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Professor A.A. Martynyuk, Institute of Mechanics,
Nesterov str.3, 03057, MSP 680, Kiev-57, Ukraine
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Convection of Polymerization Front with Solid Product under Quasi-Periodic Gravitational Modulation

Karam Allali [1], Saadia Assiyad [1] and Mohamed Belhaq [2*]

[1] *University Hassan II-Mohammedia, Department of Mathematics, Mohammadia, Morocco*
[2] *University Hassan II-Casablanca, Laboratory of Mechanics, Casablanca, Morocco*

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Abstract: The effect of quasi-periodic gravitational modulation on the convective instability of polymerization front with solid product is studied in this paper. The model we consider includes the heat equation, the concentration equation and the Navier-Stokes equations under the Boussinesq approximation. The linear stability analysis of the problem is carried out and the interface problem is established applying the narrow zone method and the matched asymptotic expansions. The convective instability threshold is determined using numerical simulation. It was shown that the frequencies ratio has a significant effect on the convective stability domain. In particular, the stability domain changes and undergoes a shift as the frequencies ratio of the quasi-periodic modulation varies.

Keywords: *convective instability; frontal polymerization; quasi-periodic modulation.*

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1 Introduction

Frontal polymerization phenomenon is the process of converting monomer to polymer via a narrow located zone, called reaction front [1]. The influence of periodic gravitational modulation on the convective instability of polymerization reaction front with solid product was studied in [2] and it was shown that the reaction front gains stability for increasing values of the modulation frequency. In this paper, we investigate the influence of quasi-periodic (QP) gravitational modulation on the convective instability of polymerization front with solid product. Such a QP modulation may result, for instance, from the existence of two simultaneous vibrations consisting of a basic vibration with

* Corresponding author: mailto:mbelhaq@yahoo.fr

a certain frequency and of an additional residual vibration having another frequency, such that the two involved frequencies are incommensurate. It is worth noticing that the influence of QP excitation on the dynamics of mechanical systems and the transition to chaos is studied in [3, 4].

To study the influence of QP gravitational modulation on the convective instability of polymerization front with solid product, we consider a system of reaction-diffusion equations coupled with incompressible Navier-Stokes equations. We notice that the case considering the influence of QP gravitational modulation on the reaction front in porous media has been examined in [5]. In this case [5], the system of reaction-diffusion equations is coupled with the equations of motion taking into account the Darcy law.

It is worthy to point out that only few works have been devoted to examine the effect of QP vibration on the convective instability. For instance, Boulal et al. [6] reported on the effect of a QP gravitational modulation on the convective instability of a heated fluid layer and it was shown that the frequencies ratio of QP vibration strongly influences the convective instability threshold. Moreover, the influence of QP gravitational modulation on convective instability in Hele-Shaw cell was analyzed in [7]. Similar study has been made to investigate the thermal instability in horizontal Newtonian magnetic liquid layer with non-magnetic rigid boundaries in the presence of a vertical magnetic field [8]. In [6, 7], the original problem was systematically reduced to a QP Mathieu equation using Galerkin method truncated to the first order. Since the Floquet theory cannot be applied in the case of QP modulation, the approach used to obtain the marginal stability curves was principally based on the harmonic balance method combined with Hill's determinants [9, 10].

Because one cannot truncate the problem under consideration to a QP Mathieu equation using Galerkin method and the Floquet theory as in [9, 10], the marginal stability curves are obtained by using the approximately narrow zone method (Frank-Kamenetskii method) and the matched asymptotic expansions. This approach leads to the interface problem which is solved by numerical simulation.

To introduce a QP gravitational modulation, we consider that the acceleration acting on the fluid is given by $g+b(t)$, where g is the gravity acceleration and $b(t) = \lambda_1 sin(\mu_1 t)+\lambda_2 sin(\mu_2 t)$ in which λ_1, λ_2 and μ_1, μ_2 are the amplitudes and the frequencies of the QP vibration, respectively.

This paper is organized as follows. In Section 2, the frontal polymerization model is introduced. The linear stability analysis is performed in Section 3, while the interface problem and the perturbation analysis are provided in Section 4. Results obtained by numerical simulations are given in Section 5 and the last section concludes the work.

2 Frontal Polymerization Model

The propagation of polymerization reaction front with solid product submitted to a QP gravitational modulation can be modeled by the system of equations

$$\frac{\partial T}{\partial t} + (v.\nabla)T = \kappa\Delta T + qW, \tag{1}$$

$$\frac{\partial \alpha}{\partial t} + (v.\nabla)\alpha = W, \tag{2}$$

$$\frac{\partial v}{\partial t} + (v.\nabla)v = -\frac{1}{\rho}\nabla p + \nu\Delta v + g(1 + \lambda_1 \sin(\mu_1 t) + \lambda_2 \sin(\mu_2 t))\beta(T - T_0)\gamma, \tag{3}$$

$$div(v) = 0, \tag{4}$$

with the following boundary conditions

$$z \to +\infty, \quad T = T_i, \quad \alpha = 0, \quad and \quad v = 0, \tag{5}$$

$$z \to -\infty, \quad T = T_b, \quad \alpha = 1, \quad and \quad v = 0, \tag{6}$$

where the gradient, divergence and Laplace operators are defined as

$$\nabla\ v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}\right), \qquad div\ \overrightarrow{\mathbf{v}} = \frac{\partial \mathbf{v}_1}{\partial x} + \frac{\partial \mathbf{v}_2}{\partial y} + \frac{\partial \mathbf{v}_3}{\partial z}, \qquad \Delta\ v = \frac{\partial^2 v}{\partial^2 x} + \frac{\partial^2 v}{\partial^2 y} + \frac{\partial^2 v}{\partial^2 z}.$$

Here (x, y, z) are the spatial coordinates, such that $-\infty < x, y, z < +\infty$, T is the temperature, α is the depth of conversion, v is the velocity of the medium, p is the pressure, κ is the coefficient of thermal diffusivity, q is the adiabatic heat release, ρ is the density, is ν the coefficient of kinematic viscosity, γ is the unit vector in the z-direction (upward), β is the coefficient of thermal expansion, g is the gravitational acceleration, T_0 is a mean value of temperature, T_i is the initial temperature and $T_b = T_i + q$ is the temperature of the burned mixture. The reaction source term is given by

$$W = k(T)\phi(\alpha), \quad \phi(\alpha) = \begin{cases} 1, \ if \ \ \alpha < 1, \\ 0, \ if \ \ \alpha = 1, \end{cases}$$

in which the temperature dependence of the reaction rate is given by the Arrhenius Law $k(T) = k_0 exp(-E/R_0T)$ [11], where k_0 is the pre-exponential factor, E is the activation energy assumed to be sufficiently large and R_0 is the universal gas constant. It is assumed that the liquid monomer and the solid polymer involved in the reaction are incompressible and the term of diffusivity in the concentration equation is neglected so that the diffusivity coefficient is very small comparing to the coefficient of thermal diffusivity.

We introduce the dimensionless spatial variables

$$x' = \frac{xc_1}{\kappa}, \quad y' = \frac{yc_1}{\kappa}, \quad z' = \frac{zc_1}{\kappa},$$

$$t' = \frac{tc_1^2}{\kappa}, \quad p' = \frac{p}{c_1^2\rho}, \quad c_1 = \frac{c}{\sqrt{2}},$$

$$v' = \frac{v}{c_1}, \quad \theta = \frac{T - T_b}{q}, \quad c^2 = \frac{2k_0\kappa R_0 T_b^2}{qE} \exp(-\frac{E}{R_0 T_b}),$$

where c denotes the stationary front velocity, which can be calculated asymptotically for large Zeldovich number [12]. For convenience, we drop the primes in variables, velocity and pressure, so that the system (1)–(6) takes the form

$$\frac{\partial \theta}{\partial t} + (v.\nabla)\theta = \Delta\theta + Z \exp\left(\frac{\theta}{Z^{-1} + \delta\theta}\right)\phi(\alpha), \tag{7}$$

$$\frac{\partial \alpha}{\partial t} + (v.\nabla)\alpha = Z \exp(\left(\frac{\theta}{Z^{-1} + \delta\theta}\right)\phi(\alpha), \tag{8}$$

$$\frac{\partial v}{\partial t} + (v.\nabla)v = -\nabla p + P\Delta v + PR(1 + \lambda_1 \sin(\sigma_1 t) + \lambda_2 \sin(\sigma_2 t))(\theta + \theta_0)\gamma, \tag{9}$$

$$div(v) = 0, \tag{10}$$

with the boundary conditions

$$z \to +\infty, \quad \theta = -1, \quad \alpha = 0, \quad and \quad v = 0, \tag{11}$$

$$z \to -\infty, \quad \theta = 0, \quad \alpha = 1, \quad and \quad v = 0. \tag{12}$$

Here $P = \dfrac{\nu}{\kappa}$ is the Prandtl number, $R = g\beta q\kappa^2/(\nu c^3)$ is the Rayleigh number, $Z = qE/R_0T_b^2$ is the Zeldovich number, $\delta = R_0T_b/E$, $\theta_0 = (T_b - T_0)/q$, $\sigma_1 = 2\kappa\mu_1/c^2$ and $\sigma_2 = 2\kappa\mu_2/c^2$.

3 Linear Stability Analysis

To perform the linear stability analysis, it is convenient to reduce the original problem to a singular perturbation one assuming that the reaction zone is infinitely narrow and the reaction term is neglected outside the zone [13]. To implement a formal asymptotic analysis, it is convenient to choose $\epsilon = Z^{-1}$ as a small parameter ensuring the reaction occurrence in a narrow zone.

