)^{\alpha} \Delta s \right]^{\gamma} \Delta p, \quad t \geq t_4.$$

By [1], the left hand side of the above inequality is finite as $t \to \infty$, but this contradicts (26). Therefore, (x, y, z) can not be a Type (a) solution. So every nonoscillatory solution of system (1) is of Type (b). This implies that $\lim_{t\to\infty} x(t)$ is finite. Then by Lemma 2.7, we have $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0$. By Theorem 3.2, $\lim_{t\to\infty} x(t) = 0$. So this completes the proof.

4 Almost Oscillatory System (1) when $\lambda = 1$

The last two results in this section are new for the discrete case.

Theorem 4.1 *Let conditions* (2), (3) *and* (14) *hold. Then system* (1) *with* $\lambda = 1$ *is almost oscillatory.*

Proof. It follows from Lemma 2.3 that nonoscillatory solutions of system (1) are either Type (a) or Type (c) solution of system (1). Assume that (x, y, z) is a Type (c) solutions of system (1). Without loss of generality, assume that there exists $t_0 \in \mathbb{T}$ such that $x(t) > 0,\ x(\tau(t)) > 0,\ y(t) > 0,\ y(\tau(t)) > 0,$ and $z(t) < 0$ for $t \geq t_0$. Since x is eventually increasing, there exist $t_1 \geq t_0,\ t_1 \in \mathbb{T}$ and $L > 0$ such that

$$h(x(\tau(t))) \geq L, \quad \tau(t) \geq t_1. \tag{29}$$

Integrating the third equation of system (1) from t_1 to t and using (29) we get

$$z(t) \geq z(t_1) + L \int_{t_1}^{t} c(s)\Delta s, \quad \tau(t) \geq t_1.$$

So (14) implies $\lim_{t\to\infty} z(t) = \infty$. This contradicts the assumptions on z. Therefore solutions of system (1) can not be of Type (c). If (x, y, z) is a Type (a) solution, then from Lemma 2.4 and equation (14), we obtain (5). This completes the proof.

For the next two theorems, we assume that

$$\int_{T}^{\infty} c(s)\Delta s < \infty, \quad T \in \mathbb{T}.$$

Theorem 4.2 *Let* (2) *and* (3) *hold. Assume that there exist positive constants* F, H *and* α, γ *such that*

$$\frac{f(u)}{\Phi_\alpha(u)} \geq F, \quad \frac{h(u)}{\Phi_\gamma(u)} \geq H \quad \textit{for large } u \neq 0, \tag{30}$$

and

$$\int_{t_3}^{\infty} c(r) \left(\int_{t_2}^{\tau(r)} a(s) \left(\int_{t_1}^{\tau(s)} b(\eta)\Delta\eta \right)^{\alpha} \Delta s \right)^{\gamma} \Delta r = \infty, \quad t_1, t_2, t_3 \in \mathbb{T}. \tag{31}$$

Then any Type (a) solution (x, y, z) *of system* (1) *with* $\lambda = 1$ *satisfies* (5).

Proof. Let (x, y, z) be a Type (a) solution of system (1) such that $x(\tau(t)) > 0,\ y(\tau(t)) > 0,\ z(\tau(t)) > 0$ for $t \geq t_0$. By (9), we have

$$y(t) \geq y(t_2) + M \int_{t_2}^{t} b(s)\Delta s \geq M \int_{t_2}^{t} b(s)\Delta s.$$

There exists $t_3 \in \mathbb{T},\ t_3 \geq t_2$ such that

$$y(\tau(t)) \geq M \int_{t_3}^{\tau(t)} b(s)\Delta s, \quad \tau(t) \geq t_3,$$

and so

$$y^\alpha(\tau(t)) \geq M^\alpha \left(\int_{t_3}^{\tau(t)} b(s)\Delta s \right)^\alpha, \quad \tau(t) \geq t_3. \tag{32}$$

By (30), there exist $t_4 \in \mathbb{T}, t_4 \geq t_3$ and $F > 0$ such that

$$f(y(\tau(t))) \geq F y^\alpha(\tau(t)) \geq F M^\alpha \left(\int_{t_3}^{\tau(t)} b(s)\Delta s \right)^\alpha, \quad \tau(t) \geq t_4, \tag{33}$$

where we used (32). Integrating the first equation of system (1) from t_4 to t and using (33) yield

$$\begin{aligned} x(t) \geq x(t) - x(t_4) &= \int_{t_4}^{t} a(s) f(y(\tau(s)))\Delta s \\ &\geq F M^\alpha \int_{t_4}^{t} a(s) \left(\int_{t_3}^{\tau(s)} b(\eta)\Delta\eta \right)^\alpha \Delta s, \quad \tau(t) \geq t_4. \end{aligned}$$

Then there exists $t_5 \in \mathbb{T},\ t_5 \geq t_4$ such that

$$x(\tau(t)) \geq F M^\alpha \int_{t_4}^{\tau(t)} a(s) \left(\int_{t_3}^{\tau(s)} b(\eta)\Delta\eta \right)^\alpha \Delta s, \quad \tau(t) \geq t_5$$

or

$$x^\gamma(\tau(t)) \geq F^\gamma M^{\alpha\gamma} \left(\int_{t_4}^{\tau(t)} a(s) \left(\int_{t_3}^{\tau(s)} b(\eta)\Delta\eta \right)^\alpha \Delta s \right)^\gamma, \quad \tau(t) \geq t_5.$$

Using the third equation of system (1), (30) and the above inequality, we have

$$\begin{aligned} z^\Delta(t) &= c(t) h(x(\tau(t))) \\ &\geq c(t) H x^\gamma(\tau(t)) \\ &\geq F^\gamma M^{\alpha\gamma} c(t) \left(\int_{t_4}^{\tau(t)} a(s) \left(\int_{t_3}^{\tau(s)} b(\eta)\Delta\eta \right)^\alpha \Delta s \right)^\gamma, \quad \tau(t) \geq t_5. \end{aligned}$$

Integrating the above inequality from t_5 to t we get

$$\begin{aligned} z(t) &> z(t) - z(t_5) \\ &\geq F^\gamma M^{\alpha\gamma} \int_{t_5}^{t} c(s) \left(\int_{t_4}^{\tau(s)} a(\eta) \left(\int_{t_3}^{\tau(\eta)} b(r)\Delta r \right)^\alpha \Delta\eta \right)^\gamma \Delta s, \quad \tau(t) \geq t_5. \end{aligned}$$

So as $t \to \infty$, $z(t) \to \infty$ by (31). The proof is complete by Lemma 2.4.

Example 4.1 Let $\mathbb{T} = \mathbb{Z}$. Then we consider system (17) with $\lambda = 1$. Assume conditions (3) and (18) hold and there exist positive constants F, H, α, γ such that (30) holds. If

$$\sum_{r=1}^{\infty} c_r \left(\sum_{s=1}^{r-l-1} a_s \left(\sum_{n=1}^{s-l-1} b_n \right)^\alpha \right)^\gamma = \infty, \tag{34}$$

then any Type (a) solution (x, y, z) of system (17) with $\lambda = 1$ satisfies (5) by Theorem 4.2.

Theorem 4.3 *Let conditions* (2), (3) *hold and* $\beta \leq 1$. *Assume that there exist positive constants* G, β *such that*

$$\frac{g(u)}{\Phi_\beta(u)} \geq G \quad \text{for large } u \neq 0, \tag{35}$$

where g *is an odd function. If*

$$\int_T^\infty c(s)\left(\int_T^{\sigma(s)} b(v)\Delta v\right)\Delta s = \infty, \quad T \in \mathbb{T}, \tag{36}$$

then every nonoscillatory solution of system (1) *with* $\lambda = 1$ *is a strongly monotone solution. In addition, if* (31) *holds, then system* (1) *with* $\lambda = 1$ *is almost oscillatory.*

Proof. Assume (x, y, z) is a Type (c) solution of system (1). Without loss of generality, assume that there exists $t_0 \in \mathbb{T}$ such that $x(t) > 0,\ x(\tau(t)) > 0,\ y(t) > 0,\ y(\tau(t)) > 0,\ z(t) < 0,\ t \geq t_0$. Since x is eventually increasing, from (3) there exist $K > 0$ and $t_1 \geq t_0,\ t_1 \in \mathbb{T}$ such that $h(x(\tau(t))) \geq K,\ \tau(t) \geq t_1$. By Lemma 2.5, $\lim_{t\to\infty} z(t) = 0$. Then integrating the third equation of system (1) from t to ∞ yields

$$-z(t) = \int_t^\infty c(s)h(x(\tau(s)))\Delta s \geq K\int_t^\infty c(s)\Delta s.$$

From (35), there exist $t_2 \geq t_1,\ t_2 \in \mathbb{T}$ and $G > 0$ such that

$$g(-z(\tau(t))) \geq G(-z(\tau(t)))^\beta \geq G(-z(t)) \geq GK\int_t^\infty c(s)\Delta s, \quad \tau(t) \geq t_2. \tag{37}$$

Integrating the second equation of system (1) from t_2 to t, we have

$$y(t) - y(t_2) = \int_{t_2}^t b(s)g(z(\tau(s)))\Delta s$$

or

$$-y(t) + y(t_2) = \int_{t_2}^t b(s)g(-z(\tau(s)))\Delta s.$$

Using (37), we have

$$-y(t) + y(t_2) \geq GK\int_{t_2}^t b(s)\left(\int_s^\infty c(v)\Delta v\right)\Delta s, \quad \tau(t) \geq t_2. \tag{38}$$

Using Remark 1.1 for (38), we get

$$-y(t) + y(t_2) \geq GK\int_{t_2}^t c(s)\left(\int_{t_2}^{\sigma(s)} b(v)\Delta v\right)\Delta s, \quad \tau(t) \geq t_2.$$

As $t \to \infty$ and using (36), we get a contradiction with the boundedness of y. The second part follows from Theorem 4.2.

Example 4.2 Let $\mathbb{T} = \mathbb{Z}$. Then we consider system (17) with $\lambda = 1$. Assume conditions (3) and (18) hold and $\beta \leq 1$. There exist positive constants G, β such that (35) holds. If

$$\sum_{s=1}^{\infty} c_s \left(\sum_{r=1}^{s-l} b_r \right) = \infty, \tag{39}$$

then every nonoscillatory solution of system (17) with $\lambda = 1$ is a strongly monotone solution. In addition, if (34) holds, then system (17) with $\lambda = 1$ is almost oscillatory by Theorem 4.3.

5 Conclusion

In this paper, we consider oscillation and asymptotic behaviour of solutions of system (1) depending on $\lambda = \pm 1$. We conclude that system (1) with $\lambda = \pm 1$ is almost oscillatory, independently of the nonlineriaties, if (14) holds. However, if (20) holds, then system (1) is almost oscillatory depending on the sign of λ and the types of nonlinearities.

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References

[1] Akın-Bohner, E., Dǒslǎ, Z., and Lawrence, B. Oscillatory properties for three dimensional dynamic systems. *Nonlinear Anal.* **69** (2) (2008) 483-494.

[2] Akın-Bohner, E., Dǒslǎ, Z., and Lawrence, B. Almost oscillatory three-dimensional dynamical system. *Adv. Difference Equ.* **46** (2012) 14 pages.

[3] Bohner, M. and Peterson, A. *Dynamic Equations on Time Scales.* Birkhäuser Boston Inc., Boston, 2001.

[4] Bohner, M. and Peterson, A. *Advances in Dynamic Equations on Time Scales.* Birkhäuser Boston Inc., Boston, 2003.

[5] Kiguradze, I. T. and Chanturia, T. A. *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations.* Translated from the 1985 Russian original. Kluwer, Dordrecht, 1993.

[6] Schmeidel, E. Oscillation of nonlinear three-dimensional difference systems with delays. *Math. Bohem.* **135** (2) (2010) 163–170.

[7] Schmeidel, E. Boundedness of solutions of nonlinear three-dimensional difference systems with delays. *Fasc. Math.* **44** (2010) 107–113.

[8] Thandapani, E. and Ponnammal, B. Oscillatory properties of solutions of three-dimensional difference systems. *Math. Comput. Modelling* **45** (5-6) (2005) 641–650.

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The Obstacle Problem Associated with Nonlinear Elliptic Equations in Generalized Sobolev Spaces

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Abstract: We prove an existence result of entropy solution to the obstacle problem associated with the equation of the type

$$-\mathrm{div}(a(x,u,\nabla u)) + g(x,u,\nabla u) = f \in L^1(\Omega)$$

in generalized Sobolev spaces, without assuming the sign condition in the nonlinearity g via penalization methods.

Keywords: *generalized Sobolev spaces; boundary value problems; truncations; penalized equations.*

Mathematics Subject Classification (2010): 35J20, 35J60, 35B30.

1 Introduction

The obstacle problem is, roughly speaking, about solving a partial differential equation with the additional constraint that the solution is required to stay above a given function, the obstacle. This leads to a variational inequality. From a minimization point of view, the problem is to find a minimizer with fixed boundary value in the set of functions lying above the obstacle function. Such a set is convex and thus, a unique minimizer exists under reasonable assumptions. The balayage concept of potential theory which is the potential theoretic viewpoint of the obstacle problem is finding the smallest superharmonic function which lies above the obstacle.

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In this paper, we deal with the obstacle problem associated with the following quasilinear elliptic equations

$$-\operatorname{div}(a(x,u,\nabla u))+g(x,u,\nabla u)=f\in L^1(\Omega) \tag{1}$$

with non-standard structural conditions which involve a variable growth exponent $p(.)$. We prove some existence result of entropy solution under the assumption that g has a constant sign. A problem like (1) was studied by Azroul, Benboubker and Rhoudaf in [1], where they proved the existence of entropy solutions by using a decomposition method of the measure μ.

The study of partial differential equations and variational problems involving $p(x)$-growth conditions has received specific attention in recent decades. This is a consequence of the fact that such equations can be used to model phenomena which arise in mathematical physics. Electrorheological fluids and elastic mechanics are two examples of physical fields which benefit from such kinds of studies. In that context, we refer to Diening [7], Ruzicka [18], and the references therein.

Most materials can be modelled with sufficient accuracy using classical Lebesgue and Sobolev spaces L^p and $W^{1,p}$, where p is a fixed constant, we recall some papers (and references therein), in which this theory is developed: [1, 5, 6, 11]. For electrorheological fluids, this is not adequate, but rather the exponent p should be able to vary. This situation leads us to the study of variable exponent Lebesgue and Sobolev spaces, $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ where $p(.)$ is a real-valued function.

The variable exponent Lebesgue Spaces $L^{p(\cdot)}$, where $p(.)$ is a real-valued function, appeared in the literature for the first time in 1931 in the paper by W.Orlicz [16]. In the 1950s, this study was carried out by Nakano [14] who made the first systematic study of spaces with a variable exponent. Later, Polish and Czechoslovak mathematicians investigated the modular function spaces (see e.g. [13] and [10]). Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. In that context, we refer to the work of Tsenov [19] and Zhikov ([22, 23]). The interested reader of the theory of Lebesgue and Sobolev spaces with a variable exponent can find numerous further references in the monograph [8]. Recently, some papers have appeared in the case of the obstacle problem with a variable exponent. See ([15, 17]) for existence and uniqueness of an entropy solution, in the framework of Lewy-Stampacchia inequalities.

A treatment of the obstacle problem (1) in the L^p-case can be found in [3] where the main goal in this work is to obtain a solution with $f\in L^1(\Omega)$ in the general settings of Orlicz-Sobolev spaces. We are interested, in this paper, in the single obstacle problem associated with equation (1), where the techniques used to study this problem are based on the following approximate problems,

$$(\mathcal{P}_\epsilon)\begin{cases} -\operatorname{div}(a(x,u_\epsilon,\nabla u_\epsilon))+g_\epsilon(x,u_\epsilon,\nabla u_\epsilon) & =f_\epsilon \ \text{ in } \Omega,\\ \qquad\qquad\qquad\qquad\qquad\qquad\quad u_\epsilon & =0 \ \text{ on } \partial\Omega,\end{cases}$$

where $g_\epsilon(x,s,\xi)=\dfrac{g(x,s,\xi)}{1+\epsilon|g(x,s,\xi)|}$ and f_ϵ is a sequence of regular functions.

Nevertheless, this approximation can not enable to obtain the a priori estimates in our case, this is due to the fact that $u_\epsilon g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)$ has no sign. To overcome this difficulty, one has introduced a doubling approximation, that is we penalized the problem $(\mathcal{P}_\epsilon)$ by

$$(\mathcal{P}^\sigma_\epsilon)\begin{cases} -\operatorname{div}(a(x,u^\sigma_\epsilon,\nabla u^\sigma_\epsilon))+g^\sigma_\epsilon(x,u^\sigma_\epsilon,\nabla u^\sigma_\epsilon)\ -\dfrac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u^{\sigma^-}_\epsilon)|^{p(x)-1} & =f_\epsilon \text{ in } \Omega,\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u^\sigma_\epsilon & =0 \text{ on } \partial\Omega,\end{cases}$$

where $g_\epsilon^\sigma(x,s,\xi) = \delta_\sigma(s) g_\epsilon(x,s,\xi)$ and where $\delta_\sigma(s)$ is some increasing Lipschitz-function (see Sections 4 and 5). Note also that the obstacle in the problem considered in this paper seems to follow the sign of the nonlinearity g.

As application to the problem considered in this paper, we have the Stefan problem which is a particular kind of boundary value problem for a partial differential equation (PDE), adapted to the case in which a phase boundary can move with time. The classical Stefan problem aims to describe the temperature distribution in a homogeneous medium undergoing a phase change, for example ice passing to water.

Our simplest model is the following $L^{p(\cdot)}$-problem,

$$-\text{div }(|\nabla u|^{p(x)-2}\nabla u) + |u|^{r(x)}|\nabla u|^{p(x)} = f \text{ in } \Omega, u = 0 \text{ on } \partial\Omega,$$

generated by the $p(x)$-Laplacian operator.

The paper is organized as follows. In Section 2, we present the preliminaries about Lebesgue and Sobolev spaces with variable exponent. In Section 3, we introduce the assumptions and prove some fundamental lemmas. In Section 4, we prove the existence of entropy solutions to the obstacle problem associated with (1) for the case of positive nonlinearity g. Finally, in Section 5, we prove the existence of entropy solutions to the obstacle problem associated with (1) for the case of negative nonlinearity g.

2 A Framework for Function Spaces

For each open bounded subset Ω of $I\!R^N$ $(N \geq 2)$, we denote

$$\mathcal{C}_+(\overline{\Omega}) = \{p | p \in \mathcal{C}(\overline{\Omega}),\ p(x) > 1 \text{ for any } \ x \in \bar{\Omega}\}.$$

For every $p \in \mathcal{C}_+(\overline{\Omega})$ we define: $p_+ = \sup_{x\in\Omega} p(x)$ and $p_- = \inf_{x\in\Omega} p(x)$.
We define the variable exponent Lebesgue space by:

$$L^{p(x)}(\Omega) = \left\{ u | u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

The Luxemburg norm on the space $L^{p(x)}(\Omega)$ is defined by

$$\|u\|_{p(x)} = \inf\left\{ \lambda > 0, \quad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \leq 1 \right\}.$$

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ (see [9], [21]). For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, the Generalized Hölder inequality

$$\left| \int_\Omega u\, v\, dx \right| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-} \Big) \|u\|_{p(x)}\, \|v\|_{p'(x)},$$

holds true.

Proposition 1 ***(see [9, 21])*** *We denote* $\rho(u) = \int_\Omega |u|^{p(x)}\, dx,\ \forall u \in L^{p(x)}(\Omega)$. *If* $u_n, u \in L^{p(x)}(\Omega)$ *and* $p^+ < +\infty$, *then the following assertions hold:*
(i) $\|u\|_{p(x)} < 1 \quad (resp, = 1, > 1) \Leftrightarrow \rho(u) < 1 \quad (resp, = 1, > 1)$,
(ii) $\|u\|_{p(x)} > 1 \Rightarrow \ \|u\|_{p(x)}^{p-} \leq \rho(u) \leq \|u\|_{p(x)}^{p+}$; $\|u\|_{p(x)} < 1 \Rightarrow \ \|u\|_{p(x)}^{p+} \leq \rho(u) \leq \|u\|_{p(x)}^{p-}$,
(iii) $\|u_n\|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho(u_n) \to 0$; $\|u_n\|_{p(x)} \to \infty \quad \Leftrightarrow \quad \rho(u_n) \to \infty$.

We define the generalized Sobolev space by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega)\}.$$

It is endowed with the following norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)} \qquad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and $p^*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$.

Proposition 2 ***(see [9]) (i)*** *Assuming $p_- > 1$, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*
(ii) *If $q \in \mathcal{C}_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous.*
(iii) *There is a constant $C > 0$, such that*

$$\|u\|_{p(x)} \leq C \, \|\nabla u\|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega), \ \textit{if } p \in \mathcal{C}(\overline{\Omega}).$$

Therefore, $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms in $W_0^{1,p(\cdot)}(\Omega)$.

3 Basic Assumptions and Some Fundamental Lemmas

Let $p \in \mathcal{C}_+(\bar{\Omega})$ such that $1 < p_- \leq p(x) \leq p_+ < \infty$ and denote $Au = -\mathrm{div}(a(x, u, \nabla u))$, where $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function satisfying the assumptions :

$$|a(x, s, \xi)| \leq \beta[k(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}], \tag{2}$$

$$[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0 \text{ for all } \ \xi \neq \eta \in I\!R^N, \tag{3}$$

$$a(x, s, \xi)\xi \geq \alpha |\xi|^{p(x)}, \tag{4}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in I\!R \times I\!R^N$, where $k(x)$ is a positive function lying in $L^{p'(x)}(\Omega)$ and $\beta, \alpha > 0$.

Furthermore, let $g : \Omega \times I\!R \times I\!R^N \to I\!R$ be a Carathéodory function having a constant sign such that for a.e. x in Ω and for all $s \in I\!R$ and $\xi \in I\!R^N$,

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + |\xi|^{p(x)}), \tag{5}$$

$$g(x, 0, \xi) = 0, \tag{6}$$

where $b : I\!R^+ \to I\!R^+$ is a continuous non-decreasing function and $c(.)$ is a positive function which belongs to $L^1(\Omega)$.

We introduce the functional spaces needed later. For $p \in \mathcal{C}_+(\bar{\Omega})$, $\mathcal{T}_0^{1,p(x)}(\Omega)$ is defined as the set of measurable functions $u : \Omega \to I\!R$ such that the truncated functions $T_k(u) \in W_0^{1,p(x)}(\Omega)$, where $T_k(s) := \max\{-k, min\{k, s\}\}$, for $s \in I\!R$ and $k > 0$.

We give the following lemma which is a generalization of Lemma 2.1 in [5] for generalized Sobolev spaces. Note that its proof is a slight modification of Lemma 2.1 in [5].

Lemma 3.1 *For every $u \in \mathcal{T}_0^{1,p(x)}(\Omega)$, there exists a unique measurable function $v : \Omega \to I\!R^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$, a.e. in Ω, for every $k > 0$.*

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$.

Lemma 3.2 *[4] Let $g \in L^{r(x)}(\Omega)$ and $g_n \in L^{r(x)}(\Omega)$ with $\|g_n\|_{L^{r(x)}(\Omega)} \leq C$ for $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ in $L^{r(x)}(\Omega)$.*

Lemma 3.3 *[4] Assume that (2)-(4) hold true, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and*

$$\int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)]\nabla(u_n - u)dx \to 0. \tag{7}$$

Then, $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$ for a subsequence.

Lemma 3.4 *[2] Let $F : I\!R \longrightarrow I\!R$ be uniformly Lipschitzian with $F(0) = 0$ and $p \in \mathcal{C}_+(\overline{\Omega})$. Let $u \in W_0^{1,p(x)}(\Omega)$. Then $F(u) \in W_0^{1,p(x)}(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial(F\circ u)}{\partial x_i} = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & a.e.\ in \quad \{x \in \Omega :\ u(x) \notin D\}, \\ \\ 0 & a.e.\ in \quad \{x \in \Omega :\ u(x) \in D\}. \end{cases}$$

Remark that the previous lemma implies that the functions in $W_0^{1,p(x)}(\Omega)$ can be truncated and as a consequence of this lemma we obtain the following result.

Lemma 3.5 *[2] Let $u \in W_0^{1,p(x)}(\Omega)$. Then, $T_k(u) \in W_0^{1,p(x)}(\Omega)$, with $k > 0$. Moreover, we have $T_k(u) \to u$ in $W_0^{1,p(x)}(\Omega)$ as $k \to \infty$.*

Definition 3.1 Let Y be a reflexive Banach space, a bounded operator B from Y to its dual Y^* is called pseudo-monotone if

$$\left.\begin{array}{l} u_n \rightharpoonup u \text{ in } Y \\ Bu_n \rightharpoonup \chi \text{ in } Y^* \\ \limsup\limits_{n\to\infty}\langle Bu_n, u_n\rangle \leq \langle \chi, u\rangle \end{array}\right\} \implies \chi = Bu \text{ and } \langle Bu_n, u_n\rangle \to \langle \chi, u\rangle.$$

Definition 3.2 Let Y be a reflexive Banach space, a bounded operator B from Y to its dual Y^* is called pseudo-monotone if

$$\left.\begin{array}{l} u_n \rightharpoonup u \text{ in } Y \\ \limsup\limits_{n\to\infty}\langle Bu_n, u_n - u\rangle \leq 0 \end{array}\right\} \implies \liminf\langle Bu_n, u_n - v\rangle \geq \langle Bu, u - v\rangle \ \text{ for all } v \in Y.$$

It is clear that the Definition 3.1 is equivalent to the well known Definition 3.2.

4 Statement of the Case of a Positive Nonlinearity g

We first consider the convex set $K_0 = \{u \in W_0^{1,p(x)}(\Omega); u \geq 0 \text{ a.e. in } \Omega\}$.

Theorem 4.1 *Assume that $(2)-(6)$ hold true and $f \in L^1(\Omega)$. Then there exists at least one solution (entropy solution) to the following unilateral problem,*

$$(\mathcal{P}) \begin{cases} u \in \mathcal{T}_0^{1,p(x)}(\Omega),\ u \geq 0 \ a.e.\ in\ \Omega,\ g(x, u, \nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x, u, \nabla u)\nabla T_k(u - v)\,dx + \int_\Omega g(x, u, \nabla u)T_k(u - v)\,dx \leq \int_\Omega f T_k(u - v)\,dx, \\ \forall v \in K_0 \cap L^\infty(\Omega), \quad \forall k > 0. \end{cases}$$

Proof of Theorem 4.1

We consider the following approximated problem

$$(\mathcal{P}_\epsilon)\begin{cases} -\mathrm{div}(a(x,u_\epsilon,\nabla u_\epsilon)) + g_\epsilon(x,u_\epsilon,\nabla u_\epsilon) &= f_\epsilon \ \text{ in } \Omega, \\ u_\epsilon &= 0 \ \text{ on } \partial\Omega, \end{cases} \tag{8}$$

where $g_\epsilon(x,s,\xi) = \dfrac{g(x,s,\xi)}{1+\epsilon|g(x,s,\xi)|}$ and $f_\epsilon = T_{\frac{1}{\epsilon}}(f)$; then $(f_\epsilon)_{\epsilon>0}$ is a sequence of bounded functions which strongly converges to f in $L^1(\Omega)$ and $\|f_\epsilon\|_1 \le \|f\|_1$, for all $\epsilon > 0$.

Note that $|g_\epsilon(x,s,\xi)| \le |g(x,s,\xi)| \le b(|s|)(c(x)+|\xi|^{p(x)})$ and $|g_\epsilon(x,s,\xi)| \le \frac{1}{\epsilon}$. Nevertheless, it seems difficult to obtain a priori estimates, due to the fact that the quantity $u_\epsilon g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)$ has no constant sign. In order to avoid this inconvenience, we approach the sign function by an increasing Lipschitz function.

Set for $\sigma > 0$,

$$\delta_\sigma(s) = \begin{cases} \frac{s-\sigma}{s}, & \text{if } s \ge \sigma > 0, \\ 0, & \text{if } |s| \le \sigma, \\ \frac{-s-\sigma}{s}, & \text{if } s < -\sigma < 0. \end{cases}$$

Now, we set

$$g_\epsilon^\sigma(x,s,\xi) = \delta_\sigma(s) g_\epsilon(x,s,\xi). \tag{9}$$

Remark that $g_\epsilon^\sigma(x,s,\xi)$ has the same sign as s.

Now, we are in a position to approximate our initial unilateral problem by the following penalized problem

$$(\mathcal{P}_\epsilon^\sigma)\begin{cases} u_\epsilon^\sigma \in W_0^{1,p(x)}(\Omega) \\ \langle Au_\epsilon^\sigma, u_\epsilon^\sigma - v\rangle + \displaystyle\int_\Omega g_\epsilon^\sigma(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)(u_\epsilon^\sigma - v)\,dx - \frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma^-})|^{p(x)-1}(u_\epsilon^\sigma - v)\,dx \\ \qquad = \displaystyle\int_\Omega f_\epsilon(u_\epsilon^\sigma - v)\,dx, \quad \forall v \in W_0^{1,p(x)}(\Omega). \end{cases} \tag{10}$$

We define the operators $G_\epsilon^\sigma, R_\epsilon^\sigma : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)$ by,

$$\langle G_\epsilon^\sigma u, v\rangle = \int_\Omega g_\epsilon^\sigma(x,u,\nabla u)v\,dx,\ \langle R_\epsilon^\sigma u, v\rangle = -\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1} v\,dx.$$

We also denote

$$\langle Au, v\rangle = \int_\Omega a(x,u,\nabla u)\nabla v\,dx.$$

Thanks to the generalized Hölder's inequality, we have for all $u, v \in W_0^{1,p(x)}(\Omega)$,

$$\begin{aligned} \left|\int_\Omega g_\epsilon^\sigma(x,u,\nabla u)v\,dx\right| &\le (\frac{1}{p_-} + \frac{1}{p'_-})\|g_\epsilon^\sigma(x,u,\nabla u)\|_{p'(x)}\|v\|_{p(x)} \\ &\le (\frac{1}{p_-} + \frac{1}{p'_-})\left(\left(1+\frac{1}{\epsilon}\right)^{\frac{p'_+}{p'_-}}(\mathrm{meas}(\Omega)+1)^{\frac{1}{p'_-}}\right)\|v\|_{p(x)} \\ &\le C\|v\|_{1,p(x)} \end{aligned} \tag{11}$$

and

$$\begin{aligned} \left| -\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1} v\,dx\right| &\le \frac{1}{\epsilon^2}\Big(\frac{1}{p_-}+\frac{1}{p'_-}\Big)\|T_{\frac{1}{\epsilon}}(u^-)^{p(x)-1}\|_{p'(x)}\|v\|_{p(x)} \\ &\le \frac{1}{\epsilon^2}\Big(\frac{1}{p_-}+\frac{1}{p'_-}\Big)\left\|\Big(\frac{1}{\epsilon}\Big)^{p(x)-1}\right\|_{p'(x)}\|v\|_{p(x)} \\ &\le C\|v\|_{1,p(x)}. \end{aligned} \tag{12}$$

We need the following lemma.

Lemma 4.1 *The operator $B^\sigma_\epsilon = A + G^\sigma_\epsilon + R^\sigma_\epsilon$ from $W_0^{1,p(x)}(\Omega)$ into $W^{-1,p'(x)}(\Omega)$ is pseudo-monotone. Moreover, B^σ_ϵ is coercive, in the following sense:*

$$\frac{\langle B^\sigma_\epsilon v, v\rangle}{\|v\|_{1,p(x)}} \to +\infty \quad \textit{if} \quad \|v\|_{1,p(x)} \to +\infty.$$

Proof of Lemma 4.1 Using the generalized Hölder's inequality and the growth condition (2) we can show that A is bounded, and by (11) and (12), B^σ_ϵ is bounded in $W_0^{1,p(x)}(\Omega)$. The coercivity follows from (4) and the fact that $g^\sigma_\epsilon(x,s,\xi)s \ge 0$ and $-\dfrac{1}{\epsilon^2}\displaystyle\int_\Omega |T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1}u\,dx \ge 0$. It remains to show that B^σ_ϵ is pseudo-monotone.

Let $(u_k)_{k>0}$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \quad \text{in} \quad W_0^{1,p(x)}(\Omega), \\ B^\sigma_\epsilon u_k \rightharpoonup \chi \text{ in } \quad W^{-1,p'(x)}(\Omega), \\ \limsup\limits_{k\to\infty}\langle B^\sigma_\epsilon u_k, u_k\rangle \le \langle \chi, u\rangle. \end{cases} \tag{13}$$

We will prove that $\chi = B^\sigma_\epsilon u$ and $\langle B^\sigma_\epsilon u_k, u_k\rangle \to \langle \chi, u\rangle$ as $k\to+\infty$.

Firstly, since $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$, then

$$u_k \to u \text{ in } L^{p(x)}(\Omega) \quad \text{for a subsequence denoted again } (u_k)_{k>0}. \tag{14}$$

As $(u_k)_{k>0}$ is a bounded sequence in $W_0^{1,p(x)}(\Omega)$, then by (2), $(a(x,u_k,\nabla u_k))_{k>0}$ is bounded in $(L^{p'(x)}(\Omega))^N$. Therefore, there exists a function $\varphi \in (L^{p'(x)}(\Omega))^N$ such that

$$a(x,u_k,\nabla u_k) \rightharpoonup \varphi \quad \text{in} \quad (L^{p'(x)}(\Omega))^N \quad \text{as} \quad k\to\infty. \tag{15}$$

Similarly, it is easy to see that $(g^\sigma_\epsilon(x,u_k,\nabla u_k))_{k>0}$ is bounded in $L^{p'(x)}(\Omega)$ with respect to k, then there exists a function $\psi^\sigma_\epsilon \in L^{p'(x)}(\Omega)$ such that

$$g^\sigma_\epsilon(x,u_k,\nabla u_k) \rightharpoonup \psi^\sigma_\epsilon \quad \text{in} \quad L^{p'(x)}(\Omega) \quad \text{as} \quad k\to\infty \tag{16}$$

and as $(-\dfrac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u_k)|^{p(x)-1})_{k>0}$ is bounded in $L^{p'(x)}(\Omega)$, then

$$-\frac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u_k^-)|^{p(x)-1} \to -\frac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1} \text{ in } L^{p'(x)}(\Omega) \text{ as } k\to\infty. \tag{17}$$

It is clear that, for all $v \in W_0^{1,p(x)}(\Omega)$, we have

$$\begin{aligned}\langle \chi, v\rangle = \lim_{k\to\infty}\langle B_\epsilon^\sigma u_k, v\rangle &= \lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla v dx + \lim_{k\to\infty}\int_\Omega g_\epsilon^\sigma(x,u_k,\nabla u_k) v dx \\ &\quad + \lim_{k\to\infty} -\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_k^-)|^{p(x)-1} v\, dx \\ &= \int_\Omega \varphi \nabla v\, dx + \int_\Omega \psi_\epsilon^\sigma v\, dx - \frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1} v\, dx.\end{aligned} \tag{18}$$

On one hand, by (14) we have

$$\int_\Omega g_\epsilon^\sigma(x,u_k,\nabla u_k)u_k\, dx \to \int_\Omega \psi_\epsilon^\sigma u\, dx \quad \text{as} \quad k\to\infty, \tag{19}$$

$$-\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_k^-)|^{p(x)-1}u_k\, dx \to -\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1}u\, dx \text{ as } \ k\to\infty. \tag{20}$$

Consequently, by the hypotheses, we have

$$\begin{aligned}\limsup_{k\to\infty}\langle B_\epsilon^\sigma(u_k), u_k\rangle &= \limsup_{k\to\infty}\Big\{\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\, dx + \int_\Omega g_\epsilon^\sigma(x,u_k,\nabla u_k)u_k\, dx \\ &\qquad - \frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_k^-)|^{p(x)-1}u_k\, dx\Big\} \\ &\le \int_\Omega \varphi\nabla u\, dx + \int_\Omega \psi_\epsilon^\sigma u\, dx - \frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1}u\, dx.\end{aligned} \tag{21}$$

Therefore,

$$\limsup_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\, dx \le \int_\Omega \varphi\nabla u\, dx. \tag{22}$$

Thanks to (3), we have

$$\int_\Omega (a(x,u_k,\nabla u_k) - a(x,u_k,\nabla u))(\nabla u_k - \nabla u)\, dx \ge 0. \tag{23}$$

Then

$$\begin{aligned}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\, dx \ge &-\int_\Omega a(x,u_k,\nabla u)\nabla u\, dx \\ &+ \int_\Omega a(x,u_k,\nabla u_k)\nabla u\, dx + \int_\Omega a(x,u_k,\nabla u)\nabla u_k\, dx.\end{aligned}$$

By (15), we get

$$\liminf_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\, dx \ge \int_\Omega \varphi\nabla u\, dx$$

which implies by using (22)

$$\lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\, dx = \int_\Omega \varphi\nabla u\, dx. \tag{24}$$

By means of (18), (19), (20) and (24), we obtain $\langle B_\epsilon^\sigma u_k, u_k\rangle \to \langle\chi, u\rangle$ as $k\to+\infty$. On the other hand, by (24) and the fact that $a(x,u_k,\nabla u)\to a(x,u,\nabla u)$ in $(L^{p'(x)}(\Omega))^N$, we can deduce that

$$\lim_{k\to+\infty}\int_\Omega (a(x,u_k,\nabla u_k) - a(x,u_k,\nabla u))(\nabla u_k - \nabla u)\, dx = 0$$

and so, by virtue of Lemma 3.3 we find $\nabla u_n \to \nabla u$ a.e. in Ω, which concludes

$$a(x,u_k,\nabla u_k) \rightharpoonup a(x,u,\nabla u) \text{ in } (L^{p'(x)}(\Omega))^N,$$

$$g_\epsilon^\sigma(x,u_k,\nabla u_k) \rightharpoonup g_\epsilon^\sigma(x,u,\nabla u) \text{ in } L^{p'(x)}(\Omega)$$

and

$$-\frac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u_k^-)|^{p(x)-1} \rightharpoonup -\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1}.$$

Thus, $\chi = B_\epsilon^\sigma u$.

