Nonlinear Dynamics and Systems Theory, **14** (4) (2014) 395–401

Chaos Synchronization Approach Based on New Criterion of Stability

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Received: December 13, 2013;   Revised: October 14, 2014

Abstract: In this paper, we propose a simple method for chaos synchronization in continuous-time based on a new criterion for stability. This criterion implies the Lyapunov stabilization criterion, and is applicable to some typical chaotic systems. Numerical simulations in 3D and 4D are presented to demonstrate the effectiveness of the synchronization results derived in this paper.

Keywords: *chaos synchronization; new criterion; dynamical system; continuous-time; Lyapunov stability.*

Mathematics Subject Classification (2010): 37B25, 37B55, 37C75.

1 Introduction

During the last decade, chaos synchronization has become an active research area, due to its potential applications in information processing such as secure communication [1, 2]. Many types of synchronization have been presented [3–6] and various methods have been developed for synchronization of chaotic systems such as active and adaptive control method [7, 8], backstepping design technique [9], sliding mode control [10], generalized Hamiltonian systems approach [11, 12], and so on. Most of synchronization methods are based on Lyapunov stability theory to guarantee zero stability of errors dynamical system between master and slave chaotic systems.

In this paper, based on some lemma derived from Halanay inequality, we introduce a new and simple stability criterion to synchronize chaotic dynamical systems in continuous-time. In [13], authors derived an important result using Halanay inequality, we give it in the following lemma:

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Lemma 1.1 *Suppose that the continuous functional h satisfies:*

$$|h(t, z_\tau)| \leq \max |z_\tau| \tag{1}$$

and

$$ess \sup |\beta(t)| = \beta \leq \alpha. \tag{2}$$

Then every solution $z(t)$ of

$$\dot{z}(t) = -\alpha z(t) + \beta(t) h(t, z_\tau) \tag{3}$$

converges to zero.

This lemma allows us to achieve synchronization without using Lyapunov theory. This paper is organized as follows: In Section 2, a new synchronization criterion for different chaotic systems is proposed. In Section 3, the case of two identical chaotic systems is investigated. In Section 4, numerical examples of 3D chaotic systems and 4D hyperchaotic systems are discussed and numerical simulations are given. In Section 5, conclusion follows.

2 Synchronization Criterion for Different Systems

Consider the chaotic system described by

$$\dot{X}(t) = f(X(t)), \tag{4}$$

where $X(t) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. We consider the system (4) as the master system, as the slave system we consider the following chaotic system described by

$$\dot{Y}(t) = BY(t) + g(Y(t)) + U, \tag{5}$$

where $Y(t) \in \mathbb{R}^n$, B is the $n \times n$ matrix of parameters system, $g : \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear part of the system (5) and U is the vector controller. The synchronization problem is to find a controller U, which stabilizes the error system

$$e(t) = Y(t) - X(t), \tag{6}$$

then the aim of synchronization is to make $\lim_{t \longrightarrow +\infty} \|e(t)\| = 0$, where $\|.\|$ is the Euclidean norm.

Remark 2.1 Most of chaotic systems, including all Lur'e nonlinear systems and Lipschitz nonlinear systems, can be described by form of (5) without the function controller U.

The error dynamical system between the master system (4) and the slave system (5), can be derived as follows

$$\dot{e}(t) = Be(k) + BX(t) + g(Y(t)) - f(X(t)) + U. \tag{7}$$

To achieve synchronization between the master system (4) and the slave system (5), we can choose the vector controller U as follows

$$U = (C - K) Y(t) + (K - C - B) X(t) + f(X(t)) - g(Y(t)), \tag{8}$$

where $C = (c_{ij}) \in \mathbb{R}^{n\times n}$ such that

$$c_{ij} = \begin{cases} -b_{ij}, & if \quad i \neq j, \\ 0, & if \quad i = j, \end{cases} \tag{9}$$

and $K = diag\,(k_1, k_2, ..., k_n)$ is unknown control diagonal matrix to be determined.