We assume that the new independent variable is given by $z_1 = z - \zeta(x, y, t)$, where $\zeta(x, y, t)$ denotes the location of the reaction zone. Upon introducing the new functions $\theta_1, \alpha_1, v_1, p_1$ such that

$$\theta(x, y, z, t) = \theta_1(x, y, z_1, t), \quad \alpha(x, y, z, t) = \alpha_1(x, y, z_1, t),$$

$$v(x, y, z, t) = v_1(x, y, z_1, t), \quad p(x, y, z, t) = p_1(x, y, z_1, t),$$

the problem (7)-(12) can be rewritten in the following form (the index 1 in the new function is omitted)

$$\frac{\partial \theta}{\partial t} - \frac{\partial \theta}{\partial z_1}\frac{\partial \zeta}{\partial t} + (v.\tilde{\nabla})\theta = \tilde{\Delta}\theta + Z \exp\left(\frac{\theta}{Z^{-1} + \delta\theta}\right)\phi(\alpha), \tag{13}$$

$$\frac{\partial \alpha}{\partial t} - \frac{\partial \alpha}{\partial z_1}\frac{\partial \zeta}{\partial t} + (v.\tilde{\nabla})\alpha = Z \exp\left(\frac{\theta}{Z^{-1} + \delta\theta}\right)\phi(\alpha), \tag{14}$$

$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial z_1}\frac{\partial \zeta}{\partial t} + (v.\tilde{\nabla})v = -\tilde{\nabla}p + P\tilde{\Delta}v + Q(1 + \lambda_1 \sin(\sigma_1 t) + \lambda_2 \sin(\sigma_2 t))(\theta + \theta_0)\gamma, \tag{15}$$

$$\frac{\partial v_x}{\partial x} - \frac{\partial v_x}{\partial z_1}\frac{\partial \zeta}{\partial x} + \frac{\partial v_y}{\partial y} - \frac{\partial v_y}{\partial z_1}\frac{\partial \zeta}{\partial y} + \frac{\partial v_z}{\partial z_1} = 0, \tag{16}$$

where

$$\tilde{\Delta} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_1^2} - 2\frac{\partial^2}{\partial x \partial z_1}\frac{\partial \zeta}{\partial x} - 2\frac{\partial^2}{\partial y \partial z_1}\frac{\partial \zeta}{\partial y} +$$

$$\frac{\partial^2}{\partial z_1^2}\left(\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2\right) - \frac{\partial}{\partial z_1}\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right),$$

$$\tilde{\nabla} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z_1}\frac{\partial \zeta}{\partial x}, \frac{\partial}{\partial y} - \frac{\partial}{\partial z_1}\frac{\partial \zeta}{\partial y}, \frac{\partial}{\partial z_1}\right), \quad Q = PR.$$

Using the matched asymptotic expansions, the outer solution of the problem (13)–(16) is sought in the form

$$\theta = \theta_0 + \epsilon\theta_1 + \dots, \quad \alpha = \alpha_0 + \epsilon\alpha_1 + \dots, \quad v = v_0 + \epsilon v_1 + \dots, \quad p = p_0 + \epsilon p_1 + \dots .$$

Moreover, to obtain the jump conditions toward the reaction zone, the inner problem is considered assuming the stretched coordinate as $\eta = z_1\epsilon^{-1}$. Then, the inner solution can be sought in the form

$$\theta = \epsilon\tilde{\theta}_1 + \dots, \quad \alpha = \tilde{\alpha}_0 + \epsilon\tilde{\alpha}_1 + \dots, \tag{17}$$

$$v = \tilde{v}_0 + \epsilon\tilde{v}_1 + \dots, \quad p = \tilde{p}_0 + \epsilon\tilde{p}_1 + \dots, \quad \zeta = \zeta_0 + \varepsilon\zeta_1 + \dots . \tag{18}$$

Substituting these expansions into (13)-(16), we obtain to the leading-order the following inner problem

$$\left(1 + \left(\frac{\partial\zeta_0}{\partial x}\right)^2 + \left(\frac{\partial\zeta_0}{\partial y}\right)^2\right)\frac{\partial^2\tilde{\theta}_1}{\partial\eta^2} + \exp\left(\tilde{\theta}_1\right)\phi(\tilde{\alpha}_0) = 0, \tag{19}$$

$$-\frac{\partial\tilde{\alpha}_0}{\partial\eta}\frac{\partial\zeta_0}{\partial t} - \frac{\partial\tilde{\alpha}_0}{\partial\eta}\left(\tilde{v}_{0x}\frac{\partial\zeta_0}{\partial x} + \tilde{v}_{0y}\frac{\partial\zeta_0}{\partial y} - \tilde{v}_{0z}\right) = \exp\left(\tilde{\theta}_1\right)\phi(\tilde{\alpha}_0), \tag{20}$$

$$\left(1 + \left(\frac{\partial\zeta_0}{\partial x}\right)^2 + \left(\frac{\partial\zeta_0}{\partial y}\right)^2\right)\frac{\partial^2\tilde{v}_0}{\partial\eta^2} = 0, \tag{21}$$

$$-\frac{\partial\tilde{v}_{0x}}{\partial\eta}\frac{\partial\zeta_0}{\partial x} - \frac{\partial\tilde{v}_{0y}}{\partial\eta}\frac{\partial\zeta_0}{\partial y} + \frac{\partial\tilde{v}_{0z}}{\partial\eta} = 0. \tag{22}$$

The matching conditions as $\eta \to +\infty$ are given by

$$\tilde{v_0} \sim v_0|_{z_1=+0},$$

$$\tilde{\theta}_1 \sim \theta_1\,|_{z_1=+0} + \left(\left.\frac{\partial\theta_0}{\partial z_1}\right|_{z_1=+0}\right)\eta, \quad \tilde{\alpha}_0 \to 0,$$

and as $\eta \to -\infty$, they read

$$\tilde{\theta}_1 \sim \theta_1\,|_{z_1=-0} \qquad \tilde{\alpha}_0 \to 1 \quad \tilde{v}_0 \sim v_0\,|_{z_1=-0}\,.$$

From (21), we obtain

$$\frac{\partial^2\tilde{v}_0}{\partial\eta^2} = 0.$$

One concludes that $\tilde{v}_0(\eta)$ is a linear function of η and identically constant because the velocity is bounded. Thus, the first term in the expression of the velocity is continuous at the front.

Since the reaction is of order zero, one obtains $\phi(\tilde{\alpha}_0) \equiv 1$. Multiplying (19) by $\dfrac{\partial\tilde{\theta}_1}{\partial\eta}$ and integrating, we get

$$\left.\left(\frac{\partial\tilde{\theta}_1}{\partial\eta}\right)^2\right|_{+\infty} - \left.\left(\frac{\partial\tilde{\theta}_1}{\partial\eta}\right)^2\right|_{-\infty} = 2A^{-1}exp(\theta_1) \tag{23}$$

Subtracting (19) from (20) and integrating, we obtain

$$\left.\frac{\partial\tilde{\theta}_1}{\partial\eta}\right|_{+\infty} - \left.\frac{\partial\tilde{\theta}_1}{\partial\eta}\right|_{-\infty} = -A^{-1}\left(\frac{\partial\zeta_0}{\partial t} + s\right), \tag{24}$$

where

$$s = \tilde{v}_{0x}\frac{\partial\zeta_0}{\partial x} + \tilde{v}_{0y}\frac{\partial\zeta_0}{\partial y} - \tilde{v}_{0z}.$$

From the last equations (23)-(24), the temperature jump conditions across the reaction front can be calculated. Indeed, using the matching conditions above and truncating the expansion as

$$\theta \approx \theta_0, \quad \theta_1\,|_{z_1=-0} \approx Z\theta\,|_{z_1=+0}\,, \quad \zeta \approx \zeta_0, \quad v \approx v_0,$$

the jump conditions read

$$\left.\left(\frac{\partial\theta}{\partial z_1}\right)^2\right|_{+0} - \left.\left(\frac{\partial\theta}{\partial z_1}\right)^2\right|_{-0} = 2Z\left(1 + \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right)^{-1} exp\,(Z\theta|_0)\,,$$

$$\left.\frac{\partial\theta}{\partial z_1}\right|_{z_1=+0} - \left.\frac{\partial\theta}{\partial z_1}\right|_{z_1=-0} = -\left(1 + \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right)^{-1}.$$

4 The Interface Problem and Perturbation

Next, we consider the case of the solid product where the velocity is zero behind the reaction zone, $v \equiv 0$ for $z < \zeta$. In this case, we obtain the interface problem:

In the liquid monomer ($z > \zeta$), we have the following system of equations

$$\frac{\partial\theta}{\partial t} + (v.\nabla)\theta = \Delta\theta, \tag{25}$$

$$\alpha = 0, \tag{26}$$

$$\frac{\partial v}{\partial t} + (v.\nabla)v = -\nabla p + P\Delta v + Q(1 + \lambda_1\sin(\sigma_1 t) + \lambda_2\sin(\sigma_2 t))(\theta + \theta_0)\gamma, \tag{27}$$

$$div(v) = 0. \tag{28}$$

In the solid polymer ($z < \zeta$), the system of equations is given by

$$\frac{\partial\theta}{\partial t} + (v.\nabla)\theta = \Delta\theta, \tag{29}$$

$$\alpha = 1, \tag{30}$$

$$v = 0. \tag{31}$$

While at the interface ($z = \zeta$), the system of equations reads

$$\theta\,|_{\zeta-0} = \theta\,|_{\zeta+0}\,, \tag{32}$$

$$\left.\frac{\partial\theta}{\partial z}\right|_{\zeta=-0} - \left.\frac{\partial\theta}{\partial z}\right|_{\zeta=+0} = \left(1 + \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right)^{-1}\left(\frac{\partial\zeta}{\partial t}\right), \tag{33}$$

$$\left(\frac{\partial\theta}{\partial z}\right)^2\bigg|_{\zeta-0}-\left(\frac{\partial\theta}{\partial z}\right)^2\bigg|_{\zeta+0}=-2Z\left(1+\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2\right)^{-1}exp(Z\theta|_\zeta), \tag{34}$$

$$v_x=v_y=v_z=0, \tag{35}$$

with the conditions at infinity

$$z=-\infty:\quad \theta=0,\quad v=0; z=+\infty:\quad \theta=-1,\quad v=0. \tag{36}$$

This problem has a travelling wave solution in the form

$$(\theta(x,y,z,t),\alpha(x,y,z,t),v)=(\theta_s(z-ut),\alpha_s(z-ut),0)$$

$$(\theta_s(z-ut),\alpha_s(z-ut))=\begin{cases}(0,1), & z_2<0,\\ (\exp(-uz_2)-1,0), & z_2>0,\end{cases} \tag{37}$$

and

$$z_2=z-ut,$$

where u is the speed of the stationary reaction front. This solution, referred to as a basic solution, is a stationary solution of (26), (28), (30)-(36) and

$$\frac{\partial\theta}{\partial t}+(v.\nabla)\theta=\Delta\theta+u\frac{\partial\theta}{\partial z_2}, \tag{38}$$

$$\frac{\partial v}{\partial t}+(v\nabla)v=-\nabla p+P\Delta v+u\frac{\partial\theta}{\partial z_2}+Q(1+\lambda_1\sin(\sigma_1 t)+\lambda_2\sin(\sigma_2 t))(\theta+\theta_0)\gamma, \tag{39}$$

for the liquid monomer, and

$$\frac{\partial\theta}{\partial t}=\Delta\theta+u\frac{\partial\theta}{\partial z_2}, \tag{40}$$

for the solid polymer.

To study the reaction front stability, we seek the solution of the problem in the form of a perturbed stationary solution as follows

$$\theta=\theta_s+\tilde{\theta},\quad p=p_s+\tilde{p},\quad v=v_s+\tilde{v}, \tag{41}$$

where $\tilde{\theta}$, $\tilde{p}$ and $\tilde{v}$ are, respectively, small perturbations of temperature, pressure and velocity.

