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Time-Fractional Generalized Equal Width Wave Equations: Formulation and Solution via Variational Methods

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Abstract: This paper presents the formulation of the time-fractional generalized Equal Width Wave (EWW) equation and generalized Equal Width Wave-Burgers (EWW-Burgers) equation using the Euler-Lagrange variational technique in the Riemann-Liouville derivative sense, and derive respectively an approximate solitary wave solution. Our results witness that He's variational-iteration method was very efficient and powerful technique in finding the solution of the proposed equation.

Keywords: *Riemann-Liouville fractional operator; Euler-Lagrange equation; fractional EWW equation; He's variational-iteration method; solitary wave.*

Mathematics Subject Classification (2010): 35R11, 35G20.

1 Introduction

The generalized EWW equation has been used to describe approximately the unidirectional propagation of the regularized long wave in certain nonlinear dispersive systems [1], and has been proposed by Benjamin, Bona and Mahony as a model for small-amplitude long waves on the surface of water in a channel [2]. In physical situations one has unidirectional waves propagating in a water channel, long-crested waves in near-shore zones and many others. This equation also serves as an alternative model to the generalized regularised long wave equation and generalized Korteweg-de Vries equation (KdV) [3–5].

During the past three decades or so, fractional calculus has gained considerable popularity and importance as generalizations of integer-order evolution equations, and is applied to model problems in neurons, hydrology, viscoelasticity and rheology, image processing, mechanics, mechatronics, physics, finance and control theory, see [6–11]. If

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the Lagrangian of conservative system is constructed using fractional derivatives, the resulting equations of motion can be nonconservative. Therefore, in many cases, the real physical processes could be modeled in a reliable manner using fractional-order differential equations rather than integer-order equations [12]. In [13], the semi-inverse method has been used to derive the Lagrangian of the KdV equation, the time operator of the Lagrangian of the KdV equation has been transformed into fractional domain in terms of the left-Riemann-Liouville fractional differential operator, the variational of the functional of this Lagrangian leads neatly to Euler-Lagrange equation. Based on the stochastic embedding theory, Cresson [14] defined the fractional embedding of differential operators and provided a fractional Euler-Lagrange equation for Lagrangian systems, then investigated a fractional Noether theorem and a fractional Hamiltonian formulation of fractional Lagrangian systems. Herzallah and Baleanu [15] presented the necessary and sufficient optimality conditions for the Euler-Lagrange fractional equations of fractional variational problems with determining in which spaces the functional must exist. Malinowska [16] proposed the Euler-Lagrange equations for fractional variational problems with multiple integrals and proved the fractional Noether-type theorem for conservative and nonconservative generalized physical systems. Riewe [17] formulated a version of the Euler-Lagrange equation for problems of calculus of variation with fractional derivatives. Wu and Baleanu [18] developed some new variational-iteration formulae to find approximate solutions of fractional differential equations and determined the Lagrange multiplier in a more accurate way. For generalized fractional Euler-Lagrange equations we can refer to the works by Odzijewicz [19, 20]. Other known results can be found in Agrawal [21–23], Baleanu et al [24], Inokuti et al [25] and Zhang [26]. In view of the fact that most of physical phenomena may be considered as nonconservative, they can be described using fractional-order differential equations. Recently, several methods have been used to solve nonlinear fractional evolution equation using techniques of nonlinear analysis, such as Adomian decomposition method [27], homotopy analysis method [28,29] and homotopy perturbation method [30]. It was mentioned that the variational-iteration method has been used successfully to solve different types of integer and fractional nonlinear evolution equations. Making use of the variational-iteration method, this work's main motivation is to formulate the time-fractional generalized EWW equation and generalized EWW-Burgers equation and to derive an approximate solitary wave solution, respectively.

This paper is organized as follows: Section 2 states some background material from fractional calculus. Section 3 presents the principle of He's variational-iteration method. Sections 4 and 5 are devoted to describing the formulation of the time-fractional generalized EWW equation and generalized EWW-Burgers equation using the Euler-Lagrange variational technique and to deriving an approximate solitary wave solution, respectively. Section 6 makes some analysis for the obtained graphs and figures and discusses the present work.

2 Preliminaries

We recall the necessary definitions for the fractional calculus (see, e.g. [31–33]) which is used throughout the remaining sections of this paper.

Definition 2.1 A real multivariable function $\varphi(x,t)$, $t > 0$ is said to be in the space C_γ, $\gamma \in \mathbb{R}$, with respect to t if there exists a real number $p > \gamma$, such that

$\varphi(x,t) = t^p \varphi_1(x,t)$, where $\varphi_1 \in C(\Omega \times T)$, $\Omega \subseteq \mathbb{R}$ and $T = [0, t_0] (t_0 > 0)$. Obviously, $C_\gamma \subset C_\delta$ if $\delta \leq \gamma$.

Definition 2.2 The left-hand side Riemann-Liouville fractional integral of a function $\varphi \in C_\gamma$, $(\gamma \geq -1)$ is defined by

$$\begin{aligned} {}_0I_t^\alpha \varphi(x,t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \varphi(x,\tau) d\tau, \quad \alpha > 0, \quad t \in T, \\ {}_0I_t^0 \varphi(x,t) &= \varphi(x,t). \end{aligned}$$

Definition 2.3 The Riemann-Liouville fractional derivatives of the order $n-1 \leq \alpha < n$ of a function $\varphi \in C_\gamma$, $(\gamma \geq -1)$ are defined as

$$\begin{aligned} {}_0D_t^\alpha \varphi(x,t) &= \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t (t-\tau)^{n-\alpha-1} \varphi(x,\tau) d\tau, \\ {}_tD_{t_0}^\alpha \varphi(x,t) &= \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_t^{t_0} (\tau-t)^{n-\alpha-1} \varphi(x,\tau) d\tau, \quad t \in T. \end{aligned}$$