By substituting Eq.(8) into Eq.(7), the error system can be written as

$$\dot{e}(t) = -e(t) + (B + C - K + I)\, e(t). \tag{10}$$

Theorem 2.1 *If the control constants $(k_i)_{1\leq i\leq n}$ are chosen such that*

$$b_{ii} < k_i < 2 + b_{ii}, \quad 1 \leq i \leq n, \tag{11}$$

then the two systems (4) and (5) are globally synchronized.

Proof. The Eq. (10) allows us to get the following scalar systems

$$\dot{e}_i(t) = -e_i(t) + (b_{ii} - k_i + 1)\, e_i(t), \quad 1 \leq i \leq n. \tag{12}$$

If we put $\tau = 0$ in Eq. (3), we can see that the Eq.(14) is the same as Eq.(3) in Lemma 1.1 with: $z\,(t) = e_i(t)$, $\alpha = 1$, $\beta\,(t) = 1$ and $h\,(t, z\,(t)) = (b_{ii} - k_i + 1)\, e_i(t)$. Now, by using condition (11), we can verify conditions of Lemma 1.1 to (12)

$$|(b_{ii} - k_i + 1)\, e_i(t)| = |b_{ii} - k_i + 1|\, |e_i(t)| \leq \max |e_i(t)| \tag{13}$$

and

$$ess \sup |\beta(t)| = \beta = 1 \leq \alpha, \tag{14}$$

hence

$$\lim_{t\to\infty} e_i(t) = 0,\ (1 \leq i \leq n)\,, \tag{15}$$

and from the fact $\lim_{t\to\infty} \|e(t)\| = 0$, we conclude that systems (4) and (5) are globally synchronized.

Proposition 2.1 *The stability criterion of Theorem 2.1 implies Lyapunov stabilization criterion.*

Proof. Assume that systems (4) and (5) are globally synchronized with the criterion of Theorem 2.1, and we consider the following Lyapunov function:

$$V\,(e(t)) = \sum_{i=1}^{n} \frac{1}{2} e_i^2(t), \tag{16}$$

we get

$$\begin{aligned} \dot{V}\,(e(t)) &= \textstyle\sum_{i=1}^{n} \dot{e}_i(t) e_i(t) \\ &= \textstyle\sum_{i=1}^{n} \left(-e_i(t) + (b_{ii} - k_i + 1)\, e_i(t)\right) e_i(t) \\ &= \textstyle\sum_{i=1}^{n} (b_{ii} - k_i)\, e_i^2(t) \end{aligned}$$

and by Theorem 2.1, we have

$$-\,2 < b_{ii} - k_i < 0, \quad 1 \leq i \leq n, \tag{17}$$

then $\dot{V}\,(e(t)) < 0$, and the implication is verified.

3 Synchronization Criterion for Identical Systems

Now, we consider the master system and the slave system in the following forms

$$X(t) = AX(t) + f(X(t)), \tag{18}$$

$$Y(t) = AY(t) + f(Y(t)) + U, \tag{19}$$

where $X(t) \in \mathbb{R}^n$, $Y(t) \in \mathbb{R}^n$ are the state vectors of the master system and the slave system, respectively, A is the $n \times n$ matrix of parameters of system, $f = (f_i\,(X(t)))_{1 \leq i \leq n}$, such that f_i are continuous nonlinear scalar functions and verifying the following condition

$$|f_i(Y(t)) - f_i(X(t))| \leq \rho_i\,|y_i(t)) - x_i(t)|\,, \quad 1 \leq i \leq n, \tag{20}$$

where $0 < \rho_i < 1$ and U is the vector controller to be determined.