Substituting (41) into (28), (38)–(40), we obtain to the first-order

for $z_2>\xi$:

$$\frac{\partial\tilde{\theta}}{\partial t}=\Delta\tilde{\theta}+u\frac{\partial\tilde{\theta}}{\partial z_2}-\tilde{v}_z\theta_s',$$

$$\frac{\partial\tilde{v}}{\partial t}=-\nabla\tilde{p}+P\Delta\tilde{v}+u\frac{\partial\tilde{\theta}}{\partial z_2}+Q(1+\lambda_1\sin(\sigma_1 t)+\lambda_2\sin(\sigma_2 t))\tilde{\theta}\gamma,$$

$$div(\tilde{v})=0,$$

for $z_2<\xi$:

$$\frac{\partial\tilde{\theta}}{\partial t}=\Delta\tilde{\theta}+u\frac{\partial\tilde{\theta}}{\partial z_2}.$$

We note
$$(\tilde{\theta},\tilde{v}_z)=\begin{cases}(\hat{\theta}_1,\hat{v}_{z_1}), & for\ z_2>\xi,\\ (\hat{\theta}_2,\hat{v}_{z_2}), & for\ z_2<\xi,\end{cases}$$
and we consider the perturbation in the form
$$\hat{\theta}_i=\theta_i(z_2,t)\exp(j(k_1x+k_2y), \tag{42}$$
$$\hat{v}_{z_i}=v_{z_i}(z_2,t)\exp(j(k_1x+k_2y), \tag{43}$$
$$\xi=\epsilon_1(t)\exp(j(k_1x+k_2y), \tag{44}$$
where k_i, $(i=1,2)$ and ϵ_1 are, respectively, the wave numbers (in x and y directions) and the amplitude of the perturbation and $j^2=-1$. Linearizing the jump conditions by taking into account that
$$\theta\,|_{\xi=\pm 0}=\theta_s(\pm 0)+\xi\theta_s'(\pm 0)+\tilde{\theta}(\pm 0),\quad \left.\frac{\partial\theta}{\partial z_2}\right|_{\xi=\pm 0}=\theta_s'(\pm 0)+\xi\theta_s''(\pm 0)+\left.\frac{\partial\tilde{\theta}}{\partial z_2}\right|_{\xi=\pm 0},$$
we obtain up to the higher-order
$$\hat{\theta}_2|_{z_2=0}-\hat{\theta}_1|_{z_2=0}=u\xi,\qquad \left.\frac{\partial\hat{\theta}_2}{\partial z_2}\right|_{z_2=0}-\left.\frac{\partial\hat{\theta}_1}{\partial z_2}\right|_{z_2=0}=-u^2\xi-\frac{\partial\xi}{\partial t},$$
$$u^2\xi+\left.\frac{\partial\hat{\theta}_2}{\partial z_2}\right|_{z_2=0}=-\frac{Z}{u}\hat{\theta}_1|_{z_2=0},$$
$$\hat{v}_{2z}\,|_{z_2=0}=\hat{v}_{1z}\,|_{z_2=0}=0,\qquad \left.\frac{\partial\hat{v}_{z_2}}{\partial z_2}\right|_{z_2=0}=\left.\frac{\partial\hat{v}_{1z}}{\partial z_2}\right|_{z_2=0}=0.$$

By applying twice the operator curl to the Navier-Stokes equations, the pressure can be eliminated. Considering only the z component in velocity in (38)–(40), one obtains the following system of equations
$$\frac{\partial\tilde{\theta}}{\partial t}=\Delta\tilde{\theta}+u\frac{\partial\tilde{\theta}}{\partial z_2}-\tilde{v}_z\theta_s', \tag{45}$$
$$\frac{\partial\Delta\tilde{v}_z}{\partial t}=P\Delta\Delta\tilde{v}_z+u\frac{\partial\tilde{v}_z}{\partial z_2}+Q(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})(1+\lambda_1\sin(\sigma_1 t)+\lambda_2\sin(\sigma_2 t))\tilde{\theta}\gamma, \tag{46}$$

Substituting the perturbation forms (42)–(43) in the two last equations (45)-(46), we obtain
$$\begin{aligned}\frac{\partial}{\partial t}\left(v''-k^2v\right)-u(v'''-k^2v')-P\left((v^{(4)}-k^2v'')-k^2(v''-k^2v)\right)&= \\ -Qk^2(1+\lambda_1\sin(\sigma_1 t)+\lambda_2\sin(\sigma_2 t))\theta&,\end{aligned} \tag{47}$$
$$\frac{\partial\theta}{\partial t}-\theta''-u\theta'+k^2\theta=u\exp(-uz_2)v, \tag{48}$$
where $k=\sqrt{k_1^2+k_2^2}$, and the boundary conditions
$$v'(0,t)=v(0,t)=0, \tag{49}$$
$$\theta'(0,t)=-u\theta(0,t)=0. \tag{50}$$

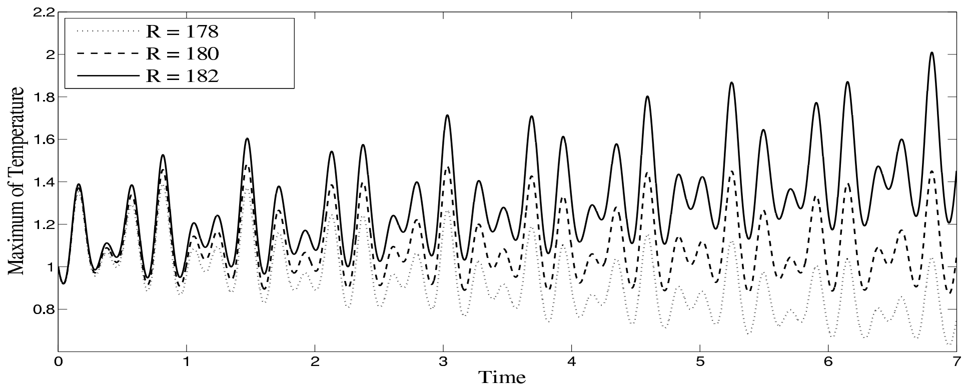

Figure 1: The evolution of the maximum of temperature versus time for $\lambda_1 = 1$, $\lambda_2 = 2$, $P = 10$, $\sigma_1 = 20$, $\dfrac{\sigma_2}{\sigma_1} = \sqrt{2}$ and different values of R.

5 Main Results

Introducing the vorticity $w = v'' - k^2 v$, the system of equations (47)–(50) becomes

$$\frac{\partial w}{\partial t} - uw' - P(w'' - k^2 w) = -Qk^2(1 + \lambda_1 \sin(\sigma_1 t) + \lambda_2 \sin(\sigma_2 t))\theta, \tag{51}$$

$$w = v'' - k^2 v, \tag{52}$$

$$\frac{\partial \theta}{\partial t} - \theta'' - u\theta' + k^2\theta = u \exp(-uz)v \tag{53}$$

with the following conditions

$$z = 0: \ \theta' = -u\theta, v' = v = 0, \tag{54}$$

$$z = L: \ \theta = v = w = 0. \tag{55}$$

In order to determine the stability threshold, we solve numerically the problem (51)–(55) using the finite-difference approximation with implicit scheme. The onset of stability is determined by evaluating the evolution of maximum of temperature versus time for different values of the Rayleigh number R. The jump between bounded and unbounded values of maximum of temperature leads precisely to the convective instability onset.

Figure 1 shows the maximum of temperature as function of time for different values of the Rayleigh number R. It can be observed that the evolution of the maximum of temperature becomes unbounded when the Rayleigh number exceeds a certain critical value.

The critical Rayleigh number as a function of the amplitude of vibration λ_2 is shown in Figure 2 for $P = 10$, $k = 1.5$, $\lambda_1 = 5$, $\sigma_1 = 5$ and for different frequencies ratio. It can be clearly seen that an increase of the frequencies ratio leads to an increase of the stability region, especially in certain interval of the amplitude λ_2 (approximately between 7 and 20). In contrast, a decrease of the frequencies ratio produces instability in the whole range of the amplitude λ_2. This result indicates that for appropriate values of parameters, a decrease in the frequencies ratio has a destabilizing effect on the reaction front.

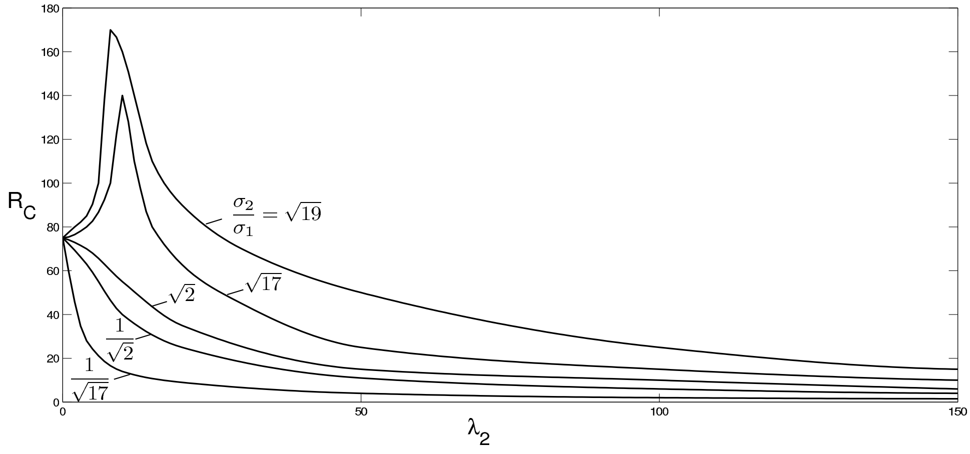

Figure 2: The critical Rayleigh number versus the amplitude of vibration λ_2 for $\lambda_1 = 5$, $P = 10$, $\sigma_1 = 5$, $k = 1.5$ and for different frequencies ratio.

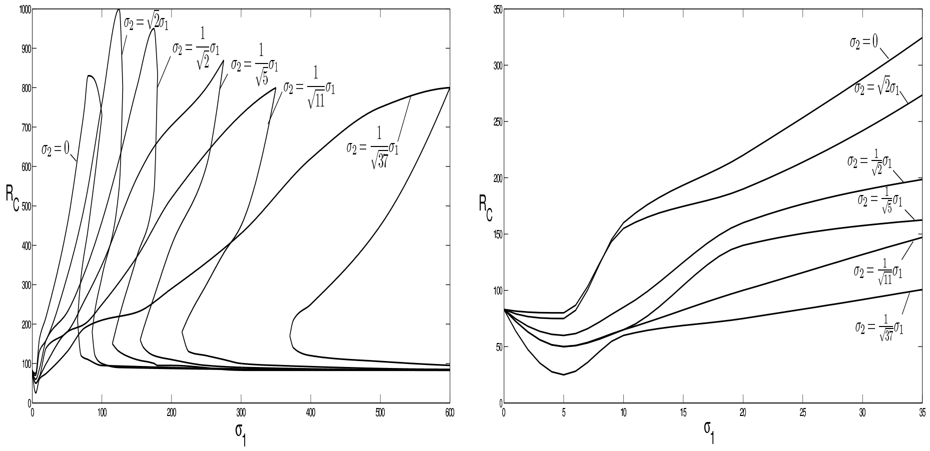

Figure 3: The critical Rayleigh number versus the frequency σ_1 for $\lambda_1 = \lambda_2 = 5$, $P = 10$, $k = 1.5$ and for different frequencies ratio (left); zoomed region (right).