Lemma 2.1 *The integration of Riemann-Liouville fractional derivative of the order $0 < \alpha < 1$ of the functions φ, ϕ, ${}_tD_{t_0}^\alpha \varphi(x,t)$ and ${}_0D_t^\alpha \phi(x,t) \in C(\Omega \times T)$ by parts are given by the rule*

$$\int_T \varphi(x,t) {}_0D_t^\alpha \phi(x,t) dt = \int_T \phi(x,t) {}_tD_{t_0}^\alpha \varphi(x,t) dt.$$

Definition 2.4 The Riesz fractional integral of the order $n-1 \leq \alpha < n$ of a function $\varphi \in C_\gamma$, $(\gamma \geq -1)$ is defined as

$${}_0^RI_t^\alpha \varphi(x,t) = \frac{1}{2}\big({}_0I_t^\alpha \varphi(x,t) + {}_tI_{t_0}^\alpha \varphi(x,t)\big) = \frac{1}{2\Gamma(\alpha)} \int_0^{t_0} |t-\tau|^{\alpha-1} \varphi(x,\tau) d\tau,$$

where ${}_0I_t^\alpha$ and ${}_tI_{t_0}^\alpha$ are respectively the left- and right-hand side Riemann-Liouville fractional integral operators.

Definition 2.5 The Riesz fractional derivative of the order $n-1 \leq \alpha < n$ of a function $\varphi \in C_\gamma$, $(\gamma \geq -1)$ is defined by

$$\begin{aligned} {}_0^RD_t^\alpha \varphi(x,t) &= \frac{1}{2}\big({}_0D_t^\alpha \varphi(x,t) + (-1)^n {}_tD_{t_0}^\alpha \varphi(x,t)\big) \\ &= \frac{1}{2\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^{t_0} |t-\tau|^{n-\alpha-1} \varphi(x,\tau) d\tau, \end{aligned}$$

where ${}_0D_t^\alpha$ and ${}_tD_{t_0}^\alpha$ are respectively the left- and right-hand side Riemann-Liouville fractional differential operators.

Lemma 2.2 *Let $\alpha > 0$ and $\beta > 0$ be such that $n-1 < \alpha < n$, $m-1 < \beta < m$ and $\alpha + \beta < n$, and let $\varphi \in L_1(\Omega \times T)$ and ${}_0I_t^{m-\alpha} \varphi \in AC^m(\Omega \times T)$. Then we have the following index rule:*

$${}_0^RD_t^\alpha \big({}_0^RD_t^\beta \varphi(x,t)\big) = {}_0^RD_t^{\alpha+\beta} \varphi(x,t) - \sum_{i=1}^m {}_0^RD_t^{\beta-i} \varphi(x,t)|_{t=0} \frac{t^{-\alpha-i}}{\Gamma(1-\alpha-i)}.$$

Remark 2.1 One can express the Riesz fractional differential operator ${}_0^RD_t^{\alpha-1}$ of the order $0 < \alpha < 1$ as the Riesz fractional integral operator ${}_0^RI_\tau^{1-\alpha}$, i.e.

$${}_0^RD_t^{\alpha-1} \varphi(x,t) = {}_0^RI_t^{1-\alpha} \varphi(x,t), \quad t \in T.$$

3 Variational-iteration Method

The variational-iteration method [34–36] provides an effective procedure for explicit and solitary wave solutions of a wide and general class of differential systems representing real physical problems. Moreover, the variational-iteration method can overcome the foregoing restrictions and limitations of approximate techniques so that it provides us with a possibility to analyze strongly nonlinear evolution equations. Therefore, we extend this method to solve the time-fractional generalized EWW equation. The basic features of the variational-iteration method are outlined as follows.

Considering a nonlinear evolution equation consists of a linear part $\mathcal{L}u$, nonlinear part $\mathcal{N}u$, and a free term $f = f(x,t)$ represented as

$$\mathcal{L}u + \mathcal{N}u = f. \tag{1}$$

According to the variational-iteration method, the $n+1$-th approximate solution of (1) can be read using iteration correction functional as

$$u_{n+1} = u_n + \int_0^t \lambda(\tau)\big(\mathcal{L}\tilde{u} + \mathcal{N}\tilde{u} - f\big)d\tau, \tag{2}$$

where $\lambda(\tau)$ is a general Lagrange's multiplier, which can be identified via the variational theory and $\tilde{u}$ is considered as a restricted variation function which means $\delta\tilde{u} = 0$. Extreming the variation of the correction functional (2) leads to the Lagrangian multiplier $\lambda(\tau)$. The initial iteration u_0 can be used as the initial value $u(x,0)$, as n tends to infinity, the iteration leads to the solitary wave solution of (1), i.e.

$$u = \lim_{n\to\infty} u_n.$$

4 Time-fractional Generalized EWW Equation

In this section, He's variational-iteration method is applied to solve time-fractional generalized EWW equation

$${}_0^R D_t^\alpha u + au^p u_x - \mu u_{xxt} = 0,$$

where $a \neq 0$, p and $\mu > 0$, $u = u(x,t)$ is a field variable, $x \in \Omega \subseteq \mathbb{R}$ is a space coordinate in the propagation direction of the field and $t \in T = [0, t_0](t_0 > 0)$ is the time, the subscripts denote the partial differentiation of the function u with respect to the parameter x and t, ${}_0^R D_t^\alpha$ is the Riesz fractional derivative.