In view of Lemma 4.1, there exists at least one solution $u_\epsilon^\sigma \in W_0^{1,p(x)}(\Omega)$ to the problem (10), by using the classical theorem in [12]. The continuation of the proof of Theorem 4.1 is divided into several steps.

4.1 Study of the approximate problem with respect to ϵ

4.1.1 A priori estimates

If we take $v = u_\epsilon^\sigma - T_k(u_\epsilon^\sigma)$ as a test function in (10), we obtain

$$\int_\Omega a(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)\nabla T_k(u_\epsilon^\sigma)\,dx + \int_\Omega g_\epsilon^\sigma(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)T_k(u_\epsilon^\sigma)\,dx$$
$$-\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma -})|^{p(x)-1}T_k(u_\epsilon^\sigma)\,dx = \int_\Omega f_\epsilon T_k(u_\epsilon^\sigma)\,dx.$$

So, as $u_\epsilon^\sigma = u_\epsilon^{\sigma +} - u_\epsilon^{\sigma -}$, then

$$\begin{aligned} -\frac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma -})|^{p(x)-1}T_k(u_\epsilon^\sigma) &= -\frac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma -})|^{p(x)-1}T_k(u_\epsilon^\sigma)\chi_{\{u_\epsilon^\sigma \le 0\}} \\ &= \frac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma -})|^{p(x)-1}T_k(u_\epsilon^{\sigma -}) \ge 0. \end{aligned} \tag{25}$$

Using the fact that $g_\epsilon^\sigma(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)T_k(u_\epsilon^\sigma) \ge 0$ and by (25) we get

$$\int_\Omega a(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)\nabla T_k(u_\epsilon^\sigma)\,dx \le k\|f\|_{L^1(\Omega)}. \tag{26}$$

So, by (4) we get

$$\alpha\|\nabla T_k(u_\epsilon^\sigma)\|_{p(x)}^\gamma \le \alpha\int_\Omega |\nabla T_k(u_\epsilon^\sigma)|^{p(x)}\,dx \le k\|f\|_{L^1(\Omega)} \tag{27}$$

with

$$\gamma = \begin{cases} p_+ & \text{if} \quad \|\nabla T_k(u_\epsilon^\sigma)\|_{p(x)} \le 1, \\ p_- & \text{if} \quad \|\nabla T_k(u_\epsilon^\sigma)\|_{p(x)} > 1. \end{cases}$$

Thanks to Poincaré inequality, we obtain

$$\|T_k(u_\epsilon^\sigma)\|_{1,p(x)} \le Ck^{\frac{1}{\gamma}}, \tag{28}$$

where C does not depend on ϵ. Consequently $(T_k(u_\epsilon^\sigma))_{\epsilon>0}$ is bounded in $W_0^{1,p(x)}(\Omega)$ uniformly on ϵ and σ.

4.1.2 Convergence in measure of u_ϵ^σ

We prove that u_ϵ^σ converges to some function u^σ in measure. To prove this, we show that u_ϵ^σ is a Cauchy sequence in measure. Let k be large enough. Combining the generalized Hölder's inequality, Poincaré's inequality and (28), one has

$$\begin{aligned} k \text{ meas}(\{|u_\epsilon^\sigma| > k\}) &= \int_{\{|u_\epsilon^\sigma|>k\}} |T_k(u_\epsilon^\sigma)|\,dx \le \int_\Omega |T_k(u_\epsilon^\sigma)|\,dx \\ &\le \big(\tfrac{1}{p_-} + \tfrac{1}{p'_-}\big)(\text{meas}(\Omega)+1)^{\frac{1}{p'_-}} \|T_k(u_\epsilon^\sigma)\|_{p(x)} \\ &\le C_1 \|T_k(u_\epsilon^\sigma)\|_{1,p(x)} \le C_2 k^{\frac{1}{\gamma}}; \end{aligned} \tag{29}$$

which yields

$$\text{meas}(\{|u_\epsilon^\sigma| > k\}) \le \frac{C_2}{k^{1-\frac{1}{\gamma}}} \quad \forall \epsilon > 0,\ \forall k > 0. \tag{30}$$

Hence

$$\text{meas}(\{|u_\epsilon^\sigma| > k\}) \to 0 \text{ as } k \to \infty \text{ (since } 1 - \frac{1}{\gamma} > 0), \tag{31}$$

uniformly in ϵ and σ. Moreover, we have, for every $\delta > 0$,

$$\begin{aligned} \text{meas } (\{|u_n^\sigma - u_m^\sigma| > \delta\}) \le \text{meas } (\{|u_n^\sigma| > k\}) + \text{meas } (\{|u_m^\sigma| > k\}) \\ + \text{meas } (\{|T_k(u_n^\sigma) - T_k(u_m^\sigma)| > \delta\}). \end{aligned} \tag{32}$$

Since $(T_k(u_\epsilon^\sigma))_{\epsilon>0}$ is bounded in $W_0^{1,p(x)}(\Omega)$, then there exists for $\sigma > 0$ fixed, $v_k^\sigma \in W_0^{1,p(x)}(\Omega)$ such that

$$T_k(u_\epsilon^\sigma) \rightharpoonup v_k^\sigma \qquad \text{in} \quad W_0^{1,p(x)}(\Omega)$$

and by the compact embedding, we have

$$T_k(u_\epsilon^\sigma) \to v_k^\sigma \qquad \text{in} \quad L^{p(x)}(\Omega) \qquad \text{and a.e. in } \Omega. \tag{33}$$

Consequently, we can assume that $(T_k(u_\epsilon^\sigma))_{\epsilon>0}$ is a Cauchy sequence in measure in Ω. Let $\eta > 0$. Then by (30) and (32), there exists some $k(\eta) > 0$ such that $\text{meas}(\{|u_n^\sigma - u_m^\sigma| > \delta\}) < \eta$ for all $n, m \ge n_0(k(\eta), \delta)$. This proves that $(u_\epsilon^\sigma)_{\epsilon>0}$ is a Cauchy sequence in measure and thus, converges almost everywhere to some measurable function u^σ. Therefore, $u_\epsilon^\sigma \to u^\sigma$ a.e. in Ω.

Furthermore,

$$\begin{aligned} & T_k(u_\epsilon^\sigma) \rightharpoonup T_k(u^\sigma) \quad \text{in } W_0^{1,p(x)}(\Omega) \\ \text{and} & \\ & T_k(u_\epsilon^\sigma) \to T_k(u^\sigma) \quad \text{in} \quad L^{p(x)}(\Omega) \text{ and a.e. in } \Omega. \end{aligned} \tag{34}$$

4.1.3 Positivity of u^σ

Taking $v = u_\epsilon^\sigma - T_{\frac{1}{\epsilon}}(u_\epsilon^\sigma)$ as a test function in (10), we obtain

$$\int_\Omega a(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) \nabla T_{\frac{1}{\epsilon}}(u_\epsilon^\sigma)\,dx + \int_\Omega g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) T_{\frac{1}{\epsilon}}(u_\epsilon^\sigma)\,dx - \\ \frac{1}{\epsilon^2} \int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma -})|^{p(x)-1} T_{\frac{1}{\epsilon}}(u_\epsilon^\sigma)\,dx = \int_\Omega f_\epsilon T_{\frac{1}{\epsilon}}(u_\epsilon^\sigma)\,dx.$$

Since $\int_\Omega a(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)\nabla T_{\frac{1}{\epsilon}}(u_\epsilon^\sigma)\,dx \geq 0$ and $g_\epsilon^\sigma(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)T_{\frac{1}{\epsilon}}(u_\epsilon^\sigma) \geq 0$, we get

$$-\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}\left(-T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma^-})\right)\,dx \leq \frac{1}{\epsilon}\|f\|_{L^1(\Omega)}.$$

Thus,

$$\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)}\,dx \leq \epsilon\|f\|_{L^1(\Omega)}.$$

Now, denote by $A = \left\{x \in \Omega \text{ such that } |T_{\frac{1}{\epsilon}}\left(u_\epsilon^{\sigma^-}\right)| = \frac{1}{\epsilon}\right\}$. As ϵ is used to tend to 0, we can take it in $(0,1)$ to get

$$\operatorname{meas}(A)\left(\frac{1}{\epsilon}\right)^{p_-} \leq \int_A |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)} \leq \epsilon\|f\|_{L^1(\Omega)};$$

which implies that (by letting ϵ go to 0)

$$\operatorname{meas}(A) = 0.$$

Hence, since $u_\epsilon^\sigma \to u^\sigma$ a.e. in Ω and the fact that $\operatorname{meas}(A) = 0$, we conclude that

$$|T_{\frac{1}{\epsilon}}\left(u_\epsilon^{\sigma^-}\right)|^{p(x)} \to |u^{\sigma^-}|^{p(x)} \text{ a.e. in } \Omega.$$

We use again the Fatou's Lemma to obtain

$$\int_\Omega |u^{\sigma^-}|\,dx \leq \liminf_{\epsilon\to 0}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)}\,dx \leq \liminf_{\epsilon\to 0}\epsilon\|f\|_{L^1(\Omega)} = 0;$$

which yields

$$u^\sigma \geq 0.$$

4.1.4 Almost everywhere convergence of the gradient

For the sake of simplicity we will write $\eta(\epsilon,h)$ for any quantity such that

$$\lim_{h\to+\infty}\lim_{\epsilon\to 0}\eta(\epsilon,h) = 0.$$

Finally, by $\eta_h(\epsilon)$ we will denote a quantity that depends on ϵ and h and is such that

$$\lim_{\epsilon\to 0}\eta_h(\epsilon) = 0,$$

for any fixed value of h.

Let $h > 2k > 0$, we shall use in (10) the test function

$$\left\{\begin{array}{lcl} v_\epsilon^{h,\sigma} &=& u_\epsilon^\sigma - \eta\varphi_k(\omega_\epsilon^{h,\sigma}) \\ \omega_\epsilon^{h,\sigma} &=& T_{2k}\big(u_\epsilon^\sigma - T_h(u_\epsilon^\sigma) + T_k(u_\epsilon^\sigma) - T_k(u^\sigma)\big) \\ \omega^{h,\sigma} &=& T_{2k}(u^\sigma - T_h(u^\sigma)). \end{array}\right. \tag{35}$$

Let $\varphi_k(t) = te^{\lambda t^2}$, $\lambda = (\frac{b(k)}{2\alpha})^2$, it's obvious to check that (see [6], Lemma 1)

$$\varphi_k'(t) - \frac{b(k)}{\alpha}|\varphi_k(t)| \geq \frac{1}{2}, \quad \forall t \in I\!R. \tag{36}$$

It follows that

$$\langle A(u_\epsilon^\sigma), \varphi_k(\omega_\epsilon^{h,\sigma})\rangle + \int_\Omega g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma})\,dx - \frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}\varphi_k(\omega_\epsilon^{h,\sigma})\,dx = \int_\Omega f_\epsilon \varphi_k(\omega_\epsilon^{h,\sigma})\,dx,$$

which is equivalent to saying that

$$\begin{aligned}&\int_\Omega a(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\nabla\omega_\epsilon^{h,\sigma}\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx + \int_\Omega g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma})\,dx \\ &-\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}\ \varphi_k(\omega_\epsilon^{h,\sigma})\,dx \\ &= \int_\Omega f_\epsilon\varphi_k(\omega_\epsilon^{h,\sigma})\,dx.\end{aligned} \tag{37}$$

Note that, $\nabla\omega_\epsilon^{h,\sigma} = 0$ on the set $\{|u_\epsilon^\sigma| > s = 4k + h\}$, therefore, we get by (37)

$$\begin{aligned}&\int_\Omega a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla\omega_\epsilon^{h,\sigma}\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx + \int_\Omega g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma})\,dx \\ &-\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}\ \varphi_k(\omega_\epsilon^{h,\sigma})\,dx \\ &= \int_\Omega f_\epsilon\varphi_k(\omega_\epsilon^{h,\sigma})\,dx.\end{aligned}$$

According to (34), we have $\varphi_k(\omega_\epsilon^{h,\sigma}) \rightharpoonup \varphi_k(\omega^{h,\sigma})$ weakly-* in $L^\infty(\Omega)$ as $\epsilon \to 0$, and then

$$\int_\Omega f_\epsilon\varphi_k(\omega_\epsilon^{h,\sigma})\,dx \to \int_\Omega f\varphi_k(\omega^{h,\sigma})\,dx.$$

Finally, by using Lebesgue's theorem, we can deduce that

$$\int_\Omega f\varphi_k(\omega^{h,\sigma})\,dx \to 0 \text{ as } h \to +\infty.$$

Therefore,

$$\int_\Omega f\varphi_k(\omega_\epsilon^{h,\sigma})\,dx = \eta(\epsilon, h). \tag{38}$$

Note that $\varphi_k(\omega_\epsilon^{h,\sigma})$ and u_ϵ^σ has the same sign in the set $\{x \in \Omega, |u_\epsilon^\sigma| > k\}$, then we have

$$g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma}) \geq 0 \quad\text{and}\quad -\frac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}\varphi_k(\omega_\epsilon^{h,\sigma}) \geq 0.$$

From (37), we deduce that

$$\begin{aligned}&\int_\Omega a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla\omega_\epsilon^{h,\sigma}\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx + \int_{\{|u_\epsilon^\sigma|<k\}} g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma})\,dx \\ &-\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}\ (u_\epsilon^\sigma - T_k(u^\sigma))\exp(\lambda(\omega_\epsilon^{h,\sigma}))^2\,dx \\ &\leq \eta(\epsilon, h).\end{aligned} \tag{39}$$

Since $u^\sigma \geq 0$, then the third term on the left-hand side of the above inequality is positive, thus,

$$\int_\Omega a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla \omega_\epsilon^{h,\sigma} \varphi_k'(\omega_\epsilon^{h,\sigma})\,dx + \int_{\{|u_\epsilon^\sigma|<k\}} g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma})\,dx \\ \leq \eta(\epsilon, h). \tag{40}$$

Splitting the first integral on the left-hand side of (40), where $|u_\epsilon^\sigma| \leq k$ and $|u_\epsilon^\sigma| > k$, we can write

$$\begin{aligned} &\int_\Omega a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla \omega_\epsilon^{h,\sigma} \varphi_k'(\omega_\epsilon^{h,\sigma})\,dx \\ &= \int_{\{|u_\epsilon^\sigma|\leq k\}} a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx \\ &+ \int_{\{|u_\epsilon^\sigma|>k\}} a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla \omega_\epsilon^{h,\sigma} \varphi_k'(\omega_\epsilon^{h,\sigma})\,dx. \end{aligned} \tag{41}$$

The first term on the right-hand side of the last inequality can be written as

$$\begin{aligned} &\int_{\{|u_\epsilon^\sigma|\leq k\}} a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx \\ &= \int_\Omega a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u_\epsilon^\sigma))[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx. \end{aligned} \tag{42}$$

For the second term on the right-hand side of (41), we can write according to (4),

$$\begin{aligned} &\int_{\{|u_\epsilon^\sigma|>k\}} a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla \omega_\epsilon^{h,\sigma} \varphi_k'(\omega_\epsilon^{h,\sigma})\,dx \\ &\geq -\varphi'(2k) \int_{\{|u_\epsilon^\sigma|>k\}} |a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))||\nabla T_k(u^\sigma)|\,dx. \end{aligned} \tag{43}$$

Since $|a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma)|$ is bounded in $(L^{p'(x)}(\Omega))^N$, if necessary we have

$$|a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma)| \rightharpoonup l_{M,\sigma} \text{ in } (L^{p'(x)}(\Omega))^N \text{ as } \epsilon \to 0, \quad \text{for a subsequence.}$$

Due to $\nabla T_k(u^\sigma)\chi_{\{|u_\epsilon^\sigma|>k\}} \to \nabla T_k(u^\sigma)\chi_{\{|u^\sigma|>k\}}$ in $L^{p(x)}(\Omega)$ as $\epsilon \to 0$, we obtain

$$-\varphi'(2k) \int_{\{|u_\epsilon^\sigma|>k\}} |a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))||\nabla T_k(u^\sigma)|\,dx \to$$

$$-\varphi'(2k) \int_{\{|u^\sigma|>k\}} l_{M,\sigma}|\nabla T_k(u^\sigma)|\,dx = 0 \text{ as } \epsilon \to 0.$$

Therefore,

$$-\varphi'(2k) \int_{\{|u_\epsilon^\sigma|>k\}} |a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))||\nabla T_k(u^\sigma)|\,dx = \eta_h(\epsilon). \tag{44}$$

Combining (41) and (44), we deduce that

$$\begin{aligned} &\int_\Omega a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla \omega_\epsilon^{h,\sigma} \varphi'(\omega_\epsilon^{h,\sigma})\,dx \\ &\geq \int_\Omega a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u_\epsilon^\sigma))[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx + \eta_h(\epsilon). \end{aligned} \tag{45}$$

It follows

$$\begin{aligned}
&\int_\Omega a(x,T_s(u_\epsilon^\sigma),\nabla T_s(u_\epsilon^\sigma))\nabla\omega_\epsilon^{h,\sigma}\varphi'(\omega_\epsilon^{h,\sigma})\,dx\\
&\geq \int_\Omega [a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u_\epsilon^\sigma))-a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))]\\
&\times[\nabla T_k(u_\epsilon^\sigma)-\nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx\\
&+\int_\Omega a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))[\nabla T_k(u_\epsilon^\sigma)-\nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx+\eta_h(\epsilon).
\end{aligned}\tag{46}$$

Concerning the second term of the right-hand side of (46) we can write

$$\begin{aligned}
&\int_\Omega a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))[\nabla T_k(u_\epsilon^\sigma)-\nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx\\
&=\int_\Omega a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))\nabla T_k(u_\epsilon^\sigma)\varphi_k'(T_k(u_\epsilon^\sigma)-T_k(u^\sigma))\,dx\\
&-\int_\Omega a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))\nabla T_k(u^\sigma)\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx.
\end{aligned}\tag{47}$$

By the continuity of Nemytskii's operator (cf. [9], [20]), we have

$$a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))\varphi_k'(T_k(u_\epsilon^\sigma)-T_k(u^\sigma))\to a(x,T_k(u^\sigma),\nabla T_k(u^\sigma))\varphi_k'(0)$$

and $a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))\to a(x,T_k(u^\sigma),\nabla T_k(u^\sigma))$ strongly in $(L^{p'(x)}(\Omega))^N$, while $\nabla T_k(u_\epsilon^\sigma)\rightharpoonup\nabla T_k(u^\sigma)$ weakly in $(L^{p(x)}(\Omega))^N$ and $\nabla T_k(u_\epsilon^\sigma)\varphi_k'(\omega_\epsilon^{h,\sigma})\to\nabla T_k(u^\sigma)\varphi_k'(0)$ strongly in $(L^{p(x)}(\Omega))^N$.

Then, the first and the second term of the right-hand side on (47) tend respectively to

$$\int_\Omega a(x,T_k(u^\sigma),\nabla T_k(u^\sigma))\nabla T_k(u^\sigma)\varphi_k'(0)\,dx \text{ as } \epsilon\to 0$$

and

$$-\int_\Omega a(x,T_k(u^\sigma),\nabla T_k(u^\sigma))\nabla T_k(u^\sigma)\varphi_k'(\omega^{h,\sigma})\,dx \text{ as } \epsilon\to 0;$$

therefore,

$$\int_\Omega a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))[\nabla T_k(u_\epsilon^\sigma)-\nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx=\eta_h(\epsilon).\tag{48}$$

Combining (46) and (48) yields

$$\begin{aligned}
&\int_\Omega a(x,T_s(u_\epsilon^\sigma),\nabla T_s(u_\epsilon^\sigma))\nabla\omega_\epsilon^{h,\sigma}\varphi'(\omega_\epsilon^{h,\sigma})\,dx\\
&\geq\int_\Omega [a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u_\epsilon^\sigma))-a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))]\\
&\times[\nabla T_k(u_\epsilon^\sigma)-\nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx+\eta(\epsilon,h).
\end{aligned}\tag{49}$$

Going back to the second term of the left hand side of (40), we have

$$\begin{aligned}
&\left|\int_{\{|u_\epsilon^\sigma|<k\}} g_\epsilon^\sigma(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma})\,dx\right|\\
&\leq b(k)\int_\Omega c(x)|\varphi_k(\omega_\epsilon^{h,\sigma})|\,dx+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u_\epsilon^\sigma))\nabla T_k(u_\epsilon^\sigma)|\varphi_k(\omega_\epsilon^{h,\sigma})|\,dx\\
&\leq\eta(\epsilon,h)+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u_\epsilon^\sigma))\nabla T_k(u_\epsilon^\sigma)|\varphi_k(\omega_\epsilon^{h,\sigma})|\,dx.
\end{aligned}\tag{50}$$

The last term of the last side of this inequality reads as

$$\begin{aligned}
&\frac{b(k)}{\alpha}\int_\Omega [a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u_\epsilon^\sigma)) - a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))]\\
&[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]|\varphi_k(\omega_\epsilon^{h,\sigma})|\,dx\\
&+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]|\varphi_k(\omega_\epsilon^{h,\sigma})|\,dx\\
&+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u_\epsilon^\sigma))\nabla T_k(u^\sigma)|\varphi_k(\omega_\epsilon^{h,\sigma})|\,dx.
\end{aligned} \tag{51}$$

Reasoning as above, it is easy to see that

$$\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]|\varphi_k(\omega_\epsilon^{h,\sigma})|\,dx = \eta_h(\epsilon)$$

and

$$\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u_\epsilon^\sigma))\nabla T_k(u^\sigma)|\varphi_k(\omega_\epsilon^{h,\sigma})|\,dx = \eta(\epsilon,h).$$

Therefore,

$$\begin{aligned}
&\left|\int_{\{|u_\epsilon^\sigma|<k\}} g_\epsilon^\sigma(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma})\,dx\right|\\
&\le \frac{b(k)}{\alpha}\int_\Omega [a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u_\epsilon^\sigma)) - a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))]\\
&\times[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]|\varphi_k(\omega_\epsilon^{h,\sigma})|\,dx + \eta(\epsilon,h).
\end{aligned} \tag{52}$$

Combining (40), (51) and (52), we obtain

$$\begin{aligned}
&\int_\Omega [a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u_\epsilon^\sigma)) - a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))]\\
&\times[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)](\varphi_k'(\omega_\epsilon^{h,\sigma}) - \frac{b(k)}{\alpha}|\varphi_k(\omega_\epsilon^{h,\sigma})|)\,dx\\
&\le \eta(\epsilon,h),
\end{aligned} \tag{53}$$

which implies by using (36) that

$$\int_\Omega [a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u_\epsilon^\sigma)) - a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))][\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\,dx \le \eta(\epsilon,h). \tag{54}$$

Letting ϵ tend to 0 and h tend to infinity, we deduce that

$$\int_\Omega [a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u_\epsilon^\sigma)) - a(x,T_k(u_\epsilon^\sigma),\nabla T_k(u^\sigma))][\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\,dx \to 0.$$

By Lemma 3.3, we get from convergence above

$$T_k(u_\epsilon^\sigma) \to T_k(u^\sigma) \text{ in } W_0^{1,p(x)}(\Omega). \tag{55}$$

Thus,

$$\nabla u_\epsilon^\sigma \to \nabla u^\sigma \text{ a.e. in } \Omega. \tag{56}$$

4.1.5 Equi-integrability of the nonlinearity g_ϵ^σ

In order to pass to the limit in the approximated equation, we now show that

$$g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) \to g^\sigma(x, u^\sigma, \nabla u^\sigma) \quad \text{in} \quad L^1(\Omega). \tag{57}$$

In particular, it is enough to prove the equi-integrability of the sequence $\{|g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)|\}$. To this purpose, we take $u_\epsilon^\sigma - T_1(u_\epsilon^\sigma - T_h(u_\epsilon^\sigma)) \geq 0$ as a test function in (10), to obtain

$$\int_{\{|u_\epsilon^\sigma| \geq h+1\}} |g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)|\, dx \leq \int_{\{|u_\epsilon^\sigma| > h\}} |f_n|\, dx.$$

Let $\eta > 0$ be fixed. Then, there exists $h(\eta) \geq 1$ such that

$$\int_{\{|u_\epsilon^\sigma| \geq h(\eta)\}} |g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)|\, dx < \frac{\eta}{2}. \tag{58}$$

For any measurable subset $E \subset \Omega$, we have

$$\begin{aligned} \int_E |g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)|\, dx \leq \int_E & b(l(\varepsilon))\big(c(x) + |\nabla T_{h(\eta)}(u_\epsilon^\sigma)|^{p(x)}\big)\, dx \\ &+ \int_{\{|u_\epsilon^\sigma| \geq h(\eta)\}} |g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)|\, dx. \end{aligned} \tag{59}$$

In view of (55), there exists $\beta(\eta) > 0$ such that

$$\int_E b(h(\eta))\big(c(x) + |\nabla T_{h(\eta)}(u_\epsilon^\sigma)|^{p(x)}\big)\, dx \leq \frac{\eta}{2} \quad \text{for all } E \text{ such that } \operatorname{meas}(E) < \beta(\eta). \tag{60}$$

Finally, by combining (58) and (60), one easily has

$$\int_E |g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)|\, dx \leq \eta \qquad \text{for all } E \text{ such that } \operatorname{meas}(E) < \beta(\eta).$$

Then, we deduce that $g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)$ is uniformly equi-integrable in Ω.

4.1.6 Passing to the limit with respect to ϵ

Let $v \in K_0 \cap L^\infty(\Omega)$, we take $u_\epsilon^\sigma - T_k(u_\epsilon^\sigma - v)$ as a test function in (10) to obtain

$$\int_\Omega a(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) \nabla T_k(u_\epsilon^\sigma - v)\, dx + \int_\Omega g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) T_k(u_\epsilon^\sigma - v)\, dx \leq \int_\Omega f_\epsilon T_k(u_\epsilon^\sigma - v)\, dx. \tag{61}$$

We deduce that

$$\int_{\{|u_\epsilon^\sigma - v| \leq k\}} a(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) \nabla (u_\epsilon^\sigma - v) dx + \int_\Omega g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) T_k(u_\epsilon^\sigma - v) dx \leq \int_\Omega f_\epsilon T_k(u_\epsilon^\sigma - v) dx, \tag{62}$$

which is equivalent to saying that

$$\begin{aligned} \int_{\{|u_\epsilon^\sigma - v| \leq k\}} a(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) \nabla u_\epsilon^\sigma\, dx - & \int_{\{|u_\epsilon^\sigma - v| \leq k\}} a(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) \nabla v\, dx \\ &+ \int_\Omega g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) T_k(u_\epsilon^\sigma - v)\, dx \\ &\leq \int_\Omega f_\epsilon T_k(u_\epsilon^\sigma - v)\, dx. \end{aligned} \tag{63}$$

By Fatou's lemma and the fact that

$$a(x, T_{k+\|v\|_\infty}(u_\epsilon^\sigma), \nabla T_{k+\|v\|_\infty}(u_\epsilon^\sigma)) \rightharpoonup a(x, T_{k+\|v\|_\infty}(u^\sigma), \nabla T_{k+\|v\|_\infty}(u^\sigma)) \text{ in } (L^{p'(x)}(\Omega))^N,$$

we get

$$\begin{aligned}\int_{\{|u^\sigma - v| \le k\}} a(x, u^\sigma, \nabla u^\sigma)\nabla u^\sigma \, dx - \int_{\{|u^\sigma - v| \le k\}} & a(x, T_{k+\|v\|_\infty}(u^\sigma), \nabla T_{k+\|v\|_\infty}(u^\sigma))\nabla v \, dx \\ &+ \int_\Omega g^\sigma(x, u^\sigma, \nabla u^\sigma) T_k(u^\sigma - v) \, dx \\ &\le \int_\Omega f T_k(u^\sigma - v) \, dx.\end{aligned} \tag{64}$$

Consequently,

$$\begin{aligned}&\int_\Omega a(x, u^\sigma, \nabla u^\sigma)\nabla T_k(u^\sigma - v) \, dx + \int_\Omega g^\sigma(x, u^\sigma, \nabla u^\sigma) T_k(u^\sigma - v) \, dx \\ &\le \int_\Omega f T_k(u^\sigma - v) \, dx, \qquad \forall v \in K_0 \cap L^\infty(\Omega) \text{ and } \forall k > 0.\end{aligned} \tag{65}$$

4.2 Study of the problem with respect to σ

4.2.1 Estimates with respect to σ

We are going to give some estimates on the sequence $(u^\sigma)_{\sigma>0}$ identical to (27). For that, we take $v = T_s(u^\sigma - T_k(u^\sigma))$ in (65) and we let $s \to \infty$; then, by the same argument as in section 4.1 we can prove that

$$\alpha \|\nabla T_k(u^\sigma)\|_{p(x)}^\gamma \le \alpha \int_\Omega |\nabla T_k(u^\sigma)|^{p(x)} \, dx \le k\|f\|_{L^1(\Omega)} \quad \text{for all} \quad k > 1. \tag{66}$$

Thus, as in section 4.1.2, there exists u such that $T_k(u) \in W_0^{1,p(x)}(\Omega)$ and

$$\begin{cases} T_k(u^\sigma) \rightharpoonup T_k(u) & \text{in } W_0^{1,p(x)}(\Omega), \\ T_k(u^\sigma) \to T_k(u) & \text{in } \ L^{p(x)}(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{67}$$

So, $u^\sigma \ge 0$ a.e. in Ω and we have also $u \ge 0$ a.e. in Ω.

4.2.2 Strong convergence of truncation with respect to σ

Here, in (65) we shall use the test function

$$\begin{cases} v &= T_s(u^\sigma - \varphi_k(\omega^{h,\sigma})), \\ \omega^{h,\sigma} &= T_{2k}\big(u^\sigma - T_h(u^\sigma) + T_k(u^\sigma) - T_k(u)\big), \\ \omega^h &= T_{2k}(u - T_h(u)), \end{cases} \tag{68}$$

where $h > 2k > 0$. It follows that for all $l > 0$,

$$\begin{aligned}&\int_\Omega a(x, u^\sigma, \nabla u^\sigma)\nabla T_l(u^\sigma - T_s(u^\sigma - \varphi_k(\omega^{h,\sigma}))) \, dx \\ &+ \int_\Omega g^\sigma(x, u^\sigma, \nabla u^\sigma) T_l(u^\sigma - T_s(u^\sigma - \varphi_k(\omega^{h,\sigma}))) \, dx \\ &\le \int_\Omega f T_l(u^\sigma - T_s(u^\sigma - \varphi_k(\omega^{h,\sigma}))) \, dx.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\{|u^\sigma-\varphi_k(\omega^{h,\sigma})|\le s\}} a(x,u^\sigma,\nabla u^\sigma)\nabla T_l(\varphi_k(\omega^{h,\sigma}))\,dx\\&+\int_\Omega g^\sigma(x,u^\sigma,\nabla u^\sigma)T_l(u^\sigma-T_s(u^\sigma-\varphi_k(\omega^{h,\sigma})))\,dx\\&\le\int_\Omega fT_l(u^\sigma-T_s(u^\sigma-\varphi_k(\omega^{h,\sigma})))\,dx.\end{aligned}$$

Letting $s\to\infty$ and choosing l large enough ($l\ge|\varphi_k(2k)|$), we deduce that

$$\int_\Omega a(x,u^\sigma,\nabla u^\sigma)\nabla\varphi_k(\omega^{h,\sigma})dx+\int_\Omega g^\sigma(x,u^\sigma,\nabla u^\sigma)\varphi_k(\omega^{h,\sigma})dx\le\int_\Omega f\varphi_k(\omega^{h,\sigma})dx. \tag{69}$$

Then, by using the same techniques as in section 4.1.4 we can deduce that

$$T_k(u^\sigma)\to T_k(u) \text{ in } W_0^{1,p(x)}(\Omega) \;\text{ and }\; \nabla u^\sigma\to\nabla u \text{ a.e. in } \Omega. \tag{70}$$

4.2.3 Equi-integrability of the nonlinearity g with respect to σ

Moreover, since g is a Carathéodory function, it is easy to see that

$$g(x,u^\sigma,\nabla u^\sigma)\to g(x,u,\nabla u) \;\text{ a.e. in }\; \Omega \;\text{ as }\; \sigma\to0.$$

Then, by assumption (6) (note that this hypothesis is only used here), it is clear that $g^\sigma(x,u^\sigma,\nabla u^\sigma)=\delta_\sigma g(x,u^\sigma,\nabla u^\sigma)\to g(x,u,\nabla u)$ a.e. in $\{x\in\Omega,u(x)\ge0\}$.

Similarly, we claim that $g^\sigma(x,u^\sigma,\nabla u^\sigma)\to g(x,u,\nabla u)$ in $L^1(\Omega)$.
Indeed, taking $u^\sigma-T_1(u^\sigma-T_l(u^\sigma))\ge0$ as test function in (65), we obtain

$$\int_{\{|u^\sigma|\ge l+1\}}|g^\sigma(x,u^\sigma,\nabla u^\sigma)|\,dx\le\int_{\{|u^\sigma|>l\}}|f|\,dx.$$

Let $\beta>0$ be fixed. Then, there exists $l(\beta)\ge1$ such that

$$\int_{\{|u^\sigma|\ge l(\beta)\}}|g^\sigma(x,u^\sigma,\nabla u^\sigma)|\,dx<\frac{\beta}{2}. \tag{71}$$

For any measurable subset $E\subset\Omega$, we have

$$\begin{aligned}\int_E|g^\sigma(x,u^\sigma,\nabla u^\sigma)|\,dx\le&\int_E b(l(\beta))\big(c(x)+|\nabla T_{l(\beta)}(u^\sigma)|^{p(x)}\big)\,dx\\&+\int_{\{|u^\sigma|\ge l(\beta)\}}|g^\sigma(x,u^\sigma,\nabla u^\sigma)|\,dx.\end{aligned} \tag{72}$$

In view of (70), there exists $\alpha(\beta)>0$ such that

$$\int_E b(l(\beta))\big(c(x)+|\nabla T_{l(\beta)}(u^\sigma)|^{p(x)}\big)\,dx\le\frac{\beta}{2}\quad\text{for all } E \text{ such that } \operatorname{meas}(E)<\alpha(\beta). \tag{73}$$

Finally, by combining (71) and (73), one easily has

$$\int_E|g^\sigma(x,u^\sigma,\nabla u^\sigma)|\,dx\le\beta\quad\text{ for all } E \text{ such that } \operatorname{meas}(E)\le\alpha(\beta).$$

Therefore, we deduce that $g^\sigma(x,u^\sigma,\nabla u^\sigma)$ is uniformly equi-integrable in Ω. So, as in section 4.1.6, we can pass to the limit in σ and conclude. This achieves the proof of Theorem 4.1.

5 Case when the Nonlinearity g is Negative

We consider the convex set $\overline{K}_0 = \{u \in W_0^{1,p(x)}(\Omega); u \leq 0 \text{ a.e. in } \Omega\}$.

Theorem 5.1 *Assume that* $(2) - (6)$ *hold true and that* $f \in L^1(\Omega)$. *Then, there exists at least one solution (entropy solution) to the following unilateral problem,*

$$(\mathcal{P}) \begin{cases} u \in \mathcal{T}_0^{1,p(x)}(\Omega), \ u \leq 0 \ a.e. \ in \ \Omega, \ g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \leq \int_\Omega fT_k(u-v)\,dx, \\ \forall v \in \overline{K}_0 \cap L^\infty(\Omega), \quad \forall k > 0. \end{cases}$$

Proof. The same proof as for Theorem 4.1 can be applied with the following changes:
i) We approach the sign function by an increasing Lipschitz function.
ii) The Lipschitz function $\delta_\sigma(s)$ is replaced by:

$$\overline{\delta}_\sigma(s) = \begin{cases} \frac{-s+\sigma}{s}, & \text{if } s \geq \sigma > 0, \\ 0, & \text{if } |s| \leq \sigma, \\ \frac{s+\sigma}{s}, & \text{if } s < -\sigma < 0. \end{cases}$$

iii) The approximated problem becomes:

$$(\overline{\mathcal{P}}_\epsilon^\sigma) \begin{cases} u_\epsilon^\sigma \in W_0^{1,p(x)}(\Omega) \\ \displaystyle\langle Au_\epsilon^\sigma, u_\epsilon^\sigma - v\rangle + \int_\Omega g_\epsilon^\sigma(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)(u_\epsilon^\sigma - v)\,dx + \frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma^+})|^{p(x)-1}(u_\epsilon^\sigma - v)\,dx \\ \displaystyle\qquad = \int_\Omega f_\epsilon(u_\epsilon^\sigma - v)\,dx, \quad \forall v \in W_0^{1,p(x)}(\Omega). \end{cases} \tag{74}$$

iv) The set K_0 is replaced by $\overline{K}_0 = \{u \in W_0^{1,p(x)}(\Omega); u \leq 0 \text{ a.e. in } \Omega\}$.

References

[1] Azroul, E., Benboubker, M.B., Rhoudaf, M. Entropy solution for some $p(x)$-Quasilinear problems with right-hand side measure. *Afr. diaspora J. Math.* **13** (2) (2012) 23–44.

[2] Azroul, E., Benboubker, M.B., Rhoudaf, M. On some $p(x)$-quasilinear problem with right-hand side measure. *Math. Comput. Simulation* **102** (2014) 117–130.