The critical Rayleigh number versus the frequency σ_1 is shown in Figure 3 for the given values $P = 10$, $k = 1.5$, $\lambda_1 = \lambda_2 = 5$ and for different frequencies ratio. It can be observed from this figure that in the absence of modulation, the modulated critical value of the Rayleigh number $R_C \simeq 83$ is found [2]. In the presence of QP vibration, the convective instability boundaries are illustrated in the figure showing that as the frequencies ratio decreases, the stability domain becomes larger and shifts toward higher values of the frequency σ_1. It can be concluded that the location of the stability domain can be controlled by tuning the frequencies ratio. It is worthy to notice that this phenomenon has not been depicted in the case where the reaction front propagates in porous media [5]. It is interesting to notice that for large values of the frequency σ_1, the critical Rayleigh number tends to the unmodulated critical value R_C which means that

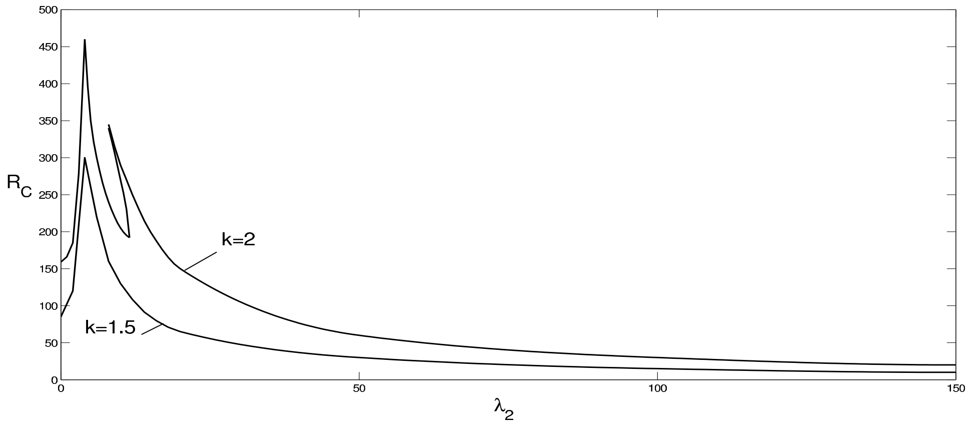

Figure 4: The critical Rayleigh number versus the amplitude of vibration λ_2 for different wave number and for $P = 10$, $\lambda_1 = 1$, $\sigma_1 = 20$, $\frac{\sigma_2}{\sigma_1} = \sqrt{2}$.

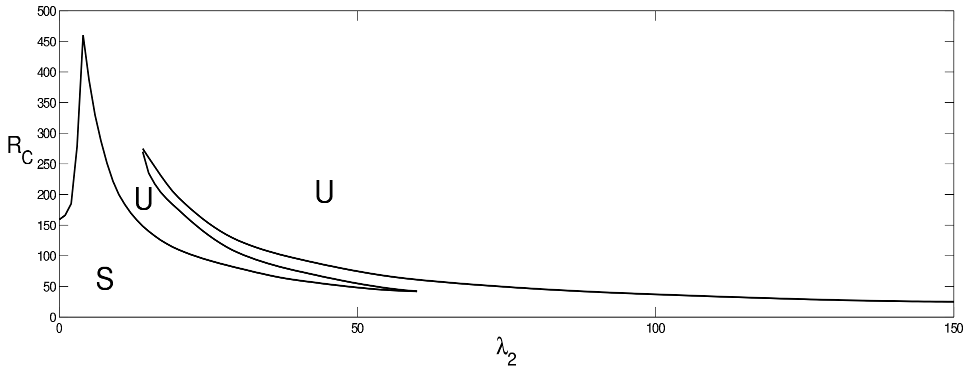

Figure 5: The critical Rayleigh number versus the amplitude of vibration λ_2 for $P = 10$, $\lambda_1 = 1$ and $\frac{\sigma_2}{\sigma_1} = \sqrt{3}$ (compare with the case $\frac{\sigma_2}{\sigma_1} = \sqrt{2}$ in Figure 4); S: stable, U: unstable.

the case of QP modulation with high frequency σ_1 is similar to the unmodulated case.

In Figure 4, we show the variation of R_c versus λ_2 for different values of the wave number. It can be observed that an increase of the wave number has a stabilizing effect and gives rise to a new domain of stability.

The influence of the frequency ratio on the convective instability boundary is shown in Figure 5 indicating that increasing the frequencies ratio increases significantly the new domain of stability.

6 Conclusion

In this work, we have studied the influence of the QP gravitational modulation on the convective instability of polymerization front with liquid reactant and solid product. The model we have considered includes the heat equation, the concentration equation and the Navier-Stokes equations under Boussinesq approximation. The Zeldovich Frank-Kamenetskii method has been applied assuming that the reaction occurs in a narrow zone.

A linear stability analysis was performed to determine the interface problem assuming that the solution is chosen as a perturbed stationary solution. To find the convective instability threshold, the reduced system of equations has been discretized using the finite difference method with implicit scheme. The obtained numerical results have shown that for fixed value of the amplitude λ_1, an increase of the frequencies ratio stabilizes the reaction front, especially for moderate values of the amplitude of vibration λ_2. Instead, a decrease of the frequencies ratio destabilizes the reaction front in the whole range of the amplitude λ_2. More interestingly, it was observed that decreasing the frequencies ratio shifts the stability domain toward higher values of the frequency σ_1. The influence of the wave number on the convective instability of the reaction front was also examined showing that, as in the periodic modulation case [2], an increase of the wave number has a stabilizing effect and gives rise to a new stability domain.

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Guidance and Control for Spacecraft Planar Re-Phasing via Input-Shaping and Differential Drag

R. Bevilacqua * and D. Perez

Mechanical and Aerospace Engineering Department, University of Florida, MAE-A Building, P.O. Box 116250, Gainesville, FL 32611-6250, USA

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Abstract: This paper proposes a solution to the problem of re-phasing circular or low eccentricity orbiting, short-distance spacecraft, by integrating existing analytical guidance solutions based on input-shaping and analytical control techniques for differential drag based on Lyapunov theory. The combined guidance and control approach is validated via numerical simulations in a full nonlinear environment using Systems Tool Kit. The results show promise for future onboard implementation on propellant-less spacecraft.

Keywords: *input-shaping; differential drag; re-phasing.*

Mathematics Subject Classification (2010): 70Q05, 93B52, 93C10, 93C40, 93D05.

1 Introduction

Small spacecraft flying in close proximity for scientific, commercial, and defense applications, are increasingly appealing to space services providers and researchers (see [1–4]). In fact, for certain applications they are preferable to larger single spacecraft, due to their lower cost, and the inherent redundancy, in general, of a multiple-spacecraft system [5]. However, spacecraft solutions, such as those based on the CubeSat format, present a new set of design challenges, mainly related to the vehicles' limited size and power. The ability to incorporate thrusters and carry on-board propellant is extremely limited on nano-spacecraft weighting a few kilograms [6]. A valid alternative for planar maneuvering of spacecraft relative motion at low Earth orbits (LEO) is represented by atmospheric differential drag, where the differential accelerations necessary to control the satellites are generated by varying the relative cross-wind surface area. C.L. Leonard [7] introduced

* Corresponding author: mailto:bevilr@ufl.edu

this method for generating the control forces that are required by rendezvous maneuvers at LEO (<600 km). The differential drag-based methodology allows for virtually propellant-free control of spacecraft relative motion on the orbital plane, since maneuverable dedicated drag surfaces can be powered by solar energy. The differential drag-based methodology was used for the ORBCOMM constellation's formation keeping [8], and it will be potentially used by the JC2Sat-FF project developed by the Canadian and Japanese Space Agencies [9, 10]. It must be noted that differential drag forces only lie in the along-track direction, limiting controllability to the orbital plane. In addition, differential drag forces are usually represented as an on-off control profile [7]. The differential drag concept holds the potential for replacing, or partially substituting, on-board thrusters and propellant tanks with clear benefits, especially for long-term, repeated relative maneuvering on the orbital plane. It should be noted that using the differential drag concept results in additional decay on the orbits of the spacecraft whenever their cross-wind surface area is increased.

In order to contribute to the field of spacecraft relative motion control and mission implementation, this paper creates a framework combining analytical guidance solutions for short distance re-phasing, based on along track, on-off control (presented in [11]) with an adaptive Lyapunov control method (presented in [12, 13]). The guidance solutions are based on a technique known as input-shaping, to be described below. Considering that the trajectories can be planned immediately, with no need for numerical iterations, the analytical nature of the solutions supports satellites with limited computing capabilities (e.g.: nano-satellites). The open loop guidance solutions obtained via input-shaping are tracked using a Lyapunov-based control strategy, also analytical and computationally inexpensive, previously developed specifically for differential drag maneuvering [12, 13].

Short distance re-phasing involves baselines up to several kilometers. This is in contrast to cases where the spacecraft may be even on the opposite side of the orbit with respect to the desired final location. The re-phasing maneuvers herein are performed with respect to a (real or virtual) circular reference orbit, with a semi-major axis equal to that of the reference orbit, and in the same orbital plane. In particular, a satellite starting from a circular orbit or a slightly eccentric one, can be re-phased to a new polar angle (if starting from a circular course) or re-phased to have a closed relative motion with respect to a desired point on the reference circular path. In general, the re-phasing solutions proposed in this paper apply to maneuvers going from an equilibrium configuration to a new equilibrium configuration, where equilibrium means a non-drifting state with respect to the final desired target location.

The analytical design of guidance for short distance re-phasing can be valuable not only for a spacecraft's relocation on its orbit but also for spacecraft proximity operations, where the target point can be actually occupied by another space vehicle. In fact, spacecraft rendezvous is an increasingly important topic given the potential for its application, for example, in on-orbit maintenance and servicing missions, spacecraft monitoring, etc. Additional applications of proximity flight and docking are seen in de-orbiting space debris, another pressing problem for future space exploitation: spacecraft capable of changing their cross-wind surface area may be envisioned docking to inactive resident space objects (RSO), and controlling their decay.