The generalized EWW equation in (1+1) dimensions is given as

$$u_t + au^p u_x - \mu u_{xxt} = 0. \tag{3}$$

Employing a potential function v on the field variable, set $u = v_x$ yields the potential equation of the generalized EWW equation (3) in the form,

$$v_{xt} + av_x^p v_{xx} - \mu v_{xxxt} = 0. \tag{4}$$

The Lagrangian of this generalized EWW equation (3) can be defined using the semi-inverse method [37, 38] as follows. The functional of the potential equation (4) can be represented as

$$J(v) = \int_\Omega dx \int_T \Big(v\big(c_1 v_{xt} + ac_2 v_x^p v_{xx} - \mu c_3 v_{xxxt}\big)\Big)dt, \tag{5}$$

with c_i $(i = 1, 2, 3)$ as an unknown constant to be determined later. Integrating (5) by parts and taking $v_x|_{\partial\Omega} = v_x|_{\partial T} = v_{xxt}|_{\partial\Omega} = 0$ yield

$$J(v) = \int_\Omega dx \int_T \Big(- c_1 v_t v_x - \frac{ac_2}{p+1} v_x^{p+2} + \mu c_3 v_{xxt} v_x \Big) dt. \tag{6}$$

The constants c_i $(i = 1, 2, 3)$ can be determined taking the variation of the functional (6) to make it optimal. By applying the variation of the functional, integrating each term by parts, and making use of the variation optimum condition of the functional $J(v)$, it yields the following representation

$$2c_1 v_{tx} + (p+2)ac_2 v_x^p v_{xx} - 2\mu c_3 v_{xxxt} = 0. \tag{7}$$

Note that the obtained result (7) is equivalent to (4), so one has that the constants c_i $(i = 1, 2, 3)$ are respectively

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{p+2}, \quad c_3 = \frac{1}{2}.$$

In addition, the functional representation given by (6) obtains directly the Lagrangian form of the generalized EWW equation,

$$L(v_t, v_x, v_{xxt}) = -\frac{1}{2} v_t v_x - \frac{a}{(p+1)(p+2)} v_x^{p+2} + \frac{\mu}{2} v_{xxt} v_x.$$

Similarly, the Lagrangian of the time-fractional version of the generalized EWW equation could be read as

$$F({}_0D_t^\alpha v, v_x, v_{xxt}) = -\frac{1}{2} {}_0D_t^\alpha v v_x - \frac{a}{(p+1)(p+2)} v_x^{p+2} + \frac{\mu}{2} v_{xxt} v_x, \quad \alpha \in]0, 1]. \tag{8}$$

Then the functional of the time-fractional generalized EWW equation will take the representation

$$J(v) = \int_\Omega dx \int_T F({}_0D_t^\alpha v_t, v_x, v_{xxt}) dt, \tag{9}$$

where the time-fractional Lagrangian $F({}_0D_t^\alpha v_t, v_x, v_{xx}, v_{xxt}, v_{xxx})$ is given by (8). Following Agrawal's method [21–23], the variation of functional (9) with respect to v leads to

$$\delta J(v) = \int_\Omega dx \int_T \Big(\frac{\partial F}{\partial {}_0D_t^\alpha v} \delta({}_0D_t^\alpha v) + \frac{\partial F}{\partial v_x} \delta v_x + \frac{\partial F}{\partial v_{xxt}} \delta v_{xxt} \Big) dt. \tag{10}$$

By Lemma 2.1, upon integrating the right-hand side of (10), one has

$$\delta J(v) = \int_\Omega dx \int_T \Big({}_tD_T^\alpha \big(\frac{\partial F}{\partial {}_0D_t^\alpha v} \big) - \frac{\partial}{\partial x} \big(\frac{\partial F}{\partial v_x} \big) - \frac{\partial^3}{\partial x^2 \partial t} \big(\frac{\partial F}{\partial v_{xxt}} \big) \Big) \delta v dt,$$

noting that $\delta v|_{\partial T} = \delta v|_{\partial\Omega} = \delta v_x|_{\partial\Omega} = \delta v_{xx}|_{\partial T} = 0$.

Obviously, optimizing the variation of the functional $J(v)$, i.e., $\delta J(v) = 0$, yields the Euler-Lagrange equation for time-fractional generalized EWW equation in the following representation

$${}_tD_T^\alpha \big(\frac{\partial F}{\partial {}_0D_t^\alpha v} \big) - \frac{\partial}{\partial x} \big(\frac{\partial F}{\partial v_x} \big) - \frac{\partial^3}{\partial x^2 \partial t} \big(\frac{\partial F}{\partial v_{xxt}} \big) = 0. \tag{11}$$

Substituting the Lagrangian of the time-fractional generalized EWW equation (8) into Euler-Lagrange formula (11) gives

$$-\frac{1}{2}{}_tD_{T_0}^{\alpha}v_x + \frac{1}{2}{}_0D_t^{\alpha}v_x + av_x^p v_{xx} - \mu v_{xxxt} = 0.$$

Once again, substituting the potential function v_x for u, yields the time-fractional generalized EWW equation for the state function u as

$$\frac{1}{2}\left({}_0D_t^{\alpha}u - {}_tD_{T_0}^{\alpha}u\right) + au^p u_x - \mu u_{xxt} = 0. \tag{12}$$

According to the Riesz fractional derivative ${}_0^RD_t^{\alpha}u$, the time-fractional generalized EWW equation represented in (12) can be written as

$${}_0^RD_t^{\alpha}u + au^p u_x - \mu u_{xxt} = 0. \tag{13}$$

Acting from the left-hand side by the Riesz fractional operator ${}_0^RD_t^{1-\alpha}$ on (13) leads to

$$\frac{\partial}{\partial t}u - {}_0^RD_t^{\alpha-1}u|_{t=0}\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + {}_0^RD_t^{1-\alpha}\Big(au^p u_x - \mu u_{xxt}\Big) = 0, \tag{14}$$

from Lemma 2.2. In view of the variational-iteration method, combining with (14), the $n+1$-th approximate solution of (13) can be read using iteration correction functional as

$$\begin{aligned} u_{n+1} = u_n + \int_0^t \lambda(\tau)\bigg(&\frac{\partial}{\partial \tau}u_n - {}_0^RD_\tau^{\alpha-1}u_n|_{\tau=0}\frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} \\ &+ {}_0^RD_\tau^{1-\alpha}\Big(a\tilde{u}_n^p\frac{\partial}{\partial x}\tilde{u}_n - \mu\frac{\partial^3}{\partial x^2\partial t}\tilde{u}_n\Big)\bigg)d\tau, \end{aligned} \tag{15}$$

where the function $\tilde{u}_n$ is considered as a restricted variation function, i.e., $\delta\tilde{u}_n = 0$. The extreme of the variation of (15) subject to the restricted variation function straightforwardly yields

$$\delta u_{n+1} = \delta u_n + \int_0^t \lambda(\tau)\delta\frac{\partial}{\partial\tau}u_n d\tau = \delta u_n + \lambda(\tau)\delta u_n|_{\tau=t} - \int_0^t \frac{\partial}{\partial\tau}\lambda(\tau)\delta u_n d\tau = 0.$$