[3] Azroul, E., Benkirane, A. and Rhoudaf, M. On some strongly nonlinear elliptic problems in L^1-data with a nonlinearity having a constant sign in Orlicz Spaces via penalization methods. *Aust. J. Math. Anal. Appl.* **7** (5) (2010) 1–25.

[4] Benboubker, M.B., Azroul, E., and Barbara, A. Quasilinear elliptic problems with non-standard growth. *Electronic J. Diff. Equ.* 62 (2011) 1–16.

[5] Bénilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M. and Vazquez, J.L. An L^1 theory of existence and uniqueness of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **22** (1995) 241–273.

[6] Boccardo, L., Murat, F. and Puel, J.P. Existence of Bounded Solutions for non linear elliptic unilateral problems. *Ann-Mat. Pura Appl.* (4) 152 (1988) 183–196.

[7] Diening, L. Theoretical and Numerical Results for Electrorheological Fluids. Ph.D. thesis, University of Freiburg, Germany, 2002.

[8] Diening, L., Harjulehto, P., Hästö, P. and *Ružička*, M. *Lebesgue and Sobolev Spaces with Variable Exponents.* Lecture Notes in Mathematics, vol. **2017**, Springer-Verlag, Heidelberg, 2011.

[9] Fan, X.L. and Zhao, D. On the generalized Orlicz-Sobolev Space $W^{k,p(x)}(\Omega)$. *J. Gansu Educ. College* **12** (1) (1998) 1–6.

[10] *Kováčik*, O. and *Rákosník*, J. On spaces $L^{p(x)}$ and $W^{1,p(x)}$. *Czechoslovak Math, J.* **41** (116) (1991) 592–618.

[11] Liang, S. and Zhang, J. Existence of positive solutions for the p-Laplacian with nonlinear boundary conditions. *Nonlinear Dynamics and Systems Theory* **12** (1) (2012) 97–103.

[12] Lions, J.L. *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires.* Dunod et Gauthiers-Villars, Paris (1969).

[13] Musielak, J. *Orlicz Spaces and Modular Spaces.* Lecture Notes in Mathematics, vol. 1034, Springer, Berlin, 1983.

[14] Nakano, H. *Modulared Semi-Ordered Linear Spaces.* Maruzen Co., Ltd., Tokyo, 1950.

[15] Ouaro, S. and Traoré, S. Entropy solutions to the obstacle problem for nonlinear elliptic problems with variable growth and L^1-data. *Pac. J. Optim.* **5** (1) (2009) 127–141.

[16] Orlicz, W. *Über konjugierte Exponentenfolgen. Studia Math.* **3** (1931) 200–212.

[17] Rodrigues, J.F., Sanchón, M. and Urbano, J.M. The obstacle problem for nonlinear elliptic equations with variable growth and L^1-data. *Monatsh. Math.* **154** (2008) 303–322.

[18] *Ružička*, M. *Electrorheological Fluids: Modeling and Mathematical Theory.* Lecture Notes in Mathematics **1748**, Springer-verlaag, Berlin, 2000.

[19] Tsenov, I. Generalization of the problem of best approximation of a function in the space L^s. *Uch. Zap. Dagestan Gos. Univ.* **7** (1961) 25–37.

[20] Zhao, D. and Fan, X.L. On the Nemytskii operators from $L^{p_1(x)}$ to $L^{p_2(x)}$. *J. Lanzhou Univ.* **34** (1) (1998) 1–5.

[21] Zhao, D., Qiang, W.J. and Fan, X.L. On generalized Orlicz spaces $L^{p(x)}(\Omega)$. *J. Gansu Sci.* **9** (2) (1997) 1–7.

[22] Zhikov, V.V. Averaging of functionals of the calculus of variations and elasticity theory. *Math. USSR Izvestiya* **29** (1) (1987) 33–66.

[23] Zhikov, V.V. On passing to the limit in nonlinear variational problem. *Math. Sb.* **183** (1992) 47–84.

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Perturbed Partial Fractional Order Functional Differential Equations with Infinite Delay in Fréchet Spaces

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Abstract: In this paper we investigate the existence of solutions of perturbed partial hyperbolic differential equations of fractional order with infinite delay and Caputo's fractional derivative by using a nonlinear alternative of Avramescu on Fréchet spaces.

Keywords: *partial functional differential equation; fractional order; solution; left-sided mixed Riemann-Liouville integral; Caputo fractional-order derivative; infinite delay; Fréchet space; fixed point.*

Mathematics Subject Classification (2010): 26A33, 34K30, 34K37, 35R11.

1 Introduction

In this paper we are concerned with the existence of solutions to fractional order initial value problem (IVP for short), for the system

$$({}^cD_0^r u)(t,x) = f(t,x,u_{(t,x)}) + g(t,x,u_{(t,x)}), \text{ if } (t,x) \in J, \tag{1}$$

$$u(t,x) = \phi(t,x), \text{ if } (t,x) \in \tilde{J}, \tag{2}$$

$$u(t,0) = \varphi(t), \ u(0,x) = \psi(x), \ (t,x) \in J, \tag{3}$$

where $\varphi(0) = \psi(0)$, $J := [0,\infty) \times [0,\infty)$, $\tilde{J} := (-\infty,+\infty) \times (-\infty,+\infty) \backslash [0,\infty) \times [0,\infty)$, ${}^cD_0^r$ is the standard Caputo's fractional derivative of order $r = (r_1, r_2) \in$

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$(0,1] \times (0,1], f, g : J \times \mathcal{B} \Rightarrow \mathbb{R}^n$ are given functions, $\phi : \tilde{J} \to \mathbb{R}^n$ is a given continuous function with $\phi(t,0) = \varphi(t),\ \phi(0,x) = \psi(x)$ for each $(t,x) \in J, \varphi : [0,\infty) \to \mathbb{R}^n$, $\psi : [0,\infty) \to \mathbb{R}^n$ are given absolutely continuous functions and $\mathcal{B}$ is called a phase space that will be specified in Section 3.

We denote by $u_{(t,x)}$ the element of $\mathcal{B}$ defined by

$$u_{(t,x)}(s,\tau) = u(t+s, x+\tau);\ (s,\tau) \in (-\infty,0] \times (-\infty,0],$$

here $u_{(t,x)}(.,.)$ represents the history of the state u.

There has been a significant development in ordinary and partial fractional differential equations in recent years. We can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [1–5]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs [6–8], and the papers [9–15] and the references therein.

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. Differential delay equations, or functional differential equations, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books [16–20], and the papers [21, 22].

In this paper, we present existence result for the problem (1)-(3). Our main result for this problem is based on a nonlinear alternative for the sum of a completely continuous operator and a contraction one in Fréchet spaces due to Avramescu [23]. To our knowledge, there are very few papers devoted to fractional differential equations with delay on Fréchet spaces. This paper can be considered as a contribution in this setting case.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $p \in \mathbb{N}$ and $J_0 = [0,p] \times [0,p]$. By $C(J_0, \mathbb{R})$ we denote the Banach space of all continuous functions from J_0 into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(t,x)\in J_0} \|w(t,x)\|,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$.
As usual, by $AC(J_0, \mathbb{R})$ we denote the space of absolutely continuous functions from J_0 into $\mathbb{R}^n$ and $L^1(J_0, \mathbb{R})$ is the space of Lebesgue-integrable functions $w : J_0 \to \mathbb{R}^n$ with the norm

$$\|w\|_{L^1} = \int_0^p \int_0^p \|w(t,x)\| dt dx.$$

Definition 2.1 [24] Let $r = (r_1, r_2) \in (0,\infty) \times (0,\infty), \theta = (0,0)$ and $u \in L^1(J_0, \mathbb{R}^n)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} u(s,\tau) d\tau ds.$$

In particular,

$$(I_\theta^\theta u)(t,x) = u(t,x),\ (I_\theta^\sigma u)(t,x) = \int_0^t \int_0^x u(s,\tau) d\tau ds;\ \text{for almost all } (t,x) \in J_0,$$

where $\sigma = (1,1)$. For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)\times(0,\infty)$, when $u \in L^1(J_0, \mathbb{R}^n)$. Note also that when $u \in C(J_0,\mathbb{R}^n)$, then $(I_\theta^r u) \in C(J_0,\mathbb{R}^n)$, moreover

$$(I_\theta^r u)(t,0) = (I_\theta^r u)(0,x) = 0;\ t,x \in J_0.$$

Example 2.1 Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,\infty)\times(0,\infty)$, then

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2},\ \text{for almost all } (t,x)\in J_0.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1)\times[0,1)$. Denote by $D_{tx}^2 := \frac{\partial^2}{\partial t \partial x}$, the mixed second order partial derivative.

Definition 2.2 [24] Let $r \in (0,1]\times(0,1]$ and $u \in L^1(J_0,\mathbb{R}^n)$. The mixed fractional Riemann-Liouville derivative of order r of u is defined by the expression

$$D_\theta^r u(t,x) = (D_{tx}^2 I_\theta^{1-r} u)(t,x)$$

and the Caputo fractional-order derivative of order r of u is defined by the expression

$$({}^cD_0^r u)(t,x) = (I_\theta^{1-r} \frac{\partial^2}{\partial t \partial x} u)(t,x).$$

The case $\sigma = (1,1)$ is included and we have

$$(D_\theta^\sigma u)(t,x) = ({}^cD_\theta^\sigma u)(t,x) = (D_{tx}^2 u)(t,x),\ \text{for almost all } (t,x)\in J_0.$$

Example 2.2 Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,1]\times(0,1]$, then

$$D_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1} x^{\omega-r_2},\ \text{for almost all } (t,x)\in J_0.$$

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 2.1 *[25] Let $\upsilon : J \to [0,\infty)$ be a real function and $\omega(.,.)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that*

$$\upsilon(t,x) \le \omega(t,x) + c\int_0^t \int_0^x \frac{\upsilon(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}} d\tau ds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\upsilon(t,x) \le \omega(t,x) + \delta c\int_0^t \int_0^x \frac{\omega(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}} d\tau ds,$$

for every $(t,x) \in J$.

3 The Phase Space $\mathcal{B}$

The notation of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato (see [22]). For further applications see for instance the books [16, 17, 19] and their references.

For any $(t, x) \in J$ denote $E_{(t,x)} := [0, t] \times \{0\} \cup \{0\} \times [0, x]$, furthermore in case $t = a,\ x = b$ we write simply E. Consider the space $(\mathcal{B}, \|(.,.)\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0] \times (-\infty, 0]$ into $\mathbb{R}^n$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

(A_1) If $y : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n$ continuous on J and $y_{(t,x)} \in \mathcal{B}$, for all $(t, x) \in E$, then there are constants $H, K, M > 0$ such that for any $(t, x) \in J$ the following conditions hold:

(i) $y_{(t,x)}$ is in $\mathcal{B}$;

(ii) $\|y(t, x)\| \leq H\|y_{(t,x)}\|_{\mathcal{B}}$,

(iii) $\|y_{(t,x)}\|_{\mathcal{B}} \leq K \sup_{(s,\tau)\in[0,t]\times[0,x]} \|y(s, \tau)\| + M \sup_{(s,\tau)\in E_{(t,x)}} \|y_{(s,\tau)}\|_{\mathcal{B}}$,

(A_2) For the function $y(.,.)$ in (A_1), $y_{(t,x)}$ is a $\mathcal{B}$-valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Now, we present some examples of phase spaces [26, 27].

Example 3.1 Let $\mathcal{B}$ be the set of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0],\ \alpha, \beta \geq 0$, with the seminorm

$$\|\phi\|_{\mathcal{B}} = \sup_{(s,\tau)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s, \tau)\|.$$

Then we have $H = K = M = 1$. The quotient space $\widehat{\mathcal{B}} = \mathcal{B}/\|.\|_{\mathcal{B}}$ is isometric to the space $C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n)$ of all continuous functions from $[-\alpha, 0] \times [-\beta, 0]$ into $\mathbb{R}^n$ with the supremum norm, this means that partial differential functional equations with finite delay are included in our axiomatic model.

Example 3.2 Let $\gamma \in \mathbb{R}$ and let C_γ be the set of all continuous functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ for which a limit $\lim_{\|(s,\tau)\|\to\infty} e^{\gamma(s+\tau)}\phi(s, \tau)$ exists, with the norm

$$\|\phi\|_{C_\gamma} = \sup_{(s,\tau)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(s+\tau)}\|\phi(s, \tau)\|.$$

Then we have $H = 1$ and $K = M = \max\{e^{-\gamma(a+b)}, 1\}$.

Example 3.3 Let $\alpha, \beta, \gamma \geq 0$ and let

$$\|\phi\|_{CL_\gamma} = \sup_{(s,\tau)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s, \tau)\| + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\gamma(s+\tau)}\|\phi(s, \tau)\|d\tau ds$$

be the seminorm for the space CL_γ of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0]$ measurable on $(-\infty, -\alpha] \times (-\infty, 0] \cup (-\infty, 0] \times (-\infty, -\beta]$, and such that $\|\phi\|_{CL_\gamma} < \infty$. Then

$$H = 1, \ K = \int_{-\alpha}^{0} \int_{-\beta}^{0} e^{\gamma(s+\tau)} d\tau ds, \ M = 2.$$

4 Some Properties in Fréchet Spaces

Let X be a Fréchet space with a family of semi-norms $\{\| \cdot \|_n\}_{n \in \mathbb{N}}$. We assume that the family of semi-norms $\{\| \cdot \|_n\}$ verifies :

$$\|u\|_1 \leq \|u\|_2 \leq \ \|u\|_3 \leq \ldots \quad for\ every\ u \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leq \overline{M}_n \quad for\ all\ y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows : For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by : $u \sim_n v$ if and only if $\|u - v\|_n = 0$ for $u, v \in X$. We denote by $X^n = (X|_{\sim_n}, \| \cdot \|_n)$ the quotient space, the completion of X^n with respect to $\| \cdot \|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $u \in X$, we denote by $[u]_n$ the equivalence class of u of subset X^n and we define $Y^n = \{[u]_n : u \in Y\}$. We denote by $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\| \cdot \|_n$ in X^n. For more information about this subject see [28].

Definition 4.1 Let X be a Fréchet space. A function $N : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in [0, 1)$ such that

$$\|N(u) - N(v)\|_n \leq k_n \|u - v\|_n \ \text{ for all } u, v \in X.$$

Theorem 4.1 *(Nonlinear Alternative of Avramescu) [23] Let $(X, |.|_n)$ be a Fréchet space and let $A, B : X \to X$ be two operators. Suppose that the following hypotheses are fulfilled:*

- (i) *A is a compact operator;*
- (ii) *B is a contraction operator with respect to a family of seminorms $||.||_n$ equivalent to the family $|.|_n$;*
- (iii) *the set $\mathcal{E} = \{u \in X : u = \lambda A(u) + \lambda B(\frac{u}{\lambda})$ for some $\lambda \in (0, 1)\}$ is bounded.*

Then there is $u \in X$ such that $u = Au + Bu$.

5 Existence of Solutions

In this section, we give our main existence result for problem (1)-(3). Before starting and proving this result, we give what we mean by a solution of this problem. Let the space

$$\Omega := \{u : \mathbb{R}^2 \to \mathbb{R}^n : u_{(t,x)} \in \mathcal{B} \text{ for } (t, x) \in E \text{ and } u|_J \in C(J, \mathbb{R}^n)\}.$$

Definition 5.1 A function $u \in \Omega$ is said to be a solution of (1)-(3) if u satisfies equations (1) and (3) on J and the condition (2) on $\tilde{J}$.

For the existence of solutions for the problem (1)–(3), we need the following lemma:

Lemma 5.1 *A function $u \in \Omega$ is a solution of problem (1)-(3) if and only if u satisfies the equation*

$$\begin{aligned} u(t,x) &= z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,u_{(s,\tau)})d\tau ds \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g(s,\tau,u_{(s,\tau)})d\tau ds, \end{aligned}$$

for all $(t,x) \in J$ and the condition (2) on $\tilde{J}$.

For each $p \in \mathbb{N}$ we consider following sets,

$$C_p = \{u : (-\infty,p] \times (-\infty,p] \to \mathbb{R}^n : u_{(t,x)} \in \mathcal{B},\ \ u_{(t,x)} = 0 \text{ for } (t,x) \in E \text{ and } u|_{J_0} \in C(J_0,\mathbb{R}^n)\},$$

and $C_0 = \{u \in \Omega : u_{(t,x)} = 0 \text{ for } (t,x) \in E\}$.

On C_0 we define the semi-norms:

$$\|u\|_p = \sup_{(t,x)\in E} \|u_{(t,x)}\| + \sup_{(t,x)\in J_0} \|u(t,x)\| = \sup_{(t,x)\in J_0} \|u(t,x)\|, \quad u \in C_p.$$

Then C_0 is a Fréchet space with the family of semi-norms $\{\|u\|_p\}$.

Theorem 5.1 *Assume:*

$(H1)$ *The functions $f, g : J \times \mathcal{B} \to \mathbb{R}^n$ are continuous.*

$(H2)$ *For each $p \in \mathbb{N}$, there exist constants $\ell_p(t,x) \in C(J_0,\mathbb{R}^n)$ such that*

$$\|g(t,x,u) - g(t,x,v)\| \le \ell_p(t,x)\|u-v\|_{\mathcal{B}}, \text{ for any } u,v \in \mathcal{B} \text{ and } (t,x) \in J_0.$$

$(H3)$ *For each $p \in \mathbb{N}$, there exist $p,q \in C(J,\mathbb{R}_+)$ such that*

$$\|f(t,x,u)\| \le p(t,x) + q(t,x)\|u\|_{\mathcal{B}}, \text{ for } (t,x) \in J_0 \text{ and each } u \in \mathcal{B}.$$

If

$$\frac{K\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{4}$$

where $\ell_p^* = \sup_{(t,x)\in J_0} \ell_p(t,x)$, then there exists a unique solution for IVP (1)-(3) on $(-\infty,+\infty) \times (-\infty,+\infty)$.

Proof. Transform the problem (1)-(3) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by,

$$(Nu)(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) & \\ +\dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,u_{(s,\tau)})d\tau ds & \\ +\dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g(s,\tau,u_{(s,\tau)})d\tau ds, & (t,x) \in J. \end{cases} \tag{5}$$

Let $v(.,.) : \mathbb{R}^n \to \mathbb{R}^n$ be a function defined by,

$$v(t,x) = \begin{cases} z(t,x), & (t,x) \in J. \\ \phi(t,x), & (t,x) \in \tilde{J}, \end{cases}$$

Then $v_{(t,x)} = \phi$ for all $(t,x) \in E$.

For each $w \in C(J, \mathbb{R}^n)$ with $w(t,x) = 0$ for each $(t,x) \in E$ we denote by $\overline{w}$ the function defined by

$$\overline{w}(t,x) = \begin{cases} w(t,x), & (t,x) \in J, \\ 0, & (t,x) \in \tilde{J}. \end{cases}$$

If $u(.,.)$ satisfies the integral equation,

$$\begin{aligned} u(t,x) &= z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,u_{(s,\tau)})d\tau ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g(s,\tau,u_{(s,\tau)})d\tau ds, \end{aligned}$$

we can decompose $u(.,.)$ as $u(t,x) = \overline{w}(t,x) + v(t,x);\ (t,x) \in J$, which implies $u_{(t,x)} = \overline{w}_{(t,x)} + v_{(t,x)}$, for every $(t,x) \in J$, and the function $w(.,.)$ satisfies

$$\begin{aligned} w(t,x) &= \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,\overline{w}_{(t,x)} + v_{(t,x)})d\tau ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g(s,\tau,\overline{w}_{(t,x)} + v_{(t,x)})d\tau ds. \end{aligned}$$

Let the operators $A, B : C_0 \to C_0$ be defined by

$$(Aw)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,\overline{w}_{(t,x)} + v_{(t,x)})d\tau ds$$

and

$$(Bw)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g(s,\tau,\overline{w}_{(t,x)} + v_{(t,x)})d\tau ds.$$

Obviously, the operator N has a fixed point which is equivalent to finding the fixed point of the operator equation $(Aw)(t,x) + (Bw)(t,x) = w(t,x), (t,x) \in J$. We shall show that the operators A and B satisfy all the conditions of Theorem 4.1.

For better readability, we break the proof into a sequence of steps.

Step 1: A is continuous.

Let $\{w_n\}$ be a sequence such that $w_n \to w$ in C_0. Then

$$\begin{aligned} \|(Aw_n)(t,x) - (Aw)(t,x)\| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\times \|f(s,\tau,\overline{w}_{n(s,\tau)} + v_{n(s,\tau)}) - f(s,\tau,\overline{w}_{(s,\tau)} + v_{(s,\tau)})\| d\tau ds. \end{aligned}$$

Since f is a continuous function, we have

$$\|(Aw_n) - (Aw)\|_p \leq \frac{p^{r_1+r_2}\|f(.,.,\overline{w_n}_{(.,.)} + v_{n(.,.)}) - f(.,.,\overline{w}_{(.,.)} + v_{(.,.)})\|_p}{\Gamma(r_1+1)\Gamma(r_2+1)} \to 0 \text{ as } n \to \infty.$$

Thus A is continuous.

Step 2: A maps bounded sets into bounded sets in C_0. Indeed, it is enough to show that, for any $\eta > 0$, there exists a positive constant $\widetilde{\ell}$ such that, for each $w \in B_\eta = \{w \in C_0 : \|w\|_p \le \eta\}$, we have $\|A(w)\|_p \le \widetilde{\ell}$.

Let $w \in B_\eta$. By $(H3)$ we have for each $(t,x) \in J_0$,

$$\begin{aligned}
\|(Aw)(t,x)\| &\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|f(s,\tau,\overline{w}_{(s,\tau)}+v_{(s,\tau)})\|d\tau ds\\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}p(s,\tau)\\
&\quad +q(s,\tau)\|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}}d\tau ds\\
&\le \frac{\|p\|_p}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds\\
&\quad +\frac{\|q\|_p\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds\\
&\le \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}p^{r_1+r_2} := \ell^*,
\end{aligned}$$

where

$$\|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}} \le \|\overline{w}_{(s,\tau)}\|_{\mathcal{B}}+\|v_{(s,\tau)}\|_{\mathcal{B}} \le K_p\eta + K_p\|\phi(0,0)\| + M_p\|\phi\|_{\mathcal{B}} := \eta^*.$$

Hence $\|(Aw)\|_p \le \ell^*$.

Step 3: A maps bounded sets into equicontinuous sets in C_0.
Let $(t_1,x_1),(t_2,x_2) \in J_0$, $t_1 < t_2$, $x_1 < x_2$, B_η be a bounded set as in Step 2, and let $w \in B_\eta$. Then

$$\begin{aligned}
&\|(Aw)(t_2,x_2)-(Aw)(t_1,x_1)\| \le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_0^{x_1}[(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}\\
&-(t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}]||f(s,\tau,\overline{w}_{(s,\tau)}+v_{(s,\tau)})||d\tau ds\\
+\;& \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau,\overline{w}_{(s,\tau)}+v_{(s,\tau)})||d\tau ds\\
+\;& \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau,\overline{w}_{(s,\tau)}+v_{(s,\tau)})||d\tau ds\\
+\;& \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_0^{x_1}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau,\overline{w}_{(s,\tau)}+v_{(s,\tau)})||d\tau ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_0^{x_1}[(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}-(t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}]d\tau ds\\
&+ \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}d\tau ds\\
&+ \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}d\tau ds\\
&+ \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_0^{x_1}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}d\tau ds\\
&\leq \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}[x_2^{r_2}(t_2-t_1)^{r_1}+t_2^{r_1}(x_2-x_1)^{r_2}\\
&\quad -(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}+t_1^{r_1}x_1^{r_2}-t_2^{r_1}x_2^{r_2}]\\
&+ \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}\\
&+ \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}[t_2^{r_1}-(t_2-t_1)^{r_1}](x_2-x_1)^{r_2}\\
&+ \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}(t_2-t_1)^{r_1}[x_2^{r_2}-(x_2-x_1)^{r_2-1}\\
&\leq \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}[2x_2^{r_2}(t_2-t_1)^{r_1}+2t_2^{r_1}(x_2-x_1)^{r_2}\\
&\quad +t_1^{r_1}x_1^{r_2}-t_2^{r_1}x_2^{r_2}-2(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}].
\end{aligned}
$$

The right-hand side of the above inequality tends to zero as $t_1 \to t_2,\ x_1 \to x_2$. The equicontinuity for the cases $t_1 < t_2 < 0,\ x_1 < x_2 < 0$ and $t_1 \leq 0 \leq t_2,\ x_1 \leq 0 \leq x_2$ is obvious.

As a consequence of steps 1 to 3 together with Arzela-Ascoli theorem, we can conclude that $A : C_0 \to C_0$ is a compact operator.

Step 4: B is a contraction.
Let $w, w^* \in C_0$. Then we have for each $(t,x) \in J_0$

$$
\begin{aligned}
\|(Bw)(t,x) \;-\;& (Bw^*)(t,x)\| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}\\
\times\;& \|g(s,\tau,\overline{w}_{(s,\tau)}+v_{(s,\tau)})-g(s,\tau,\overline{w^*}_{(s,\tau)}+v_{(s,\tau)})\|d\tau ds\\
\leq\;& \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_p(s,\tau)\|\overline{w}_{(s,\tau)}-\overline{w^*}_{(s,\tau)}\|_{\mathcal{B}}\\
\leq\;& \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}K\ell_p(s,\tau)\\
\times\;& \sup_{(s,\tau)\in[0,t]\times[0,x]}\|\overline{w}(s,\tau)-\overline{w^*}(s,\tau)\|d\tau ds\\
\leq\;& \frac{K\ell_p^*(s,\tau)}{\Gamma(r_1)\Gamma(r_2)}\int_0^p\int_0^p(t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds\|\overline{w}-\overline{w^*}\|_p.
\end{aligned}
$$

Therefore,

$$\|(Bw)-(Bw^*)\|_p \leq \frac{K\ell_p^*p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\|\overline{w}-\overline{w^*}\|_p,$$

since by (4), B is a contraction.

Step 5: (A priori bounds)

Now it remains to show that the set

$$\mathcal{E}=\{w \in C(J,\mathbb{R}) : w = \lambda A(w) + \lambda B(\tfrac{w}{\lambda}) \text{ for some } \lambda \in (0,1)\}$$

is bounded. Let $w \in \mathcal{E}$, then $w = \lambda A(w) + \lambda B(\frac{w}{\lambda})$ for some $0 < \lambda < 1$. Thus for each $(t,x) \in J_0$, we have

$$\begin{aligned} w(t,x) &= \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,\overline{w}_{(s,\tau)} + v_{(s,\tau)})d\tau ds \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g(s,\tau,\frac{\overline{w}_{(s,\tau)} + v_{(s,\tau)}}{\lambda})d\tau ds. \end{aligned}$$

This implies by $(H2)$ and $(H3)$ that, for each $(t,x) \in J_0$, we have

$$\begin{aligned} \|w(t,x)\| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}[p(s,\tau) \\ &\quad +q(s,\tau)\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\|_{\mathcal{B}}]d\tau ds \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad \left|g(s,\tau,\frac{\overline{w}_{(s,\tau)} + v_{(s,\tau)}}{\lambda}) - g(s,\tau,0)\right| d\tau ds \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}|g(s,\tau,0)|d\tau ds \\ &\leq \frac{p^{r_1+r_2}\|p\|_p}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{p^{r_1+r_2}g^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &+ \frac{||q||_p}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\|_{\mathcal{B}}d\tau ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_p(s,\tau)||\overline{w}_{(s,\tau)} + v_{(s,\tau)}||_{\mathcal{B}}d\tau ds \\ &\leq \frac{p^{r_1+r_2}(\|p\|_p + g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &+ \frac{(||q||_p + \ell_p^*)}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\|_{\mathcal{B}}d\tau ds, \end{aligned}$$

where $g^* = \sup_{(s,\tau)\in J_0} |g(s,\tau,0)|$ and

$$\begin{aligned} \|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\|_{\mathcal{B}} &\leq \|\overline{w}_{(s,\tau)}\|_{\mathcal{B}} + \|v_{(s,\tau)}\|_{\mathcal{B}} \\ &\leq K\sup\{w(\tilde{s},\tilde{\tau}) : (\tilde{s},\tilde{\tau}) \in [0,s]\times[0,\tau]\} \\ &\quad + M\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\|. \end{aligned} \tag{6}$$

If we name $y(s,\tau)$ the right hand side of (6), then we have

$$\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\|_{\mathcal{B}} \leq y(t,x),$$

and therefore, for each $(t,x) \in J_0$ we obtain

$$\|w(t,x)\| \leq \frac{p^{r_1+r_2}(\|p\|_p + g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}$$

$$+ \frac{||q||_p + \ell_p^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} y(s,\tau) d\tau ds. \tag{7}$$

Using the above inequality and the definition of y for each $(t,x) \in J_0$ we have

$$\begin{aligned} y(t,x) \quad &\leq \quad M\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\| + \frac{Kp^{r_1+r_2}(\|p\|_p + g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\quad + \frac{K(\|q\|_p + \ell_p^*)}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} y(s,t) d\tau ds. \end{aligned}$$

Then by Lemma 2.1, there exists $\delta = \delta(r_1, r_2)$ such that we have

$$\|y(t,x)\| \quad \leq \quad R + \delta \frac{K(\|q\|_p + \ell_p^*)}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} R d\tau ds,$$

where

$$R = M\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\| + \frac{Kp^{r_1+r_2}(\|p\|_p + g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

Hence

$$\|y\|_p \leq R + \frac{R\delta Kp^{r_1+r_2}(\|q\|_p + \ell_p^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} := \widetilde{R}.$$

Then, (7) implies that

$$\|w\|_p \leq \frac{p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}[\|p\|_p + g^* + \widetilde{R}(\|q\|_p + \ell_p^*)] := R_p^*.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 4.1 we deduce that $A + B$ has a fixed point which is a solution of problem (1)-(3). □

6 An Example

As an application of our results we consider the following partial perturbed hyperbolic functional differential equations of the form

$$({}^cD_0^r u)(t,x) = \frac{2 + e^{t+x}(|u(t-2,x-3)| + 3)}{c_p e^{t+x}(2 + |u(t-2,x-3)|)}, \text{ if } (t,x) \in J := [0,\infty) \times [0,\infty), \tag{8}$$

$$u(t,0) = t, \ u(0,x) = x^2, \ (t,x) \in J, \tag{9}$$

$$u(t,x) = t + x^2, \ (t,x) \in \tilde{J}, \tag{10}$$

where $\tilde{J} := \mathbb{R}^2 \backslash [0, \infty) \times [0, \infty)$.
Set

$$f(t, x, u_{(t,x)}) = \frac{|u(t-2, x-3)| + 3}{c_p(2 + |u(t-2, x-3)|)}, \quad \text{if } (t, x) \in J,$$

$$g(t, x, u_{(t,x)}) = \frac{2}{c_p e^{t+x}(2 + |u(t-2, x-3)|)}, \quad \text{if } (t, x) \in J,$$

and

$$c_p = \frac{3p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

Let $\gamma > 0$, and consider the following phase space

$$\mathcal{B}_\gamma = \{u \in C((-\infty, 0] \times (-\infty, 0], \mathbb{R}) : \lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta+\eta)} u(\theta, \eta) \text{ exists} \in \mathbb{R}\}.$$

The norm of $\mathcal{B}_\gamma$ is given by

$$\|u\|_\gamma = \sup_{(\theta,\eta) \in (-\infty,0] \times (-\infty,0]} e^{\gamma(\theta+\eta)} |u(\theta, \eta)|.$$

Let

$$E := [0, 1] \times \{0\} \cup \{0\} \times [0, 1],$$

and $u : (-\infty, 1] \times (-\infty, 1] \to \mathbb{R}$ such that $u_{(t,x)} \in \mathcal{B}_\gamma$ for $(t, x) \in E$, then

$$\lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta+\eta)} u_{(t,x)}(\theta, \eta) = \lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta-t+\eta-x)} u(\theta, \eta)$$

$$= e^{\gamma(t+x)} \lim_{\|(\theta,\eta)\| \to \infty} u(\theta, \eta) < \infty.$$

Hence $u_{(t,x)} \in \mathcal{B}_\gamma$. Finally we prove that

$$\|u_{(t,x)}\|_\gamma = K \sup\{|u(s, \tau)| : (s, \tau) \in [0, t] \times [0, x]\}$$
$$+ M \sup\{\|u_{(s,\tau)}\|_\gamma : (s, \tau) \in E_{(t,x)}\},$$

where $K = M = 1$ and $H = 1$.

If $t + \theta \leq 0,\ x + \eta \leq 0$ we get

$$\|u_{(t,x)}\|_\gamma = \sup\{|u(s, \tau)| : (s, \tau) \in (-\infty, 0] \times (-\infty, 0]\},$$

and if $t + \theta \geq 0,\ x + \eta \geq 0$, then we have

$$\|u_{(t,x)}\|_\gamma = \sup\{|u(s, \tau)| : (s, \tau) \in [0, t] \times [0, x]\}.$$

Thus for all $(t + \theta, x + \eta) \in [0, 1] \times [0, 1]$, we get

$$\begin{aligned} \|u_{(t,x)}\|_\gamma &= \sup\{|u(s, \tau)| : (s, \tau) \in (-\infty, 0] \times (-\infty, 0]\} \\ &\quad + \sup\{|u(s, \tau)| : (s, \tau) \in [0, t] \times [0, x]\}. \end{aligned}$$

Then

$$\|u_{(t,x)}\|_\gamma = \sup\{\|u_{(s,\tau)}\|_\gamma : (s, \tau) \in E\} + \sup\{|u(s, \tau)| : (s, \tau) \in [0, t] \times [0, x]\}.$$

$(\mathcal{B}_\gamma, \|.\|_\gamma)$ is a Banach space. We conclude that $\mathcal{B}_\gamma$ is a phase space.

For each $u,\ \overline{u} \in \mathcal{B}_\gamma$ and $(t,x) \in J$, we have

$$|g(t,x,u_{(t,x)}) - g(t,x,\overline{u}_{(t,x)})| \leq \frac{1}{c_p e^{t+x}} \|u-\overline{u}\|_{\mathcal{B}_\gamma}.$$

Hence condition $(H2)$ is satisfied with $\ell_p e^{t+x} = \frac{1}{c_p e^{t+x}}$. Since

$$\ell_p^* = \sup\left\{\frac{1}{c_p e^{t+x}}, \quad (t,x) \in J \times \mathbb{R}\right\} \leq \frac{1}{c_p}$$

and $K = 1$, we get

$$\frac{k\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{3} < 1.$$

Hence condition (4) holds for each $(r_1, r_2) \in (0,1] \times (0,1]$ and all $p \in \mathbb{N}^*$. Also, the function f is continuous on $[0,\infty) \times [0,\infty) \times [0,\infty)$ and

$$|f(t,x,w)| \leq |w| + 3, \text{ for each } (t,x,w) \in [0,\infty) \times [0,\infty) \times \mathcal{B}_\gamma.$$

Thus conditions $(H1)$ and $(H3)$ hold. Consequently Theorem 5.1 implies that problem (8)-(10) has at least one solution defined on $(-\infty,+\infty) \times (-\infty,+\infty)$.

References

[1] Baleanu, D. Diethelm, K. Scalas, E. and Trujillo, J.J. *Fractional Calculus Models and Numerical Methods.* World Scientific Publishing, New York, 2012.

[2] Hilfer, R. *Applications of Fractional Calculus in Physics.* World Scientific, Singapore, 2000.

[3] Mainardi, F. *Fractional Calculus and Waves in Linear Viscoelasticity. An introduction to mathematical models.* Imperial College Press, London, 2010.

[4] Tarasov, V.E. *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media.* Springer, Heidelberg; Higher Education Press, Beijing, 2010.

[5] Podlubny, I. *Fractional Differential Equations.* Academic Press, San Diego, 1999.

[6] Abbas, S. Benchohra, M. and N'Guérékata, G.M. *Topics in Fractional Differential Equations.* Springer, New York, 2012.

[7] Kilbas, A.A. Srivastava, H. M. and Trujillo, J.J. *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[8] Lakshmikantham, V. Leela, S. and Vasundhara, J. *Theory of Fractional Dynamic Systems.* Cambridge Academic Publishers, Cambridge, 2009.

[9] Agarwal, R.P., Belmekki, M. and Benchohra, M. A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative. *Adv Differ. Equat.* **2009** (2009) Article ID 981728, 1-47.

[10] Agarwal, R.P., Benchohra, M. and Hamani, S. A survey on existence result for boundary value problems of nonlinear fractional differential equations and inclusions. *Acta. Appl. Math.* **109** (3) (2010) 973-1033.

[11] Belarbi, A. Benchohra, M. and Ouahab, A. Uniqueness results for fractional functional differential equations with infinite delay in Fréchet spaces. *Appl. Anal.* **85** (2006) 1459-1470.

[12] Benchohra, M. Henderson, J. Ntouyas, S.K. and Ouahab, A. Existence results for functional differential equations of fractional order. *J. Math. Anal. Appl.* **338** (2008) 1340-1350.

[13] Denton, Z. Ng, P.W. and Vatsala, A.S. Quasilinearization method via lower and upper solutions for Riemann-Liouville fractional differential equations. *Nonlinear Dynamics and Systems Theory* **11** (2011) 239–251.

[14] Ramrez, J.D. and Vatsala, A.S. Monotone method for nonlinear Caputo fractional boundary value problems. *Dynam. Systems Appl.* **20** (2011) 73–88.

[15] Stutson, D. and Vatsala, A.S. Generalized monotone method for Caputo fractional differential systems via coupled lower and upper solutions. *Dynam. Systems Appl.* **20** (2011) 495–503.

[16] Hale, J. and Verduyn Lunel, S. *Introduction to Functional -Differential Equations* Springer-Verlag, New York, 1993.