Input-shaping is a convolution technique based on the knowledge of a system's natural frequencies of oscillation. Given a feed-forward control signal, which is designed to perform a desired maneuver but not to take into account potential excitation of undesired oscillations, input-shaping consists of the convolution of the signal itself and a specified train of impulses so that the system's resulting behavior presents minimal residual vibrations at the end of the maneuver. The impulses and their locations in time are computed based on the frequencies that need to be suppressed, i.e., the modes one wants to limit in amplitude. The majority of input-shaping applications fall under the category of flexible structures control, such as space manipulators control, as seen in References [14–21]. It is important to emphasize that input-shaping is not intended to reduce the energy of a system, i.e., existing oscillations cannot be damped. However, maneuvers from an equilibrium condition to a new equilibrium are possible. In addition, appropriate modifications of the input-shaping parameters can inject energy into the system, and lead it to a new equilibrium configuration, with desired higher oscillations, as shown in [11]. In addition, to differential drag maneuvering, input-shaping can be applied to on-off thrust profiles, maintaining the nature of the control signal.

The main contribution of this paper consists in demonstrating the feasibility of differential drag for rephasing maneuvers, combining an analytical guidance technique (developed in [11]) and a control method (developed in [12, 13]), and simulating their use in a realistic spacecraft relative maneuvering scenario. Thus, illustrating how such analytical approaches could be orchestrated and used in real time, during a real space flight.

2 Spacecraft Relative Motion Dynamics and Input-Shaping Analytical Guidance

2.1 Spacecraft relative motion dynamics

Spacecraft relative motion dynamics is used to model how a spacecraft moves with respect to the final desired point, regardless of the presence of a reference spacecraft at the re-phasing desired location. Thus, the re-phasing target point can be represented by the origin of a Local Vertical Local Horizontal (LVLH) reference frame. In such a frame, x points from Earth to the reference spacecraft (virtual or real), y points along the track (direction of motion), and z completes the right-handed frame (see Figure 1). For this paper, the origin of the LVLH frame moves on a circular orbit, with a semi-major axis equal to that of the active spacecraft's orbit.

The out-of-plane z and in-plane xy motions are usually assumed to be decoupled. In this paper it is assumed that the spacecraft's and its target's re-phasing location lie in the same orbital plane, and the out-of-plane motion will be neglected. Furthermore, it is assumed that the commands to the drag surfaces are on-off, i.e., instantaneously changing from open to close and vice versa (see [12, 13, 22–25]).

The atmospheric differential drag control concept is based on the assumption that two spacecraft can change their respective cross-wind surface area, generating differential values of drag acceleration along track y, as depicted in Figure 1. In this example, one spacecraft increases its drag by opening a surface, thus lowering its orbit and increasing its speed with respect to the other spacecraft. The main limitations of this propellant-less control are that only planar motion can be addressed x and y, and that the orbits decay faster whenever the surfaces are opened.

In the equations presented in this paper, bolded symbols represent vectors, while underlining refers to matrices.

The in-plane, linearized equations of spacecraft relative motion, or HCW equations, described in References [26, 27], with along-track control only, are given by (1). The assumptions to derive these equations are: two-body force, circular reference orbit, and

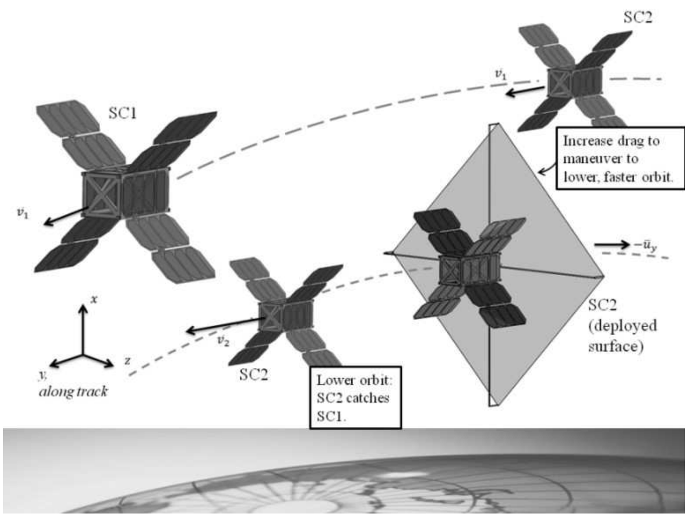

Figure 1: Conceptual sketch explaining differential drag control. Spacecraft 2 increases its drag, thus lowering its orbit and increasing its speed, to catch up with spacecraft 1 in terms of orbital polar angle.

close proximity:

$$\begin{aligned} &\dot{\mathbf{x}} = \underline{\mathbf{A}}\mathbf{x} + \mathbf{B}u_y,\ A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & -2\omega & 0 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},\ \omega = \frac{2\pi}{T}, \\ &\mathbf{x} = \begin{bmatrix} x & y & \dot{x} & \dot{y} \end{bmatrix}^T, \end{aligned} \tag{1}$$

where T is the circular orbital period. When drag is used as the control variable, the expression for u_y depends on the atmospheric density, the spacecraft cross-wind surface area, its drag coefficient, mass, and the velocity of the spacecraft relative to the medium. This velocity can be assumed to be equal to the orbital velocity, since the relative velocity between the spacecraft and the origin of the LVLH frame is negligible and the medium can be assumed to rotate with the Earth. As an approximation, the differential is only driven by changes in cross-wind surface area (see [22, 23]).

2.2 Input-shaped control

Input-shaping is based on the concept of providing and then removing energy to/from an oscillatory system. A train of specific impulses, based on the system's natural frequency and damping ratio, are used in convolution with an original control signal, shaping it to achieve the desired final state with minimal residual vibration, as seen in [21].

The train of impulses used herein is defined as a function of the variables y_{fd} (along-track desired final location) and Δt (duration of coasting phases) in (2). The final analytical solution, that takes into account the HCW dynamics and drives the state to the desired final value y_{fd} is obtained by solving for Δt and an adjusted value of y_{fd}.

The control signal to be shaped is chosen as a bang-bang profile of amplitude $\bar{u}$, and a three-impulse shaper as described in (2) (originally presented in [11])

$$\begin{aligned}
&u_y = A_1 f_{t_1} + A_2 f_{t_2} + A_3 f_{t_3}, \;\; A_1 = \frac{1}{4}, A_2 = \frac{1}{2}, A_3 = \frac{1}{4}, \\
&f_{t_1} = \begin{cases} \bar{u}\, sign\left(y_{fd} - y\left(t_0\right)\right), if\, t \le t^*/2, \\ -\bar{u}\, sign\left(y_{fd} - y\left(t_0\right)\right), if\, t^*/2 \; < t \le t^*, \\ 0, if\, t > t^*, \end{cases} f_{t_2} = f_{t_1}\left(t - \Delta t\right), \;\; f_{t_3} = f_{t_1}\left(t - 2\Delta t\right), \\
&t^* = \sqrt{2\left|y_{fd} - y\left(t_0\right)\right| / \bar{u}}\,.
\end{aligned} \tag{2}$$

In particular, the quantities A_i in (2) represent the three impulses, convoluted with an original signal. They are given in Reference [19] as $1/(1+K)^2$, $2K/(1+K)^2$, and $K^2/(1+K)^2$, respectively, with $K = exp(-\zeta/pi/(1-\zeta^2)^{1/2})$. ζ indicates the damping ratio of the given dynamic system. The assumed model presents $\zeta = 0$, leading to the A_i values in Equation (2).

Control profiles as the one represented in (2) can be tracked using Pulse Width Modulation (PWM) by on-off, single magnitude engines or differential drag devices. Continuously changing profiles are harder to reproduce with PWM. A more effective option is given in previous work using Lyapunov theory to control a nonlinear system with on-off actuation only (see [12, 13]), as will be shown in the remainder of the paper.

As outlined in [11], the control profile of Equation (2) can be applied on the HCW relative motion equations, obtaining several analytical solutions for rephasing from point to a different point, point to equilibrium relative motion and equilibrium relative motion to another equilibrium relative motion.

2.3 Analytical solution for leader-follower re-phasing

Re-phasing, in the linear approximation of the LVLH frame means maneuvering the spacecraft from an initial stationary y location, to a final, also stationary new y. For the remainder of the paper such configurations will be called leader-follower, and so the related re-phasing maneuvers will be named.

In [11], the control signal shown in (2) was applied to the dynamics of Equation (1), starting from an equilibrium leader-follower initial condition ($\mathbf{x}(t_0) = \begin{bmatrix} 0 & y_0 & 0 & 0 \end{bmatrix}^T$), and considering a variable Δt. This resulted in an analytical expression for the final state, which is not included in this paper for brevity, but can be found in [11]. The resulting trajectory will have the center located at the desired along-track location $y_f d$, if a new desired virtual location $y_f d$ (given in (3)) is selected and combined with the expressions for the center of the ellipse representing the final relative orbit ($\bar{y}$ and $\bar{x}$ in (3))

$$\begin{aligned}
&{y_{fd}}' = (2/3\,)\, y_{fd} + (1/3\,)\, y_0, \\
&\bar{x} = 4x_f + 2\dot{y}_f/\omega \; = 0, \bar{y} = y_f - 2\dot{x}_f/\omega \; = -0.5y_0 + 1.5y_{fd}.
\end{aligned} \tag{3}$$

Using the expression for the final state, Equation (3), and the relative eccentricity (e_{rel}, which represents the physical dimension of the obtained closed orbit), the direct dependency of e_{rel} from Δt was obtained:

$$
e_{rel} = 0.5\sqrt{2}\frac{\bar{u}}{\omega^2}\sqrt{\begin{array}{c} 6\,\mathrm{c}\left(\frac{2}{3}\alpha_2\right) - 4\,\mathrm{c}\left(-\frac{1}{3}\alpha_2 + 2\omega\Delta t\right) - 4\,\mathrm{c}\left(2\omega\Delta t + \frac{1}{3}\alpha_2\right) \\ +c\left(-\frac{2}{3}\alpha_2 + 2\omega\Delta t\right) + 4\,\mathrm{c}\left(\omega\Delta t - \frac{2}{3}\alpha_2\right) - 16\,\mathrm{c}\left(\frac{1}{3}\alpha_2 + \omega\Delta t\right) \\ -24\,\mathrm{c}\left(\frac{1}{3}\alpha_2\right) - 16\,\mathrm{c}\left(\omega\Delta t - \frac{1}{3}\alpha_2\right) + 18 + \mathrm{c}\left(2\omega\Delta t + \frac{2}{3}\alpha_2\right) \\ +6\,\mathrm{c}\left(2\omega\Delta t\right) + 24\,\mathrm{c}\left(\omega\Delta t\right) + 4\,\mathrm{c}\left(\omega\Delta t + \frac{2}{3}\alpha_2\right) \end{array}}, \quad \alpha_2 = \omega\sqrt{\frac{3y_{fd} - 3y_0}{\bar{u}}}. \tag{4}
$$

For a detailed derivation of this expression refer to [11].

If classical input-shaping is applied, with $\Delta t = 0.5T = \pi/\omega$ (Equation (2)), the resulting relative eccentricity is zero, and the final state is obtained as $\mathbf{x}(t_f) = [0 \quad y_{fd} \quad 0 \quad 0]^T$, that is, the initial leader-follower condition (both spacecraft on the same orbit) is reproduced at the end of the maneuver, and the desired along-track baseline is achieved.