This representation reduces the following stationary conditions

$$\frac{\partial}{\partial\tau}\lambda(\tau) = 0, \quad 1 + \lambda(\tau) = 0,$$

which converted to the Lagrangian multiplier at $\lambda(\tau) = -1$. Therefore, the correction functional (15) takes the following form

$$u_{n+1} = u_n - \int_0^t\bigg(\frac{\partial}{\partial\tau}u_n - {}_0^RI_\tau^{1-\alpha}u_n|_{\tau=0}\frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} + {}_0^RD_\tau^{1-\alpha}\Big(au_n^p\frac{\partial}{\partial x}u_n - \mu\frac{\partial^3}{\partial x^2\partial t}u_n\Big)\bigg)d\tau, \tag{16}$$

since $\alpha - 1 < 0$, the fractional derivative operator ${}_0^RD_t^{\alpha-1}$ reduces to fractional integral operator ${}_0^RI_t^{1-\alpha}$ by Remark 2.1.

In view of the right-hand side Riemann-Liouville fractional derivative is interpreted as a future state of the process in physics. For this reason, the right-derivative is usually neglected in applications, when the present state of the process does not depend on the results of the future development, and so the right-derivative is used equal to zero in the following calculations. The zero order solitary wave solution can be taken as the initial value of the state variable, which is taken in this case as

$$u_0(x,t)=u(x,0)=\Big(\frac{(p+1)(p+2)c}{2a}\operatorname{sech}^2\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\Big)^{\frac{1}{p}},$$

where c and x_0 are constants.

Substituting this zero order solitary wave solution into (16) and using the Definition 2.5 lead to the first order solitary wave solution

$$\begin{aligned}u_1(x,t)=&\Big(\frac{(p+1)(p+2)c}{2a}\operatorname{sech}^2\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\Big)^{\frac{1}{p}}+\frac{t^\alpha}{\Gamma(\alpha+1)}\frac{a}{\sqrt{\mu}}\Big(\frac{(p+1)(p+2)c}{2a}\Big)^{\frac{1+p}{p}}\\&\times\sinh\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\operatorname{sech}^{\frac{2+3p}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big).\end{aligned}$$

Substituting first order solitary wave solution into (16) and using the Definition 2.5 then lead to the second order solitary wave solution in the following form

$$\begin{aligned}u_2(x,t)=&\Big(\frac{(p+1)(p+2)c}{2a}\operatorname{sech}^2\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\Big)^{\frac{1}{p}}\\&+\frac{t^\alpha}{\Gamma(\alpha+1)}\frac{(p+1)^2(p+2)^2c^2}{2ap}\Big(\frac{(p+1)(p+2)c}{2a}\operatorname{sech}^2\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\Big)^{\frac{1-p}{p}}\\&\times\sinh\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\operatorname{sech}^5\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\\&-\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big(\frac{(p+1)(p+2)c}{2a}\Big)^{\frac{1+p}{p}}\Big(\frac{acp(p+1)(p+2)}{4\mu}\operatorname{sech}^{\frac{2+4p}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\\&-\frac{ac(p+1)(p+2)(2+3p)}{4\mu}\sinh^2\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\operatorname{sech}^{\frac{2+6p}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\\&-\frac{a^2c(p+1)(p+2)}{\sqrt{\mu}}\sinh^2\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\operatorname{sech}^{\frac{2+6p}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\Big)\\&-\frac{t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)}\frac{ap(p+1)^2(p+2)^2c^2}{8\mu\sqrt{\mu}}\Big(\frac{(p+1)(p+2)c}{2a}\Big)^{\frac{1+p}{p}}\\&\times\Big(p\operatorname{sech}^{\frac{2}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)-(2+3p)\sinh^2\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\\&\times\operatorname{sech}^{\frac{2+2p}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\Big)\sinh\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\operatorname{sech}^7\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\end{aligned}$$

$$+\frac{t^{2\alpha-1}a}{2\sqrt{\mu}\Gamma(2\alpha)}\Big(\frac{(p+1)(p+2)c}{2a}\Big)^{\frac{1+p}{p}}\bigg((-3p-4p^2)\sinh\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)$$
$$\times\operatorname{sech}^{\frac{2+3p}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)+(1+2p)(2+3p)\sinh^3\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)$$
$$\times\operatorname{sech}^{\frac{2+5p}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\bigg).$$

Making use of Definition 2.5 and the Maple or Mathematics and substituting $n-1$ order solitary wave solution into (16), lead to the solitary wave solution u_3, u_4, $\ldots$, u_n, $\ldots$. As n tends to infinity, the iteration leads to the solitary wave solution of the time-fractional generalized EWW equation

$$u(x,t)=\lim_{n\to\infty}u_n=\Big(\frac{(p+1)(p+2)c}{2a}\operatorname{sech}^2\big(\frac{p}{2\sqrt{\mu}}(x-ct-x_0)\big)\Big)^{\frac{1}{p}}.$$

Selecting the appropriate values of p, a, μ, c and x_0, we can present the distribution function u as a 3-dimensions graph and 2-dimensions graph to the approximate solitary wave solution.