[17] Hino, Y. Murakami, S. and Naito, T. *Functional Differential Equations with Infinite Delay,* in: Lecture Notes in Mathematics, 1473, Springer-Verlag, Berlin, 1991.

[18] Kolmanovskii, V. and Myshkis, A. *Introduction to the Theory and Applications of Functional-Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[19] Lakshmikantham, V. Wen, L. and Zhang, B. *Theory of Differential Equations with Unbounded Delay.* Kluwer Academic Publishers, Dordrecht, 1994.

[20] Wu, J. *Theory and Applications of Partial Functional Differential Equations.* Springer-Verlag, New York, 1996.

[21] Corduneanu, C and Lakshmikantham, V. Equations with unbounded delay. *Nonlinear Anal.* **4** (1980) 831-877.

[22] Hale, J. and Kato, J. Phase space for retarded equationswith infinite delay, *Funkcial. Ekvac.* **21** (1978) 11-41.

[23] Avramescu, C. Some remarks on a fixed point theorem of Krasnoselskii. *Electron. J. Qual. Theory Differ. Equ.* **5** (2003) 1-15.

[24] Vityuk, A.N. and Golushkov, A.V. Existence of solutions of systems of partial differential equations of fractional order. *Nonlinear Oscil.* **7** (3) (2004) 318-325.

[25] Henry, D. *Geometric theory of Semilinear Parabolic Partial Differential Equations.* Springer-Verlag, Berlin-New York, 1989.

[26] Czlapinski, T. On the Darboux problem for partial differential-functional equations with infinite delay at derivatives. *Nonlinear Anal.* **44** (2001) 389–398.

[27] Czlapinski, T. Existence of solutions of the Darboux problem for partial differential-functional equations with infinite delay in a Banach space. *Comment. Math. Prace Mat.* **35** (1995) 111–122.

[28] Frigon, M. and Granas, A. Résultats de type Leray-Schauder pour des contractions sur des espaces de Fréchet, *Ann. Sci. Math. Québec* **22** (2) (1998) 161-168.

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Existence and Multiplicity of Periodic Solutions for a Class of the Second Order Hamiltonian Systems

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Abstract: In this paper, we study the existence and multiplicity of periodic solutions of the following second-order Hamiltonian systems

$$\ddot{x}(t) + V'(t, x(t)) = 0,$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^N$ and $V \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. By using a symmetric mountain pass theorem, we obtain a new criterion to guarantee that second-order Hamiltonian systems has infinitely many periodic solutions. We generalize and improve recent results from the literature. Some examples are also given to illustrate our main theoretical results.

Keywords: *periodic solutions; Hamiltonian systems; mountain pass theorem; symmetric mountain pass theorem.*

Mathematics Subject Classification (2010): 34C25, 58E05, 70H05.

1 Introduction

Consider the second-order Hamiltonian systems

$$\ddot{x}(t) + V'(t, x(t)) = 0, \tag{HS}$$

where $x = (x_1, ..., x_N)$, $V \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $V'(t, x) = \nabla_x V(t, x)$. The existence and multiplicity of periodic solutions for system (HS) have been studied in many papers via critical point theory, see the classical monographs [8] and [10] and the recent papers [5, 6, 12, 13, 15, 18]. In [10], Rabinowitz established the existence of periodic solutions for (HS) under the well known Ambrosetti-Rabinowitz condition:

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(AR) there is a constant $\mu > 2$ such that

$$0 \ < \ \mu V(t,x) \ \leq \ V'(t,x) \ . \ x$$

for all $t \in [0,T]$, $T > 0$, and $x \in \mathbb{R}^N \setminus \{0\}$.

The potential $V(t,x)$ in (HS) is of the following form:

$$V(t,x) \ = \ -\frac{1}{2}L(t)x \ . \ x \ + \ W(t,x),$$

where $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix valued function and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and satisfy:

(W_1) there exist constants $\alpha_0 > 0$ and $d_0 > 0$ such that

$$|W'(t,x)| \ \leq \ d_0 \ (|x|^{\alpha_0} + 1) \ \forall \ t \in [0,T], \ x \in \mathbb{R}^N,$$

He and Wu [6] have obtained some results of the existence of nontrivial $T-$periodic solutions for (HS). See also Fei [5].

Motivated by the ideas of [5–7, 10, 12, 14–18], in this paper we will further study the existence of $T-$periodic solutions for (HS) under some general conditions.

Here and in the following $x \ . \ y$ denotes the inner product of $x, y \in \mathbb{R}^N$ and $|.|$ denotes the associated norm.

Our main results are the two following theorems.

Theorem 1.1 *Assume that V satisfies*

(V$_1$) $V(t,x) \ = \ -K(t,x) \ + \ W(t,x)$, *where $K, W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are C^1-maps and are $T-$periodic in its first variable with $T > 0$, and $V(t,0) = 0$,*

(V$_2$) $\limsup\limits_{|x| \to 0} \dfrac{V(t,x)}{|x|^2} \ < \ 0$ *uniformly in $t \in [0,T]$,*

(V$_3$) *there exist constants $\mu > 2$, $\theta \in [2, \mu)$, $\lambda \in (1,2]$ and $b > 0$ such that*

$$K(t,x) \ \geq \ b\,|x|^{\lambda}, \ K'(t,x) \ . \ x \ \leq \ \theta K(t,x), \ \forall \ (t,x) \in [0,T] \times \mathbb{R}^N,$$

(V$_4$) *there exist constants $\sigma \in (1, \ \lambda)$ and $C \in \mathbb{R}$ such that*

$$0 \ \leq \ \mu W(t,x) \ \leq \ W'(t,x) \ . \ x + C\,|x|^{\sigma}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^N$,

(V$_5$) *there exist $\alpha_0(t) > 0$ and constants $\alpha_1 > \theta$, $R > 0$ such that*

$$W(t,x) \ \geq \ \alpha_0(t)\,|x|^{\alpha_1} \ \ \forall (t,x) \in [0,T] \times \mathbb{R}^N, \ |x| \geq R.$$

Then the system (HS) has a nontrivial $T-$periodic solution.

Moreover, if $V(t,x)$ is symmetric in x, i.e. V satisfies

(V_6) $\quad V(t,-x) \ = \ V(t,x), \quad \forall (t,x) \in [0,T]\times\mathbb{R}^N;$

then we obtain the following result by using the symmetric mountain pass theorem.

Theorem 1.2 *Assume that V satisfies $(V_1)-(V_6)$, then the system (HS) has an unbounded sequence of $T-$periodic solutions and, in particular, infinite $T-$periodic solutions.*

Remark 1.1 There are functions K and W which satisfy the hypotheses of Theorem 1.1 and Theorem 2.2, but do not satisfy the corresponding results in [4–7, 10, 12, 14–18].

For example, define a function $K \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ as follows

$$K(t,x) = \begin{cases} |x|^{\frac{5}{4}}\, exp(|x|^{\frac{1}{4}}) + |x|^2\,, \ if \ \ |x| \leq 1, \\ \\ exp(1)\,|x|^{\frac{3}{2}} + |x|^2\,, \ if \ \ |x| > 1. \end{cases}$$

An easy computation shows that K satisfies the condition (V_3) but do not satisfy the corresponding results in [4–7, 10, 12, 14–18]. Define a function $W \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ as follows

$$W(t,x) = |x|^{\frac{5}{4}}\, exp(|x|^{\frac{1}{4}}).$$

Then we have

$$\begin{aligned} W'(t,x)\ .\ x &= \frac{5}{4}|x|^{\frac{5}{4}}\, exp(|x|^{\frac{1}{4}}) + \frac{1}{4}\,|x|^{\frac{1}{4}}\,|x|^{\frac{5}{4}}\, exp(|x|^{\frac{1}{4}}) \\ &= (\frac{5}{4} + \frac{1}{4}\,|x|^{\frac{1}{4}})\,|x|^{\frac{5}{4}}\, exp(|x|^{\frac{1}{4}}). \end{aligned}$$

So, W does not satisfy (W_1).

Moreover, for any constant $\mu > 2$, we have

$$\mu W(t,x) - W'(t,x)\ .\ x = (\mu - \frac{5}{4} - \frac{1}{4}\,|x|^{\frac{1}{4}})\,|x|^{\frac{5}{4}}\, exp(|x|^{\frac{1}{4}})$$

which yields that

$$0 < \mu W(t,x) - W'(t,x)\ .\ x \leq (\mu - \frac{5}{4})\,|x|^{\frac{5}{4}}\, exp(4\mu - 5)$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ and $0 < |x| < (4\mu-5)^4$, i.e. the condition (AR) does not hold for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^N\backslash\,\{0\}$ and

$$\mu W(t,x) - W'(t,x).x \leq 0, \forall\ (t,x) \in \mathbb{R}\times\mathbb{R}^N, |x| > (4\mu-5)^4;$$

then (V_4) holds.

Corollary 1.1 *Assume that V satisfies $(V_1), (V_3) - (V_5)$ and*

(V_2') $\quad W(t,x) = o(|x|^2)$ *as* $|x| \to 0$ *uniformly in* $t \in [0,T]$.

Then the system (HS) has a nontrivial $T-$periodic solution.

Moreover, if V satisfies (V_6) then the system (HS) has an unbounded sequence of $T-$periodic solutions.

2 Proof of the Main Results

Let

$$H_T^1 = \{x : [0,T] \to \mathbb{R}^N,\ x \textit{ is absolutely continuous},\ x(0) = x(T),\ \textit{and}\ \dot{x} \in L^2([0,T],\mathbb{R}^N)\}$$

Then H_T^1 is a Hilbert space with the norm defined by

$$\|x\| = \left(\int_0^T (|x(t)|^2 + |\dot{x}(t)|^2)dt\right)^{\frac{1}{2}}$$

for $x \in H_T^1$. Consider the functional $\phi : H_T^1 \to \mathbb{R}$ defined by

$$\phi(x) = \int_0^T \left(\frac{1}{2}|\dot{x}(t)|^2 + K(t,x(t)) - W(t,x(t))\right) dt\ . \tag{1}$$

It is well known that $\phi \in C^1(H_T^1,\mathbb{R})$ and for all $x, y \in H_T^1$

$$\phi'(x)y = \int_0^T (\dot{x}(t).\dot{y}(t) + K'(t,x(t)).y(t) - W'(t,x(t)).y(t))\, dt\ . \tag{2}$$

It is well known that the $T-$periodic solution of system (HS) corresponds to the critical points of ϕ in H_T^1. We will obtain the critical point of ϕ by using the mountain pass theorem and the symmetric mountain pass theorem. We say that ϕ satisfies the Palais-Smale condition if every bounded sequence $\{u_k\}$ in the space H such that $\lim_{k\to\infty} \phi'(u_k) = 0$ contains a convergent subsequence. Therefore we state these theorems.

Theorem 2.1 *[10] Let H be a real Banach space and $\phi \in C^1(H,\mathbb{R})$ satisfying the Palais-Smale condition. If ϕ satisfies the following conditions:*

(i) $\phi(0) = 0$,

(ii) there exist constants $\rho,\ \alpha > 0$ such that $\phi_{/\partial B_\rho(0)} \geq \alpha$,

(iii) there exists $e \in H\backslash\overline{B}_\rho(0)$ such that $\phi(e) \leq 0$.

Then ϕ possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} \phi(g(s)),$$

where $B_\rho(0)$ is the open ball in H centered in 0, with radius ρ, $\partial B_\rho(0)$ its boundary and

$$\Gamma = \{g \in C([0,1],H) : g(0) = 0,\ g(1) = e\}\,.$$

Theorem 2.2 *[10] Let H be a real Banach space, ϕ is even and $\phi \in C^1(H,\mathbb{R})$ satisfyies the Palais-Smale condition. If ϕ satisfies (i) and (ii) of Theorem 2.1 and the following condition:*

(iii') For each finite dimensional subspace $E \subset H$, there is $r = r(E)$ such that $\phi(x) \leq 0$ for $x \in E\backslash B_r(0)$ where $B_r(0)$ is an open ball in H centered in 0, with radius r.

Then ϕ possesses an unbounded sequence of critical values.

In the following, we denote C_i $(i = 1, 2, 3...)$ for different positive constants.

Lemma 2.1 *[7] For all $x \in H_T^1$*

$$\|x\|_\infty \leq C_\infty \|x\| \ . \tag{3}$$

where $\|x\|_\infty = \max_{0\leq t\leq T} |x(t)|$.

2.1 Proof of Theorem 1.1

Let $\gamma_T : H_T^1 \to [0, +\infty)$ be given by

$$\gamma_T(x) = \left(\int_0^T (|\dot{x}(t)|^2 + 2K(t, x(t)))dt\right)^{\frac{1}{2}} \ . \tag{4}$$

By (1) and (4) we have

$$\phi(x) = \frac{1}{2}\ \gamma_T^2(x) - \int_0^T W(t, x(t))dt \ . \tag{5}$$

Moreover, using (V_3) and (2) we obtain

$$\phi'(x)x \leq \int_0^T \left(|\dot{x}(t)|^2 + \theta K(t, x(t))\right) dt - \int_0^T W'(t, x(t)).x(t)dt. \tag{6}$$

It is clear that $\phi(0) = 0$. Firstly, we will show that ϕ satisfies the Palais-Smale condition. Let $(y_j) \subset H_T^1$ be a sequence such that $(\phi(y_j))_{j\in\mathbb{N}}$ is bounded and $\phi'(y_j) \to 0$ as $j \to +\infty$. Then, there exists C_0 such that

$$\phi(y_j) \leq C_0, \quad \|\phi'(y_j)\|_{H_T^{1\ *}} \leq C_0, \tag{7}$$

for every $j \in \mathbb{N}$. Without loss of generality, we can assume that $\|y_j\| \neq 0$. Then from (3), (4) and (V_3), we obtain for $j \in \mathbb{N}$

$$\begin{aligned} \gamma_T^2(y_j) &= \int_0^T \left(|\dot{y}_j(t)|^2 + 2K(t, y_j(t))\right) dt \\ &\geq \int_0^T \left(|\dot{y}_j(t)|^2 + 2b\,|y_j(t)|^\lambda\right) dt \\ &\geq \int_0^T |\dot{y}_j(t)|^2\, dt + 2b\,(C_\infty \|y_j\|)^{\lambda-2} \int_0^T |y_j(t)|^2\, dt \\ &\geq \min\left\{1,\ 2b(C_\infty \|y_j\|)^{\lambda-2}\right\}\ \|y_j\|^2 \\ &= \min\left\{\|y_j\|^2, 2bC_\infty^{\lambda-2} \|y_j\|^\lambda\right\}. \end{aligned} \tag{8}$$

By (4), (6) and (V_4) we have

$$-\frac{\theta}{\mu}\ \gamma_T^2(y_j) \leq \frac{2}{\mu}\ \|\phi'(y_j)\|\ \|y_j\| - \frac{2}{\mu}\int_0^T W'(t, y_j(t))\ .\ y_j(t)\ dt. \tag{9}$$

By Sobolev's embedding theorem, $(5),(7),(9)$ and (V_4) we obtain

$$\left(\frac{\mu-\theta}{\mu}\right)\gamma_T^2(y_j) \leq 2\phi(y_j)+\frac{2}{\mu}\left\|\phi'(y_j)\right\|\left\|y_j\right\|+\frac{2}{\mu}\int_0^T C\left|y_j(t)\right|^\sigma dt$$
$$\leq 2C_0 + C_1\left\|y_j\right\| + C_2\left\|y_j\right\|^\sigma. \tag{10}$$

Combining (8) with (2.1), we obtain

$$\min\left\{\left\|y_j\right\|^2, 2bC_\infty^{\lambda-2}\left\|y_j\right\|^\lambda\right\} \leq \frac{\mu}{\mu-\theta}(C_0+C_1\left\|y_j\right\|+C_2\left\|y_j\right\|^\sigma). \tag{11}$$

It follows from (11) that $\|y_j\|$ is bounded in H_T^1. In a similar way as in Proposition 4.3 in [8], we can prove that (y_j) has a convergent subsequence in H_T^1. Hence, ϕ satisfies the Palais-Smale condition. Now, let us show that ϕ satisfies assumption (ii) of Theorem 2.1. By (V_2), there exist constants $\alpha_0, \rho_0 > 0$ such that

$$V(t,x) \leq -\alpha_0\left|x\right|^2 \tag{12}$$

for all $|x| \leq \rho_0$ and $t \in [0,T]$. Choose $\rho = \frac{\rho_0}{C_\infty}$ and let $S = \left\{x \in H_T^1,\ \|x\| = \rho\right\}$. By 3, we have $\|x\|_\infty \leq \rho_0$, for all $x \in S$, which together with (12) implies

$$\begin{array}{rcl}\phi(x) &=& \frac{1}{2}\int_0^T \left|\dot{x}(t)\right|^2 dt - \int_0^T V(t,x(t))\ dt\\ &\geq& \frac{1}{2}\int_0^T \left|\dot{x}(t)\right|^2 dt + \ \alpha_0\int_0^T \left|x(t)\right|^2\ dt\\ &\geq& \min\left\{\frac{1}{2},\alpha_0\right\}\rho^2 := \alpha.\end{array}$$

for every $x \in S$.

It remains to prove that ϕ satisfies assumption (iii) of Theorem 2.1. By (V_3) we have

$$K(t,x) \leq C_3\left|x\right|^\theta + C_4 \quad \forall\ (t,x) \in [0,T]\times\mathbb{R}^N, \tag{13}$$

where $C_3 = \sup\limits_{t\in[0,T],|x|=1} K(t,x)$ and $C_4 = \sup\limits_{t\in[0,T],|x|\leq 1} K(t,x)$. By (1) and (13) we have, for every $s \in \mathbb{R}\setminus\{0\}$ and $x \in H_T^1\setminus\{0\}$,

$$\phi(sx) \leq \frac{s^2}{2}\int_0^T \left|\dot{x}(t)\right|^2 dt + C_3 s^\theta \int_0^T \left|x(t)\right|^\theta dt + C_5 - \int_0^T W(t,sx(t))\ dt. \tag{14}$$

Take some $Q \in H_T^1$ such that $\|Q\| = 1$. Then there exists a subset Ω of positive measure of $[0,T]$ such that $Q(t) \neq 0$ for $t \in \Omega$. Take $s > 1$ such that $s\left|Q(t)\right| \geq R$ for $t \in \Omega$. Then by $(V_4),(V_5)$ and (14)

$$\phi(sQ) \leq C_6 s^\theta - s^{\alpha_1}\int_\Omega \alpha_0(t)\left|Q(t)\right|^{\alpha_1} dt. \tag{15}$$

Since $\alpha_0(t) > 0$ and $\alpha_1 > \theta$, (15) implies that $\phi(sQ) < 0$ for some $s > 1$ such that $s\left|Q(t)\right| \geq R$ for $t \in \Omega$ and $s\left\|Q\right\| > \rho$. By Theorem 1.1, ϕ possesses a critical value $c \geq \alpha > 0$ given by

$$c = \inf_{g\in\Gamma}\max_{s\in[0,1]}\phi(g(s)),$$

where

$$\Gamma = \left\{g \in C([0,1],H) : g(0) = 0,\ g(1) = e\right\}.$$

Hence, there is $x \in H_T^1$ such that $\phi(x) = c, \phi'(x) = 0$. The proof of Theorem 1.1 is complete.

2.2 Proof of Theorem 1.2

(V_6) implies that ϕ is even. By Theorem 2.1 and the proof of Theorem 1.1, it suffices to prove that ϕ satisfies (iii') of Theorem 2.2.

Let $E \subset H^1_T$ be a finite dimensional subspace. From the proof of Theorem 1.1 we know that for any $Q \in E \subset H^1_T$ such that $\|Q\| = 1$, there is $s_Q > 1$ such that $\phi(sQ) < 0$, for every $|s| \geq s_Q > 1$. Since $E \subset H^1_T$ is a finite dimensional subspace, we can choose $r = r(E) > 0$ such that

$$\phi(x) < 0, \ \forall\ x \in E \backslash\ B_r(0).$$

Hence, by Theorem 2.1, ϕ possesses an unbounded sequence of critical values $(c_n)_{n\in\mathbb{N}}$ with $c_n \to +\infty$. The proof of Theorem 1.2 is complete.

2.3 Proof of Corollary 1.1.

It follows from (V_3) and (V_2')

$$\limsup_{|x|\to 0} \frac{V(t,x)}{|x|^2} \ \leq\ \limsup_{|x|\to 0} \left(\frac{W(t,x)}{|x|^2} - b\,|x|^{\lambda-2} \right) < 0$$

uniformly in $t \in [0,T]$, which implies the conditions (V_2). An easy application of Theorem 2.1 and Theorem 2.2 will show that Corollary 1.1 holds.

References

[1] Abbondandolo, A. Morse theory for asymptotically linear Hamiltonian systems. *Nonlinear Anal.* **39** (2000) 997–1049.

[2] Bahri,A. and Beresttycki, H. Existence of forced osciallations for some nonlinear differential equations. *Commun. Pure Appl. Math.* **37** (4) (1984) 403–442.

[3] Benhassine, A. and Timoumi, M. Homoclinic Orbits for a class of second order Hamiltonian systems. *Nonlinear Dynamics and Systems Theory.* **12** (2) (2012) 145–156.

[4] Ding, Y.H. Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems. *Nonlinear Anal.* **25**(11) (1995) 1095–1113.

[5] Fei, G.H. On periodic solutions of superquadratic Hamiltonian systems. *Electron. J. Differential Equations* **2002** (8) (2002) 1–12.

[6] He, X.M. and Wu, X. Periodic solutions for class of nonautonomous second order Hamiltonian systems. *J. Math. Anal. Appl.* **341**(2) (2008) 1354–1364.

[7] Izydorek, M. and Janczewska, J. Homoclinic solutions for a class of second order Hamiltonian systems. *J. Differential Equations* **219** (2) (2005) 375–389.

[8] Mawhin, J. and Willem, M. Critical Point Theory and Hamiltonian Systems. *Applied Mathematical Sciences*, Vol. **74**, Springer-Verlag, New Yorek. 1989.

[9] Qu, Z.Q. and Tang, C.L. Existence of homoclinic orbits for the second order Hamiltonian systems. *J. Math. Anal. Appl.* **291** (1) (2004) 203–213.

[10] Rabinowitz, P.H. Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Regional Conference Series in Mathematics, Vol.65, American Mathematical Society, Providence, RI, 1986. Published for the Conference Board of the Mathematical Sciences, Washington, DC.

[11] Rabinowitz, P.H. Periodic solutions of Haliltonians systems. *Comm. Pure Appl. Math.* **31** (1978) 157–184.

[12] Schechter, M. Periodic solutions of second order non-autonoumous dynamical systems. *J. Differential Equations* **223**(25) (2006) 290–302.

[13] Schechter, M. Periodic solutions of second order non-autonoumous dynamical systems. *Bound. Value Probl.* (2006) 1–9.

[14] Sun, J., Chen, H. and Nieto, J.J. Homoclinic solutions for a class of subquadratic second-order Hamiltonian systems. *J. Math. Anal. Appl.* **373** (2011) 20–29.

[15] Sun, J., Chen, H. and Nieto, J.J. Infinitely many solutions for second-order Hamiltonian system with impulsive effects. *Mathematical and Computer Modelling* **54** (2011) 544–555.

[16] Sun, J., Chen, H. and Nieto, J.J. Homoclinic orbits for a class of first-order nonperiodic asymptotically quadratic Hamiltonian systems with spectrum point zero. *J. Math. Anal. Appl.* **378** (2011) 117–127.

[17] Sun, J., Chen, H. and Nieto, J.J. On ground state solutions for some non-autonomous Schredinger-Poisson systems. *Journal of Differential Equations* **252** (2012) 3365–3380.

[18] Sun, J., Chen, H., Nieto, J.J. and Otero-Novoa, M. The multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects. *Nonlinear Analysis, Theory, Methods and Applications* **72** (2010) 4575–4586.

[19] Timoumi, M. Periodic and subharmonic solutions for a class of noncoercive superquadratic Hamiltonian systems. *Nonlinear Dynamics and Systems Theory* **11** (3) (2011) 319–336.

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Approximate Controllability of a Functional Differential Equation with Deviated Argument

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Abstract: This paper deals with the approximate controllability of a functional differential equation with deviated argument and finite delay. Sufficient condition for approximate controllability is proved under the assumption that the linear control system is approximately controllable; thereby removing the need to assume the invertibility of a controllability operator which fails to exist in infinite dimensional space if the generated semigroup is compact. Schauder fixed point theorem is used and the C_0 semigroup associated with mild solution has been replaced by the fundamental solution.

Keywords: *deviated argument; approximate controllability; fundamental solution; semilinear control system; delay; reachable set; Schauder fixed point theorem.*

Mathematics Subject Classification (2010): 34K30, 34K35, 93C25.

1 Introduction

In certain real world problems, delay depends not only on time but also on the unknown quantity. The differential equations with deviated arguments are generalization of delay differential equations in which the unknown quantity and its derivative appear in different values of their arguments. Functional differential equations with deviated argument model various control problems arising in the field of engineering, physics and so on. Many partial differential systems can be reduced to functional differential equations with deviated arguments, see for instance [3, 8, 15, 16]. Aftereffect, hereditary systems, equations with deviated arguments, etc. feature in several mathematical models. As a matter of fact delay differential systems are still resistant to many classical controllers.

In recent years, controllability of infinite dimensional systems has been extensively studied for various applications. The papers of Benchohra et al. [10] and Chang [19]

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discuss the exact controllability of functional systems with infinite delay. However, in these papers the invertibility of a controllability operator is assumed. As a consequence their approach fails in infinite dimensional spaces whenever the generated semigroup is compact. Also it is practically difficult to verify their condition directly. This is one of the motivations of our paper.

Controllability results are available in overwhelming majority of investigations for abstract differential delay systems (see [4–6, 9–11, 18–20]); rather than for functional differential equations with deviated arguments. It is interesting to note that approximate controllability problem for nonlinear dynamical systems with deviated argument has not been investigated thoroughly in literature. In an attempt to fill this gap we study the approximate controllability of the following control system using fixed point approach which removes the above restrictions.

However C.G. Gal [1] studied the existence and uniqueness of local and global solutions for initial value problem with deviated argument

$$u'(t) = Au(t) + f(t, u(t), u[\alpha(u(t), t)]), t \in R_+, u(0) = u_0.$$

Muslim and Bahuguna [12] studied a neutral differential equation with the same type of deviated argument as studied by C.G. Gal [1]. Haloi, Pandey and Bahuguna [17] studied a system with the same deviated argument. Fractional operators, analyticity and compactness are mostly used to establish these results which impose more restriction on the semigroup and the nonlinear part of the semilinear system. Thus, in this paper the C_0 semigroup associated with mild solution has been replaced by the fundamental solution.

Several papers studied the approximate controllability of semilinear control systems, see for instance [2, 7, 14] and references therein. Generally these papers proposed conditions on the systems operators by assuming the corresponding linear system is approximately controllable. For instance, Naito [7] proved that a semilinear system is approximately controllable under range condition on the control operator and uniform boundedness of the nonlinear operator. Sukavanam [14] proved sufficient conditions for approximate controllability where the nonlinear function satisfies growth conditions.

Motivated by results in [7] and [14] the purpose of this paper is to study the existence and uniqueness of mild solution and approximate controllability of a functional differential equation with deviated argument and finite delay using Schuader fixed point theorem. However we proceed by establishing a relation between the reachable set of linear control problem and that of the semilinear delay control problem.

In this work we study the approximate controllability of the functional differential equation with finite delay and deviated argument, which is illustrated as follows.

$$\begin{aligned} &\frac{dx(t)}{dt} = Ax(t) + A_1x_t + Bu(t) + f(t, x_t, x(a(x(t), t))), t \in J = [0, \tau], \\ &x(t) = \phi(t), -h \le t \le 0, \end{aligned} \tag{1}$$

where $x(t) \in X$ and $u(t) \in U$, X and U being Hilbert spaces. Let $Z = L_2([0, \tau]; X)$, $Z_h = L_2([-h, \tau]; X)$, $0 < h < \tau$ and $Y = L_2([0, \tau]; U)$ be the corresponding function spaces. $A : D(A) \subset X \to X$ is a closed linear operator which generates a strongly continuous semigroup $T(t)$. A_1 is a bounded linear operator from $C([-h, \tau]; X)$ to $L_2([0, \tau], X)$. $B : Y \to Z$ is a bounded linear operator. When $x : [-h, \tau] \to X$ is a continuous function then $x_t(.)$ is denoted by $x_t(\theta) = x(t + \theta)$, $\theta \in [-h, 0]$ and $\phi \in C([-h, 0]; X)$. $x_t \in$

$C([-h,0],X)$ a Banach space of all continuous functions from $[-h,0]$ to X with norm

$$\|x_t\|_C := sup_{\theta\in[-h,0]}\|x_t(\theta)\|_X \quad for\ t\in(0,\tau].$$

$C_L(J,X)=\{u\in C(J,X): \exists l>0 \ such\ that\ \|u(t)-u(s)\|\leq l|t-s|, \forall t,s\in J\}$.

Simple Lipschitz conditions are required to study the differential equation with deviated argument in Section 3.

2 Preliminaries and Assumptions

Some basic definitions and lemmas are stated which are used in proving the existence and uniqueness of the mild solution and approximate controllability of (1). In equation (1) if we put $f\equiv 0$ the resulting equation without the delay term is called the corresponding linear system (2)

$$\begin{aligned}\frac{dx(t)}{dt} &= Ax(t)+Bu(t),\ t\in[0,\tau],\\ x(0) &= \phi(0)\ \in[-h,0].\end{aligned} \tag{2}$$

Let us consider the linear delayed system

$$\begin{aligned}\frac{dx(t)}{dt} &= Ax(t)+A_1x_t,\ t\in[0,\tau],\\ x_0 &= \phi\ \in[-h,0].\end{aligned} \tag{3}$$

Let $x^{\phi}(t)$ be the unique solution of system (3). Define a map $S: J\to\mathcal{L}(X)$ by

$$S(t)\phi(0)=\begin{cases} x^{\phi}(t), & t\geq 0,\\ 0, & t<0.\end{cases} \tag{4}$$

Then $S(t)$ is called the fundamental solution of (3) satisfying

$$\begin{aligned}&S(t)=T(t)\phi(0)+\int_0^t T(t-s)A_1S(s+\theta)ds,\ t>0,\\ &S(0)=I, S(t)=0,\ -h\leq t<0.\end{aligned} \tag{5}$$

It follows from [9] that $S(t)$ is the unique solution of (3). It can be easily shown that

$$S(t)=K_0\exp(K_0||A_1||\tau):=M,$$

where $||T(t)||=K_0$. Therefore the mild solution of semilinear control system (1) is defined as

Definition 2.1 The function $x:(-h,\tau]\to X$ is said to be a mild solution of (1) if $x(.)\in C_L(J,X)$, $x(t)=\phi(t)$ for $t\in[-h,0]$ and it satisfies the integral equation.

$$x(t)=S(t)\phi(0)+\int_0^t S(t-s)Bu(s)ds+\int_0^t S(t-s)f(s,x_s,x(a(x(s),s)))ds,\ t\in J, \tag{6}$$

and the mild solution of the corresponding linear system with delay and control term (7)

$$\begin{aligned}\frac{dx(t)}{dt} &= Ax(t) + A_1 x_t + Bu(t),\ t \in [0, \tau],\\ x_0 &= \phi\ \in [-h, 0],\end{aligned} \tag{7}$$

is defined as

$$\begin{aligned}x(t) =&S(t)\phi(0) + \int_0^t S(t-s)Bu(s)ds, t \in [0, \tau],\\ x(t) =&\phi(t), -h \le t < 0.\end{aligned} \tag{8}$$

Definition 2.2 The set given by $K_\tau(f) = \{x(T) \in X : x \in Z_h\}$ is called reachable set of the system (1). $K_\tau(0)$ is the reachable set of the corresponding linear control system (7).

Definition 2.3 The system (1) is said to be approximately controllable if $K_\tau(f)$ is dense in X. The corresponding linear system is approximately controllable if $K_\tau(0)$ is dense in X.

Let us assume that:

(H1) The nonlinear function $f : J \times X \times X \to X$ satisfies Lipschitz condition,

$$\|f(t, x_1, z_1) - f(t, x_2, z_2)\| \le P(\|x_1 - x_2\| + \|z_1 - z_2\|)$$

for all $x_1, x_2, z_1, z_2 \in X,\ t \in (0, \tau]$ and $\exists$ a constant $g > 0$, such that $\|f(s, 0, x(a(x(0), 0)))\| \le g,\ \forall\ s \in J$.

(H2) Let $a : X \times R^+ \to R^+$ satisfy the Lipschitz condition $|a(x_1, s) - a(x_2, s)| \le L_a\|x_1 - x_2\|$ and $a(., 0) = 0$.

Lemma 2.1 *The fundamental solution $S(t)$ is bounded.*

Proof. Since

$$\begin{aligned}||S(t)|| \le&K_0 + K_0||A_1|| \int_0^t ||S(s+\theta)||ds\\ \le&K_0 + k_0||A_1|| \int_0^{t+\theta} ||S(\sigma)||d\sigma\\ \le&K_0 + ||A_1||K_0 \int_{-h}^t ||S(\sigma)||d\sigma\\ \le&K_0 + K_0||A_1|| \int_0^{t+h} ||S(\sigma)||d\sigma\\ ||S(t)|| \le&K_0 \exp K_0||A_1||(t+h) \le K_0(1+d) \exp K(\tau+h) = M\\ &max\{||S(t)|| : t \in [0, \tau]\} = M,\end{aligned} \tag{9}$$

the fundamental solution is bounded.

Lemma 2.2 *If the C_0-semigroup $T(t)$ is compact then the fundamental solution $S(t)$ is compact.*

Proof. Let us define the sequence of operators $S_n(t)$ on $[-h,\tau]$. From the compactness of $T(t)$ and boundedness of $||A_1||$ we conclude that S_n is compact. Let $||A_1|| = K_1$. To prove $S_n(t) \to S(t)$ in $\mathcal{L}(X)$ we first show that $\{S_n(t)\}$ is a Cauchy sequence in $\mathcal{L}(X)$. Let us define

$$\begin{aligned} S_1(t) &= T(t), t \in [0,\tau], \\ &= 0, t \in [-h,0], \\ S_{n+1}(t) &= T(t) + \int_0^t T(t-s)S_n(s+\theta)ds, t \in (0,\tau], \theta \in [-h,0], \\ &= 0, t \in [-h,0], \end{aligned} \tag{10}$$

for $n = 1, 2, ...$

Therefore,

$$\begin{aligned} ||S_2(t) - S_1(t)|| &\leq \int_0^t ||T(t-s)||||A_1||||S(s+\theta)||ds \leq K_0K_1Mt, \\ ||S_{n+1}(t) - S_n(t)|| &\leq \frac{1}{n!}K_0^nK_1^nM_1\tau^n \to 0 \ \ as \ \ n \to 0. \end{aligned} \tag{11}$$

Thus $\{S_n(t)\}$ is a Cauchy sequence. As $\mathcal{L}(X)$ is the Banach space of all bounded linear operators on X, $\exists$ an operator $S(t) \in \mathcal{L}(X)$ such that $S_n(t) \to S(t)$ uniformly on $[0,\tau]$ and hence $S(t)$ is compact $\forall t \in [0,\tau]$. It is easy to check that $S(t)$ is unique.

2.1 Existence and uniqueness of mild solution

The equation (6) is verified to be the unique mild solution of the semilinear delay control system (1).