Equation (4) also enables the design of different types of re-phasing by adjusting the value of Δt to obtain a final closed relative orbit around the along-track point y_{fd}, with desired relative eccentricity. These types of maneuvers may be envisioned for close approach to a target and fly-around for monitoring purposes. In doing this, an oscillation at the end of the maneuver is added, in a quantifiable and desired fashion.

It must be noted that (4) shows 2ω as the highest frequency. The Nyquist-Shannon sampling theorem (see [28]) can be used to determine how many points are needed to approximate the function in (4). By computing (4) at Δt points spaced by a $1/(4\omega)$ time distance, that is, theoretically 8π points total (i.e. at least 26) an entire orbital period is approximated. A desired e_{rel} value can be then interpolated using these points (e.g. using splines), posing minimal computational burden. The equilibrium-to-equilibrium e_{rel} case presented later in the paper shows an example of how to set up such approximation and interpolation.

2.4 Analytical solution for equilibrium-to-equilibrium closed relative orbit re-phasing

Any maneuver re-phasing an eccentric periodic relative orbit of the active spacecraft with respect to a center point along-track (in the linear LVLH environment) will be called equilibrium-to-equilibrium. Re-phasing in this case implies shifting the center of this equilibrium relative motion, justifying the choice of the equilibrium-to-equilibrium nomenclature.

The control signal (2) was applied to on the dynamics of system (1), starting from an equilibrium closed relative orbit $\mathbf{x}(t_0) = [x_0 \quad y_0 \quad \dot{x_0} \quad -2\omega x_0]^T$ [26], and considering a variable Δt, thus yielding an expression for the final state (see [11] for details). The center of the ellipse representing the final relative orbit, computed as in (3), is obtained as:

$$
\bar{x} = 0, \quad \bar{y} = y_{fd} - \frac{2}{\omega}\dot{x}_0. \tag{5}
$$

Equation (5) shows that re-phasing to a final equilibrium relative orbit, with center at a desired location, is possible. In fact, starting from t_0, and waiting for any instant when $\dot{x} = 0$ (there are two positions along the closed relative orbit that correspond to this condition), the input-shaped control signal can be applied then. The wait time is given by:

$$t_{wait} = \frac{1}{\omega}\tan^{-1}\left(\frac{\dot{x}_0}{\omega x_0}\right) + k\pi, k = 0, 1, 2, \dots \tag{6}$$

Reference [11] shows how following the same reasoning for the re-phasing from a leader-follower configuration, the direct dependency of e_{rel} from can be obtained as:

$$\begin{aligned} \Delta e_{rel} &= \frac{1}{2}\sqrt{\begin{array}{c} \frac{4}{\omega^4}\left(\begin{array}{c} x_0\omega^2\,\mathrm{c}\left(\frac{2}{3}\alpha_2 + 2\omega\Delta t\right) + \dot{x}_0\omega\,\mathrm{s}\left(\frac{2}{3}\alpha_2 + 2\omega\Delta t\right) \\ +\bar{u}\,\mathrm{s}\left(\frac{2}{3}\alpha_2 + \omega\Delta t\right) - \bar{u}\,\mathrm{s}\left(\frac{1}{3}\alpha_2 + 2\omega\Delta t\right) - 2\bar{u}\,\mathrm{s}\left(\frac{1}{3}\alpha_2 + \omega\Delta t\right) \\ +\frac{1}{2}\bar{u}\,\mathrm{s}\left(\frac{2}{3}\alpha_2 + 2\omega\Delta t\right) - \bar{u}\,\mathrm{s}\left(\frac{1}{3}\alpha_2\right) \\ \frac{1}{2}\bar{u}\,\mathrm{s}\left(\frac{2}{3}\alpha_2\right) + \frac{1}{2}\bar{u}\,\mathrm{s}\left(2\omega\Delta t\right) + \bar{u}\,\mathrm{s}\left(\omega\Delta t\right) \end{array}\right)^2 \\ +\frac{1}{\omega^4}\left(\begin{array}{c} -2x_0\omega^2\,\mathrm{s}\left(\frac{2}{3}\alpha_2 + 2\omega\Delta t\right) + 2\dot{x}_0\omega\,\mathrm{c}\left(\frac{2}{3}\alpha_2 + 2\omega\Delta t\right) \\ -4\bar{u}\,\mathrm{c}\left(\frac{1}{3}\alpha_2 + \omega\Delta t\right) + 2\bar{u}\,\mathrm{c}\left(\frac{2}{3}\alpha_2 + \omega\Delta t\right) - 2\bar{u}\,\mathrm{c}\left(\frac{1}{3}\alpha_2 + 2\omega\Delta t\right) \\ +\bar{u}\,\mathrm{c}\left(\frac{2}{3}\alpha_2 + 2\omega\Delta t\right) - 2\bar{u}\,\mathrm{c}\left(\frac{1}{3}\alpha_2\right) + \bar{u}\,\mathrm{c}\left(\frac{2}{3}\alpha_2\right) \\ +2\bar{u}\,\mathrm{c}\left(\omega\Delta t\right) + \bar{u}\,\mathrm{c}\left(2\omega\Delta t\right) + \bar{u} \end{array}\right)^2 \end{array}}, \\ \alpha_2 &= \omega\sqrt{\frac{3y_{fd} - 3y_0}{\bar{u}}}. \end{aligned} \tag{7}$$

The Nyquist-Shannon sampling theorem [28] must be invoked again, to find the number of points to approximate the function in (7), and then interpolation to compute the correct Δt for a desired Δe_{rel}.

3 The Lyapunov-Based Nonlinear Controller for Differential Drag

The problem of designing a real-time controller using differential drag consists of finding an analytical expression to command the opening or closing of the drag surfaces (see Figure 1) that will force the spacecraft to follow the desired guidance. In particular, the following assumptions are commonly made when using atmospheric differential drag control:

1. The control is only along the y direction, as described earlier.
2. The opening/closing of the drag surfaces is instantaneous, i.e., their actuation time is negligible with respect to the duration of the maneuver, resulting in an on-off sequence for commands for opening or closing the drag surfaces.
3. Atmospheric density is known with poor accuracy (30%, as suggested by previous work [22]).

The poor knowledge of atmospheric density requires the design of a command logic, capable of dealing with an unknown and continuously variable control magnitude. The authors previously devised such a command strategy, using an adaptive Lyapunov approach. The fine details of the methodology are presented in [12,13], while only the most important results are presented here, along with a discussion on the expected behavior of the atmospheric density.

The controller is based on the idea of being conservative and maintaining a sufficient margin of control authority on the system. In particular, at the initial time of the maneuver the atmospheric density is underestimated (30% less than what is provided by atmospheric models, see [29]), underestimating the available differential drag. At the same time, the initial adjustable parameters for the controller are chosen such that the initial underestimated differential drag is above a critical, or minimum, value that guarantees Lyapunov stability. From that instant on, the controllers parameters are adapted to maintain a low critical value (shown in (11)). This critical value is the minimum amount of differential drag acceleration that will ensure Lyapunov stability for the controller. This conservative procedure relies on the assumption that in average the atmospheric density will only increase throughout the maneuver, since the orbits of the spacecraft are decaying. The critical differential drag value is maintained low, or possibly reduced throughout the maneuver, by adapting the controller. With this methodology, a positive control margin is maintained between real differential drag and minimum differential drag for Lyapunov stability.

The controller is devised as follows. A quadratic Lyapunov function of the tracking error between the spacecraft state and the desired state (e.g., the input-shaping-designed guidance) is defined as:

$$V_L = \mathbf{e}^T \underline{\mathbf{P}} \mathbf{e},\ \mathbf{e} = \mathbf{x_n} - \mathbf{x},\ \underline{\mathbf{P}} > \mathbf{0}, \tag{8}$$

where $\underline{\mathbf{P}}$ is a symmetric positive definite matrix, e is the tracking error vector, $\mathbf{x}_n$ and $\mathbf{x}$ are the actual spacecraft relative state vector and a reference desired state vector (the guidance obtained controlling system (1) with the input (2), solved with the solutions in (3) or (5) and (6), depending on the type of maneuver), respectively. The drag surfaces activation strategy is obtained by differentiating (8) with respect to time, and imposing a negative sign in this time derivative, leading to an expression for the signal, indicating the open/closed condition for the drag surfaces (1 = open; 0 = closed; -1 = other S/C opens). See References [12, 13] for details to obtain the formula

$$\hat{u} = -sign(\mathbf{e}^T \underline{\mathbf{P}} \mathbf{B}). \tag{9}$$

The same steps leading to Equation (9) (see References [12, 13]) define the matrix $\underline{\mathbf{P}}$ as the solution of the Lyapunov equation

$$\underline{\mathbf{A}}_d^T \underline{\mathbf{P}} + \underline{\mathbf{P}}\, \underline{\mathbf{A}}_d = -\underline{\mathbf{Q}}, \tag{10}$$

where $\underline{\mathbf{Q}}$ is a symmetric positive definite matrix and $\underline{\mathbf{A}}_d$ is a Hurwitz matrix. These two matrices are user defined, and represent the controllers adjustable parameters, affecting the Lyapunov function and thus the systems behavior.

The information needed to command the drag surfaces (tracking error and matrix $\underline{\mathbf{P}}$ and vector $\mathbf{B}$ in (9)) would be available in real-time onboard a spacecraft, and the command is a straightforward instruction that poses no issues in terms of onboard computer implementation. In addition, there is no information about the actual density value required by the control law. The Lyapunov algebraic developments also lead to the expression of a critical value (a_{Dcrit}) of differential drag that is needed to maintain stable Lyapunov control (see References [12, 13] for details). This critical value is given as:

$$a_{Dcrit} = \frac{\mathbf{e}^T \underline{\mathbf{P}} \left(\underline{\mathbf{A}}_d \mathbf{x}_n - f(\mathbf{x}_n) + \mathbf{B} u_d \right)}{|\mathbf{e}^T \underline{\mathbf{P}} \mathbf{B}|} \tag{11}$$

with $f(\mathbf{x}_n)$ representing the nonlinear relative motion dynamics. $f(\mathbf{x}_n)$ can be as accurate as the number of higher order gravitational terms that can be expressed analytically. u_d is a desired control, i.e. the acceleration profile generated in the guidance. The analytical expressions for the partial derivatives of the critical value with respect to the adjustable matrices were developed in [12, 13]. A real-time adaptation of the matrices themselves (shown in (12)), with the intent to maintain the critical value as low as possible (see

References [12, 13] for details) was designed based on the partial derivatives. In (12) δ_A and δ_Q are increments in the matrices components, chosen such that $\underline{\mathbf{A}}_d$ remains Hurwitz, and $\underline{\mathbf{Q}}$ positive definite. The adaptation occurs at discrete time steps, as explained in the simulations section

$$\begin{aligned} &\Delta A_{ij} = \kappa_A \left[-sign(\frac{\partial a_{Dcrit}}{\partial A_{ij}})\delta_A \right], \ \Delta Q_{ij} = \kappa_Q \left[-sign(\frac{\partial a_{Dcrit}}{\partial Q_{ij}})\delta_Q \right], \\ &\kappa_A = \begin{cases} 1, \text{ if } \left| \frac{\partial a_{Dcrit}}{\partial A_{ij}} \right| > \left| \frac{\partial a_{Dcrit}}{\partial A_{kl}} \right| \text{ for } i,j \neq k,l, \\ 0, \text{ else} \end{cases} \\ &\kappa_Q = \begin{cases} 1, \text{ if } \left| \frac{\partial a_{Dcrit}}{\partial Q_{ij}} \right| > \left| \frac{\partial a_{Dcrit}}{\partial Q_{kl}} \right| \text{ for } i,j \neq k,l, \\ 0, \text{ else} \end{cases} \end{aligned} \tag{12}$$

Depending on the spacecraft computing capabilities the non-adaptive or the adaptive controller can be chosen. The adaptive controller requires the additional computation of the matrix derivative expressions, and the adaptation rule of (12). Once again, all these expressions are analytical, and can be computed provided knowledge of the spacecraft's state vector. Both types of controller perform satisfactorily, as shown in the next section, with expected increased performance when adapting the parameters $\underline{\mathbf{A}}_d$ and $\underline{\mathbf{Q}}$.