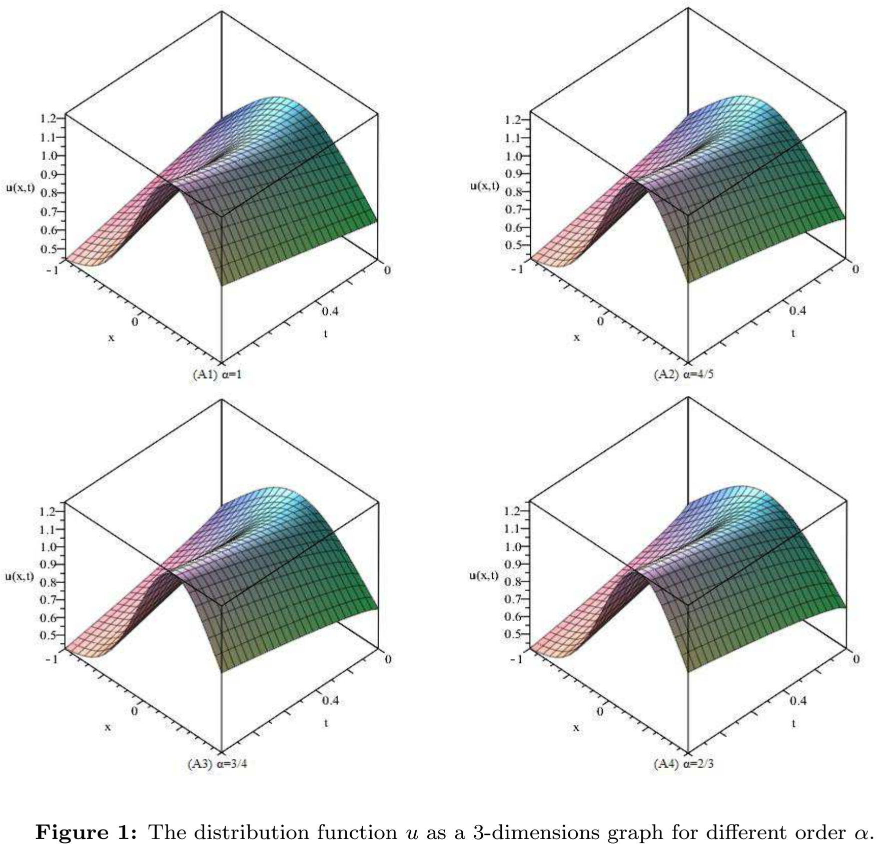


Figure 1: The distribution function u as a 3-dimensions graph for different order α.

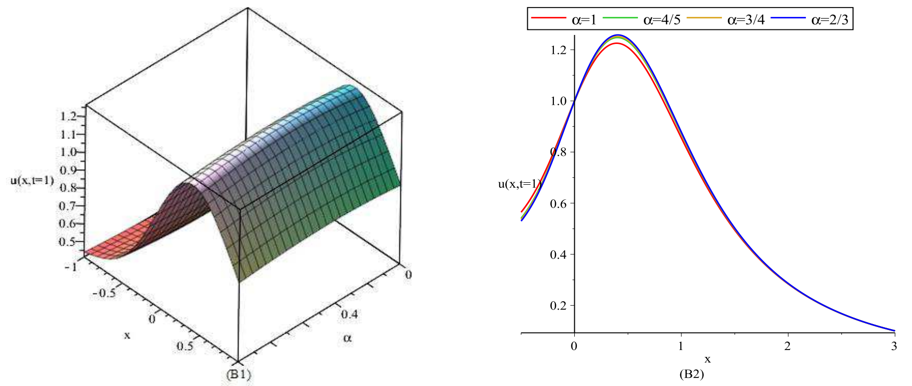


Figure 2: The distribution function u as a function of space x at time $t = 1$ for different order α: (B1) 3-dimensions graph, (B2) 2-dimensions graph.

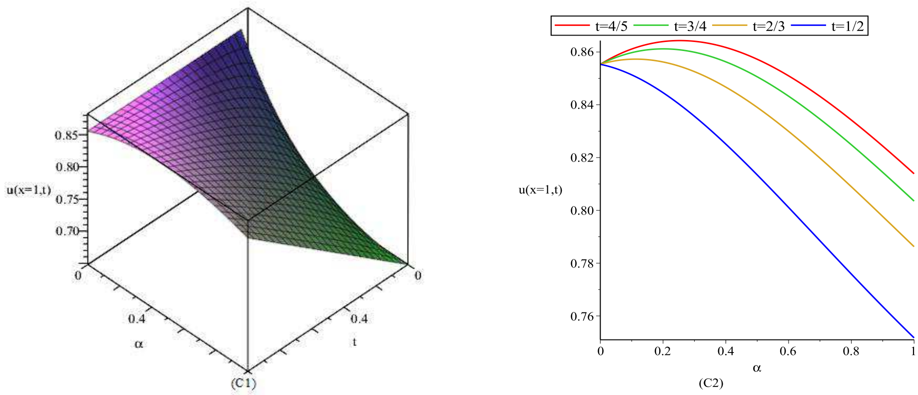


Figure 3: The distribution function u as a function of time t at space $x = 1$ of the different order α: (C1) 3-dimensions graph, (C2) 2-dimensions graph.