Theorem 2.1 *The system (1) has a unique mild solution in $C_L(J,X)$ for each control $u \in L_2([0,T];U)$ if assumptions $(H1)$ and $(H2)$ are satisfied.*

Proof. Define the space $C_{L_0}([-h,\tau],X) = \{x \in C([-h,\tau],X) : x \in C_L([0,\tau],X)\}$. Fix $0 < t_1 < T$ such that

$$PMt_1(l + 2lL_a)R < M\|\phi\| + MM_BT\|u\| + MTg + 1.$$

Define the mapping $\Phi : C_{L_0}([-h,t_1],X) \to C_{L_0}([-h,t_1],X)$ as

$$\begin{aligned} (\Phi x)(t) &= S(t)\phi(0) + \int_0^t S(t-s)[Bu(s) + f(s,x_s,x(a(x(s),s)))]ds, \ t \in (0,t_1], \\ &= \phi(\theta), \ \ \theta \in [-h,0]. \end{aligned} \tag{12}$$

Let us consider the space $B_R = \{x(.) \in C_{L_0}([-h,t_1],X) : \|x\|_{C([-h,t_1],X)} \leq R, \ \ x(0) = \phi(0)\}$ endowed with the norm of uniform convergence. For any $x \in B_R$ and $0 \leq t \leq t_1$,

$$\|x_t\|_C = sup_{-h\leq\theta\leq 0}\|x_t(\theta)\|_X \leq sup_{-h\leq\zeta\leq t_1}\|x(\zeta)\|_X \leq R.$$

Then

$$
\begin{aligned}
\|(\Phi x)(t)\| &\leq M\|\phi(0)\| + MM_B T\|u\| \\
&+ \int_0^t M[\|f(s,x_s,x(a(x(s),s))) - f(s,0,x(a(x(0),0)))\| \\
&+ \|f(s,0,x(a(x(0),0)))\|]ds \\
&\leq M\|\phi\| + MM_B T\|u\| \\
&+ \int_0^t M[P(\|x(s+\theta)-0\| + lL_a\|x(s)-x(0)\|) + g]ds \\
&\leq M\|\phi(0)\| + MM_B t_1\|u\| \\
&+ \int_{-h}^{t_1} MP(\|x(\sigma)\|d(\sigma) + \int_0^{t_1}[MlL_a\|x(s)-x(0)\| + g]ds \\
&\leq M\|\phi(0)\| + MM_B t_1\|u\| + M(t_1+h)P\|x\| + 2Mt_1PlL_a\|x\| + gt_1 \\
&\leq M\|\phi(0)\| + MM_B t_1\|u\| + M(t_1+h)PR + 2Mt_1PlL_aR + gt_1.
\end{aligned}
$$

Let

$$M\|\phi\| + MM_B t_1\|u\| + M(t_1+h)PR + 2Mt_1PlL_aR + gt_1 < R.$$

Then

$$M\|\phi\| + MM_B t_1\|u\| + gt_1 < R(1 - M(t_1+h)P - 2Mt_1PlL_a).$$

RHS is positive if

$$
\begin{aligned}
t_1(PM + 2MPlL_a) &< M(t_1+h)P + 2Mt_1PlL_a < 1, \\
t_1 &< \frac{1}{(PM + 2MPlL_a)}. \qquad (13)
\end{aligned}
$$

Hence Φ maps B_R into itself when t_1 satisfies (13). Next it is shown that Φ is a contraction. Let $x_1,\ x_2 \in B_R$

$$
\begin{aligned}
\|(\Phi x_1)(t) - (\Phi x_2)(t)\| &\leq \int_0^t M\|f(s,(x_1)_s,x_1(a(x_1(s),s))) \\
&- f(s,(x_1)_s,x_1(a(x_2(s),s))) - f(s,(x_2)_s,x_2(a(x_2(s),s))) \\
&+ f(s,(x_1)_s,x_1(a(x_2(s),s)))\|ds \\
&\leq tMP[\|x_1(a(x_1(s),s)) - x_1(a(x_2(s),s))\| \\
&+ (\|(x_2)_s - (x_1)_s\| \\
&+ \|x_2(a(x_2(s),s) - x_1(a(x_2(s),s)))\|)] \\
&\leq tMP[l|a(x_1(s),s) - a(x_2(s),s)| \\
&+ \|x_2(s+\theta) - x_1(s+\theta)\| + (\|x_2 - x_1\|_{C([-h,t_1];X)})] \\
&\leq tM(lPL_a\|x_1(s) - x_2(s)\|_{C([-h,t_1],X)} \\
&+ P\|x_2(t_1) - x_1(t_1)\| + P\|x_2 - x_1\|_{C([-h,t_1],X)}) \\
&\leq Mt(lPL_a + 2P)\|x_2 - x_1\|_{C([-h,t_1],X)}. \qquad (14)
\end{aligned}
$$

So, $\|\Phi x_1 - \Phi x_2\|_{C([-h,t_1],X)} \leq Mt(lPL_a + 2P)\|x_1 - x_2\|_{C([-h,t_1],X)}$. Thus Φ is a contraction mapping. Therefore, Φ has a fixed point in B_R. Hence (6) is the mild solution on $[-h,t_1]$.

Similarly it can be shown that (6) is the mild solution on the interval $[t_1, t_2],\ t_1 < t_2$ Repeating the above process we get that

$$\|\Phi^n x_1 - \Phi^n x_2\|_{C([-h,t_1],X)} \leq \frac{Mt^n}{n!}(lPL_a + 2P)\|x_1 - x_2\|_{C([-h,t_1];X)}.$$

Thus (6) is the mild solution on the maximal existence interval $[-h, t^*],\ t^* < \tau$.

Now it is shown that x is well defined in $[-h, \tau]$.

$$\begin{aligned}
\|x(t)\| &\leq M\|\phi\| + M\int_0^t [M_B\|u(s)\| + P\|x_s - 0\| \\
&+ P|x(a(x(s), s) - x(a(x(0), 0)\| + g]ds \\
&\leq M\|\phi\| + MM_B\tau\|u(s)\| \\
&+ M\int_0^t P[\|x_s\| + lL_a\|x(s) - x(0)\| + g] \\
&\leq M\|\phi\| + MM_B\tau\|u(s)\| \\
&+ M\tau P(\|x(0)\| + g) + M\int_0^t l\|x(s)\|ds. \qquad (15)
\end{aligned}$$

By Gronwall's inequality $\|x(t)\| \leq \|x_t\|_C \leq [M\|\phi\| + MM_B\tau\|u(s)\| + MTP(\|x(0)\| + g)]\exp(M\tau P)$. So $\|x(t)\|$ is bounded on $[-h, t^*]$. Thus x is well defined on $[-h, T]$. To prove the uniqueness of solution let x_1 and x_2 be any two mild solutions of (6) such that for $t \in [-h, 0],\ x_1(t) = x_2(t) = \phi$. For $t \in [0, t^*)$

$$\begin{aligned}
\|x_1(t) - x_2(t)\| &\leq M\int_0^t \|f(s, (x_1)_s, x_1(a(x_1(s), s))) \\
&- f(s, (x_2)_s, x_2(a(x_1(s), s)))\|ds + f(s, (x_2)_s, x_2(a(x_1(s), s))) \\
&- f(s, (x_2)_s, x_2(a(x_2(s), s)))\| \\
&\leq M\int_0^t P\{\|(x_1)_s - (x_2)_s\| + \|x_1(s) - x_2(s)\| \\
&+ lL_a\|x_1(s) - x_2(s)\|\}ds \\
&\leq M\int_{-h}^t P\|x_1(\eta) - x_2(\eta)\|d\eta + M\int_0^t P\|x_1(s) - x_2(s)\|ds \\
&+ M\int_0^t PlL_a\|x_1(s) - x_2(s)\|ds \\
&\leq M\int_{-h}^0 P\|x_1(\eta) - x_2(\eta)\|d\eta + M\int_0^t P(2 + lL_a)\|x_1(s) - x_2(s)\|ds.
\end{aligned}$$

Since uniqueness of the mild solution is proved on $[-h, 0]$, we get

$$\|x_1(t) - x_2(t)\| \leq MP(2 + lL_a)\int_0^t \|x_1(s) - x_2(s)\|ds.$$

Hence by Gronwall's inequality $x_1(t) = x_2(t)$ for all $t \in [-h, \tau]$.

3 Main Result

Define a linear operator L from Z to $C_L([0, \tau], X)$ by $Lx = \int_0^\tau S(t - s)x(s)ds, t \in [0, \tau]$. Let $Kx(t) = \int_0^t S(t - s)x(s)ds, t \in [0, \tau]$.

Z can be decomposed uniquely as $Z = N_0(L) \oplus N_0^{\perp}(L)$ where $N_0(L)$ is the null space of the operator L and $N_0(L)$ is its orthogonal space.

Let us assume

(H3) $\forall\, p \in Z$, $\exists$ a function $q \in \overline{R(B)}$ such that $Lp = Lq$.

The approximate controllability of the corresponding linear system (2) follows from the hypothesis $(H3)$. Then it is to be proved that the linear system (7) with finite delay is approximately controllable. Next by assuming that the linear system with delay (7) is approximately controllable, the system (1) is to be proved to be approximately controllable using Schauder fixed point theorem. Define the operator $F : C_{L_0}([0,\tau], X) \to L_2([0,\tau], X)$ as

$$F(x)(t) = f(t, x_t, x(a(x(t), t))); 0 < t \leq \tau.$$

From hypotheses $(H1), (H2)$ we conclude that F is a continuous map. From hypothesis $(H3)$ it follows that for any $p \in Z$, there exists a $q \in \overline{R(B)}$ such that $L(p-q) = 0$. Therefore $p - q = n \in N_0(L)$ which implies that $Z = N_0(L) \oplus \overline{R(B)}$. Therefore, it implies the existence of a linear and continuous mapping Q from $N_0^{\perp}(L)$ into $\overline{R(B)}$ which is defined as $Qu^* = v$ where v is the unique minimum norm element $v \in (u^* + N_0(L)) \bigcap \overline{R(B)}$, i.e. $\|Qu^*\| = \|v\| = min\{\|v\| : v \in \{(u^* + N_0(L)) \bigcap \overline{R(B)}\}$. By $(H3)$, $\forall v \in \{u^* + N_0^{\perp}\} \cap \overline{R(B)}$ is not empty and $\forall z \in Z$ has a unique decomposition $z = n + q$. Hence the operator Q is well defined. Moreover, $\|Q\| = c$ for some constant c.

Let us consider the subspace M_0 of $C_{L_0}([0,\tau], X)$ which is defined as

$$M_0 = \begin{cases} m \in C_{L_0}([0,\tau], X) : m(t) = Kn(t), & n \in N_0(L); 0 \leq t \leq \tau, \\ m(t) = 0, & -h \leq t \leq 0; \end{cases} \tag{16}$$

Let

$$f_x : \overline{M_0} \to \overline{M_0}$$

defined by

$$f_x = \begin{cases} Kn, & 0 < t \leq \tau; \\ 0, & -h \leq t \leq 0; \end{cases} \tag{17}$$

where n is given by the unique decomposition of $F(x+m)(t) = n(t) + q(t)$, $n \in N_0(L)$ and $q \in \overline{R(B)}$.

The following assumption is made

$(A1)$ $\overline{R(A_1)} \subset \overline{R(B)}$.

Theorem 3.1 *The operator f_x has a fixed point in M_0 if $M(1+c)P\tau < 1$.*

Proof. Since $S(t)$ is compact, K is compact and f_x is compact. Let $z \in Z$ then $z = q + n$, $n \in N_0(L)$, $q \in \overline{R(B)}$. Also $\|n\|_Z \leq (1+c)\|z\|_Z$ for some constant c. Let

$$B_r = \{v \in \overline{M_0} : ||v|| \leq r\}.$$

Let $m \in B_r$ and $\|f(0,0,(x+m)(a(m(s),0)\| \leq l_f$. Suppose on the other hand

$$
\begin{aligned}
r < \|f_x(m)\| =& \|Kn\| \leq \int_0^t \|S(t-s)n(s)\|ds \\
\leq& \int_0^t M(1+c)\|F(x+m)\|_Z ds \\
\leq& \int_0^t M(1+c)[\|f(s,(x+m)_s,(x+m)(a((x+m)(s),s)))\| \\
-&\|f(0,0,(x+m)(a(m(s),0))))\| + \|f(0,0,(x+m)(a(m(s),0))))\|] \\
\leq& M(1+c)\int_0^t P[\|(x+m)(s+\theta)-0\| \\
+&\|(x+m)(a((x+m)(s),s)) - (x+m)(a((m)(s),0))\| + l_f]ds \\
\leq& M(1+c)\int_0^t P[\|x\| + \|m\| + l|a((x+m)(s),s) - a(m(s),0)| + l_f]ds \\
\leq& M(1+c)\int_0^t P[\|x\| + r + lL_a\|(x+m)(s) - m(s)\| + l_f]ds \\
\leq& M(1+c)\int_0^t P[\|x\| + r + lL_a\|x\| + l_f]ds \\
\leq& M(1+c)P(\|x\|T + r\tau + lL_a\|x\|T + l_f T).
\end{aligned}
\tag{18}
$$

Dividing by r and taking limit as r tends to ∞ we get a contradiction. So f_x maps B_r into itself. Therefore, by Schauder fixed point theorem it has a fixed point.

Theorem 3.2 *Suppose the linear control system* (2)

$$
\begin{aligned}
\frac{dx(t)}{dt} &= Ax(t) + Bu(t), \\
x(0) &= \phi(0),
\end{aligned}
\tag{19}
$$

is approximately controllable then the linear delay control system (7)

$$
\begin{aligned}
\frac{dx(t)}{dt} &= Ax(t) + A_1 x_t + Bu(t), \\
x(t) &= \phi(t),\ -h \leq t \leq 0,
\end{aligned}
$$

is controllable if assumptions $(A1)$ *hold.*

Proof. Consider

$$
\begin{aligned}
y'(t) &= Ay(t) + Bu(t), t\ \in [0,\tau], \\
y(t) &= \phi(t),\ t \in [-h,0].
\end{aligned}
\tag{20}
$$

The mild solution of equation (20) is as follows

$$
\begin{aligned}
y(t) &= T(t)\phi(0) + \int_0^t T(t-s)Bu(s)ds,\ t > 0, \\
y(t) &= \phi(t),\ t \in [-h,0].
\end{aligned}
\tag{21}
$$

Since $\overline{R(A_1)} \subset \overline{R(B)}$, $\forall\ \epsilon > 0$, $\exists\ w \in U$ such that

$$\|A_1 y_s - Bw\|_Z \leq \epsilon.$$

Let $x(t)$ be a solution of linear delay control system corresponding to control $(u - w)$ satisfying

$$\begin{aligned} x(t) &= T(t)\phi(0) + \int_0^t T(t-s)\{B(u-w) + A_1 x_s\}ds, t > 0, \\ x(t) &= \phi(t), t \in [-h, 0]. \end{aligned} \tag{22}$$

If $t \in [-h, 0]$, then

$$x_0(t) - y_0(t) = 0$$

and if $t \in (0, \tau]$ then we get

$$\begin{aligned} x(t) - y(t) &= \int_0^t T(t-s)[-Bw(s) + A_1 x_s] \\ &= \int_0^t T(t-s)[-Bw(s) + A_1 y_s]ds \\ &+ \int_0^t T(t-s)[A_1 x_s - A_1 y_s]ds. \end{aligned} \tag{23}$$

Take the norm on both sides

$$\begin{aligned} \|x(t) - y(t)\| &\leq K_0 \int_0^t \|Bw(s) - A_1 x_s\|ds \\ &+ K_0 \int_0^t \|A_1 x_s - A_1 y_s\|ds \\ &\leq K_0 \tau \|Bw(s) - A_1 x_s\|_Z + K_0 \int_0^t K_1 \|x_s - y_s\|ds \\ &\leq K_0 \epsilon \tau + K_0 \int_0^t K_1 \|x_s - y_s\|ds \\ &\leq K_0 \epsilon \tau + K_0 \int_{-h}^t K_1 \|x(\eta) - y(\eta)\|d\eta, \end{aligned} \tag{24}$$

where $\|A_1\| \leq K_1$, since A_1 is bounded linear operator from $C_{L_0}([-h, \tau], X)$ to $L_2([0, \tau], X)$ and $\widetilde{A} : L_2([0, \tau], X) \to C_0([0, \tau], X)$ defined by $\widetilde{A}(x) = \int_0^t T(t-s)A_1 x_s ds$ This implies

$$||x(t) - y(t)|| \leq K_0 \epsilon \tau + K_0 K_1 \int_{-h}^t \|x(\eta) - y(\eta)\|d\eta. \tag{25}$$

Using Gronwall's inequality

$$||x(t) - y(t)|| \leq K_0 \epsilon \tau \exp(K_0 K_1 \{\tau + h\}).$$

Since RHS depends on ϵ, it can be made as small as possible. This implies that the reachable set of linear delay control system is dense in the reachable set of the linear control system (2) which in turn is dense in X as (7) is apprroximately controllable. Hence the linear delay control system is controllable.

Theorem 3.3 *The semilinear control system (1) is approximately controllable if the linear delay control system (7)*

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + A_1 x_t + Bu(t), \\ x(t) &= \phi(t), \ -h \le t \le 0, \end{aligned}$$

is approximately controllable.

Proof. Let $x(.)$ be the mild solution of the linear delay control system (7) given by

$$x(t) = S(t)\phi(0) + KBu(t), t \in (0, \tau],$$

$$x(t) = \phi(t), t \in [-h, 0].$$

We prove

$$y(t) = x(t) + m_0(t)$$

to be mild solution of semilinear problem (1). Since

$$KF_h(x + m_0)(t) = Kn(t) + Kq(t),$$

operating K on both sides at $m = m_0$, fixed point of f_x,

$$\begin{aligned} KF_h(x + m_0)(t) &= Kn(t) + Kq(t) \\ &= m_0(t) + Kq(t). \end{aligned} \tag{26}$$

Add $x(.)$ to both sides and using $y(t) = x(t) + m_0(t)$, we have

$$\begin{aligned} x(t) + KF_h(x + m_0)(t) &= x(t) + m_0(t) + Kq(t), \\ x(t) + KF_h(y)(t) &= y(t) + Kq(t), \\ \Rightarrow y(t) &= x(t) + KF_h(y)(t) - Kq(t), \\ \Rightarrow y(t) &= S(t)\phi(0) + K(Bu - q)(t) + KF_h(y)(t). \end{aligned} \tag{27}$$

This is the mild solution of semilinear problem with control $(Bu - q)$. By following the same proof in [13] we get the following conclusion that since $q \in \overline{R(B)}$, there exists a $v \in U$ such that $\|Bv - q\| < \epsilon$ for any given $\epsilon > 0$. Let x_v be a solution of the given semilinear delay control system (1.1) corresponding to the control v. Then as shown by [7] we have $||y(\tau) - x_v(\tau)|| = ||x(\tau) - x_v(\tau)|| \le \epsilon$. This implies that $x(\tau) \in \overline{K_\tau(f)}$. Then it follows that $\overline{K_\tau(0)} \subset \overline{K_\tau(f)}$. Thus (1) is approximately controllable, since the corresponding linear system (7) is approximately controllable.

4 Example

Let us consider the heat control system with finite delay

$$\begin{aligned} \frac{\partial y(t,x)}{\partial t} &= \frac{\partial^2 y(t,x)}{\partial x^2} + y(t+\theta, x) + Bu(t,x) + f(t, x(t+\theta), x(a(x(s), s)))ds \\ &\quad 0 < t < T, \ -h < \theta < 0, \ 0 < x < \pi, \\ y(t, 0) &= y(t, \pi) = 0, \ 0 \le t \le T, \\ y(t, x) &= \xi(x), \ \ -h \le t \le 0, \ \ 0 \le x \le \pi. \end{aligned} \tag{28}$$

Let $X = L_2(0,\pi)$ and $A = -\frac{d^2}{dx^2}$. Define

$$\begin{aligned} D(A) = \{y &\in X : y, \frac{dy}{dx} \; are \; absolutely \; continuous, \\ \frac{d^2y}{dx^2} &\in X \; and \; y(0) = y(\pi) = 0\}. \end{aligned}$$

For $y \in D(A)$, $y = \sum_{n=1}^{\infty} < y, \phi_n > \phi_n$ and $Ay = -\sum_{n=1}^{\infty} n^2 < y, n > \phi_n$. where $\phi_n(x) = \frac{2}{\pi}^{\frac{1}{2}} \sin nx,\ 0 \le x \le \pi,\ n = 1, 2, 3...$ is the eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator A. ϕ_n is an orthonormal base. A will generate a compact semigroup $T(t)$ such that $T(t)y = \sum_{n=1}^{\infty} e^{-n^2 t} < y, \phi_n > \phi_n,\ n = 1, 2, ...\ \forall\ y \in X$. Let the infinite dimensional control space be defined as $U = \{u : u = \sum_{n=2}^{\infty} u_n\phi_n,\ \sum_{n=2}^{\infty} u_n^2 < \infty\}$ with norm $\|u\|_U = (\sum_{n=2}^{\infty} u_n^2)^{\frac{1}{2}}$. Thus U is a Hilbert space. Let $\widetilde{B} : U \to X :\ \widetilde{B}u = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n \ \ for\ u = \sum_{n=2}^{\infty} u_n\phi_n \in U$. The bounded linear operator $B : L_2(0,T : U) \to L_2(0,T;X)$ is defined by $(Bu)(t) = \widetilde{B}u(t)$. Then this problem (28) can be reformulated into an abstract semilinear differential equation with deviated argument and finite delay by substituting $I = A_1$. If the hypotheses $(H1) - (H3)$ and assumption $(A1)$ are satisfied then it can be shown that this system (28) is approximately controllable.

5 Conclusion

Thus, we prove the existence and uniqueness and approximate controllability of the functional differential equation (1) with deviated argument and finite delay by using Schuader fixed point theorem and fundamental solution instead of C_0 semigroup.

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References

[1] Gal, C. G. Semilinear Abstract Differential Equations with Deviated Argument. *Research on Evolution Equation Compendium* **333** (16) (2009) 381–386.

[2] Dauer, J.P. and Mahmudov, N.I. Approximate controllability of semilinear functional equations in Hilbert spaces. *J. Math. Anal. Appl.* **273** (2002) 310-327.

[3] Pandey, D. N., Ujlayan, A. and Bahuguna, D. On nonlinear abstract neutral differential equations with deviated argument. *Nonlinear Dynamics and Systems Theory* **10**(3) (2010) 283-294.

[4] Jeong, J.M., Kim, J.R. and Roh, H.H. Controllability for semilinear retarded control systems in Hilbert spaces, *Journal of Dynamical and Control Systems* **13**(4) (2007) 577-591.

[5] Balachandran, K. and Anandhi, E. R. Boundary controllability of integrodifferential systems in Banach spaces. In: *Proc. Indian Acad. Sci. (Math. Sci.)* **111**(1) (2001) 127-135.

[6] Balachandran, K. and Park, J. Y. Existence of solutions and controllability of nonlinear integrodifferential systems in Banach spaces. *Mathematical Problems in Engineering* **2** (2003) 65–79.

[7] Koichiro Naito. Controllability of Semilinear Control Systems Dominated by Linear Part. *Siam J. Control and Optimization* **25**(3) (1987) 715–722.

[8] El'sgol'ts, L. E. and Norkin, S. B. Introduction to the Theory of Differential Equations with Deviating Arguments. *Academic Press*, 1973.

[9] Wang, L.W. Approximate controllability of delayed semilinear control of control system. *J. Appl. Math. Stoch. Anal.* **1** (2005) 67-76.

[10] Benchohra, M., Gorniewicz, L., Ntouyas, S. K. and Ouahab, A. Controllability results for impulsive functional differential inclusions. *Rep. Math. Phys.* **54** (2004) 211–228.

[11] Li, M., Wang, M. and Zhang, F. Controllability of impulsive functional differential systems in Banach spaces. *Chaos, Solitons and Fractals* **29** (2006) 175-181.

[12] Muslim, M. and Bahuguna, D. Existence of solutions to neutral differential equations with deviated argument. *Electronic Journal of Qualitative Theory of Differential Equations* (27) (2008) 1–12.

[13] Sukavanam, N. Approximate controllability of semilinear control of control system with growing nonlinearity. In: *Math. Theory of Control Proc. Int. Conf.* Marcel Dekker, New York, 1993, 353-357.

[14] Sukavanam, N. and Divya. Exact and Approximate controllability of abstract semilinear control systems. *Indian Journal of Pure and Applied Mathematics* **33**(13) (2002) 1835–1837.

[15] Pradeep Kumar. On the New Concepts of Solutions and Existence Results for Impulsive Integro-Differential Equations with a Deviating Argument. *Nonlinear Dynamics and Systems Theory* **14** (1) (2014) 58–75.

[16] Rajib Haloi. On Solutions to a Nonautonomous Neutral Differential Equation with Deviating Arguments. *Nonlinear Dynamics and Systems Theory* **13**(3) (2013) 242–257.

[17] Haloi, R., Pandey, D. N. and Bahuguna, D. Existence, Uniqueness and Asymptotic Stability of Solutions to Non-Autonomous Semi-Linear Differential Equations with Deviated Arguments. *Nonlinear Dynamics and Systems Theory* **12**(2) (2012) 179–191.

[18] Sakthivel, R. and Anandhi, E.R. Approximate controllability of impulsive differential equations with state-dependent delay. *International Journal of Control* **83** (2) (2009) 387–393.

[19] Chang, Y.K. Controllability of impulsive functional differential systems with infinite delay in Banach spaces. *Chaos, Solitons and Fractals* **33** (2007) 1601–1609.

[20] Tai, Z. and Wang, X. Controllability of fractional-order impulsive neutral functional infinite delay integrodifferential systems in Banach spaces. *Applied Mathematics Letters* **22** (2009) 1760–1765.

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Stability Conditions for a Class of Nonlinear Time Delay Systems

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Abstract: In this paper, stability analysis for a class of nonlinear time delay system is done. A state space representation of the class of system under consideration is used and a transformation is carried out to represent the system by an arrow form matrix. Taking advantage of this representation and applying the Kotelyanski lemma in combination with properties of M-matrices, some new sufficient stability conditions are determined. An illustrative example is presented to show the effectiveness of the proposed approach.

Keywords: *nonlinear time delay systems; arrow matrix; stability analysis.*

Mathematics Subject Classification (2010): 34K20.

1 Introduction

Time delay exists in many practical systems. This includes chemical processes, teleoperators, mechanical systems, network control systems etc. see [2, 3, 8, 11]. The delay can be an inherent part of the dynamics of the system or can be a result of actuators and sensors used and the time needed for transmission of control signals. Presence of delay complicates the analysis of such systems and can even cause instability [6,10,11]. In many situations industrial models have to represent nonlinear phenomena for the delay or the system itself. This is justified by the insufficiency of the first order linear approximations to explain the typically nonlinear problem of instability linked to excessive initial conditions or perturbations. Difficulties are greater when delays appear in nonlinear systems, see [1,3–5] for an excellent exposition of nonlinear delay equations. For all these reasons, there has been an extensive literature on stability of time delay systems [7,19,21]. In this

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paper, we determine sufficient stability conditions for nonlinear systems with constant delay.

There are mainly two main approaches in determining stability conditions for time delay systems, namely, delay independent conditions and delay dependent conditions. To this extent most of the existing results are delay-independent [6, 9, 12, 20] and few are delay-dependent, see [13, 18, 22] and the references therein. Even fewer give practical results which can be applied to nonlinear systems. In this paper, we determine sufficient delay dependent stability conditions for nonlinear systems with a constant delay.

The paper is organized as follows. In Section 2, the main result is given. Delay dependent sufficient conditions for stability of the nonlinear system with delay are derived. Section 3 is devoted to the application of the obtained result to delayed Lurie systems. An illustrative example is given in Section 4. We finish this paper by some concluding remarks in Section 5.

2 Sufficient Stability Conditions

Our work consists of determining stability conditions for systems described by the following equation:

$$\tilde{S}:\left\{\begin{array}{l} y^{(n)}(t)+\sum_{i=0}^{n-1}\tilde{f}_i(t,x_t,\wp)y^{(i)}(t)+\sum_{i=0}^{n-1}\tilde{g}_i(t,x_t,\wp)y^{(i)}(t-\tau)=0,\\ y^{(i)}(t)=\phi_i(t),\forall t\in[-\tau\ \ 0],\, i=0,\dots,n-1, \end{array}\right. \tag{1}$$

where τ is a constant delay and $\tilde{f}_i$, $\tilde{g}_i$ $i=0,\dots,n-1$ are nonlinear functions. Let us fix the notation used. Let $C_n = C([-\tau\ 0]\ ,R^n)$ be the Banach space of continuous functions mapping the interval $[-\tau\ 0]$ into R^n with the topology of uniform convergence. Let $x_t\in C_n$ be defined by $x_t(\theta)=x(\theta)$, $\theta\in[-\tau\ 0]$. For a given $\phi\in C_n$, we define $\|\phi\|=\sup_{-\tau\le\theta\le 0}\|\phi(\theta)\|$, $\phi(\theta)\in R^n$. Let $x_t\in C_n$ be defined by $x_t(\theta)=x(\theta)$, $\theta\in[-\tau\ 0]$. The functions $\tilde{f}_i$, $\tilde{g}_i$, $i=0,1,..,n-1$ are completely continuous mapping the set $J_a\times C_n^H\times S_\wp$ into R, where $C_n^H=\{\phi\in C_n,\|\phi\|<H\}$, $H>0$, $J_a=[a\ +\infty)$, $a\in R$ and $S_\wp=\{\wp\in R,\underline{\wp}\le\wp\le\overline{\wp}$ where $\underline{\wp}\le\overline{\wp}\in R\}$. Finally we say that the function g satisfies the finite sector condition if $g\in E([k_1\ ,\ k_2])=\{g\mid g(0)=0,k_1\sigma^2<\sigma g(\sigma)<k_2\sigma^2,\sigma\neq 0$ and $k_1<k_2\}$. In the sequel, we denote $(t,x_t,\wp)=(.)$. We start by making the following changes:

$$x_{i+1}(t)=y^{(i)}(t),\, i=0,\dots,n-1$$

which implies that

$$\dot{x}_i(t)=x_{i+1}(t),\, i=0,\dots,n-1,$$

therefore,

$$\dot{x}_n(t)=-\sum_{i=1}^{n}\tilde{f}_{i-1}(.)x_i(t)-\sum_{i=1}^{n}\tilde{g}_{i-1}(.)x_i(t-\tau).$$

The studied system is described by the following state space representation:

$$\left\{\begin{array}{l}\dot{x}(t)=\tilde{F}(.)x(t)+\tilde{G}(.)x(t-\tau),\\ x(t)=\phi(t),\forall t\in[-\tau\ \ 0],\end{array}\right. \tag{2}$$

where

$$x(t)=\left(\begin{array}{ccccc} x_1(t) & x_2(t) & \dots & x_{n-1}(t) & x_n(t)\end{array}\right)',$$

$$\phi(t) = \begin{pmatrix} \phi_1(t) & \phi_2(t) & \dots & \phi_{n-1}(t) & \phi_n(t) \end{pmatrix}'.$$

The matrices $\tilde{F}(.)$ and $\tilde{G}(.)$ are given by

$$\tilde{F}(.) = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -\tilde{f}_0(.) & -\tilde{f}_1(.) & \cdots & -\tilde{f}_{n-2}(.) & -\tilde{f}_{n-1}(.) \end{pmatrix} \tag{3}$$

and

$$\tilde{G}(.) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ -\tilde{g}_0(.) & -\tilde{g}_1(.) & \dots & -\tilde{g}_{n-2}(.) & -\tilde{g}_{n-1}(.) \end{pmatrix}. \tag{4}$$

Applying the following transformation:

$$x = Pz, \tag{5}$$

where

$$P = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \alpha_1^{n-2} & \alpha_2^{n-2} & \cdots & \alpha_{n-1}^{n-2} & 0 \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_{n-1}^{n-1} & 1 \end{pmatrix} \quad \alpha_i \neq \alpha_j \quad \forall i, j \tag{6}$$

leads to the following state representation

$$S : \dot{z}(t) = F(.)z(t) + \Delta(.)z(t - \tau) \tag{7}$$

which describes the dynamics of system (1) by using the new state vector z. The matrix $F(.)$ is given by

$$F(.) = P^{-1}\tilde{F}(.)P = \begin{pmatrix} \alpha_1 & & & & \beta_1 \\ & \alpha_2 & & & \beta_2 \\ & & \ddots & & \vdots \\ & & & \alpha_{n-1} & \beta_{n-1} \\ \gamma_1(.) & \gamma_2(.) & \cdots & \gamma_{n-1}(.) & \gamma_n(.) \end{pmatrix}. \tag{8}$$

Elements of the matrix $F(.)$ are defined in [15] by

$$\gamma_i(.) = -D(\alpha_i, .) \quad i = 1...n-1, \tag{9}$$

where

$$D(s, .) = s^n + \sum_{i=0}^{n-1} \tilde{f}_i(.)s^i \tag{10}$$

and

$$\gamma_n(.) = -\tilde{f}_{n-1}(.) - \sum_{i=1}^{n-1} \alpha_i, \tag{11}$$

$$\beta_i = \left.\frac{\alpha_i - \lambda}{Q(\lambda)}\right|_{\lambda=\alpha_i} \quad i = 1...n-1, \tag{12}$$

where

$$Q(\lambda) = \prod_{j=1}^{n-1} (\lambda - \alpha_j). \tag{13}$$

The matrix $\Delta(.)$ is given by

$$\Delta(.) = P^{-1}\tilde{G}(.)P = \begin{pmatrix} O_{n-1,n-1} & O_{n-1,1} \\ \delta_1(.)\cdots\delta_{n-1}(.) & \delta_n(.) \end{pmatrix} \tag{14}$$

with

$$\delta_i(.) = -N(\alpha_i, .), \; i = 1, \ldots, n-1, \tag{15}$$

where

$$N(s, .) = \sum_{i=0}^{n-1} \tilde{g}_i(.) s^i \tag{16}$$

and

$$\delta_n(.) = -\tilde{g}_{n-1}(.). \tag{17}$$

Based on this transformation and the arbitrary choice of parameters α_i, $i = 1, \ldots, n-1$ which play an important role in simplifying the use of aggregate techniques, we give now the main result. Let us start by writing our system in another form. By using the Newton-Leibniz formula

$$z(t - \tau) = z(t) - \int_{t-\tau}^{t} \dot{z}(\theta) d\theta, \tag{18}$$

equation (7) becomes

$$\dot{z}(t) = (F(.) + \Delta(.))z(t) - \Delta(.) \int_{t-\tau}^{t} \dot{z}(\theta) d\theta. \tag{19}$$

Let Ω be a domain of R^n, containing a neighborhood of the origin, and $\sup\limits_{J_\tau,\, \Omega, S_\wp}$ the suprema calculated for $t \in J_\tau$ (i.e $t \geq \tau$), for functions x with values in Ω, and for $\wp$ in $S_\wp$. Next, using the special form of system (1) and applying the notation $\sup_{[.]} = \sup\limits_{J_\tau,\, \Omega, S_\wp}$, we can announce the following theorem.

Theorem 2.1 *The system (1) is asymptotically stable, if there exist distinct parameters $\alpha_i < 0$ $i = 1, ..., n-1$, such that the matrix $T(.)$ is the opposite of an M-matrix, where $T(.)$ is given by*

$$T(.) = \begin{pmatrix} \alpha_1 & & & & |\beta_1| \\ & \alpha_2 & & & |\beta_2| \\ & & \ddots & & \vdots \\ & & & \alpha_{n-1} & |\beta_{n-1}| \\ t_1(.) & t_2(.) & \cdots & t_{n-1}(.) & t_n(.) \end{pmatrix} \tag{20}$$

and the elements $t_i(.)$, $i = 1, ..., n$ *are given by*

$$t_i(.) = \frac{|\gamma_i(.) + \delta_i(.)| + \tau|\alpha_i| \sup_{[.]} |\delta_i(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|} \tag{21}$$

and

$$t_n(.) = \gamma_n(.) + \delta_n(.) + \frac{\tau \sup_{[.]} |\delta_n(.)||\gamma_n(.) + \delta_n(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|} + \frac{\tau \sum_{i=1}^{n-1} |\beta_i| \sup_{[.]} |\delta_i(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|}. \tag{22}$$

Proof. We use the following vector norm

$$p(z) = (\ p_1(z) \ \ p_2(z) \ \ p_3(z) \ \ ... \ \ p_n(z)\ )', \tag{23}$$

where $p_i(z) = |z_i|$, $i = 1, \ldots, n-1$ and $p_n(z)$ is given by

$$p_n(z) = |z_n| + \frac{\sum_{i=1}^{n} \sup_{[.]} |\delta_i(.)|}{1 - \tau \left(\sup_{[.]} |\delta_n(.)| \right)} \int_{-\tau}^{0} \int_{t+\theta}^{t} |\dot{z}_i(\vartheta)| d\vartheta d\theta \tag{24}$$

with the condition

$$\tau \left(\sup_{[.]} |\delta_n(.)| \right) < 1. \tag{25}$$

Let $V(t)$ be a radially unbounded Lyapunov function given by (26).

$$V(t) = \left\langle (p(z(t)))', w \right\rangle = \sum_{i=1}^{n} w_i p_i(z(t)), \tag{26}$$

where $w \in R_+^n$, $w_i > 0$, $i = 1, \ldots, n$. First, note that

$$V(t_0) \leq \sum_{i=1}^{n-1} w_i \, |z_i(t_0)| + w_n \left(|z_n(t_0)| + \frac{\sup_{[.]}(|\delta_n(.)|)}{1 - \tau \sup_{[.]}(|\delta_n(.)|)} \sup_{[-\tau,0]} |\dot{\phi}_n| \frac{\tau^2}{2} \right) := r < +\infty \tag{27}$$

and

$$V(t) \geq \sum_{i=1}^{n} w_i \, |z_i(t)| \, . \tag{28}$$

The right Dini derivative of $V(t)$, along the solution of (19), gives

$$D^+V(t) = \sum_{i=1}^{n} w_i \frac{d^+ p_i(z(t))}{dt^+}. \tag{29}$$

For clarification reasons, each element of $\frac{d^+ p_i(z(t))}{dt^+}$, $i = 1, ..., n$ is calculated separately. Let us begin with the first $(n-1)$ elements. Because $|z_i| = z_i sign(z_i)$, we can write, for

$i = 1, ..., n-1,$

$$\begin{aligned} \frac{d^+ p_i(z(t))}{dt^+} &= \frac{d^+ |z_i(t)|}{dt^+} \\ &= \frac{d^+ z_i(t)}{dt^+} sign(z_i(t)) \\ &= (\alpha_i z_i(t) + \beta_i z_n(t)) sign(z_i(t)) \\ &\le \alpha_i |z_i(t)| + |\beta_i| |z_n(t)| \end{aligned} \tag{30}$$

and $\frac{d^+ p_n(z)}{dt^+}$ is given by

$$\tfrac{d^+ p_n(z)}{dt^+} = \frac{d^+ |z_n|}{dt^+} + \frac{\sum\limits_{i=1}^{n} \sup\limits_{[.]} |\delta_i(.)|}{1 - \tau \sup\limits_{[.]} |\delta_n(.)|} \frac{d^+}{dt^+} \left[\int_{-\tau}^{0} \int_{t+\theta}^{t} |\dot{z}_i(\vartheta)| d\vartheta d\theta \right]. \tag{31}$$

Finally, it is easy to see that equation (31) can be overvalued by the following one

$$\frac{d^+ p_n(z)}{dt^+} \le \sum_{i=1}^{n} t_i(.) \, |z_i| \, , \tag{32}$$

where elements $t_i(.)$, $i = 1, ..., n$ are given by

$$\begin{aligned} t_i(.) &= |\gamma_i(.) + \delta_i(.)| + \frac{\tau \sup_{[.]} |\delta_n(.)| |\gamma_i(.) + \delta_i(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|} + \frac{\tau |\alpha_i| \sup_{[.]} |\delta_i(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|} \\ &= \frac{|\gamma_i(.) + \delta_i(.)| + \tau |\alpha_i| \sup_{[.]} |\delta_i(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|} \end{aligned} \tag{33}$$

and

$$t_n(.) = \gamma_n(.) + \delta_n(.) + \frac{\tau \sup_{[.]} |\delta_n(.)| |\gamma_n(.) + \delta_n(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|} + \frac{\tau \sum\limits_{i=1}^{n-1} |\beta_i| \sup\limits_{[.]} |\delta_i(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|}. \tag{34}$$

Then the inequality (29) becomes

$$D^+ V(t) < \left\langle T^{'}(.) w, |z| \right\rangle , \tag{35}$$

where $T(.)$ is given by (36)

$$T(.) = \begin{pmatrix} \alpha_1 & & & & |\beta_1| \\ & \alpha_2 & & & |\beta_2| \\ & & \ddots & & \vdots \\ & & & \alpha_{n-1} & |\beta_{n-1}| \\ t_1(.) & t_2(.) & \cdots & t_{n-1}(.) & t_n(.) \end{pmatrix}. \tag{36}$$

Because the nonlinear elements of $T(.)$ are isolated in the last row, the eigenvector $v(t, x_t, \wp)$ relative to the eigenvalue λ_m is constant [17], where λ_m is such that $Re(\lambda_m) = \max\{Re(\lambda), \lambda \in \lambda(T(.))\}$. Then, in order to have $D^+ V(t) < 0$, it is sufficient to have $T(.)$ as the opposite of an M$-$matrix. Indeed, according to properties of

M−matrices, we have $\forall\, \sigma \in R_+^{*n}$, $\exists\, w \in R_+^{*n}$ such that $-\left(T^{'}(.)\right)^{-1}\sigma = w$. This enables us to write the following equation

$$\left\langle T^{'}(.)w, |z(t)|\right\rangle = \langle -\sigma, |z(t)|\rangle = -\sum_{i=1}^{n} \sigma_i |z_i(t)| \tag{37}$$

which yields

$$D^+V(t) \leq -\sum_{i=1}^{n} \sigma_i |z_i(t)|. \tag{38}$$

This completes the proof of theorem.