4 Numerical Simulations

This section starts by presenting the different types of maneuvers achievable with the analytical guidance, using illustrations obtained from numerical simulations of the linear dynamics, and concludes by illustrating the closed-loop nonlinear simulations and a discussion of the results. In particular, the first subsection shows several leader-follower maneuvers obtained by varying Δt. The second subsection shows the equilibrium-relative-orbit-to-equilibrium-relative-orbit approach, while changing Δt to show how the final relative eccentricity can be varied. The Lyapunov closed-loop control is then used to track the guidance in a full nonlinear environment available in Systems Tool Kit (STK). The analytical guidance assumes a maximum control acceleration of approximately $2 * 10^{-5}$ m/s^2, typical of atmospheric differential drag at the simulations' given altitude.

It is important to underline that relative navigation is beyond the scope of this paper, and that robust estimation techniques will be needed to accurately compute the analytical guidance and use the closed-loop controller. In the following, perfect knowledge of the relative state between the two spacecraft is assumed, envisioning, for example, a high precision differential GPS technique running on the two spacecraft (example: [30]).

4.1 Leader-follower re-phasing guidance

The initial conditions in Table 1, in terms of orbital parameters, are used, with the goal of re-phasing the S/C position to match a desired one. The initial location and desired final location are in the same orbit, with different polar angles. In particular, backward and forward re-phasing maneuvers are presented.

With the parameters in Table 1 the correct initial S/C state vectors in the LVLH frame centered at the desired target locations are $\mathbf{x}(t_0) = 10^3[-0.0013 \quad -4.2588 \quad 0 \quad 0]^T$ for the 27.216 degrees case, and $\mathbf{x}(t_0) = 10^3[-0.0009 \quad 3.5490 \quad 0 \quad 0]^T$ for the 27.15 degrees case, with units in meters and meters per second. In the linearized environment, a leader-follower configuration does not present any cross-track displacement nor any along-track velocity component. The linear approximation to obtain the analytical solutions described earlier requires the use of $\mathbf{x}(t_0) = 10^3[0 \quad -4.2588 \quad 0 \quad 0]^T$ and $\mathbf{x}(t_0) = 10^3[0 \quad 3.590 \quad 0 \quad 0]^T$, respectively.

Orbital Parameter	**Desired**	**S/C initial**
Semi-major axis a	$6,778.1\,km$	$6,778.1\,km$
Eccentricity e	0	0
Inclination i	$97.9908\,deg$	$97.9908\,deg$
Right Ascension of the Ascending Node ($RAAN$) Ω	$261.621\,deg$	$261.621\,deg$
Argument of Perigee ω_p	$30\,deg$	$30\,deg$
Polar Angle ν	$27.15\,deg$ and $27.216\,deg$	$27.18\,deg$

Table 1: Initial Orbital parameters for S/C and desired location for Leader-Follower case, plus general data for simulations.

Figure 2 shows the backwards maneuver, that is, re-phasing to a smaller polar angle using the input shaping technique of Equation (2). A value of $\Delta = 0.5T$ is used, corresponding to a new leader-follower configuration.

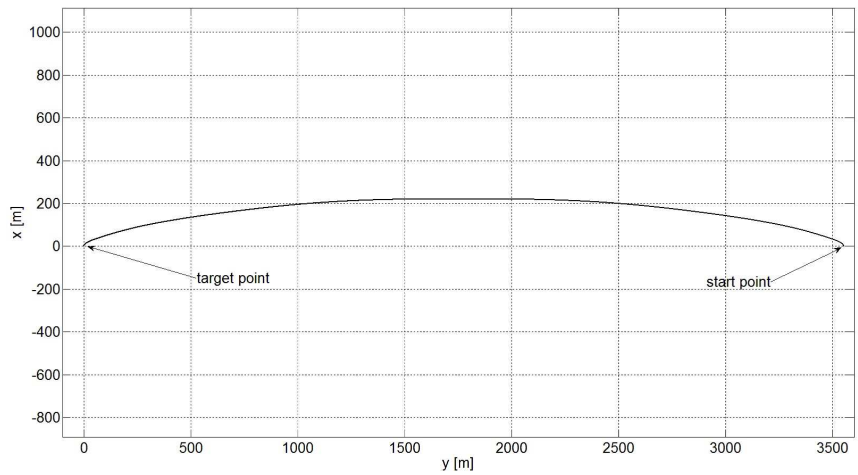

Figure 2: Re-phasing to a lower polar angle, with $\Delta t = 0.5T$, obtaining a new leader-follower configuration (linear dynamics case).

In Figure 3 the forward maneuver is shown for three different values of Δt. For $\Delta = 0.5T$, an input-shaped control is applied, with no residual oscillation at the target point (the LVLH origin). The maximum relative eccentricity is obtained for $\Delta = 0$, while $\Delta = 0.25T$ is an example of intermediate relative eccentricity (see (4)) The simulation is propagated beyond the end of the control signal, to show the closed relative motion about the target along-track point.

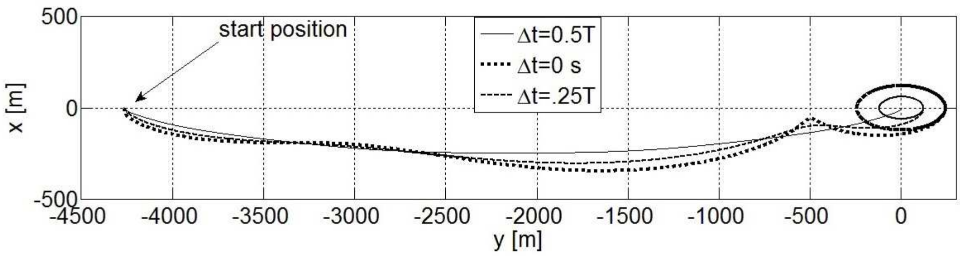

Figure 3: Re-phasing to a higher polar angle. 1) $\Delta t = 0.5T$, obtaining a new leader-follower configuration; 2) $\Delta t = 0$, obtaining the maximum relative eccentricity for the final equilibrium orbit around the target point; 3) $\Delta t = 0.25T$, obtaining an intermediate value of relative eccentricity for the final equilibrium relative orbit around the target point (linear dynamics case).

The initial conditions in Table 2, in terms of orbital parameters, are used with the goal of re-phasing the S/C, from an equilibrium relative orbit about an initial along-track point, to a final equilibrium relative orbit about a desired final along-track point. In this case, a small eccentricity is given to the S/C, to generate an equilibrium initial relative orbit. The semi-major axes are the same to guarantee boundedness of the relative motion. Only a forward re-phasing maneuver is presented for this case.

Orbital Parameter	**Desired**	**S/C initial**
Semi-major axis a	$6,778.1\,km$	$6,778.1\,km$
Eccentricity e	0	0.0001
Inclination i	$97.9908\,deg$	$97.9908\,deg$
Right Ascension of the Ascending Node ($RAAN$) Ω	$261.621\,deg$	$261.621\,deg$
Argument of Perigee ω_p	$30\,deg$	$30\,deg$
Polar angle ν	$27.216\,deg$	$27.18\,deg$

Table 2: Initial Orbital parameters for S/C and desired location for Leader-Follower case, plus general data for simulations.

With the parameters in Table 2 the correct initial S/C state vectors in the LVLH frame, centered at the desired target locations, are $\mathbf{x}(t_0) = 10^3[-0.6043 \quad -4.2584 \quad 0.0004 \quad 0.0014]^T$, where the units are m and m/sec. In the linearized environment, an equilibrium configuration requires the modification of this initial condition to $\mathbf{x}(t_0) = 10^3[-0.6043 \quad -4.2584 \quad 0.0004 \quad -2\omega x_o]^T$. From the above initial modified condition for the linear model, a waiting time (coasting) is used (Equation (6)), with $k = 0$, before applying the control signal.

Figure 4 represents equilibrium-relative-orbit-to-equilibrium-relative-orbit maneuvers with the same target point as center (origin of LVLH), varying the Δt value. The simulations are propagated beyond the end of the control signal to show the closed relative motion about the target along-track point.

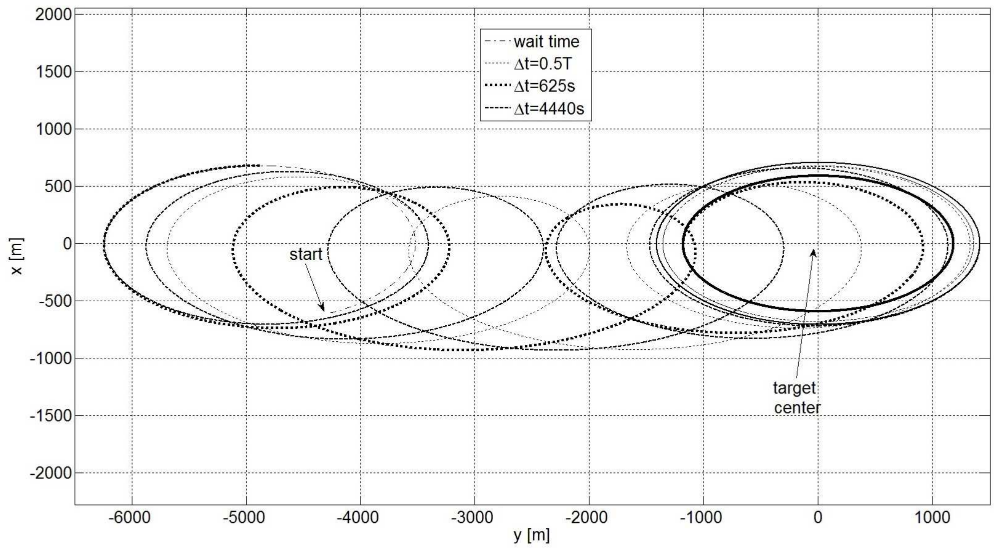

Figure 4: Re-phasing to a higher polar angle for equilibrium-to-equilibrium maneuver. 1) $\Delta t = 0.5T$, obtaining an intermediate relative eccentricity (between initial and maximum achievable) on final relative orbit; 2) $\Delta t = 625s$, obtaining the minimum relative eccentricity for the final equilibrium orbit around the target point; 3) $\Delta t = 4440s$, obtaining the maximum relative eccentricity for the final equilibrium relative orbit around the target point.