The error system between the master system (18) and the slave system (19), can be derived as folow:

$$\dot{e}(t) = Ae(t) + f(Y(t)) - f(X(t)) + U, \tag{21}$$

To achieve synchronization between systems (18) and (19), we choose the vector controller as

$$U = (C - K)\,e(t), \tag{22}$$

where $C = (c_{ij}) \in \mathbb{R}^{n \times n}$, such that:

$$c_{ij} = \begin{cases} -a_{ij}, & if \quad i \neq j, \\ 0, & if \quad i = j, \end{cases} \tag{23}$$

and $K = diag\,(k_1, k_2, ..., k_n) \in \mathbb{R}^{n \times n}$ is unknown control diagonal matrix to be designed later.

By substituting Eq.(22) into Eq.(21), one can obtain the following formula for the error system:

$$\dot{e}(t) = (A + C - K)\,e(t) + f(Y(t)) - f(X(t)). \tag{24}$$

Theorem 3.1 *If the control constants* $(k_i)_{1 \leq i \leq n}$ *are chosen such that*

$$a_{ii} + \rho_i < k_i < 2 + a_{ii} - \rho_i, \quad 1 \leq i \leq n, \tag{25}$$

then the two systems (18) and (19) are globally synchronized.

Proof. According to the same procedure as in the proof of Theorem 2.1, the Eq. (24) provides the following scalar equations

$$\dot{e}_i(t) = -e_i(t) + (a_{ii} - k_i + 1)\,e_i(t) + f_i(Y(t)) - f_i(X(t)), \quad 1 \leq i \leq n, \tag{26}$$

and we can see that Eq.(26) is the same as Eq.(3) with: $z\,(t) = e_i(t)$, $\alpha = 1$, $\beta\,(t) = 1$ and $h\,(t, z\,(t)) = (a_{ii} - k_i + 1)\,e_i(t) + f_i(Y\,(t)) - f_i(X\,(t)$.

Thus, we apply Lemma 1.1 to Eq.(26) and by using Theorem 3.1, we obtain

$$|h\,(t, z\,(t))| \quad \leq (|a_{ii} - k_i + 1| + \rho_i)\,|e_i\,(t)| \quad \leq \max\left(|e_i\,(t)|\right), \tag{27}$$

and we have also

$$ess\sup|\beta(t)| = 1 \leq \alpha, \tag{28}$$

Hence

$$\lim_{t \to \infty} e_i\,(t) = 0, \quad 1 \leq i \leq n, \tag{29}$$

implying $\lim_{t \to \infty} \|e\,(t)\| = 0$, i.e., systems (18) and (19) are globally synchronized.

4 Numerical Examples and Simulations

In this section, to demonstrate the use of chaos synchronization criterion proposed herein, two numerical examples are considered.

4.1 Example 1

Here, as the master system we consider the Chen system [14] described by

$$\begin{cases} \dot{x}_1 = a\left(x_2 - x_1\right), \\ \dot{x}_2 = \left(c - a\right) x_1 + c x_2 - x_1 x_3, \\ \dot{x}_3 = -b x_3 + x_1 x_2, \end{cases} \tag{30}$$

when $a = 35$, $b = 3$ and $c = 28$, the Chen system has chaotic attractor.

As the slave system, we consider the controlled Lü system [15] described by

$$\begin{cases} \dot{y}_1 = \alpha(y_2 - y_1) + u_1, \\ \dot{y}_2 = \beta y_2 - y_1 y_3 + u_2, \\ \dot{y}_3 = -\gamma y_3 + y_1 y_2 + u_3, \end{cases} \tag{31}$$

where u_1, u_2, u_3 are synchronization controllers and when $\alpha = 36$, $\beta = 3$, $\gamma = 20$, the Lü system is chaotic.