Remark 2.1 If the couple $(D(s,.) + N(s,.), Q(s))$ forms a positive pair, then there exist distinct negative parameters α_i, $i = 1, ..., n-1$, verifying the condition $(\gamma_i(.) + \delta_i(.))\,\beta_i > 0$ for $i = 1, ..., n-1$.

Using Theorem 2.1 and Remark 2.1, the obtained supremum is a function of α_i values, $i = 1, ..., n-1$. As a result, a sufficient condition for asymptotic stability of our system is when values of the time delay are less than this supremum.

Corollary 2.1 *If the couple $(D(s,.) + N(s,.), Q(s))$ forms a positive pair and there exist distinct negative parameters α_i, $i = 1, ..., n-1$, such that:*

$$2\tau\left((\gamma_n(.) + \delta_n(.))\sup_{[.]} |\delta_n(.)| - \nu(.)\right) + \frac{D(0,.) + N(0,.)}{Q(0)} > 0, \tag{39}$$

then the system (1) is asymptotically stable.

Proof. According to Remark 2.1, we find that

$$\begin{aligned} \gamma_n(.) + \delta_n(.) - \sum_{j=1}^{n-1} \frac{|\gamma_j(.) + \delta_j(.)||\beta_j|}{\alpha_j} &= \gamma_n(.) + \delta_n(.) - \sum_{j=1}^{n-1} \frac{(\gamma_j(.) + \delta_j(.))\beta_j}{\alpha_j} \\ &= -\frac{D(0,.) + N(0,.)}{Q(0)}. \end{aligned}$$

The result of Theorem 2.1 becomes

$$-2\tau(\gamma_n(.) + \delta_n(.))\sup_{[.]} |\delta_n(.)| + 2\tau\nu(.) - \frac{D(0,.) + N(0,.)}{Q(0)} < 0$$

which is equivalent to

$$2\tau\left((\gamma_n(.) + \delta_n(.))\sup_{[.]} |\delta_n(.)| - \nu(.)\right) + \frac{D(0,.) + N(0,.)}{Q(0)} > 0.$$

This completes the proof of corollary.

Remark 2.2

- Theorem 2.1 depends on the new basis change, where parameters α_i of the matrix P are arbitrary chosen such that matrix $T(.)$ is the opposite of an M-matrix. The appropriate choice of the set of free parameters α_i makes the given stability conditions satisfied.

- The theorem takes into account the fact that delayed terms may stabilize our system [22]. Theorem 2.1 can hold even if $D(s,.)$ is unstable. This is another advantage as the majority of previously published results assume that $D(s)$ is linear and stable.
- The theorem can easily be extended to the study of systems with multiple time-delays and can generalize the work of [14] in the case of fuzzy TS systems with time-delay and the work of [16] in the case of discrete time delay system.

3 Application to Delayed Nonlinear n-th Order All Pole Plant

Consider the complex system S given in Figure 1.

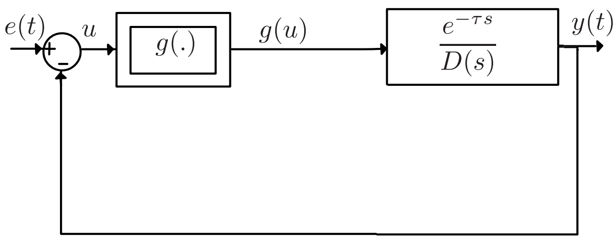

Figure 1: Block representation of the studied system.

$D(s)$ is defined by (10) and $N(s) = 1$, respectively. In this case $\tilde{f}_i(.)$ are constants and g is a function satisfying the finite sector condition. Let $\hat{g}$ be a function defined as follows

$$\hat{g}(e(\theta), y(\theta)) = \frac{g(e(\theta) - y(\theta))}{e(\theta) - y(\theta)}, e(\theta) \neq y(\theta) \ \ \forall\theta \in [-\tau \ \ +\infty[, \tag{40}$$

$$\sup_{[.]} |\hat{g}(e(t), y(t))| = \bar{g} \in R_+^*.$$

The presence of delay in the system of Figure 1 makes stability study difficult. The following steps show how to represent this system in the form of system (1). Then we can write

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i \frac{d^i y(t)}{dt^i} = -\hat{g}(e(t-\tau), y(t-\tau))y(t-\tau) + \hat{g}(e(t-\tau), y(t-\tau))e(t-\tau). \tag{41}$$

We use the following notation

$$\hat{g}(.) = \hat{g}(e(t-\tau), bx(t-\tau)),$$

therefore,

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i y^{(i)}(t) + \hat{g}(.)y(t-\tau) = \hat{g}(.)e(t-\tau). \tag{42}$$

It is clear that system (42) is equivalent to system (1) in the special cases $e(\theta) = 0$ and $e(\theta) = -Kx(\theta)$, $x(t) = \left(y(t), \dot{y}(t), ..., y^{(n)}(t)\right)'$, $\forall\ \theta \in [-\tau \ \ +\infty[$. We will now consider each case separately.

3.1 Case $e(t) = 0$

In the case $e(t) = 0 \ \ \forall t \in [-\tau \ +\infty[$, the description of the system becomes

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i y^{(i)}(t) + \hat{g}(.)y(t-\tau) = 0. \tag{43}$$

This is a special representation of system (1) where $\tilde{f}_i(.) = a_i$,$\tilde{g}_1(.) = \hat{g}(.)$ $\tilde{g}_i(.) = 0$ $\forall \ i = 2, ..., n-1$, $D(s,.) = D(s)$, $N(s,.) = \hat{g}(.)$, $\gamma_n(.) = \gamma_n = -a_{n-1} - \sum_{i=1}^{n-1} \alpha_i$ and $\delta_n(.) = 0$. A sufficient stability condition for this system is given in the following proposition.

Proposition 1 *If there exist distinct $\alpha_i < 0$ $i = 1, ..., n-1$, such that the following conditions*

$$\begin{cases} \gamma_n < 0, \\ \mu_1(.) + 2\tau\nu_1(.) - \xi_1(.) < 0, \end{cases} \tag{44}$$

where

$$\begin{cases} \mu_1(.) = \gamma_n, \\ \nu_1(.) = \bar{g}, \\ \xi_1(.) = \frac{|D(\alpha_1)+\hat{g}(.)||\beta_1|}{\alpha_1} + \sum_{i=2}^{n-1} \frac{|D(\alpha_i)||\beta_i|}{\alpha_i}, \end{cases} \tag{45}$$

are satisfied. Then the system S is asymptotically stable.

Suppose that $D(s)$ admits n distinct real roots p_i, $\ i = 1, ..., n$ among which there are $n-1$ negative ones. We use the fact that $a_{n-1} = -\sum_{i=1}^{n} p_i$, then the choice $\alpha_i = p_i$, $\forall i = 1, .., n-2$ and $\alpha_{n-1} = p_{n-1} + \varepsilon$ permits us to write $\gamma_n = -a_{n-1} - \sum_{i=1}^{n-1} p_i = p_n - \varepsilon$. In this case the last proposition becomes

Proposition 2 *If $D(s)$ admits $n-1$ distinct real negative roots such that the following conditions*

$$\begin{cases} p_n - \varepsilon < 0, \\ \mu_2(.) + 2\tau\nu_2(.) - \xi_2(.) < 0, \end{cases} \tag{46}$$

are satisfied, where

$$\begin{cases} \mu_2(.) = p_n - \varepsilon, \\ \nu_2(.) = \bar{g}, \\ \xi_2(.) = \frac{|\hat{g}(.)||\beta_1|}{\alpha_1} + \frac{|D(\alpha_{n-1})||\beta_{n-1}|}{\alpha_{n-1}}, \end{cases} \tag{47}$$

then the system S is asymptotically stable.

3.2 Case $e(t) = -Kx(t)$

In this case, take $e(t) = -Kx(t)$ with $K = (k_0, k_1, \ldots, k_{n-1})$, then the obtained system has the same form as (1), with $\hat{g}_1^K(.) = \hat{g}^K(.)\,(k_0 + 1)$ and $\hat{g}_i^K(.) = \hat{g}^K(.)k_{i-1}$, $i = 2, \ldots, n$. The stabilizing values of K can be obtained by making the following changes: $\gamma_n = -a_{n-1} - \sum_{i=1}^{n-1} \alpha_i$, $\delta_n^K(.) = -\hat{g}^K(.)k_{n-1}$, $\nu_1^K(.) = \bar{g}^K \sum_{i=1}^{n-1} \left|\tilde{N}(\alpha_i)\right|$ where $\bar{g}^K = \sup_{[.]} |\hat{g}^K(.)|$ and $\tilde{N}(\alpha) = (1 + k_0) + \sum_{i=1}^{n-1} (b_i + k_i)\alpha^i$.

Proposition 3 *If there exist distinct $\alpha_i < 0$, $i = 1, ..., n-1$, such that the following conditions*

$$\left\{ \begin{array}{l} \gamma_n - \hat{g}^K(.)k_{n-1} < 0, \\ \tau < \dfrac{1}{2\bar{g}^K |k_{n-1}|}, \\ \mu_1^K(.) + 2\tau\nu_1^K(.) - \xi_1^k(.) < 0, \end{array} \right. \tag{48}$$

where

$$\left\{ \begin{array}{l} \mu_1^K(.) = (1 - 2\bar{g}^K\tau|k_{n-1}|)(\gamma_n + \delta_n^K(.)), \\ \nu_1^K(.) = \bar{g}^K \sum_{i=1}^{n-1} |\beta_i||\tilde{N}(\alpha_i)|, \\ \xi_1^K(.) = \sum_{i=1}^{n-1} \frac{|D(\alpha_i) + \hat{g}^K(.)\tilde{N}(\alpha_i)||\beta_i|}{\alpha_i}, \end{array} \right. \tag{49}$$

are satisfied. Then the system S is asymptotically stable.

By a special choice of K the result of Proposition 3 can be simplified. In fact, if the conditions of this proposition are verified we can choose the vector K such that $D(p_i) = \tilde{N}(p_i)$. In this case we obtain $D(p_i) = \tilde{N}(p_i) = 0$, $\forall$, $i = 1, ..., n-1$ and $\nu_1(.) = \xi_1(.) = 0$ which yields the following new proposition.

Proposition 4 *If $D(s)$ admits $n-1$ distinct real negative roots p_i such that the following conditions*

$$\left\{ \begin{array}{l} p_n - \bar{g}^K(.)k_{n-1} < 0, \quad p_n \text{ is the } n\text{-th root of } D(s), \\ \tau < \dfrac{1}{2\bar{g}^K |k_{n-1}|}, \end{array} \right. \tag{50}$$

are satisfied. Then the system S is asymptotically stable.

4 Illustrative Example

Let us consider the block diagram in Figure 2 which describes the dynamics of a time-delayed DC motor speed control system with nonlinear gain, where:

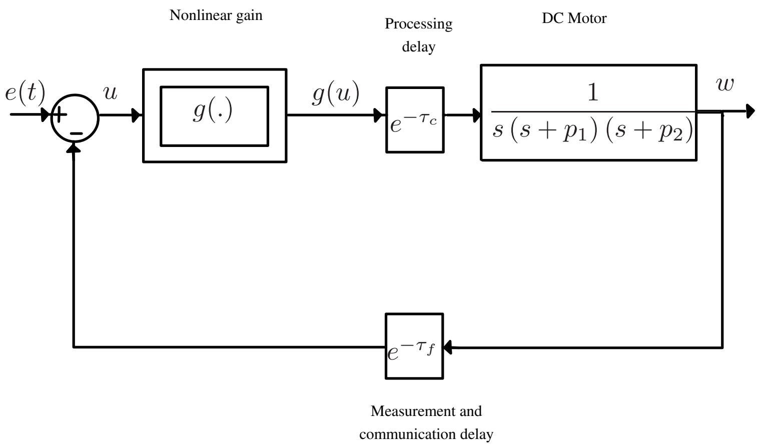

Figure 2: Block diagram of time-delayed DC motor speed control system with nonlinear gain.

- $p_1 = \frac{1}{T_e}$ and $p_2 = \frac{1}{T_m}$ where T_e and T_m are respectively electrical constant and mechanical constant.

- τ_f present the feedback delay between the output and the controller. This delay represents the measurement and communication delays (sensor-to-controller delay).

- τ_c the controller processing and communication delay (controller-to-actuator delay) is placed in the feedforward part between the controller and the DC motor.

- $g(.) : R \to R$ is a function that represents a nonlinear gain.

The process of Figure 2 can also be modeled by Figure 3, where $\tau = \tau_f + \tau_c$.

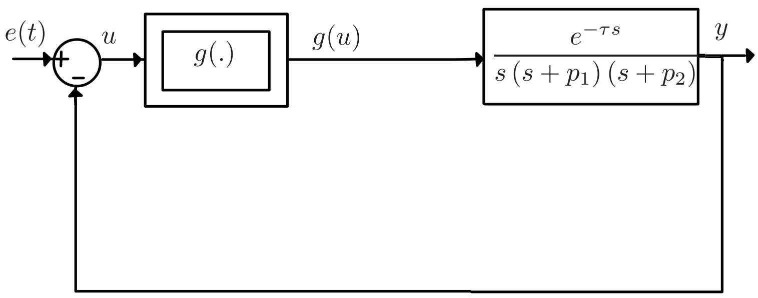

Figure 3: Delayed nonlinear model of DC motor speed control.

It is clear that model of Figure 3 is a particular form of delayed Lurie system in the case where $D(s) = s(s+p_1)(s+p_2) = s^3 + (p_1+p_2)s^2 + p_1 p_2 s$ and $N(s) = 1$. Thereafter, applying the result of Theorem 2.1, a stability condition of the system is that the matrix $T(.)$ given by

$$T(.) = \begin{pmatrix} \alpha_1 & 0 & |(\alpha_1 - \alpha_2)^{-1}| \\ 0 & \alpha_2 & |(\alpha_2 - \alpha_1)^{-1}| \\ t_1(.) & t_2(.) & t_3(.) \end{pmatrix}, \tag{51}$$

where

$$t_1(.) = |\gamma_1 + \hat{g}(.)| + \tau|\alpha_1|\bar{g}, \quad t_2(.) = |\gamma_2|, \quad t_3(.) = \gamma_3 + \tau|\beta_1|\bar{g},$$

must be the opposite of an M-matrix. By choosing α_i, $i = 1, 2$, negative real and distinct, we get the following stability condition:

$$\gamma_3 + 2\tau|\beta_1|\bar{g} - \frac{|\beta_1||\gamma_1 + \hat{g}(.)|}{\alpha_1} - \frac{|\beta_2||\gamma_2|}{\alpha_2} < 0. \tag{52}$$

For the particular choice of $\alpha_1 = -p_1$ and $\alpha_2 = -p_2 + \varepsilon$, $\varepsilon > 0$ yields $|\beta_1| = |\beta_2| = |(\varepsilon + p_1 - p_2)^{-1}|$ and we obtain the new stability condition:

$$2\tau\bar{g} + |p_1|^{-1}|\hat{g}(.)| + |\alpha_2|^{-1}|D(\alpha_2)| < \varepsilon|\varepsilon + p_1 - p_2|. \tag{53}$$

Assume that we have this inequality:

$$\bar{g} < |D(\alpha_2)|.$$

We can find from (53) the stabilizing delay given by the following condition:

$$\tau < \frac{1}{2}\left(\frac{\varepsilon|\varepsilon + p_1 - p_2|}{|D(\alpha_2)|} - |p_1|^{-1}| - |\alpha_2|^{-1}\right).$$

By applying the control $e(t) = -Kx(t)$ with $K = (k_0, k_1, k_2)$, we can determine the stabilizing values of K that can be obtained by making the following changes: $\gamma_3 = -(p_1 + p_2) - \sum_{i=1}^{2} \alpha_i$, $\delta_1^K(.) = -\hat{g}^K(.)\,(k_0 + 1)\,, \delta_i^K(.) = -\hat{g}^K(.)k_{i-1},\ \ i = 2, 3.$ $\nu_1^K(.) = \bar{g}^K \sum_{i=1}^{2} |\beta_i||\tilde{N}(\alpha_i)|$ where $\bar{g}^K = \sup_{[.]} |\hat{g}^K(.)|$ and $\tilde{N}(\alpha) = 1 + k_0 + \sum\limits_{i=1}^{2} k_i\alpha^i$.

If we choose $\alpha_i < 0$, $i = 1, 2$, such that the following conditions $D(\alpha_i) = \tilde{N}(\alpha_i) = 0$, $\forall$, $i = 1, 2$ hold, we get $\dfrac{1 + k_0}{k_2} = p_1 + p_2$, $\dfrac{k_1}{k_2} = p_1 p_2$ and from Proposition 3 the stabilizing gain values satisfying the following relations:

$$\begin{cases} 0 - \bar{g}^K(.)k_2 < 0, \\ |k_2| < \dfrac{1}{2\tau\bar{g}^K}. \end{cases}$$

Finally we find the domain of stabilizing $k_0,\ k_1,\ k_2$ as follows

$$\begin{cases} 0 < k_2 < \dfrac{1}{2\tau\bar{g}^K}, \\ k_1 = p_1 p_2 k_2, \\ k_0 = (p_1 + p_2)\,k_2 - 1. \end{cases}$$

5 Conclusion

In this paper, new sufficient stability conditions for a class of time delay systems are derived. The proposed method is based on a specific choice of a Lyapunov function. The obtained conditions are explicit and easy to apply. Indeed, the proposed approach is successfully applied to nonlinear n-th order all pole plant that is a particular form of delayed Lurie Postnikov systems. In addition, the simplicity of the application of these criteria is demonstrated on model of time-delayed DC motor speed control.

References

[1] Richard, J.P. Time-delay systems: an overview of some recent advances and open problems. *Automatica* **39** (10) (2003) 1667–1694.

[2] Nuño, E., Basañez, L. and Ortega, R. Control of Teleoperators with Time-Delay: A Lyapunov Approach. In: *Topics in Time Delay Systems Analysis, Algorithms and Control*, (eds: J.J. Loiseau, W. Michiels, S.I. Niculescu and R. Sipahi). Berlin, Heidelberg, New York: Springer-Verlag, 2009, 371–382.

[3] Hahn, W. *Stability of the Motion*. New York: Springer-Verlag, 1967.

[4] Hale, J.K. *Theory of Functional Differential Equations*. New York: Springer-Verlag, 1977.

[5] Bellman, R. and Cooke, K.L. *Differential-Difference Equations*. New York: Academic Press, 1963.

[6] Gu, K., Kharitonov, V. L. and Chen, J. *Stability of Time-Delay Systems*. Boston, MA: Birkhuser, 2003.

[7] Li, H., Chen, B., Zhou, Q. and Su, Y. New results on delay-dependent robust stability of uncertain time delay systems. *International Journal of Systems Science* **41** (2) (2010) 627–634.

[8] Silvia, G.J., Datta, A. and Bhattacharyya, S.P. *PID Controllers for Time Delay Systems*. Springer, Birkhauser: Boston, 2005.

[9] Bliman, P.A. Extension of Popov absolute stability criterion to non-autonomous systems with delays. *International Journal on Control* **73** (15) (2000) 1349–1361.

[10] Richard, J.P., Goubet-Bartholomefis, A., Tchangani, P.A. and Dambrine, M. Nonlinear Delay Systems: Tools for a Quantitative Approach to Stabilization. In: *Stability and Control of Time-delay Systems*, (eds: Dugard L., Verriest E.I.) London, Springer-Verlag, 1998, 218–239.

[11] Niculescu, S. I., Verriest, E. I., Dugard, L. and Dion, J. M. Stability and Robust Stability of Time-Delay Systems: A Guided Tour. In: *Stability and Control of Time-delay Systems*, (eds: Dugard L., Verriest E.I.). Springer-Verlag, London, 1998, 1–71.

[12] Altshuller, D. A. A Partial Solution of the Aizerman Problem for Second-Order Systems With Delays. *IEEE Transaction on Automatic Control* **53** (9) (2008) 2158–2160.

[13] Rãsvan, V. , Danciu, D. and Popescu, D. Frequency domain stability inequalities for nonlinear time delay systems. In: *15th IEEE Mediterranean Electrotechnical Conference*. Valletta, Malta, 2010, 1398–1401.

[14] Benrejeb, M., Gasmi, M. and Borne, P. New stability conditions for TS fuzzy continous nonlinear models. *Nonlinear Dynamics and Systems Theory* **5** (4) (2005) 369–379.

[15] Benrejeb, M. On an algebraic stability criterion for non linear processes. Interpretation in the frequency domain. In: *Proc. of MECO Congress*, 1978, 678–682.

[16] Sfaihi, B., Benrejeb, M. and Borne, P. On stability conditions of singularly perturbed nonlinear Lur'e discrete-time systems. *Nonlinear Dynamics and Systems Theory* **5** (4) (2013) 369–379.

[17] Gentina, J.C., Borne, P., Burgat, C., Bernoussou, J. and Grujic, L.T. Sur la stabilite des systemes de grande dimension. Normes vectorielles. *RAIRO Automatic Systems Analysis and Control* **13** (1979) 57–75.

[18] Elmadssia, S., Saadaoui, K. and Benrejeb, M. New delay-dependent stability conditions for linear systems with delay. *Systems Science & Control Engineering: An Open Access Journal* **1** (1) (2013) 2–11.

[19] Shujaee, K. and Lehman, B. Vibrational Feedback Control of Time Delay Systems. *IEEE Transactions Automatic Control* **42** (11) (1997) 1529–1545.

[20] Han, Q. L. Absolute stabilityof time-delaysy stems with sector-bounded nonlinearity. *Automatica* **41** (12) (2005) 2171–2176.

[21] Michiels, W., Assche V. V. and Niculescu, S. I. Stabilization of Time-Delay Systems With a Controlled Time-Varying Delay and Applications. *IEEE Transactions Automatic Control* **50** (4) (2005) 493–504.

[22] Elmadssia, S., Saadaoui, K. and Benrejeb, M. New stability conditions for nonlinear time delay systems. In: *International Conference on Control, Decision and Information Technologies (CoDIT)*. Hammamet, Tunisia, 2013, 399–403.

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Observer Design for Descriptor Systems with Lipschitz Nonlinearities: an LMI Approach

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Abstract: In this paper, a method is proposed to design asymptotic observers for a class of semilinear descriptor systems satisfying the complete detectability condition on the corresponding linear part. The method is based on the properties of restricted system equivalent, derived here from a given descriptor system by means of simple matrix theory. Using restricted system equivalent form, coefficient matrices of the proposed observer have been synthesized by linear matrix inequality (LMI) approach based on the Lyapunov stability theory.

Keywords: *descriptor system; Lipschitz continuity; observer design; detecability.*

Mathematics Subject Classification (2010): 93C10, 93C35, 93D20.

1 Introduction

In the last three decades, considerable amount of research was focused on the analysis, design, and numerical simulation techniques for descriptor systems, which arise in modeling of many real and practical systems, e.g. electrical network analysis, power systems, constrained mechanics, economic systems, chemical process control, see, [1–7] and the references therein. Depending on the area, descriptor systems are termed by a variety of names, *viz.* differential algebraic equations (DAEs), singular, implicit, generalized state space, noncanonic, degenerate, semi-state and nonstandard systems. In this paper, we consider the following semilinear system:

$$E^*\dot{x} = A^*x + B^*u + D^*f(Hx, u, t), \tag{1a}$$
$$y = Cx, \tag{1b}$$

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where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, are the state vector, the input vector and the output vector, respectively, $E^* \in \mathbb{R}^{n\times n}$ $A^* \in \mathbb{R}^{n\times n}$, $B^* \in \mathbb{R}^{n\times m}$, $D^* \in \mathbb{R}^{n\times n_d}$, $H \in \mathbb{R}^{n_h\times n}$, and $C \in \mathbb{R}^{p\times n}$ are known constant matrices, $\mathrm{rank}(E^*) = r < n$. Without loss of generality, we assume that $\mathrm{rank}(B^*) = m$, $\mathrm{rank}(D^*) = n_d$, $\mathrm{rank}(C) = p$, and $\mathrm{rank}(H) = n_h$. If E is nonsingular or $E \equiv I$, then the system is called normal system.

To design a controller, the knowledge about the states of the system is important. But it is not always possible or necessary to measure all the state variables. In such cases, the states can be estimated from the output of another dynamical system, which is called an observer for the given system. An observer is a mathematical realization which uses the input and output information of a given system and its output asymptotically approaches to the true state values of the given system.

Observer design problem for normal linear systems has received a great attention in the literature [8–11] and the techniques used for them have been extended successfully to descriptor linear systems, see [6, 7, 12, 13] and references therein. For normal nonlinear systems, in general, literature concerned with the design of observers could broadly be classified into two categories based on the solution approach. In the first approach, the states are transformed in such a way that the given nonlinear system converts into a system, where linear theory is applicable [14–17]. In another approach, the observers are designed for nonlinear systems without any state transformation [18–21]. For a comparison of these approaches, we refer to [22]. On the other hand literature on observers for descriptor nonlinear systems is not so rich. However some researchers have extended the approaches mentioned above to descriptor nonlinear systems [23–32].

In [23], authors extended linearization technique to design state observers for descriptor nonlinear systems and illustrated its application to AC/DC converter model. Boutayeb *et al.* [24] extended the results of [23] to the rectangular descriptor systems. In [25], a method for observing the states of continuous quasilinear descriptor systems is developed by casting the given system as an equivalent system of explicit differential equations on a restricted manifold. In [26], authors considered a nonlinear observer for a class of continuous nonlinear descriptor systems with unknown inputs and faults. In last few years, due to availability of computationally fast and reliable algorithms for solving convex optimization problems subjected to LMI constraints (like MATLAB LMI tool box [33]), researchers developed LMI based approaches to design controllers and observers for normal [34, 35] and descriptor systems [27–32]. Semilinear descriptor systems with the Lipschitz nonlinearities and arbitrary unknown inputs with or without disturbances were considered in [26–32] and existence conditions were derived for full-order, reduced-order, minimal-order, or H_∞ observers in the form of LMIs.

In this paper, we develop a method for full-order state observer design for a class of Lipschitz nonlinear descriptor systems. Contrary to the results available in [31, 32], the observer presented in this paper has normal system form. The sufficient condition for the stability of error dynamics is given in terms of an LMI. In square system case, the proposed method is simple, easy to understand and implement compared to the methods available in the literature. Numerical examples are provided in the last section to illustrate our results.

2 Problem Description and Design Approach

Let us make the following conditions on the system (1):

(H1) $\text{rank}\begin{bmatrix} E^* \\ C \end{bmatrix} = n$,

(H2) nonlinear function $f(Hx, u, t)$ satisfies the Lipschitz property in its first argument, *i.e.* there exists a $\lambda > 0$ such that

$$\|f(Hx_1, u, t) - f(Hx_2, u, t)\| \leq \lambda \|H(x_1 - x_2)\|, \tag{2}$$

(H3) $\text{rank}\begin{bmatrix} \lambda E^* - A^* \\ C \end{bmatrix} = n \ \forall \ \lambda \ \in \ \bar{\mathbb{C}}^+$, where $\mathbb{C}$ represents the set of complex numbers. $\bar{\mathbb{C}}^+ = \{s | s \in \mathbb{C}, Re(s) \geq 0\}$ is the closed right half complex plane.

The problem is to design the matrices N, L, M, and D of compatible dimensions such that the following normal system becomes a full-order state observer (*i.e.*, $\hat{x} \to x$ as $t \to \infty$) for system (1)

$$\begin{aligned} \dot{z} &= Nz + Bu + Ly + Df(H\hat{x}, u, t), && \text{(3a)} \\ \hat{x} &= z + My. && \text{(3b)} \end{aligned}$$

Our approach is as follows.

First, using the algorithm, which we have designed in the Appendix of this paper, a nonsingular matrix $R \in \mathbb{R}^{n \times n}$ is constructed such that the system (1) is restricted system equivalent to the following descriptor system:

$$\begin{aligned} E\dot{x} &= Ax + Bu + Df(Hx, u, t) && \text{(4a)} \\ y &= Cx, && \text{(4b)} \end{aligned}$$

where $E = RE^*$, $A = RA^*$, $B = RB^*$, and $D = RD^*$. It is easy to verify that if the system (1) satisfies (H1), then the system (4) satisfies the following property:

$$rank \begin{bmatrix} I - E \\ C \end{bmatrix} = p. \tag{5}$$

It should be noted that this matrix R is not unique. The proof of the existence of such matrix R can be found in [36].

Second, solution of a system does not change by multiplying a nonsingular matrix, observer for the system (4) works for the system (1). From equations (3) and (4) the error

$$\begin{aligned} e &= x - \hat{x} \\ &= x - z - MCx \\ &= (I - MC)x - z \\ &= Ex - z \end{aligned} \tag{6}$$

gives the dynamics

$$\begin{aligned} \dot{e} &= E\dot{x} - \dot{z} \\ &= Ax + Bu + Df(Hx, u, t) \\ &\quad -(Nz + Bu + LCx + Df(H\hat{x}, u, t)) \\ &= (A - LC)x - N(Ex - e) + D\Delta f \\ &= Ne + (A - LC - NE)x + D\Delta f \\ &= Ne + (A - LC - N + NMC)x + D\Delta f, \\ &= Ne + D\Delta f \end{aligned} \tag{7}$$

where $\Delta f = f(Hx, u, t) - f(H\hat{x}, u, t)$. Moreover, in the construction of equations (6) and (7), we have assumed the existence of matrices M, K, N, and L of compatible orders such that

$$\begin{aligned} I - MC &= E, & (8)\\ N &= A - KC, & (9)\\ K &= L - NM. & (10) \end{aligned}$$

Finally, the problem of designing the state observer (3) boils down to determining the matrices M, K, N, and L such that the equations (8)–(10) are satisfied with the stability of error dynamics (7).

3 Main Result

Theorem 3.1 *Suppose the assumptions $(H1)$ and $(H2)$ hold for the system (1). Then system (3) is observer for the system (1) if the following LMI has a solution for any $P > 0$*

$$\begin{bmatrix} PA + A^TP - \tilde{K}C - C^T\tilde{K}^T + \lambda^2 H^T H & PD \\ D^TP & -I \end{bmatrix} < 0, \qquad (11)$$

where $\tilde{K} = PK$.

Proof. Equation (5) implies that there exists a matrix M such that (8) is satisfied. Now, we show the existence of matrix K such that matrix N (in equation (9)) and the error dynamics (7) are stable if the LMI (11) has a solution for $P > 0$. Considering a Lyapunov function $V = e^TPe$, and using (7) and (9) we have

$$\begin{aligned} \dot{V} &= \dot{e}^TPe + e^TP\dot{e} \\ &= (Ne + D\Delta f)^TPe + e^TP(Ne + D\Delta f) \\ &= e^T(N^TP + PN)e + \Delta f^TD^TPe + e^TPD\Delta f \\ &\le e^T(N^TP + PN)e + \Delta f^TD^TPe + e^TPD\Delta f \\ &\quad + \lambda^2 e^TH^THe - \Delta f^T\Delta f \\ &= \begin{bmatrix} e^T & \Delta f^T \end{bmatrix} \begin{bmatrix} N^TP + PN + \lambda^2H^TH & PD \\ D^TP & -I \end{bmatrix} \begin{bmatrix} e \\ \Delta f \end{bmatrix} \\ &= \begin{bmatrix} e^T & \Delta f^T \end{bmatrix} \begin{bmatrix} N^TP + PN + \lambda^2H^TH & PD \\ D^TP & -I \end{bmatrix} \begin{bmatrix} e \\ \Delta f \end{bmatrix}. \end{aligned}$$

According to the stability theory, the error dynamics (7) is stable if

$$\begin{aligned} &\begin{bmatrix} N^TP + PN + \lambda^2H^TH & PD \\ D^TP & -I \end{bmatrix} < 0 \\ \Rightarrow &\begin{bmatrix} (A-KC)^TP + P(A-KC) + \lambda^2H^TH & PD \\ D^TP & -I \end{bmatrix} < 0 \\ \Rightarrow &\begin{bmatrix} PA + A^TP - \tilde{K}C - C^T\tilde{K}^T + \lambda^2H^TH & PD \\ D^TP & -I \end{bmatrix} < 0. \end{aligned}$$

Hence by a solution of LMI (11), we can find a matrix K, and hence matrix N, such that the error dynamics (7) is stable. Finally using the equation (10), we can find the matrix L.

Remark 3.1 If the LMI (11) is solvable then it is clear that

$$PA + A^T P - \tilde{K}C - C^T \tilde{K}^T + \lambda^2 H^T H < 0.$$

That implies

$$PA + A^T P - \tilde{K}C - C^T \tilde{K}^T < 0,$$

which is equivalent to the detectability of matrix pair (A, C). It can be proved easily that under assumption $(H1)$, condition $(H3)$ is equivalent to the detectability of matrix pair (A, C). Hence the condition $(H3)$ is a necessary condition for the solvability of LMI (11).

4 Numerical Examples

Example 4.1 Consider the descriptor system (1) described by the following matrices. (This example is taken from [28] with zero disturbance vector.)

$$E^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B^* = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T,$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D^* = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T, \quad H = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},$$

$u(t) = sin(2t)$. The nonlinearity function $f(x, u, t) = sin(x_2(t))$. Since rank$\begin{bmatrix} E^* \\ C \end{bmatrix} = 3$ and rank$\begin{bmatrix} I - E^* \\ C \end{bmatrix} \neq 1$, using the algorithm given in the Appendix, we calculate

$$R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T.$$

Now, we can check that rank$\begin{bmatrix} I - E \\ C \end{bmatrix} = 1$ and $M = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.

By using MATLAB LMI tool box we solve (11) and find

$$K = \begin{bmatrix} 4.4252 & 4.3573 & 4.5410 \end{bmatrix}^T.$$

Thus

$$N = \begin{bmatrix} -3.4252 & 0 & 1.0000 \\ -3.3573 & 0 & 0 \\ -4.5410 & 1.0000 & 0 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T.$$

If we take

$$x(0) = \begin{bmatrix} -2.8415 & 1 & 2 \end{bmatrix}^T,$$

$$z(0) = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}^T,$$

then the truth and estimated states are plotted in Figure 1. The graph of the error vector is shown in Figure 2, which clearly shows the efficiency of the proposed observer.

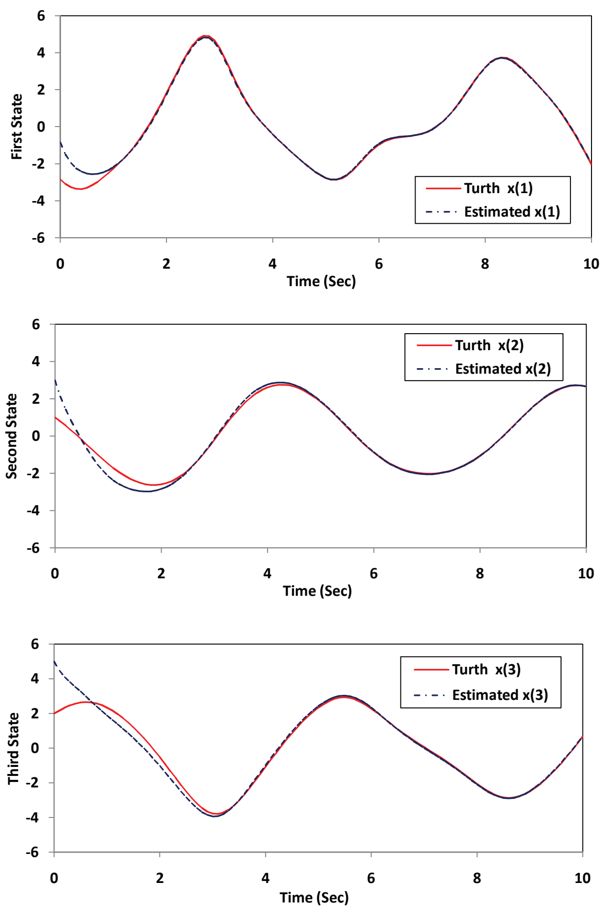

Figure 1: Plot of the true and estimated values of the states in Example 4.1.