5 Time-fractional Generalized EWW-Burgers Equation

In this section, He's variational-iteration method is applied to solve time-fractional generalized EWW-Burgers equation

$$ {}_0^R D_t^\alpha u + a u^p u_x - \lambda u_{xx} - \mu u_{xxt} = 0. $$

The generalized EWW-Burgers equation in (1+1) dimensions is given as

$$ u_t + a u^p u_x - \lambda u_{xx} - \mu u_{xxt} = 0. \tag{17} $$

Employing a potential function v on the field variable, and setting $u = v_x$ yield the potential equation of the generalized EWW-Burgers equation (17) in the form,

$$ v_{xt} + a v_x^p v_{xx} - \lambda v_{xxx} - \mu v_{xxxt} = 0. \tag{18} $$

The Lagrangian of this generalized EWW-Burgers equation (17) can be defined using the semi-inverse method as follows. The functional of the potential equation (18) can be represented as

$$J(v)=\int_{\Omega}dx\int_{T}\Big(v\big(d_1v_{xt}+ad_2v_x^pv_{xx}-\lambda d_3v_{xxx}-\mu d_4v_{xxxt}\big)\Big)dt, \tag{19}$$

with d_i $(i=1,2,3,4)$ as an unknown constant to be determined later. Integrating (19) by parts and taking $v_x|_{\partial\Omega}=v_x|_{\partial T}=v_{xxt}|_{\partial\Omega}=0$ yield

$$J(v)=\int_{\Omega}dx\int_{T}\Big(-d_1v_tv_x-\frac{ad_2}{p+1}v_x^{p+2}+\lambda d_3v_{xx}v_x+\mu d_4v_{xxt}v_x\Big)dt. \tag{20}$$

The constants d_i $(i=1,2,3,4)$ can be determined taking the variation of the functional (20) to make it optimal. By applying the variation of the functional, integrating each term by parts, and making use of the variation optimum condition of the functional $J(v)$, yield the following representation

$$2d_1v_{tx}+(p+2)ad_2v_x^pv_{xx}-2\lambda d_3v_{xxx}-2\mu d_4v_{xxxt}=0. \tag{21}$$

Notice that the obtained result (21) is equivalent to (18), so one has that the constants d_i $(i=1,2,3,4)$ are respectively

$$d_1=\frac{1}{2},\quad d_2=\frac{1}{p+2},\quad d_3=d_4=\frac{1}{2}.$$

In addition, the functional representation given by (20) obtains directly the Lagrangian form of the generalized EWW-Burgers equation,

$$L(v_t,v_x,v_{xx},v_{xxt})=-\frac{1}{2}v_tv_x-\frac{a}{(p+1)(p+2)}v_x^{p+2}+\frac{\lambda}{2}v_{xx}v_x+\frac{\mu}{2}v_{xxt}v_x.$$

Similarly, the Lagrangian of the time-fractional version of the generalized EWW-Burgers equation could be read as

$$F({}_0D_t^{\alpha}v,v_x,v_{xx},v_{xxt})=-\frac{1}{2}{}_0D_t^{\alpha}vv_x-\frac{a}{(p+1)(p+2)}v_x^{p+2}+\frac{\lambda}{2}v_{xx}v_x+\frac{\mu}{2}v_{xxt}v_x,\quad \alpha\in]0,1]. \tag{22}$$

Then the functional of the time-fractional generalized EWW-Burgers equation will take the form

$$J(v)=\int_{\Omega}dx\int_{T}F({}_0D_t^{\alpha}v_t,v_x,v_{xx},v_{xxt})dt, \tag{23}$$

where the time-fractional Lagrangian $F({}_0D_t^{\alpha}v_t,v_x,v_{xx},v_{xxt},v_{xxx})$ is given by (22). Following Agrawal's method, the variation of functional (23) with respect to v leads to

$$\delta J(v)=\int_{\Omega}dx\int_{T}\Big(\frac{\partial F}{\partial_0D_t^{\alpha}v}\delta({}_0D_t^{\alpha}v)+\frac{\partial F}{\partial v_x}\delta v_x+\frac{\partial F}{\partial v_{xx}}\delta v_{xx}+\frac{\partial F}{\partial v_{xxt}}\delta v_{xxt}\Big)dt. \tag{24}$$

By Lemma 2.1, upon integrating the right-hand side of (24), one has

$$\delta J(v)=\int_{\Omega}dx\int_{T}\Big({}_tD_T^{\alpha}\big(\frac{\partial F}{\partial_0D_t^{\alpha}v}\big)-\frac{\partial}{\partial x}\big(\frac{\partial F}{\partial v_x}\big)+\frac{\partial^2}{\partial x^2}\big(\frac{\partial F}{\partial v_{xx}}\big)-\frac{\partial^3}{\partial x^2\partial t}\big(\frac{\partial F}{\partial v_{xxt}}\big)\Big)\delta v dt,$$

noting that $\delta v|_{\partial T} = \delta v|_{\partial\Omega} = \delta v_x|_{\partial\Omega} = \delta v_{xx}|_{\partial T} = 0$.

Obviously, optimizing the variation of the functional $J(v)$, i.e., $\delta J(v) = 0$, yields the Euler-Lagrange equation for time-fractional generalized EWW-Burgers equation in the following form

$$ {}_tD_T^\alpha\Big(\frac{\partial F}{\partial_0 D_t^\alpha v}\Big) - \frac{\partial}{\partial x}\Big(\frac{\partial F}{\partial v_x}\Big) + \frac{\partial^2}{\partial x^2}\Big(\frac{\partial F}{\partial v_{xx}}\Big) - \frac{\partial^3}{\partial x^2 \partial t}\Big(\frac{\partial F}{\partial v_{xxt}}\Big) = 0. \tag{25} $$

Substituting the Lagrangian of the time-fractional generalized EWW-Burgers equation (22) into Euler-Lagrange formula (25) one obtains

$$ -\frac{1}{2}{}_tD_{T_0}^\alpha v_x + \frac{1}{2}{}_0D_t^\alpha v_x + av_x^p v_{xx} - \lambda v_{xxx} - \mu v_{xxxt} = 0. $$

Once again, substituting the potential function v_x for u, yields the time-fractional generalized EWW-Burgers equation for the state function u as

$$ \frac{1}{2}\big({}_0D_t^\alpha u - {}_tD_{T_0}^\alpha u\big) + au^p u_x - \lambda u_{xx} - \mu u_{xxt} = 0. \tag{26} $$