Corollary 4.1 *For the two coupled Chen system and Lü system, if $(k_i)_{1 \le i \le 3}$ are chosen such that the inequalities: $-36 < k_1 < -34$, $3 < k_2 < 5$ and $-20 < k_3 < -18$, holds. Then they are globally synchronized.*

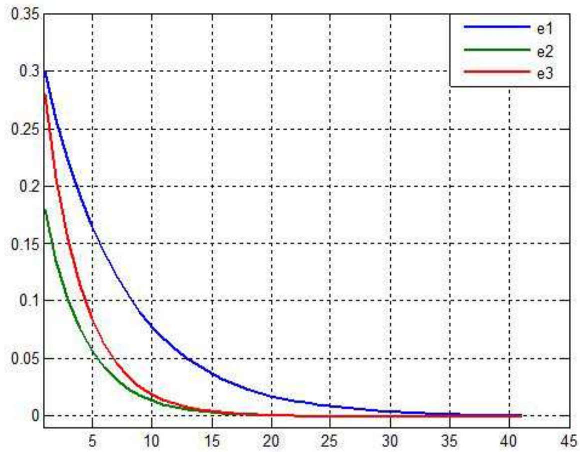

Figure 1: Time evolution of synchronization errors between the master system (30) and the slave system (31).

4.2 Example 2

Now, as the master system we consider the hyperchaotic Lü system [16], described by

$$\begin{cases} \dot{x}_1 = a\left(x_2 - x_1\right) + x_4, \\ \dot{x}_2 = cx_2 - x_1x_3, \\ \dot{x}_3 = -bx_3 + x_1x_2, \\ \dot{x}_4 = x_1x_3 + dx_4, \end{cases} \tag{32}$$

when $a = 36$, $b = 3$, $c = 20$ and $-0.35 < d \leq 1.3$, the 4D Lü system has hyperchaotic attractor.

As the slave system, we consider the controlled hyperchaotic Chen system [17] described by

$$\begin{cases} \dot{y}_1 = b_1\left(y_2 - y_1\right) + u_1, \\ \dot{y}_2 = 4\left(y_1 + y_4\right) + b_2y_2 - 10y_1y_3 + u_2, \\ \dot{y}_3 = -b_3y_3 + y_2^2 + u_3, \\ \dot{y}_4 = -b_4y_1 + u_4, \end{cases} \tag{33}$$

where u_1, u_2, u_3 and u_4.are synchronization controllers. The 4D Chen system is hyperchaotic when the parameter values are taken as $b_1 = 35$, $b_2 = 21$, $b_3 = 3$, $b_4 = 2$.

Corollary 4.2 *For the two coupled, hyperchaotic Lü system and hyperchaotic Chen system, if $(k_i)_{1\leq i\leq 4}$ are chosen such that the inequalities: $-35 < k_1 < -33$, $21 < k_2 < 23$, $-3 < k_3 < -1$ and $0 < k_4 < 2$, hold. Then they are globally synchronized.*

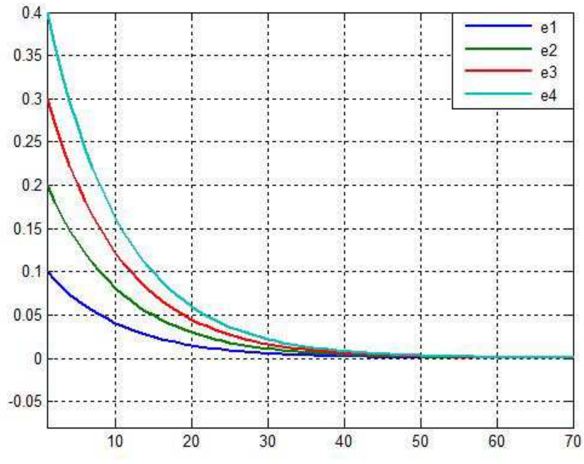

Figure 2: Time evolution of synchronization errors between the master system (32) and the slave system (33).

5 Conclusion

In this paper, using Lemma 1.1, a new criterion was derived. It was also demonstrated that this criterion can be applied to some chaotic and hyperchaotic systems. Finally, we remark that in the first case when the chaotic systems are different the controller is taken in a nonlinear form, but in the case of identical systems the controller is linear.

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