Example 4.2 Consider the descriptor system (1) described by the following matrices:

$$E^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^* = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}, \quad B^* = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}^T, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

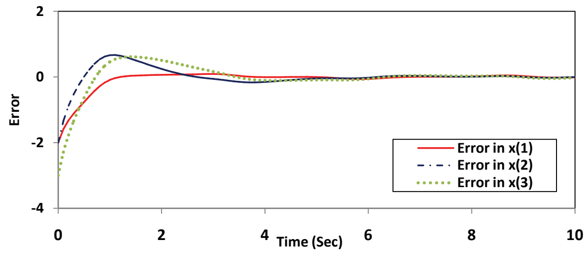

Figure 2: Estimation performance in Example 4.1.

$$D^* = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T, \quad H = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

$u(t) = t^2$. The nonlinearity function $f(x, u, t) = cos(x_3(t))$. Since rank $\begin{bmatrix} I - E^* \\ C \end{bmatrix} = 2$, $R = I_3$ and $M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. By using MATLAB LMI tool box we solve (11) and find

$$K = \begin{bmatrix} -172.2387 & 132.1813 \\ -386.6106 & 180.5962 \\ -193.7974 & 103.5193 \end{bmatrix}.$$

Thus,

$$N = \begin{bmatrix} 173.2387 & -130.1813 & 0 \\ 386.6106 & -182.5962 & 0 \\ 194.7974 & -103.5193 & -3.0000 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 1.0000 & 132.1813 \\ 0 & 180.5962 \\ 1.0000 & 103.5193 \end{bmatrix}.$$

If we take

$$x(0) = \begin{bmatrix} -1.5839 & 1 & 2 \end{bmatrix}^T,$$
$$z(0) = \begin{bmatrix} 10 & 12 & 15 \end{bmatrix}^T,$$

then the truth and estimated states are plotted in Figure 3. The graphs of the errors are plotted in Figure 4, which clearly shows that error vector converges to zero.

5 Conclusions

A method has been developed to design the state observers for a class of semilinear descriptor systems. This class is characterized by two properties: (i) the linear part of each member system is completely detectable, and (ii) the nonlinear part satisfies the Lipschitz property. The sufficient condition for the stability of error dynamics is given in terms of an LMI. A new restricted equivalent system which follows the same state representation as the given descriptor system, has been made with the help of an

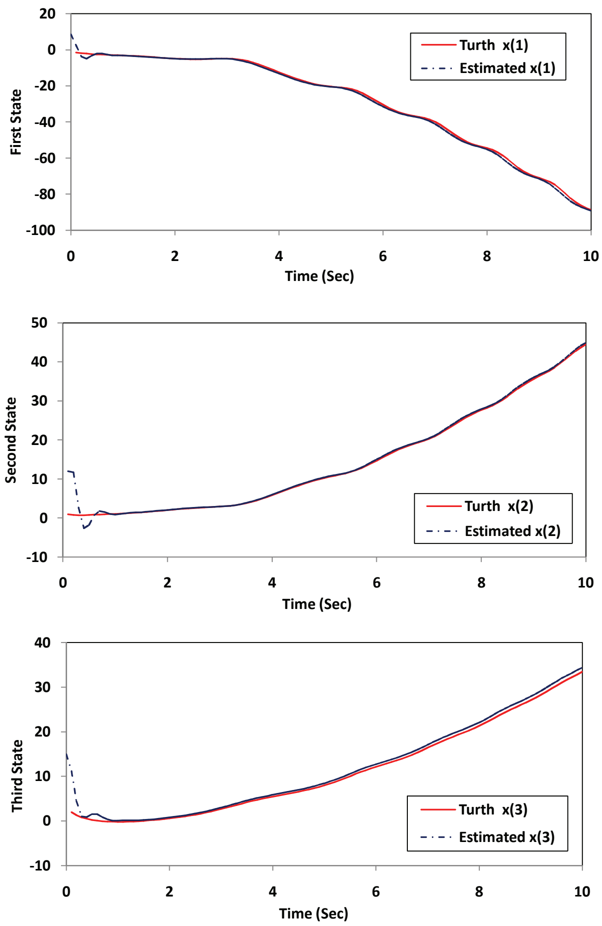

Figure 3: Plot of the true and estimated values of the states in Example 4.2

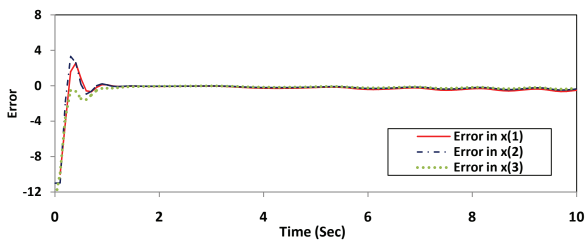

Figure 4: Estimation performance in Example 4.2

invertible matrix R. The advantage of using this equivalent system is the fact that the detectability of its corresponding normal system is equivalent to the detectability of the given descriptor system, and this fact gave necessary condition for the solution of the proposed LMI (see Remark 3.1). The extension of this work to rectangular semilinear and nonlinear descriptor system is under construction.

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Appendix

Algorithm to find the matrix R:

1. Determine
$p :=$ rank of matrix C,
$n :=$order of matrix E^*.

2. Check
(i) If rank $\begin{bmatrix} I - E^* \\ C \end{bmatrix} = p$. Take $R = I_n$ and stop.
(ii) If rank $\begin{bmatrix} E^* \\ C \end{bmatrix} = n$, then go to steps 3-8.

3. Carry out the singular value decomposition (SVD) of matrix $C = U_1 \begin{bmatrix} D_1 & 0 \end{bmatrix} V_1^T$.

4. Calculate $P = V_1 \begin{bmatrix} D_1^{-1} U_1^T & 0 \\ 0 & I_{n-p} \end{bmatrix}$.

5. Calculate $\tilde{E} = E^* P \begin{bmatrix} 0 \\ I_{n-p} \end{bmatrix}$.

6. Carry out the SVD of matrix $\tilde{E} = U_2 \begin{bmatrix} D_2 \\ 0 \end{bmatrix} V_2$.

7. Calculate $R_0 = \begin{bmatrix} 0 & I_p \\ V_2^T D_2^{-1} & 0 \end{bmatrix} U_2^T$.

8. Calculate $R = P R_0$.

References

[1] Campbell, S. L. *Singular Systems of Differential Equations – II*. Pitman, London, 1982.

[2] Luenberger, D. G. Non-linear descriptor systems. *J. Econ. Dynam. Control* **1**(3) (1979) 219–242.

[3] Lewis, F. L. A survey of linear singular systems. *Circuits, Systems and Signal Processing* **5**(1) (1986) 3–36.

[4] Brenan, K. E., Campbell, S. L. and Petzold, L. R. *Numerical Solution of Initial-value Problems in Differential-algebraic Equations*. SIAM, 1996.

[5] Campbell, S. L. *Singular Systems of Differential Equations*. Pitman, 1980.

[6] Dai, L. *Singular Control Systems*. Springer-Verlag, 1989.

[7] Duan, G.-R. *Analysis and Design of Descriptor Linear systems*. Springer, 2010.

[8] Luenberger, D. G. An introduction to observers. *IEEE Trans. Autom. Control* **16**(6) (1971) 596–602.

[9] Carroll, R. L. and Lindorff, D. An adaptive observer for single-input single-output linear systems. *IEEE Trans. Autom. Control* **18**(5) (1973) 428–435.

[10] O'Reilly, J. *Observers for Linear Systems*. Academic Press, 1983.

[11] Spurgeon, S. K. Sliding mode observers: a survey. *Int. J. Systems Science* **39**(8) (2008) 751–764.

[12] Ren, J. and Zhang, Q. PD Observer design for descriptor system: An LMI approach. *Int. J. Control Autom. Syst.* **8**(4) (2010) 735–740.

[13] Chen, X. and Zhai, G. Observation for the descriptor systems with disturbances. *Nonlinear dynam. syst. theory* **7**(2) (2007) 121–139.

[14] Krener, A. J. and Respondek, W. Nonlinear observers with linearizable error dynamics. *SIAM J Control Optim.* **23**(2) (1985) 197–216.

[15] Keller, H. Non-linear observer design by transformation into a generalized observer canonical form. *Int. J. Control* **46**(6) (1987) 1915–1930.

[16] Zeitz, M. The extended Luenberger observer for nonlinear systems. *Syst. Control Lett.* **9**(2) (1987) 149–156.

[17] Röbenack, K. Observer design for a class of nonlinear systems with non-full relative degree. *Nonlinear Dynamics and Systems Theory* **7**(4) (2007) 399–408.

[18] Thau, F. Observing the state of non-linear dynamic systems. *Int. J. Control* **17**(3) (1973) 471–479.

[19] Kou, S. R., Elliott, D. L. and Tarn, T. J. Exponential observers for nonlinear dynamic systems. *Inf. Control* **29**(3) (1975) 204–216.

[20] Arcak, M. and Kokotović, P. Nonlinear observers: a circle criterion design and robustness analysis. *Automatica* **37**(12) (2001) 1923–1930.

[21] Aloui, R. and Braiek, N. B. Optimal State Observer Design for Nonlinear Dynamical Systems. *Nonlinear Dynamics and Systems Theory* **12**(1) (2012) 37–48.

[22] Walcott, B., Corless, M. and Żak, S. Comparative study of non-linear state-observation techniques. *Int. J. Control* **45**(6) (1987) 2109–2132.

[23] Kaprielian, S. and Turi, J. An observer for a nonlinear descriptor system. In: *Decision and Control*, IEEE, 1992, 975–976.

[24] Boutayeb, M., Darouach, M., Rafaralahy, H. and Krzakala, G. Asymptotic observers for a class of nonlinear singular systems. In: *American Control Conference,* IEEE, 1994, 1440–1441.

[25] Zimmer, G. and Meier, J. On observing nonlinear descriptor systems. *Syst. Control Lett.* **32**(1) (1997) 43–48.

[26] Shields, D. N. Observer design and detection for nonlinear descriptor systems. *Int. J. Control* **67**(2) (1997) 153–168.

[27] Darouach, M. and Boutat-Baddas, L. Observers for a class of nonlinear singular systems. *IEEE Trans. Autom. Control* **53**(11) (2008) 2627–2633.

[28] Zerrougui, M., Boutat-Baddas, L. and Darouach, M. H_∞ observers design for a class of continuous time nonlinear singular systems. *J. Control Sci. Eng.* **2011** (2011) 1–8.

[29] Darouach, M., Boutat-Baddas, L. and Zerrougui, M. H_∞ observers design for a class of nonlinear singular systems. *Automatica* **47**(11) (2011) 2517–2525.

[30] Koenig, D. Observer design for unknown input nonlinear descriptor systems via convex optimization. *IEEE Trans. Autom. Control* **51**(6) (2006) 1047–1052.

[31] Yang, C., Zhang, Q., Chou, J.-H. and Yin, F. H_∞ Observer design for descriptor systems with solpe-restricted nonlinearities. *Asian J. Control* **14** (2012) 1–8.

[32] Lu, G. and Ho, D. W. Full-order and reduced-order observers for Lipschitz descriptor systems: the unified LMI approach. *IEEE Trans. Circuits Syst. Express Briefs* **53**(7) (2006) 563–567.

[33] Gahinet, P., Nemirovskii, A., Laub, A. J. and Chilali, M. The LMI control toolbox. In: *Decision and Control,* IEEE, 1994, 2038–2038.

[34] Kanev, S., Scherer, C., Verhaegen, M. and De Schutter, B. Robust output-feedback controller design via local BMI optimization. *Automatica* **40**(7) (2004) 1115–1127.

[35] Chen, W. and Saif, M. Unknown input observer design for a class of nonlinear systems: An LMI approach. In: *American Control Conference,* IEEE, 2006, 834–838.

[36] Gupta, M. K., Tomar, N. K. and Bhaumik, S. Detectability and observer design for linear descriptor system. In: *22nd Mediterranean Conference on Control and Automation,* IEEE, 2014, 1094–1098.

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Stabilizing Sliding Mode Control for Homogeneous Bilinear Systems

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Abstract: The stabilization of homogeneous bilinear systems constitutes the main interest of this paper. A sliding mode control is suggested and a stability study is held leading to sufficient conditions of global stabilization. The sliding surface is determined through the resolution of the nonlinear constraints of stabilization. Simulations on numerical examples are presented proving the effectiveness of the proposed approach.

Keywords: *homogeneous bilinear systems; sliding mode control; stabilization.*

Mathematics Subject Classification (2010): 11D09, 93D20, 39B62.

1 Introduction

Bilinear systems constitute an important class of nonlinear systems. Since their introduction in the early sixties, they have got great interest and have been used to model processes in several fields; biologic, ecologic, economic, social ... [4, 16, 17]. As they are partially linear in state and in input without being jointly linear in both, they constitute a gateway between linear and nonlinear systems and that's why they need special attention in their study. In the last decades, many researchers investigated the control design and the stability analysis of this special category of systems [1, 9, 13, 15, 19, 20].

Many results in this field are yet demonstrated, since the stabilization by linear or quadratic state feedback has been widely treated especially for non homogeneous bilinear systems. However it was shown that there exists a large class of homogeneous bilinear systems which can not be stabilized by a continuous feedback even in planar case [6]. In fact for this type of systems the relative degree isn't defined in zero and the linearized

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system is control independent. For such systems, R. Chabour and al. proposed, in the case of second order dimension, zero degree homogeneous positive controls. For three dimensional systems, Celikovsky investigated in [5] the global asymptotic stabilization by constant feedback and the practical stabilization by a family of linear feedbacks for a special class of single input homogeneous bilinear system ($\dot{x} = Ax + uNx$), where A is a diagonal matrix with a negative trace and N is a skew-symmetric matrix. This work was extended by O. Chabour for n dimensional systems where matrix A has not to be diagonal and its trace has not to be negative, [7]. An integrated overview of bilinear system research presented by Mohler and al. in [18] deals with the efficiency of the optimal control and the variable structure control such as bang-bang control. Later, in [2] Amato and al. suggested a procedure to design a stabilizing state feedback controller formulated in a convex optimisation problem involving LMIs.

In this paper, we interest in the stabilization of homogenous bilinear systems of any dimension. No restriction on the system's structure are imposed. The sliding mode approach is adopted to design a variable structure control. Stability study is investigated leading to sufficient conditions of global stabilization formulated in computationally resolvable nonlinear matrix constraints. Besides a simplified algorithm is provided making use of the linear quadratic control approach. The resolution of the stabilization constraints system enables to provide the matrix C characterising the sliding surface ($S = Cx$). The proposed approach is successfully applied to homogeneous bilinear systems of different orders.

In the following section the control design procedure is detailed for homogeneous bilinear systems leading to the definition of two control laws; the switching control needed in the reaching phase toward the sliding surface, and the equivalent control required while the system slides on the surface. In Section 4 an extended stabilization study is carried out based on quadratic Lyapunov function. To formulate the global stabilization conditions during the sliding mode in resolvable matrix constraints the "vec" operator and the tensor product are employed. Finally two numeric examples are considered in Section 6 to underline the effectiveness of the proposed approach.

2 Homogeneous Bilinear Systems and Sliding Mode Control Design

Bilinear systems are generally represented by a state equation of the form:

$$\dot{x} = Ax(t) + Bu(t) + \sum_{j=1}^{m} N_j x(t) u_j(t), \tag{1}$$

where $x \in X \subset \Re^n$ is the state vector, $u = [u_1 \dots u_m]^T \in U \subset \Re^m$ is the control input, A, B and N_j, $j = 1 \dots m$, are matrices of suitable dimensions.

When the matrix B is not null, this general form characterises non-homogeneous bilinear systems, and if B is null, the represented system is said to be homogeneous.

As we are interested in this paper in this last class of systems, we will consider the state space equations of the form:

$$\dot{x} = Ax(t) + \sum_{j=1}^{m} N_j x(t) u_j(t). \tag{2}$$

The sliding mode approach consists in bringing the system's state up to a well defined surface where it will slide toward the equilibrium point. Thus the sliding mode control is usually constituted by two parts, the switching control and the equivalent control. The first is discontinuous and it is needed during the reaching phase until the system's state attend the sliding surface, and the second is continuous and aims to keep the state on this surface while sliding.

2.1 Reaching condition and switching control

Let define the sliding surface with $C \in \Re^{p\times n}$:

$$S(t) = Cx(t) = 0. \tag{3}$$

The reaching mode to the sliding surface is guaranteed if [1]

$$\frac{d}{dt}(S^T S) = 2x^T C^T C\dot{x} < 0. \tag{4}$$

When substituting $\dot{x}$ by its expression (2) one gets:

$$2x^T C^T C(Ax + \sum_{i=1}^{m} N_i x u_i) < 0. \tag{5}$$

So the controls u_i $(i = 1...m)$ must be designed such that to satisfy the inequality above. We consider a switching control law defined by:

$$u_{i_s} = \begin{cases} -\alpha \frac{x^T N_i^T C^T Cx|x^T C^T CAx|}{\|x^T C^T CN_i x\|^2}, & if \quad S \neq 0 \; and \; x^T C^T CN_i x \neq 0, \\ 0, & else. \end{cases} \tag{6}$$

Let Ł be the set of the indices i such that $x^T C^T CN_i x \neq 0, \forall x \neq 0$, and let l be the number of its elements, then when substituting u_i by u_{i_s}, the left hand term of the inequality (5) will be reduced to:

$$x^T C^T CAx - \alpha l|x^T C^T CAx| \tag{7}$$

which is negative for all $\alpha > 1$ and $l \geq 1$.

2.2 Sliding mode and equivalent control

In order to keep the system's state on the surface S during the sliding mode, the following condition must be fulfilled:

$$\dot{S} = 0 \quad when \quad S = 0, \tag{8}$$

$$\dot{S} = CAx + \sum_{i=1}^{m} CN_i x u_i. \tag{9}$$

Let Ψ be the set of the indices i such that $CN_i x \neq 0$, $\forall x \neq 0$, and let s be the number of its elements, so we can write

$$\dot{S} = \sum_{i=1}^{s} [\frac{1}{s} CAx + CN_i x u_i]. \tag{10}$$

[1] In the following we will omit the time symbol $'(t)'$ of dynamic variables for the aim of simplification

Thus $\dot{S} = 0$ if we have $u_i = u_{i_{eqv}}$ for all $i = 1, \ldots, m$, where

$$u_{i_{eqv}} = \begin{cases} -\frac{1}{s}\frac{(CN_i x)^T CAx}{(CN_i x)^T (CN_i x)}, & if \quad S = 0 \; and \; CN_i x \neq 0, \forall x \neq 0, \\ 0, & else. \end{cases} \tag{11}$$

The homogeneous bilinear system (2) can then be efficiently controlled by the sliding mode control defined by:

$$u = [u_1 \ldots u_m]^T \tag{12}$$

where for all $i = 1, \ldots, m$

$$u_i = u_{i_s} + u_{i_{eqv}}, \tag{13}$$

The switching control u_{i_s} and the equivalent control $u_{i_{eqv}}$ are those defined by (6) and (11).

3 Stability Analysis

As the considered bilinear system is controlled by the sliding mode control, its behavior depends on two phases: the reaching mode and the sliding mode. The stability of the controlled system is guarantied unless the reaching condition is fulfilled and the system remains stable on the sliding surface. The first condition is already verified ($\frac{d}{dt}(S^T S) < 0$ when $S \neq 0$), so we must prove the stability during the sliding mode.

On the sliding surface the function $S(x) = Cx(t) = 0$ where C is a matrix of dimension $p \times n$. One can write $C = [C_1 \;\; C_2]$ where $C_1 \in \Re^{p \times p}$ and $C_2 \in \Re^{p \times (n-p)}$ then we have: $Cx = C_1 x_1 + C_2 x_2 = 0$ with $x_1 \in \Re^p$ and $x_2 \in \Re^{n-p}$.

Suppose that [1] $C_1 = I_p$, so we obtain a relationship between x_1 and x_2:

$$x_1 = -C_2 x_2. \tag{14}$$

Thanks to the above relationship, the convergence of the system's state to the zero equilibrium point can be demonstrated by the convergence of its second part x_2. Then we can eliminate x_1 from the system and the control formulations. For this consider the following notations:

$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $N_i = \begin{bmatrix} N_{i_{11}} & N_{i_{12}} \\ N_{i_{21}} & N_{i_{22}} \end{bmatrix}$, $\forall i = 1 \ldots m$, with $A_{11}, N_{i_{11}} \in \Re^{p \times p}$,

A_{12}, $N_{i_{12}} \in \Re^{p \times (n-p)}$, A_{21}, $N_{i_{21}} \in \Re^{(n-p) \times p}$, A_{22} and $N_{i_{22}} \in \Re^{(n-p) \times (n-p)}$. So the equation (2) can be detailed as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{i_{11}} x_1 + A_{i_{12}} x_2 \\ A_{i_{21}} x_1 + A_{i_{22}} x_2 \end{bmatrix} + \sum_{j=1}^{m} (\begin{bmatrix} N_{ij_{11}} x_1 + N_{ij_{12}} x_2 \\ N_{ij_{21}} x_1 + N_{ij_{22}} x_2 \end{bmatrix}) u_{i_j}. \tag{15}$$

Replacing x_1 by its expression in (14), the derivative of x_2 can be expressed by:

$$\dot{x}_2 = (A_{22} - A_{21} C_2) x_2 + \sum_{i=1}^{m} (N_{i_{22}} - N_{i_{21}} C_2) x_2 u_i. \tag{16}$$

[1] I_p denotes the identity matrix of dimension p

The control u_i on the sliding surface is equal to the equivalent control $u_{i_{eqv}}$ and it can also be expressed as function of x_2:

$$u_i = u_{i_{eqv}} = -\frac{1}{s}\frac{(CN_i x)^T CAx}{(CN_i x)^T (CN_i x)}, \quad \forall i = 1, \dots, s. \tag{17}$$

It is easy to obtain:

$$CN_i x(t) = G_i x_2(t), \quad CAx(t) = Hx_2(t),$$

where

$$\begin{array}{rcl} G_i & = & C_2(N_{i_{22}} - N_{i_{21}}C_2) + N_{i_{12}} - N_{i_{11}}C_2, \\ H & = & C_2(A_{22} - A_{21}C_2) + A_{12} - A_{11}C_2, \end{array}$$

so

$$u_i = u_{i_{eqv}} = -\frac{1}{s}\frac{x_2^T G_i^T H x_2}{x_2^T G_i^T G_i x_2}. \tag{18}$$

Consider the Lyapunov function $V(x_2) = x_2^T P x_2$ where P is a positive definite symmetric matrix, we have to prove that $\dot{V}(x_2) < 0$ for all $x \in X \subset \Re^n$.

$$\dot{V}(x_2) = x_2^T P \dot{x}_2 + \dot{x}_2^T P x_2. \tag{19}$$

Let

$$\mathbb{A} = A_{22} - A_{21}C_2. \tag{20}$$

$$\mathbb{N}_i = N_{i_{22}} - N_{i_{21}}C_2, \ \ \forall i = 1 \dots m. \tag{21}$$

Then the derivative of the Lyapunov function becomes:

$$\dot{V}(x_2) \ = \ x_2^T P[\mathbb{A} - \frac{1}{s}\sum_{i=1}^{s}\frac{x_2^T G_i^T H x_2}{x_2^T G_i^T G_i x_2}\mathbb{N}_i]x_2 \ + \ x_2^T[\mathbb{A} - \frac{1}{s}\sum_{i=1}^{s}\frac{x_2^T G_i^T H x_2}{x_2^T G_i^T G_i x_2}\mathbb{N}_i]^T P x_2. \tag{22}$$

Noting that $\dot{V}(x_2)$ can be rearranged in the following form

$$\dot{V}(x_2) = x_2^T(P\mathbb{A} + \mathbb{A}^T P)x_2 - \frac{1}{s}\sum_{i=1}^{s}\frac{x_2^T G_i^T H x_2}{x_2^T G_i^T G_i x_2}x_2^T(P\mathbb{N}_i + \mathbb{N}_i^T P)x_2 \tag{23}$$

and since the term $x_2^T G_i^T G_i x_2$ is usually positive, then we can deduce that $\dot{V}(x_2) < 0$ if for all $i = 1, \dots, s$ we verify:

$$\left\{ \begin{array}{l} x_2^T(P\mathbb{A} + \mathbb{A}^T P)x_2 < 0, \\ x_2^T G_i^T H x_2 x_2^T(P\mathbb{N}_i + \mathbb{N}_i^T P)x_2 \geq 0, \end{array} \right. \forall \ \ x_2 \neq 0. \tag{24}$$

The first inequality is equivalent to the definite negativity of the matrix $(P\mathbb{A} + \mathbb{A}^T P)$ while the second necessitates additional developments. This latter represents a product of two scalars:

$$(x_2^T \mathbb{V}_i x_2)(x_2^T \mathbb{W}_i x_2), \tag{25}$$

where

$$\left\{ \begin{array}{l} \mathbb{V}_i = G_i^T H, \\ \mathbb{W}_i = P\mathbb{N}_i + \mathbb{N}_i^T P. \end{array} \right. \tag{26}$$

Using the relation between the 'vec' operator and the tensor product ($\otimes$) [3]:

$$vec(AXB) = (B^T \otimes A)vec(X), \tag{27}$$

where A, X and B are any matrices of coherent dimensions, the expression (26) can be reformulated as follows:

$$x_2^T \mathbb{V}_i x_2 x_2^T \mathbb{W}_i x_2 = vec(x_2^T \mathbb{V}_i x_2 x_2^T \mathbb{W}_i x_2) = (x_2^{[2]})^T (\mathbb{W}_i^T \otimes \mathbb{V}_i)(x_2^{[2]}). \tag{28}$$

This expression is strictly positive for all $x_2 \neq 0$ if the matrix $\mathbb{W}_i^T \otimes \mathbb{V}_i$ is positive definite. However, since the vector $x_2^{[2]}$ has redundant terms, it might exist a solution to this problem even with non-positive definite matrix. Therefore, it is possible to relax this condition by eliminating the redundancy in the vector $x_2^{[2]}$. For that a transition matrix T can be introduced, [3],such that:

$$x_2^{[2]} = T\tilde{x}_2^{[2]}. \tag{29}$$

Hence the expression(28) becomes:

$$x_2^T \mathbb{V}_i x_2 x_2^T \mathbb{W}_i x_2 = (\tilde{x}_2^{[2]})^T T^T (\mathbb{W}_i^T \otimes \mathbb{V}_i) T \tilde{x}_2^{[2]}. \tag{30}$$

Finally we can confirm that the derivative of the Lyapunov function (23) is negative definite if we have:

$$\begin{cases} P\mathbb{A} + \mathbb{A}^T P < 0, \\ T^T(\mathbb{W}_i^T \otimes \mathbb{V}_i)T \geq 0, \forall i = 1 \dots s. \end{cases} \tag{31}$$

The above results are then summarized in the following theorem.

Theorem 3.1 *The homogeneous bilinear system* (2) *is stabilizable by the sliding mode control* (12), (13), (6), (11) *for all real* $\alpha > 1$ *if there exist a positive definite symmetric matrix* P *and a matrix* C_2 *verifying the conditions* (31), *with all defined notations respected.*

The conditions (31) constitute nonlinear matrix inequalities system which can be solved via a multi-objective optimization function such as 'fgoalattain' or 'fmincon' of the Matlab optimization toolbox. The resolution of this problem will provide the matrix C_2 and the positive definite symmetric matrix P if there exist any.

One way to get round this nonlinear optimization problem is to search C_2 that stabilizes the pair (A_{22}, A_{21}), for example by the linear quadratic regulator function 'lqr' (which ensures the negativity of the first inequality), while verifying the positivity of the second inequality of the system (31).

Consider the linear system

$$\dot{z}(t) = A_{22}z(t) + A_{21}u(t). \tag{32}$$

If the pair (A_{22}, A_{21}) is stabilizable then we can calculate C_2 as the optimal gain matrix such that the state-feedback law $u = -C_2 z$ minimizes the quadratic cost function:

$$J(u) = \int_0^\infty (z^T Q z + u^T R u)dt, \tag{33}$$

while verifying the Riccati equation:

$$PA_{22} + A_{22}^T P - PA_{21}R^{-1}A_{21})^T P + Q = 0, \tag{34}$$

where Q and R are matrices satisfying:

$$\begin{cases} R > 0, \\ Q \geq 0, \\ Q \text{ and } A_{22} \text{ have no unobservable mode on the imaginary axis.} \end{cases}$$

The gain matrix C_2 is then deduced by the expression:

$$C_2 = R^{-1}A_{21}^T P. \tag{35}$$

When choosing $R = I_n$,the Riccati equation and the matrix gain become:

$$\begin{array}{l} PA_{22} + A_{22}^T P = PA_{21}A_{21}^T P - Q, \\ C_2 = A_{21}^T P. \end{array}$$

So the constraint $(P(A_{22} - A_{21}C_2) + (A_{22} - A_{21}C_2)^T P < 0)$ will be satisfied for whatever $Q \geq 0$. Then it will be easy to find a C_2 fulfilling the constraints (31) by adjusting the matrix parameter Q.

4 Simulation Examples

4.1 Second order bilinear system

In the case of second order homogeneous bilinear systems the state subvector x_2 is scalar and so does C_2, so all the matrices involved in the inequality system (31) are also scalar terms. Hence this latter leads to the following conditions of global stability:

$$\begin{cases} P(A_{22} - A_{21}C_2) < 0, \\ G_i H_i P(N_{i_{22}} - N_{i_{21}}C_2) \geq 0, \end{cases} \forall i = 1, ..., s, \tag{36}$$

where P is a positive scalar.

Since G_i and H do not depend on P, the problem can be reduced to the search for only one unknown variable which is C_2 such that:

$$\begin{cases} A_{22} - A_{21}C_2 < 0, \\ G_i H_i (N_{i_{22}} - N_{i21}C_2) \geq 0, \end{cases} \forall i = 1, ..., s. \tag{37}$$

Consider the second order homogeneous bilinear system defined by (2) where $m = 2$ and

$$A = \begin{pmatrix} 13 & -12 \\ 10 & -10 \end{pmatrix}, N_1 = \begin{pmatrix} 0.7 & 0.1 \\ 0.1 & 0.7 \end{pmatrix}, N_2 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

In free run mode, the systems' states are divergent. The sliding mode control law is designed according to the expressions $(12), (13), (6), (11)$. The sliding surface is defined by $S = Cx = 0$ with $C = [1\ C_2]$.

To search C_2 that guarantees the stability of the controlled system, we solve the matrix inequalities system (36) and we obtain $P = 0.1449$ and $C_2 = 2.3$.

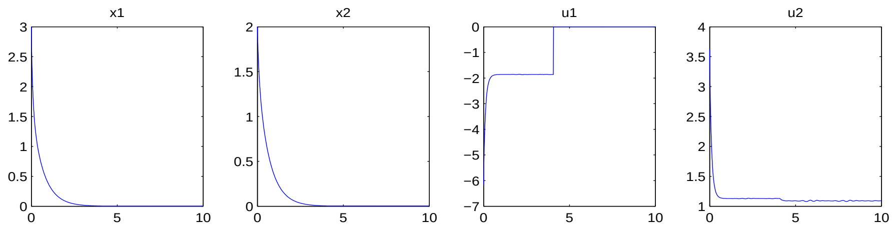

Figure 1: Second order system responses with sliding mode control, $X(0) = [3\ 2]^T$ and $\alpha = 1.01$.

When implementing the proposed control law with the sliding surface $C = [1\ \ 2.3]$ for $\alpha = 1.01$ and the initial conditions $x(0) = [3\quad 2]^T$, we obtain the simulation results presented in Figure 1. We note that the states converge to zero before 3s and with low control levels (between -8 and +4).

Even when trying to enlarge the initial conditions values or the uncertainties domains, the designed control ensure the convergence of the system's states, as shown in Figure 2.

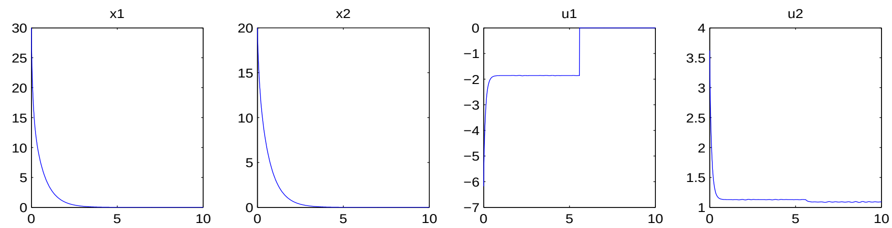

Figure 2: Second order system responses with large initial values $X(0) = [30\ 20]^T$ and $\alpha = 1.01$.

4.2 Third order bilinear system

Consider the third order bilinear system defined by (2), where $m = 1$ and

$$A = \begin{pmatrix} -2 & 3 & 1 \\ 1 & -7 & 1 \\ 2 & 1 & 0.5 \end{pmatrix},\ N = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The resolution of the stabilization constraints system (31) gives the symmetric positive definite matrix P and the vector C_2 defined by: $P = \begin{pmatrix} 0.0777 & 0.0661 \\ 0.0661 & 0.6308 \end{pmatrix}$, $C_2 = [0.2100\quad 1.3278]$.

Simulations of the so controlled system are presented in Figures 3 and 4 respectively for small and large initial conditions, with $\alpha = 2.5$. We notice that the states converge to zero within two seconds at least. The control amplitude doesn't exceed four units.

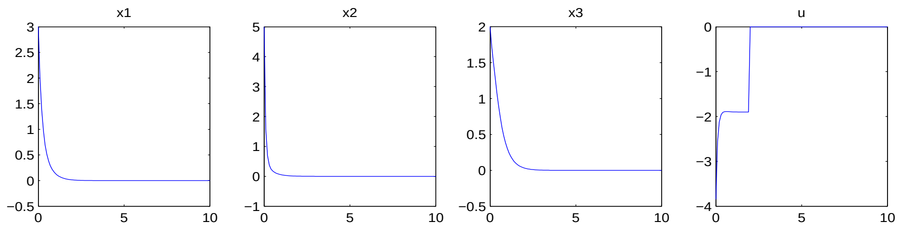

Figure 3: Third order system responses with sliding mode control, $X(0) = [3\ 5\ 2]^T$ and $\alpha = 2.5$.

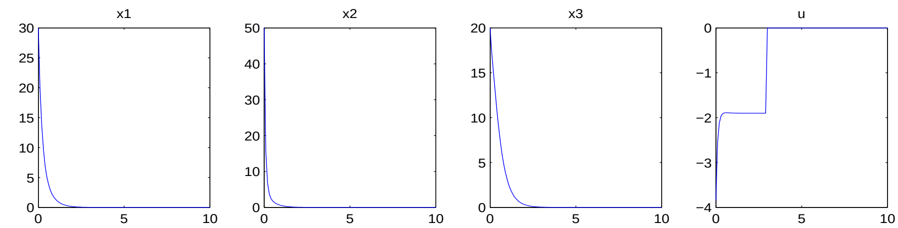

Figure 4: Third order system responses with large initial values $X(0) = [30\ 50\ 20]^T$ and $\alpha = 2.5$.

5 Conclusions

A sliding mode control approach is proposed for homogeneous bilinear systems. The control design strategy detailed in this paper enabled to provide an efficient sliding mode control constituted by two components: a switching control law basically built so as to ensure the system stability during the reaching phase, and an equivalent control law deduced from the condition of keeping the system's state quietly on the sliding surface once reached. The meticulous study held on the system's closed loop stability during this sliding phase allowed to provide sufficient conditions of global stabilization formulated in a set of linear and nonlinear matrix inequalities. The sliding surface can be automatically defined through the resolution of the stability constraints problem. Analytical and numerical cleverness have permitted to facilitate the resolution of so complex optimisation problem. In fact, for the second order systems, simplified form of the stabilization constraints is retrieved showing that the problem can be reduced to the search for only one unknown variable. On the other hand, for higher order systems the linear quadratic based algorithm suggested enables to obtain feasible solutions to the nonlinear constrained problem if there exist ones.

References

[1] AL-Shamali, S. A., Crisalle, O. D. and Latchman, H. A. Sliding Mode Control for a class of Bilinear Systems. In: *Proceeding of the 46th IEEE Conference on Decision an Control*. New Orleans, LA, USA, 2007, 1981–1985.

[2] Amato, F., Cosentino, C. and Merola, A. Stabilization of Bilinear Systems via Linear State Feedback Control. In: *Mediterranean Conference on Control and Automation (MED '07)*, 2007.

[3] Braiek, Benhadj N. and Rotella, F. Stabilization of Nonlinear Systems using a Kronecker Product Approach. In: *European Control Conference (ECC'95)*. September, 1995, 2304–2309.

[4] Bruni, C., Di Pillo, G. and Koch, G. Bilinear Systems: An Appealing Class of Nearly Linear Systems in Theory and Applications. *IEEE Transactions on Automatic Control* **19** (1974) 334–348.

[5] Celikovsky, S. On the stablization of the homogeneous bilinear systems. *Systems Control Letters* **21** (1993) 503–510.

[6] Chabour, R., Sallet, G. and Vivalda, J.C. Stabiization of nonlinear two dimentional systems: a bilinear approach. *Math. Control Signals Systems* **6** (1993) 224–246.

[7] Chabour, O. and Vivalda, J.C. Remark on local and global stabilization of homogeneous bilinear systems. *Systems and Control Letters* **41**(2) (2000) 141–143.