According to the Riesz fractional derivative ${}_0^RD_t^\alpha u$, the time-fractional generalized EWW-Burgers equation represented in (26) can be written as

$$ {}_0^RD_t^\alpha u + au^p u_x - \lambda u_{xx} - \mu u_{xxt} = 0. \tag{27} $$

Acting from the left-hand side by the Riesz fractional operator ${}_0^RD_t^{1-\alpha}$ on (27) leads to

$$ \frac{\partial}{\partial t}u - {}_0^RD_t^{\alpha-1}u|_{t=0}\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + {}_0^RD_t^{1-\alpha}\Big(au^p u_x - \lambda u_{xx} - \mu u_{xxt}\Big) = 0, \tag{28} $$

from Lemma 2.2. In view of the variational-iteration method, combining with (28), the $n+1$-th approximate solution of (27) can be read using iteration correction functional as

$$ \begin{aligned} u_{n+1} = u_n + \int_0^t \lambda(\tau)\Big(&\frac{\partial}{\partial\tau}u_n - {}_0^RD_\tau^{\alpha-1}u_n|_{\tau=0}\frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} \\ &+ {}_0^RD_\tau^{1-\alpha}\Big(a\tilde{u}_n^p\frac{\partial}{\partial x}\tilde{u}_n - \lambda\frac{\partial^2}{\partial x^2}\tilde{u}_n - \mu\frac{\partial^3}{\partial x^2\partial t}\tilde{u}_n\Big)\Big)d\tau, \end{aligned} \tag{29} $$

where the function $\tilde{u}_n$ is considered as a restricted variation function, i.e., $\delta\tilde{u}_n = 0$. By the same argument as in Section 4, it is converted to the Lagrangian multiplier at $\lambda(\tau) = -1$. Therefore, the correction functional (29) takes the following form

$$ \begin{aligned} u_{n+1} = u_n - \int_0^t \Big(&\frac{\partial}{\partial\tau}u_n - {}_0^RI_\tau^{1-\alpha}u_n|_{\tau=0}\frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} \\ &+ {}_0^RD_\tau^{1-\alpha}\Big(au_n^p\frac{\partial}{\partial x}u_n - \lambda\frac{\partial^2}{\partial x^2}u_n - \mu\frac{\partial^3}{\partial x^2\partial t}u_n\Big)\Big)d\tau, \end{aligned} \tag{30} $$

since $\alpha - 1 < 0$, the fractional derivative operator ${}_0^RD_t^{\alpha-1}$ reduces to fractional integral operator ${}_0^RI_t^{1-\alpha}$ by Remark 2.1.

The zero order solitary wave solution can be taken as the initial value of the state variable, which is taken in this case as

$$u_0(x,t) = \Big(A - A\tanh\kappa(x-x_0) - \frac{A}{2}\mathrm{sech}^2\kappa(x-x_0)\Big)^{\frac{1}{p}}.$$

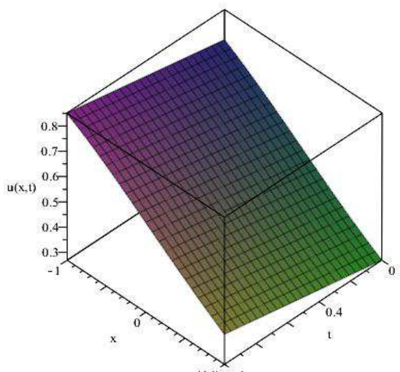
Substituting zero order solitary wave solution into (30) and using the Definition 2.5 lead to the first order solitary wave solution

$$\begin{aligned}
u_1(x,t) = &\Big(A - A\tanh\kappa(x-x_0) - \frac{A}{2}\mathrm{sech}^2\kappa(x-x_0)\Big)^{\frac{1}{p}}\\
&- \frac{t^\alpha}{\Gamma(\alpha+1)}\bigg[\frac{a}{p}\Big(A - A\tanh\kappa(x-x_0) - \frac{A}{2}\mathrm{sech}^2\kappa(x-x_0)\Big)^{\frac{1}{p}}\\
&\times\Big(A\kappa\mathrm{sech}^2\kappa(x-x_0)\tanh\kappa(x-x_0) - A\kappa\mathrm{sech}^2\kappa(x-x_0)\Big)\\
&- \frac{\lambda(1-p)}{p^2}\Big(A - A\tanh\kappa(x-x_0) - \frac{A}{2}\mathrm{sech}^2\kappa(x-x_0)\Big)^{\frac{1-2p}{p}}\\
&\times\Big(A\kappa\mathrm{sech}^2\kappa(x-x_0)\tanh\kappa(x-x_0) - A\kappa\mathrm{sech}^2\kappa(x-x_0)\Big)^2\\
&- \frac{\lambda}{p}\Big(A - A\tanh\kappa(x-x_0) - \frac{A}{2}\mathrm{sech}^2\kappa(x-x_0)\Big)^{\frac{1-p}{p}}\\
&\times\Big(2A\kappa^2\mathrm{sech}^2\kappa(x-x_0)\tanh\kappa(x-x_0) - 2A\kappa^2\mathrm{sech}^2\kappa(x-x_0)\tanh^2\kappa(x-x_0)\\
&+ A\kappa^2\mathrm{sech}^4\kappa(x-x_0)\Big)\bigg].
\end{aligned}$$

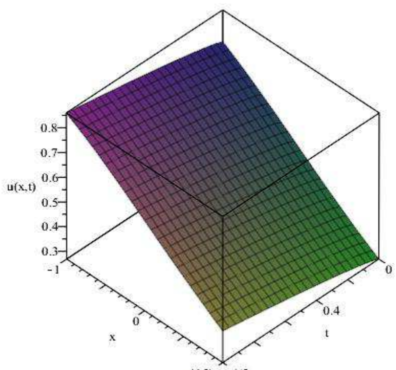
Substituting first order solitary wave solution into (30) and using the Definition 2.5 then lead to the solitary wave solution u_2, u_3, …, u_n, …. As n tends to infinity, the iteration leads to the solitary wave solution of the time-fractional generalized EWW-Burgers equation

$$u(x,t) = \lim_{n\to\infty} u_n = \Big(A - A\tanh\kappa(x-ct-x_0) - \frac{A}{2}\mathrm{sech}^2\kappa(x-ct-x_0)\Big)^{\frac{1}{p}}.$$

Selecting the appropriate values of $p, a, \lambda, \mu, A, \kappa, c$ and x_0, we can present the distribution function u as a 3-dimensions graph and 2-dimensions graph to the approximate solitary wave solution.