[8] Chen, T. T., Tsai, S. H., Tsai, K. L. and Li, C. L. Fuzzy Control for T-S Fuzzy Bilinear Systems with Time-Delay in State and Control Input. In: *International Conference on System Science and Engineering*. Macau, China, June 2011, 346–351.

[9] Gao, H., Lam, J., Mao, X. and Shi, P. H^{∞} Filtering for Uncertain Bilinear Stochastic Systems. *Nonlinear Dynamics and Systems Theory* **7**(2) (2007) 151–168.

[10] Haitao, W. and Xianlai, W. Sliding Mode Control of Uncertain Bilinear System. *Transactions of Tianjin University* **5**(1) (1999) 57–60.

[11] Jerbi, H. Global Feedback Stabilization of New Class of Bilinear Systems. *Systems and Control Letters* **42** (2001) 313–320.

[12] Khapalov, A. Y. and Mohler, R. R. Reachable Sets and Controllability of Bilinear Time-Invariant Systems: A Qualitative Approach. *IEEE Transactions on Automatic Control* **41** (1996) 1342–1346.

[13] Kim, B.S., Kim, Y.J. and Lim, M.T. Robust H^{∞} State Feedbak Control Method for Bilinear Systems. *IEEE Proceeding, Control Theory Applications* **152** (2005) 553–559.

[14] Li, T. S. and Tsai, S. H. T-S Fuzzy Bilinear Model and Fuzzy Controller Design for a Class of Nonlinear Systems. *IEEE Transaction on Fuzzy Systems* **15**(3) (2007) 494–506.

[15] Mahout, V., Tarbouriech, S. and Garcia, G. Controller Design For Unstable Bilinear Systems. In: *Proceedings of the 10th Mediterranean Conference on Control and Automation - MED2002*. Lisbon, Portugal, July 2002, 9–12.

[16] Mohler, R. R. *Bilinear Control Processes*. Academic Press, New York, NY, 1973.

[17] Mohler, R. R., Bruni, C. and Gandol, A. A Systems Approach to Immunology. *Proceedings of the IEEE* **68** (1980) 964–990.

[18] Mohler, R. R. and Khapalov, A. Y. Bilinear Control and Application to Flexible a.c. Transmission Systems. *Journal of Optimization Theory and Applications* **105**(3) (2000) 621-637.

[19] Pardalos, P.M. and Yatsenko, V. Optimization and Control of Bilinear Systems: Theory, Algorithm and Applications. Springer Science+Business Media, LLC 2008.

[20] Sayem, H., Braiek, Benhadj N. and Hammouri, H. Trajectory Planning and Tracking of Bilinear Systems Using Orthogonal Functions. *Nonlinear Dynamics and Systems Theory* **10**(3) (2010) 295-304.

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On the Convergence of Solutions of Some Nonlinear Differential Equations of Fourth Order

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Abstract: In this paper, we consider a nonlinear differential equation of fourth order. By the Lyapunov function approach, we discuss the convergence of the solutions of the equation considered. Our findings generalize some well known results in the literature.

Keywords: *convergence of solutions; nonlinear fourth order equation; Routh-Hurwitz interval; Lyapunov functions.*

Mathematics Subject Classification (2010): 34D20, 34C11.

1 Introduction

As we know the qualitative theory refers to the investigation of the behaviors of solutions of differential equations such as the stability, instability, boundedness,convergence of solutions etc. without determining explicit formulas for the solutions.The relative works can be summarized as follows:

In [1, 15, 16], the authors investigated the asymptotic behaviour of the solutions of certain fourth-order differential equations. In [11, 13, 19–25], the authors considered the stability, instability and boundedness properties of the solutions of some nonlinear third, fourth and fifth-order differential equations (see, also, [10, 14]). In [7], Afuwape studied the existence of a limiting regime in the sense of Demidovic for a certain fourth-order nonlinear differential equations. These studies were done using the Lyapunov's second method. In [2, 5, 8, 9], the authors created conditions for the existence of periodic, almost periodic, exponential stability and dissipative solutions by using the frequency domain

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method. In [3, 4, 6, 12], the authors discussed the convergence of solutions. In [17], Tejumola studied periodic solutions of boundary value problems for some fifth, fourth and third order ordinary differential equations. In [18], Tiryaki and Tunc created Lyapunov functions for certain fourth-order autonomous differential equations.

This paper is concerned with differential equations of the form

$$x^{(iv)} + f_1(x, x', x'', x''') + f_2(x, x', x'') + f_3(x, x') + f_4(x) = p(t, x, x', x'', x'''), \tag{1}$$

where the functions f_1, f_2, f_3, f_4 and p are real valued and continuous in their respective arguments such that the uniqueness theorem is valid, the solutions are continuously dependent on the initial conditions. The function $p(t, x, x', x'', x''')$ is assumed to have the form

$$p(t, x, x', x'', x''') = q(t) + r(t, x, x', x'', x''')$$

with the functions q and r depending explicitly on the arguments displayed and being continuous in their respective arguments. Furthermore, it is assumed that $r(t, 0, 0, 0, 0) = 0$ for all t.

Definition 1.1 Any two solutions $x_1(t)$, $x_2(t)$ of Eq.(1) are said to converge (to each other) if $x_1 - x_2 \to 0$, $x_1' - x_2' \to 0$, $x_1'' - x_2'' \to 0$, $x_1''' - x_2''' \to 0$ as $t \to \infty$.

Our results assert the existence of convergence of solutions with the functions f_1, f_2, f_3 and f_4 not necessarily differentiable. Here, the functions f_4 are only required to satisfy the increment ratio

$$\frac{f_4(\xi + \eta) - f_4(\xi)}{\eta} \in I_0,$$

where I_0 is closed sub-interval of the Routh -Hurwitz interval defined by

$$I_0 = \left[\Delta_0, K_0 \left[\frac{(ab - c)\, c}{a^2}\right]\right],$$

for some positive constants $a, b, c, d, D, \Delta_0, K_0$, and $(ab - c)\, c - a^2 d > 0, ab - c > 0$.

2 Main Results

Theorem 2.1 *In addition to the basic assumptions imposed on the functions f_1, f_2, f_3 and f_4, we assume that $f_1(x, y, z, 0) = f_2(x, y, 0) = f_3(x, 0) = f_4(0) = 0$ and that:*

(i) there are positive constants δ, δ_0, γ, γ_0, β and β_0 such that

$$\begin{aligned} &\delta \le \frac{f_1(x_2, y_2, z_2, u_2) - f_1(x_1, y_1, z_1, u_1)}{u_2 - u_1} \le \delta_0, && (u_2 \ne u_1), \\ &\gamma \le \frac{f_2(x_2, y_2, z_2) - f_2(x_1, y_1, z_1)}{z_2 - z_1} \le \gamma_0, && (z_2 \ne z_1), \\ &\beta \le \frac{f_3(x_2, y_2) - f_3(x_1, y_1)}{y_2 - y_1} \le \beta_0, && (y_2 \ne y_1), \end{aligned} \tag{2}$$

(ii) for any ξ, η $(\eta \ne 0)$, the increment ratios for f_4 satisfy

$$\frac{f_4(\xi + \eta) - f_4(\xi)}{\eta} \in I_0,$$

(iii) there is a continuous function $\phi(t)$ such that

$$\begin{aligned} & |r(t,x_2,y_2,z_2,u_2) - r(t,x_1,y_1,z_1,u_1)| \\ \leq\ & \phi(t)\left\{|x_2-x_1| + |y_2-y_1| + |z_2-z_1| + |u_2-u_1|\right\} \end{aligned} \tag{3}$$

holds for arbitrary $t, x_1, y_1, z_1, u_1, x_2, y_2, z_2, u_2$.

Then,there exists a constant D_1 such that if

$$\int_0^t \phi^v(\tau)d\tau \leq D_1 t \tag{4}$$

for some v in the range $1 \leq v \leq 2$, then all solutions of Eq.(1) converge.

Theorem 2.2 *Let $x_1(t), x_2(t)$ be any two solutions of Eq.(1). Suppose that all the conditions of Theorem 2.1 hold. Then, for each fixed v in the range $1 \leq v \leq 2$, there exist constants $D_2, D_3,$ and D_4 such that*

$$S(t_2) \leq D_2 S(t_1) \exp\left\{-D_3\left(t_2-t_1\right) + D_4 \int_{t_1}^{t_2} \phi^v(\tau)d\tau\right\} \quad \text{for } t_2 \geq t_1, \tag{5}$$

where

$$S(t) \;=\; \left[x_2(t) - x_1(t)\right]^2 + \left[x_2'(t) - x_1'(t)\right]^2 + \left[x_2''(t) - x_1''(t)\right]^2 + \left[x_2'''(t) - x_1'''(t)\right]^2.$$

We have the following corollaries when $x_1(t) = 0$ and $t_1 = 0$.

Corollary 2.1 *Suppose that $p = 0$ in Eq.(1) and assumptions (i) and (ii) of Theorem 2.1 hold for arbitrary $\eta \neq 0$. Then the trivial solution of Eq.(1) is exponentially stable.*

Corollary 2.2 *If $p \neq 0$ and assumptions (i) and (ii) of Theorem 2.1 hold for arbitrary $\eta \neq 0$ and $\xi = 0$, then there exists a constant $D_5 > 0$ such that every solution $x(t)$ of Eq.(1) satisfies*

$$|x(t)| \leq D_5, \quad |x'(t)| \leq D_5, \quad |x''(t)| \leq D_5, \quad |x'''(t)| \leq D_5.$$

Proof of Theorem 2.2 Writing Eq.(1) as a system of first order equations, we obtain

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= u, \\ u' &= -f_1(x,y,z,u) - f_2(x,y,z) - f_3(x,y) - f_4(x) + r(t,x,y,z,u) + q(t). \end{aligned} \tag{6}$$

Let $(x_i(t), y_i(t), z_i(t), u_i(t))$, $(i = 1,2)$, be two solutions of (1), such that

$$\Delta_0 \leq \frac{f_4(x_2) - f_4(x_1)}{x_2 - x_1} \leq K_0\left[\frac{(ab-c)\,c}{a^2}\right]$$

hold. For the proof of the convergence theorem, we define a function

$$\begin{aligned} 2V \;=\; & \left[\beta(1-\epsilon)x + \gamma y + \delta z + u\right]^2 + \left[(1-\epsilon)D - 1\right](\delta z + u)^2 \\ & + \beta\delta\left[\epsilon + (1-\epsilon)D - 1\right]y^2 + \gamma\,(D-1)\,z^2 + \epsilon D u^2 \\ & + \beta^2\epsilon\,(1-\epsilon)\,x^2 + 2\gamma\delta\left[(1-\epsilon)^2 D - 1\right]yz, \end{aligned} \tag{7}$$

where $0 < \epsilon < \frac{1}{2}$, $\frac{\gamma\delta}{\beta} > (1-\epsilon)$, β, γ, δ are positive real numbers and $D = 1 + \frac{\beta(1-\epsilon)[\gamma\delta - \beta(1-\epsilon)]}{\gamma\delta - \beta\epsilon}$ with $D > \frac{1}{(1-\epsilon)^2}$ always. Indeed, we can rearrange the terms in (7) to obtain

$$2V = 2V_1 + 2V_2, \tag{8}$$

where

$$\begin{aligned} 2V_1 &= [\beta(1-\epsilon)x + \gamma y + \delta z + u]^2 + [(1-\epsilon)D - 1](\delta z + u)^2 \\ &\quad + \epsilon D u^2 + \beta^2 \epsilon (1-\epsilon) x^2 + \epsilon\beta\delta y^2, \\ 2V_2 &= \beta\delta[(1-\epsilon)D - 1] y^2 + 2\gamma\delta\left[(1-\epsilon)^2 D - 1\right] yz + \gamma(D-1) z^2. \end{aligned}$$

We note that V_1 is obviously positive definite. This follows from the condition above. Also V_2 can be regarded as quadratic form in y and z, and is always positive.

Let us recall that a real 2×2 matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

is positive definite $\Leftrightarrow a_1 > 0$, $a_4 > 0$ and $a_1 a_4 - a_2 a_3 > 0$. Thus we can rearrange the terms in V_2 as

$$(y, z) \begin{pmatrix} \beta\delta[(1-\epsilon)D - 1] & \gamma\delta\left[(1-\epsilon)^2 D - 1\right] \\ \gamma\delta\left[(1-\epsilon)^2 D - 1\right] & \gamma(D-1) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

Hence V is positive definite. We can therefore find a constant $D_6 > 0$, such that

$$D_6(x^2 + y^2 + z^2 + u^2) \leq V. \tag{9}$$

Furthermore, by using the Schwartz inequality $|y|\,|z| \leq \frac{1}{2}(y^2 + z^2)$, we obtain the following estimate:

$$2|V_2| \leq D^*(y^2 + z^2), \qquad D^* = D^*(\beta, \gamma, \delta, D, \epsilon) > 0.$$

Thus there exists a constant $D_7 > 0$ such that

$$V \leq D_7(x^2 + y^2 + z^2 + u^2), \tag{10}$$

Using inequalities (9) and (10), we obtain

$$D_6(x^2 + y^2 + z^2 + u^2) \leq V \leq D_7(x^2 + y^2 + z^2 + u^2). \tag{11}$$

The following lemma can be easily verified for $W \equiv V$. $\square$

Lemma 2.1 *Let the function $W(t) = W(x_2 - x_1, y_2 - y_1, z_2 - z_1, u_2 - u_1)$ be defined by*

$$\begin{aligned} 2W &= [\beta(1-\epsilon)(x_2 - x_1) + \gamma(y_2 - y_1) + \delta(z_2 - z_1) + (u_2 - u_1)]^2 \\ &\quad + [(1-\epsilon)D - 1](\delta(z_2 - z_1) + (u_2 - u_1))^2 \\ &\quad + \beta\delta[\epsilon + (1-\epsilon)D - 1](y_2 - y_1)^2 + \gamma(D-1)(z_2 - z_1)^2 \\ &\quad + \epsilon D(u_2 - u_1)^2 + \beta^2\epsilon(1-\epsilon)(x_2 - x_1)^2 \\ &\quad + 2\gamma\delta\left[(1-\epsilon)^2 D - 1\right](y_2 - y_1)(z_2 - z_1), \end{aligned}$$

where $0 < \epsilon < \frac{1}{2}$, $\frac{\gamma\delta}{\beta} > (1-\epsilon)$, β, γ, δ are positive real numbers and $D = 1 + \frac{\beta(1-\epsilon)[\gamma\delta - \beta(1-\epsilon)]}{\gamma\delta - \beta\epsilon}$ with $D > \frac{1}{(1-\epsilon)^2}$ always.

i) $W(0,0,0,0) = 0$.

ii) There exist finite positive constants D_6, D_7 such that

$$\begin{aligned} W &\geq D_6 \left\{ (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (u_2 - u_1)^2 \right\}, \\ W &\leq D_7 \left\{ (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (u_2 - u_1)^2 \right\}. \end{aligned} \tag{12}$$

The solutions (x_i, y_i, z_i, u_i), $(i = 1, 2)$ satisfy the system (6). Then $S(t)$ as defined in (6) becomes

$$S(t) = [x_2(t) - x_1(t)]^2 + [y_2(t) - y_1(t)]^2 + [z_2(t) - z_1(t)]^2 + [u_2(t) - u_1(t)]^2.$$

Next we prove a result on the derivative of $W(t)$ with respect to t.

Lemma 2.2 *Assume that conditions (i) and (ii) of Theorem 2.1 hold. Then there exist positive constants D_8 and D_9 such that*

$$\frac{dW}{dt} \leq -2D_8 S + D_9 S^{\frac{1}{2}} |\theta|, \tag{13}$$

where $\theta = r(t, x_2, y_2, z_2, u_2) - r(t, x_1, y_1, z_1, u_1)$.

Proof of Lemma 2.2 Using the system (6), a direct computation of $\frac{dW}{dt}$ gives after simplification

$$\dot{W} = \frac{dW}{dt} = -W_1 + W_2, \tag{14}$$

where

$$\begin{aligned} W_1 = {} & \beta(1-\epsilon)F_4(x_2 - x_1)^2 + \gamma\left[F_3 - \beta(1-\epsilon)\right](y_2 - y_1)^2 \\ & + (1-\epsilon) D\delta \left[F_2 - \gamma(1-\epsilon)\right](z_2 - z_1)^2 + D\left[F_1 - \delta(1-\epsilon)\right](u_2 - u_1)^2 \\ & + \left\{\beta(1-\epsilon)\left[F_3 - \beta\right] + \gamma F_4\right\}(x_2 - x_1)(y_2 - y_1) \\ & + \left\{\beta(1-\epsilon)\left[F_2 - \gamma\right] + (1-\epsilon) D\delta F_4\right\}(x_2 - x_1)(z_2 - z_1) \\ & + \left\{\beta(1-\epsilon)\left[F_1 - \delta\right] + DF_4\right\}(x_2 - x_1)(u_2 - u_1) \\ & + \left\{\gamma\left[F_2 - \gamma\right] + (1-\epsilon) D\delta\left[F_3 - \beta\right]\right\}(y_2 - y_1)(z_2 - z_1) \\ & + \Big\{\gamma\left[F_1 - \delta\right] + D\left[F_3 - \beta\right] + \gamma\delta + D\beta - \beta(1-\epsilon) \\ & - \gamma\delta(1-\epsilon)^2 D\Big\}(y_2 - y_1)(u_2 - u_1) \\ & + \left\{D\left[F_2 - \gamma\right] + (1-\epsilon) D\delta\left[F_1 - \delta\right]\right\}(z_2 - z_1)(u_2 - u_1), \end{aligned}$$

$$\begin{aligned} W_2 = {} & \theta(t)\left\{\beta(1-\epsilon)(x_2 - x_1) + \gamma(y_2 - y_1) + (1-\epsilon) D\delta(z_2 - z_1) \right. \\ & \left. + D(u_2 - u_1)\right\}, \end{aligned} \tag{15}$$

$$F_1 = \frac{f_1(x_2, y_2, z_2, u_2) - f_1(x_1, y_1, z_1, u_1)}{u_2 - u_1}, \qquad (u_2 \neq u_1),$$
$$F_2 = \frac{f_2(x_2, y_2, z_2) - f_2(x_1, y_1, z_1)}{z_2 - z_1}, \qquad (z_2 \neq z_1),$$
$$F_3 = \frac{f_3(x_2, y_2) - f_3(x_1, y_1)}{y_2 - y_1}, \qquad (y_2 \neq y_1),$$
$$F_4 = \frac{f_4(x_2) - f_4(x_1)}{x_2 - x_1}, \qquad (x_2 \neq x_1),$$

and $\lambda_i, \mu_i, \tau_i, \sigma_i$ are strictly positive constants such that

$$\sum_{i=1}^{7}\lambda_i = 1, \quad \sum_{i=1}^{8}\mu_i = 1, \qquad \sum_{i=1}^{7}\tau_i = 1, \sum_{i=1}^{8}\sigma_i = 1.$$

Then W_1 can be rearranged as

$$\begin{aligned} W_1 \;\; = \;\; & W_{11} + W_{12} + W_{13} + W_{14} + W_{15} + W_{16} + W_{17} + W_{18} + W_{19} \\ & + W_{20} + W_{21} + W_{22} + W_{23} + W_{24}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} W_{11} \;\; = \;\; & \lambda_1\beta(1-\epsilon)F_4(x_2 - x_1)^2 + \{\gamma\,[F_3 - \beta] + \mu_1\gamma\beta\epsilon\}\,(y_2 - y_1)^2 \\ & + \{(1-\epsilon)\,D\delta\,[F_2 - \gamma] + \tau_1\,(1-\epsilon)\,D\delta\gamma\epsilon\}\,(z_2 - z_1)^2 \\ & + \{D\,[F_1 - \delta] + \sigma_1 D\delta\epsilon\}\,(u_2 - u_1)^2, \end{aligned}$$

$$W_{12} = \lambda_2\beta(1-\epsilon)F_4(x_2 - x_1)^2 + \beta(1-\epsilon)\,[F_3 - \beta]\,(x_2 - x_1)\,(y_2 - y_1) + \mu_2\gamma\beta\epsilon\,(y_2 - y_1)^2,$$

$$W_{13} = \lambda_3\beta(1-\epsilon)F_4(x_2 - x_1)^2 + \gamma F_4\,(x_2 - x_1)\,(y_2 - y_1) + \mu_3\gamma\beta\epsilon\,(y_2 - y_1)^2,$$

$$\begin{aligned} W_{14} \;\; = \;\; & \lambda_4\beta(1-\epsilon)F_4(x_2 - x_1)^2 + \beta(1-\epsilon)\,[F_2 - \gamma]\,(x_2 - x_1)\,(z_2 - z_1) \\ & + \tau_2\,(1-\epsilon)\,D\delta\gamma\epsilon\,(z_2 - z_1)^2, \end{aligned}$$

$$\begin{aligned} W_{15} \;\; = \;\; & \lambda_5\beta(1-\epsilon)F_4(x_2 - x_1)^2 + (1-\epsilon)\,D\delta F_4\,(x_2 - x_1)\,(z_2 - z_1) \\ & + \tau_3\,(1-\epsilon)\,D\delta\gamma\epsilon\,(z_2 - z_1)^2, \end{aligned}$$

$$W_{16} = \lambda_6\beta(1-\epsilon)F_4(x_2 - x_1)^2 + \beta(1-\epsilon)\,[F_1 - \delta]\,(x_2 - x_1)\,(u_2 - u_1) + \sigma_2 D\delta\epsilon\,(u_2 - u_1)^2,$$

$$W_{17} = \lambda_7\beta(1-\epsilon)F_4(x_2 - x_1)^2 + DF_4\,(x_2 - x_1)\,(u_2 - u_1) + \sigma_3 D\delta\epsilon\,(u_2 - u_1)^2,$$

$$W_{18} = \mu_4\gamma\beta\epsilon\,(y_2 - y_1)^2 + \gamma\,[F_2 - \gamma]\,(y_2 - y_1)\,(z_2 - z_1) + \tau_4\,(1-\epsilon)\,D\delta\gamma\epsilon\,(z_2 - z_1)^2,$$

$$\begin{aligned} W_{19} \;\; = \;\; & \mu_5\gamma\beta\epsilon\,(y_2 - y_1)^2 + (1-\epsilon)\,D\delta\,[F_3 - \beta]\,(y_2 - y_1)\,(z_2 - z_1) \\ & + \tau_5\,(1-\epsilon)\,D\delta\gamma\epsilon\,(z_2 - z_1)^2, \end{aligned}$$

$$W_{20} = \mu_6\gamma\beta\epsilon\,(y_2 - y_1)^2 + \gamma\,[F_1 - \delta]\,(y_2 - y_1)\,(u_2 - u_1) + \sigma_4 D\delta\epsilon\,(u_2 - u_1)^2,$$

$$W_{21} = \mu_7\gamma\beta\epsilon\,(y_2 - y_1)^2 + D\,[F_3 - \beta]\,(y_2 - y_1)\,(u_2 - u_1) + \sigma_5 D\delta\epsilon\,(u_2 - u_1)^2,$$

$$\begin{aligned} W_{22} &= \mu_8\gamma\beta\epsilon\,(y_2-y_1)^2 + \left\{\gamma\delta + D\beta - \beta(1-\epsilon) - \gamma\delta\,(1-\epsilon)^2 D\right\}(y_2-y_1)(u_2-u_1) \\ &\quad + \sigma_6 D\delta\epsilon\,(u_2-u_1)^2, \end{aligned}$$

$$W_{23} = \tau_6\,(1-\epsilon)\,D\delta\gamma\epsilon\,(z_2-z_1)^2 + D\,[F_2-\gamma]\,(z_2-z_1)(u_2-u_1) + \sigma_7 D\delta\epsilon\,(u_2-u_1)^2,$$

$$\begin{aligned} W_{24} &= \tau_7\,(1-\epsilon)\,D\delta\gamma\epsilon\,(z_2-z_1)^2 + (1-\epsilon)\,D\delta\,[F_1-\delta]\,(z_2-z_1)(u_2-u_1) \\ &\quad + \sigma_8 D\delta\epsilon\,(u_2-u_1)^2. \end{aligned}$$

It is clear that $W_{11} \geq 0$. Since each $W_{12}, W_{13}, ..., W_{23}, W_{24}$ are quadratic forms in their respective variables, then from the fact that any quadratic of the form $Ap^2 + Bpq + Cq^2$ is non negative if $4AC - B^2 \geq 0$, it follows that

$$W_{12} \geq 0 \;\text{ if }\; [F_3-\beta]^2 \leq 16\lambda_3\mu_3\lambda_2\mu_2(\epsilon\beta)^2,$$

$$W_{13} \geq 0 \;\text{ if }\; F_4 \leq \frac{4\lambda_3\mu_3\epsilon\beta^2\,(1-\epsilon)}{\gamma},$$

$$W_{14} \geq 0 \;\text{ if }\; [F_2-\gamma]^2 \leq 16\lambda_4\lambda_5\tau_2\tau_3\,(\gamma\epsilon)^2,$$

$$W_{15} \geq 0 \;\text{ if }\; F_4 \leq \frac{4\lambda_5\tau_3\epsilon\beta\gamma}{D\delta},$$

$$W_{16} \geq 0 \;\text{ if }\; [F_1-\delta]^2 \leq 16\lambda_6\lambda_7\sigma_2\sigma_3\,(\delta\epsilon)^2,$$

$$W_{17} \geq 0 \;\text{ if }\; F_4 \leq \frac{4\lambda_7\sigma_3\beta\,(1-\epsilon)\,\delta\epsilon}{D},$$

$$W_{18} \geq 0 \;\text{ if }\; [F_2-\gamma]^2 \leq 4\mu_4\tau_4\beta D\delta\epsilon^2\,(1-\epsilon),$$

$$W_{19} \geq 0 \;\text{ if }\; [F_3-\beta]^2 \leq \frac{4\mu_5\tau_5\beta\,(\gamma\epsilon)^2}{(1-\epsilon)\,D\delta},$$

$$W_{20} \geq 0 \;\text{ if }\; [F_1-\delta]^2 \leq \frac{4\mu_6\sigma_4\beta\epsilon^2 D\delta}{\gamma},$$

$$W_{21} \geq 0 \;\text{ if }\; [F_3-\beta]^2 \leq \frac{4\mu_7\sigma_5\beta\epsilon^2\gamma\delta}{D},$$

$$W_{22} \geq 0 \;\text{ if }\; 4\mu_8\gamma\beta\epsilon\sigma_6 D\delta\epsilon \geq \left\{\gamma\delta + D\beta - \beta(1-\epsilon) - \gamma\delta\,(1-\epsilon)^2 D\right\}^2,$$

$$W_{23} \geq 0 \;\text{ if }\; [F_2-\gamma]^2 \leq 4\tau_6\sigma_7\gamma\,(1-\epsilon)\,(\delta\epsilon)^2,$$

$$W_{24} \geq 0 \;\text{ if }\; [F_1-\delta]^2 \leq \frac{4\tau_7\sigma_8\gamma\epsilon^2}{(1-\epsilon)}.$$

That is,

$$[F_1-\delta]^2 \leq \min\left\{\frac{4\tau_7\sigma_8\gamma\epsilon^2}{(1-\epsilon)},\; \frac{4\mu_6\sigma_4\beta\epsilon^2 D\delta}{\gamma},\; 16\lambda_6\lambda_7\sigma_2\sigma_3\,(\delta\epsilon)^2\right\},$$

$$[F_2-\gamma]^2 \leq \min\left\{16\lambda_4\lambda_5\tau_2\tau_3\,(\gamma\epsilon)^2,\, 4\mu_4\tau_4\beta D\delta\epsilon^2\,(1-\epsilon),\; 4\tau_6\sigma_7\gamma\,(1-\epsilon)\,(\delta\epsilon)^2\right\},$$

$$[F_3-\beta]^2 \leq \min\left\{16\lambda_3\mu_3\lambda_2\mu_2(\epsilon\beta)^2, \frac{4\mu_5\tau_5\beta\,(\gamma\epsilon)^2}{(1-\epsilon)\,D\delta}, \frac{4\mu_7\sigma_5\beta\epsilon^2\gamma\delta}{D}\right\},$$

$$F_4 \leq \min\left\{\frac{4\lambda_3\mu_3\epsilon\beta^2\,(1-\epsilon)}{\gamma}, \frac{4\lambda_5\tau_3\epsilon\beta\gamma}{D\delta}, \frac{4\lambda_7\sigma_3\beta\,(1-\epsilon)\,\delta\epsilon}{D}\right\},$$

Because of $W_{12} \geq 0$, $W_{13} \geq 0, ..., W_{24} \geq 0$, we obtain $W_1 \geq W_{11}$. Then we find a constant D_8 such that

$$W_1 \geq W_{11} \geq 2D_8 S(t), \tag{17}$$

where

$$2D_8 = \min\left\{\beta(1-\epsilon)\Delta_0, \gamma\beta\epsilon, (1-\epsilon)\,D\delta\gamma\epsilon, D\delta\epsilon\right\}.$$

Similarly, we can find from the value of W_2, a constant $D_9 > 0$ small enough such that

$$W_2 \leq D_9 S^{\frac{1}{2}}\,|\theta|\,, \tag{18}$$

where $D_9 = \max\left\{\beta(1-\epsilon), \gamma, (1-\epsilon)\,D\delta, D\right\}.$

Writing (17) and (18) in (14), we get

$$\frac{dW}{dt} \leq -2D_8 S + D_9 S^{\frac{1}{2}}\,|\theta|\,.$$

Let v be any constant in the range $1 \leq v \leq 2$ and $2\mu = 2 - v$, so that $0 \leq \mu \leq 1/2$. One can arrange the estimate in (13) as

$$\frac{dW}{dt} + D_8 S \leq -D_8 S + D_9 S^{1/2}\,|\theta| = D_{10} S^{\mu} W^*,$$

where

$$W^* = \left(|\theta| - D_{11} S^{1/2}\right) S^{1/2-\mu}, \tag{19}$$

with $D_{11} = D_8 D_{10}^{-1}$. We consider the following two cases:
a) $|\theta| < D_{11} S^{1/2}$, b) $|\theta| \geq D_{11} S^{1/2}$.

If $|\theta| < D_{11} S^{1/2}$, then $W^* < 0$. On the other hand, if $|\theta| \geq D_{11} S^{1/2}$, then the definition of W* in (19) gives at least

$$W^* \leq S^{1/2-\mu}\,|\theta|\,,$$

and also $S^{1/2} \leq |\theta|\,/D_{11}$. The foregoing inequality leads to

$$S^{1/2(1-2\mu)} \leq \left[\frac{|\theta|}{D_{11}}\right]^{(1-2\mu)},$$

so that

$$S^{1/2(1-2\mu)}\,|\theta| \leq \left[\frac{|\theta|}{D_{11}}\right]^{(1-2\mu)} |\theta|\,.$$

The above estimate implies

$$W^* \leq D_{12}\,|\theta|^{2(1-\mu)}\,,$$

where $D_{12} = D_{11}^{(2\mu-1)}$. Hence, it is clear that

$$\frac{dW}{dt} + D_8 S \le D_{10} D_{12} S^{\mu} |\theta|^{2(1-\mu)} \le D_{13} S^{\mu} \phi^{2(1-\mu)} S^{(1-\mu)},$$

where $D_{13} = S^{1-\mu} D_{10} D_{12}$ which follows from

$$\begin{aligned} |\theta| &= |r(t, x_2, y_2, z_2, u_2) - r(t, x_1, y_1, z_1, u_1)| \\ &\le \phi(t) \{|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1| + |u_2 - u_1|\}. \end{aligned}$$

Using the estimate $v = 2(1-\mu)$, we obtain

$$\frac{dW}{dt} \le -D_8 S + D_{13} \phi^v S.$$

By the inequality (12), we find

$$\frac{dW}{dt} + (D_{14} - D_{15} \phi^v(t)) W \le 0 \tag{20}$$

for some positive constants D_{14} and D_{15}. Integrating (20) from t_1 to t_2 $(t_2 \ge t_1)$, we have

$$W(t_2) \le W(t_1) \exp\left\{-D_{14}(t_2 - t_1) + D_{15} \int_{t_1}^{t_2} \phi^v(\tau) d\tau\right\}.$$

Again, using Lemma 2.1, we obtain (5) with $D_2 = D_7 D_6^{-1}$, $D_3 = D_{14}$, and $D_4 = D_{15}$. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.1 Choose $D_1 = D_3 D_4^{-1}$ in (4). From the estimate (5), if

$$\int_{t_1}^{t_2} \phi^v(\tau) d\tau \le D_3 D_4^{-1} (t_2 - t_1),$$

then the exponential index remains negative for all $t_2 - t_1 \ge 0$. Then, as $t = t_2 - t_1 \to \infty$, we have $S(t) \to 0$, and this gives

$$x_2 - x_1 \to 0, \quad y_2 - y_1 \to 0, \quad z_2 - z_1 \to 0, \quad u_2 - u_1 \to 0$$

as $t \to \infty$. This completes the proof of Theorem 2.1. □

References

[1] Abou-El-Ela, A. M. A. and Sadek, A. I. On the asymptotic behaviour of solutions of some differential equations of the fourth-order. *Ann. Differential Equations.* **8** (1) (1992) 1–12.

[2] Adesina, O. A. On the qualitative behaviour of solutions of some fourth-order non-linear differential equations. In: *Ordinary differential equations.* Abuja, 2000, 87–93.

[3] Adesina, O. A. and Ogundare, B. S. New results on the convergence of solutions for a certain fourth order nonlinear differential equation. *Journal of Computational Analysis and Appl.* **14** (1) (2012) 10–18.

[4] Afuwape, A. U. On the convergence of solutions of certain fourth order differential equations. *An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat.* (N.S.) **27** (1) (1981) 133–138.

[5] Afuwape, A. U. On some properties of solutions for certain fourth-order nonlinear differential equations. *Analysis.* **5** (2) (1985) 175–183.

[6] Afuwape, A. U. Convergence of the solutions for the equation $x^{(4)} + ax''' + bx'' + g(x') + h(x) = p(t, x, x', x'', x''')$. *Internat. J. Math. Math. Sci.* **11** (4) (1988) 727–733.

[7] Afuwape, A. U. On the existence of a limiting regime in the sense of Demidovic for a certain fourth-order nonlinear differential equation. *J. Math. Anal. Appl.* **129** (2) (1988) 389–393.

[8] Afuwape, A. U. Uniform dissipativity of some fourth-order nonlinear differential equations. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat.* **35** (1) (1989) 9–16.

[9] Afuwape, A. U. Uniform dissipative solutions for some fourth order nonlinear differential equations. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat.* 37 (4) (1991) 431–445.

[10] Ezeilo, J. O. C. New properties of the equation $x''' + ax'' + bx' + h(x) = p(t, x, x', x'')$ for certain special values of the incrementary ratio $\mathrm{y}^{-1}\{h(x+y) - h(x)\}$. In: *Équations différentielles et fonctionnelles non linéaires (Actes Conférence Internat. "Equa-Diff 73", Brussels/Louvain-la-Neuve*, 1973, 447–462.

[11] Ogundare, B. S. Boundedness of solutions to fourth order differential equations with oscillatory restoring and forcing terms. Electron. *J. Differential Equations.* (6) (2006) 6 pp.

[12] Ogundare, B. S. and Okecha, G. E. Convergence of solutions of certain fourth-order nonlinear differential equations. *Int. J. Math. Math. Sci.* (2007), Art. ID 12536, 13 pp.

[13] Ogundare, B. S. and Okecha, G. E. Boundedness and stability properties of solution to certain fourth order non-linear differential equation. *Nonlinear Stud.* **15** (1) (2008) 61–70.

[14] Reissig, R.; Sansone, G. and Conti, R. Non-linear Differential Equations of Higher Order. *Translated from the German.* Noordhoff International Publishing, Leyden, 1974. xiii+669 pp.

[15] Sadek, A. I. On the asymptotic behaviour of solutions of certain fourth-order ordinary differential equations. *Math. Japon.* **45** (3) (1997) 527–540.

[16] Sadek, A. I. and AL-Elaiw, A. S. Asymptotic behaviour of the solutions of a certain fourth-order differential equation. *Ann. Differential Equations* **20** (3) (2004) 221–234.

[17] Tejumola, H. O. Periodic boundary value problems for some fifth, fourth and third order ordinary differential equations. *J. Nigerian Math. Soc.* **25** (2006) 37–46.

[18] Tiryaki, A. and Tunç, C. Constructing Lyapunov functions for certain fourth-order autonomous differential equations. *Indian J. Pure Appl. Math.* **26** (3) (1995) 225–232.

[19] Tiryaki, A. and Tunç, C. Boundedness and stability properties of solutions of certain fourth order differential equations via the intrinsic method. *Analysis.* **16** (4) (1996) 325–334.

[20] Tunc, C. A note on the stability and boundedness results of solutions of certain fourth order differential equations. *Appl. Math. Comput.* **155** (3) (2004) 837–843.

[21] Tunc, C. Some stability and boundedness results for the solutions of certain fourth order differential equations. *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* **44** (2005) 161–171.

[22] Tunc, C. About stability and boundedness of solutions of certain fourth order differential equations. *Nonlinear Phenom. Complex Syst.* **9** (4) (2006) 380–387.

[23] Tunc, C. The boundedness of solutions to nonlinear third order differential equations. *Nonlinear Dynamics and systems Theory* **10** (1) (2010) 97–102.

[24] Tunc, C. Instability for nonlinear differential equations of fifth order subject to delay. *Nonlinear Dynamics and systems Theory* **12** (2) (2012) 207–214.

[25] Wu, X. and Xiong, K. Correspondence remarks on stability results for the solutions of certain fourth-order autonomous differential equations. *Internat. J. Control* **69** (2) (1998) 353–360.