(A1) α=1



(A2) α=4/5

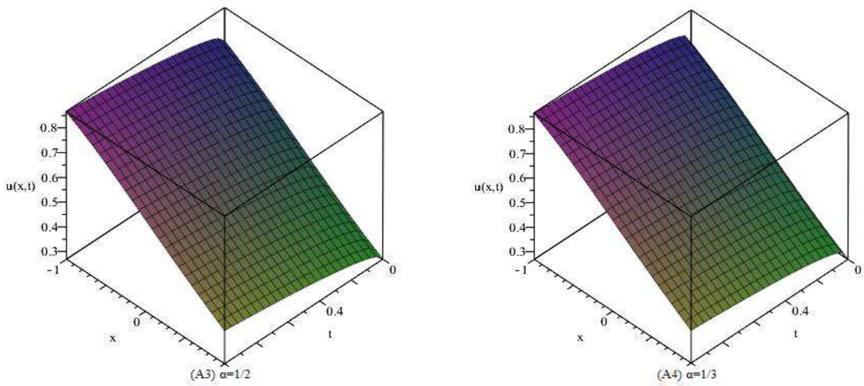


Figure 4: The distribution function u as a 3-dimensions graph for different order α.

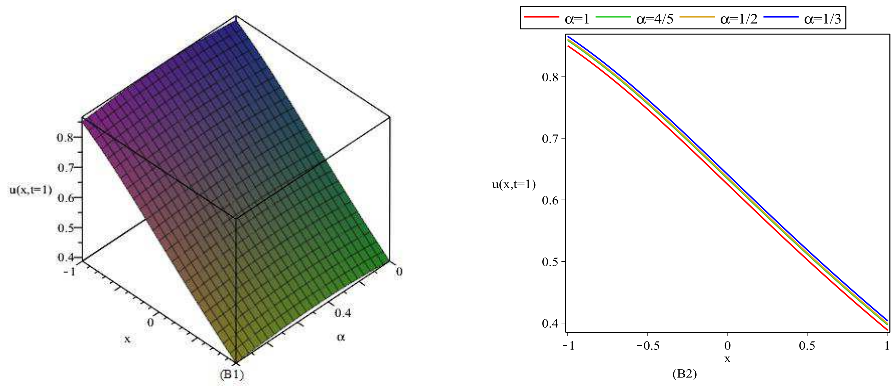


Figure 5: The distribution function u as a function of space x at time $t = 1$ for different order α: (B1) 3-dimensions graph, (B2) 2-dimensions graph.

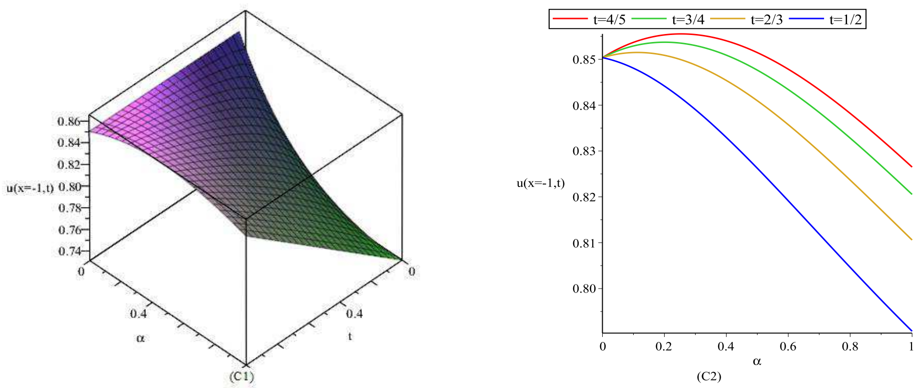


Figure 6: The distribution function u as a function of time t at space $x = -1$ for different order α: (C1) 3-dimensions graph, (C2) 2-dimensions graph.

6 Discussion

The purpose of the present work is to explore the effect of the fractional order derivative on the structure and propagation of the resulting solitary waves obtained from time-fractional generalized EWW equation. We derive the Lagrangian of the generalized EWW equation by the semi-inverse method, then take a similar form of Lagrangian to the time-fractional generalized EWW equation. Using the Euler-Lagrange variational technique, we continue our calculations until the high-order iteration. During this period, our approximate calculations are carried out concerning the solution of the time-fractional generalized EWW equation as well as generalized EWW-Burgers equation. The results of approximate solitary wave solution of time-fractional generalized EWW equation and generalized EWW-Burgers equation are obtained. In addition, 3-dimensional representation of the solution u for the time-fractional generalized EWW equation and generalized EWW-Burgers equation with space x and time t for different values of the order α are presented respectively in Figures 1 and 4, the solution u is still a single soliton wave solution for all values of the order α. It shows that the balancing scenario between nonlinearity and dispersion is still valid. Figures 2 and 5 present respectively the change of amplitude and width of the soliton due to the variation of the order α, 2- and 3-dimensional graphs depicted the behavior of the solution u at time $t = 1$ corresponding to different values of the order α. This behavior indicates that the increases of the value α increasing both the height and the width of the solitary wave solution. That is, the order α can be used to modify the shape of the solitary wave without change of the nonlinearity and the dispersion effects in the medium. Figures 3 and 6 are respectively devoted to studying the representation between the amplitude of the soliton and the fractional order at different time values, these figures show that at the same time, the increasing of the fractional α decreases the amplitude of the solitary wave to some value of α.

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