The above examples are valid in the simplified linear dynamics case. In order to implement these solutions on a real spacecraft, a closed-loop controller is needed, to track the analytical guidance profiles. This controller is used for the simulations in the following subsection.

4.2 Closed-loop control in the full nonlinear case

In this section, the Lyapunov controller described earlier, both the non-adaptive and adaptive versions, is used to track the following guidance:

- **CASE 1**: re-phasing and generation of closed relative orbit at target (Figure 3 with $\Delta t = 0$).
- **CASE 2**: pure re-phasing (Figure 3 with $\Delta t = 0.5T$).
- **CASE 3**: intermediate change of the size of the relative orbit, and re-phasing it (Figure 4 with $\Delta t = 0.5T$).

To reduce the frequency of actuation and allow the drag forces enough time to change the orbits, the controllers are activated every 10 minutes. The same simulations are also run activating the drag devices every 5 minutes to show improvement in accuracy in guidance tracking as the control frequency increases. Numerical simulations are run using the High Precision Orbital Propagator (HPOP) in STK and Matlab. Matlab extracts the relative state vectors from STK, and generates the command to the drag surfaces, going back to

STK. An STK scenario with full gravitational field model, variable atmospheric density (using NRLMSISE-00 available in STK) and solar pressure radiation effects is used.

Two identical maneuvering spacecraft are considered, with one at the origin of LVLH, masses of $2kg$, maximum surface of $0.5m^2$, and minimum of $10cm^2$ (representing what is depicted in Figure 1), and a drag coefficient of 2.2. The initial adaptable matrix $\underline{\mathbf{A}}_d$ is chosen as $\underline{\mathbf{A}} - \mathbf{BK}$, where $\underline{\mathbf{A}}$ represents the dynamics matrix of the spacecraft relative motion linear equations, stabilized through a LQR-based $\mathbf{K}$ vector, to make $\underline{\mathbf{A}}_d$ Hurwitz. In the LQR problem $\mathbf{K}$ is obtained from $\underline{\mathbf{Q}}_{LQR} = \underline{\mathbf{I}}_{4x4}$, and $R_{LQR} = 1.5 * 10^8$. The initial adaptable matrix is chosen to be $\underline{\mathbf{I}}_{4x4} * 10^{-2}$. The chosen increments for the adaptable matrices in Equation (12) are the values $\delta_A = 10^{-6}$ for $\underline{\mathbf{A}}_d$ and $\delta_Q = 10^{-6}$ for $\underline{\mathbf{Q}}$.

The ultimate goals of these simulations are a critical comparison between the two controllers and a discussion helping a potential spacecraft developer in choosing what type of guidance and control should be used on the spacecraft.

4.2.1 CASE 1: re-phasing from leader-follower, and generating a closed relative motion at the target

Figure 5 shows the results of a nonlinear STK simulation using the Lyapunov controllers to track a re-phasing guidance with final desired closed motion about the target (origin of the LVLH frame) (Figure 3). The simulation is stopped when the guidance reaches its end. The bottom image clearly shows the benefit of using the adaptive controller versus the non-adaptive. The non-adaptive approach cannot reach the final desired motion, while the adaptation does reach a final motion very close to the desired one. Likewise, the adaptation allows for increased accuracy in tracking the guidance, especially in the last phases of the maneuver, as depicted by the bottom image.

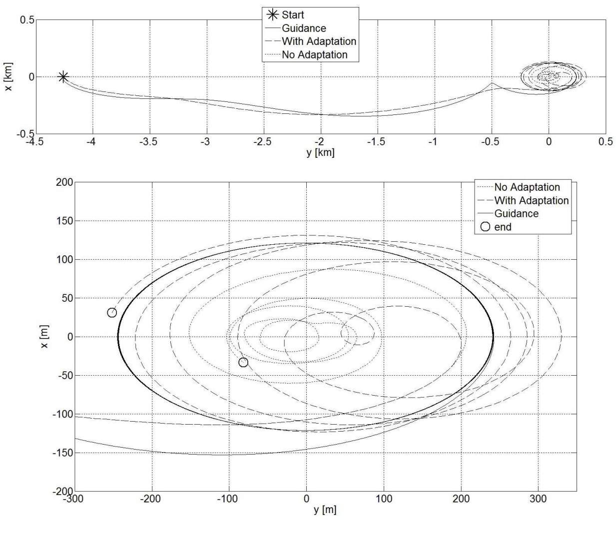

Figure 5: Nonlinear Simulation result (control update every 10 minutes): re-phasing to higher polar angle from leader-follower initial condition and generation of a closed relative motion around the target point. Guidance from Figure 3, with $\Delta t = 0$. (TOP) full trajectory; (BOTTOM) zoom of last phase.

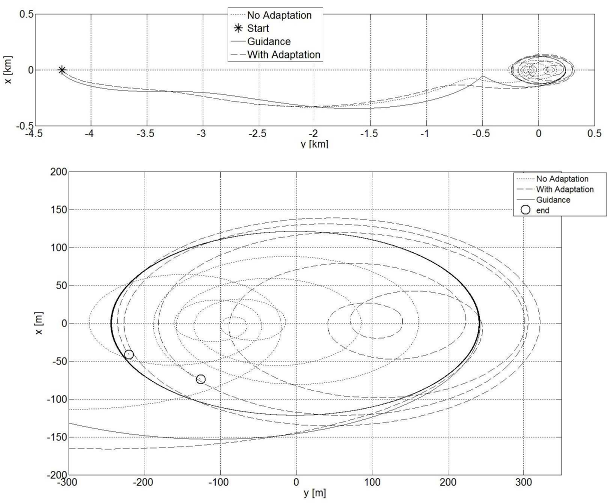

Figure 6: Nonlinear Simulation result (control update every 5 minutes): re-phasing to higher polar angle from leader-follower initial condition and generation of a closed relative motion around the target point. Guidance from Figure 3, with $\Delta t = 0$. (TOP) full trajectory; (BOTTOM) zoom of last phase.

Figure 6 shows the same scenario as Figure 5, with an increased control frequency (from 10 to 5 minutes). While an improvement in performance and accuracy is observed for both the adaptive and non-adaptive controllers, the increase in frequency particularly benefits the non-adaptive solution, but it still does not achieve performance equal to that of the adaptive case. This additional result further supports the thesis of preferring adaptation since similar performance can be achieved without the need of increasing frequency of actuation.

4.2.2 CASE 2: re-phasing from leader-follower to leader-follower

Figure 7 shows the results of a nonlinear STK simulation using the Lyapunov controllers to track a pure re-phasing guidance with final desired location at the origin of the LVLH frame (Figure 3). The simulation is stopped when the guidance reaches its end. As in CASE 1, the bottom image shows that the adaptation allows for better accuracy in tracking the guidance.

Figure 8 shows the same scenario as Figure 7, with an increased control frequency (from 10 to 5 minutes). In this case both controllers enhance their performance significantly. In particular, the final distance from the desired location reached with the adaptive controller, makes the differential drag approach a viable candidate for very close proximity operations. In fact, such distances are in the order of magnitude of the reach envelope for existing space robotic arms (Canadarm [31]). The maneuver is stopped when the guidance reaches its end, but additional control could be performed via differential drag, at a higher frequency of actuation, to move the spacecraft even closer to the target location, or small thrusters could be used for the very final approach for rendezvous and grappling.

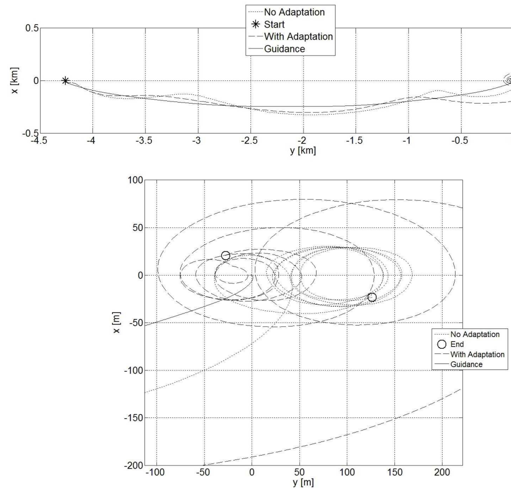

Figure 7: Nonlinear Simulation result (control update every 10 minutes): re-phasing to higher polar angle from leader-follower initial condition to leader-follower final condition. Guidance from Figure 3, with $\Delta t = 0.5T$. (TOP) full trajectory; (BOTTOM) zoom of last phase.

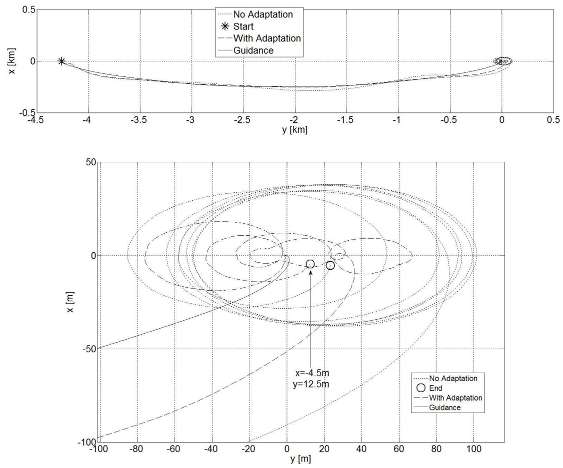

Figure 8: Nonlinear Simulation result (control update every 5 minutes): re-phasing to higher polar angle from leader-follower initial condition to leader-follower final condition. Guidance from Figure 3, with $\Delta t = 0.5T$. (TOP) full trajectory; (BOTTOM) zoom of last phase.

4.2.3 CASE 3: re-phasing from equilibrium-relative-orbit-to-equilibrium-relative-orbit

Figure 9 shows the results of a nonlinear STK simulation using the Lyapunov controllers to track a re-phasing guidance starting from an initial closed relative motion with a final goal of creating a new closed motion around the origin of the LVLH frame (Figure 4). The simulation is stopped when the guidance reaches its end. The bottom plot shows how the adaptation allows for more precise tracking of the guidance towards the end of the maneuver.

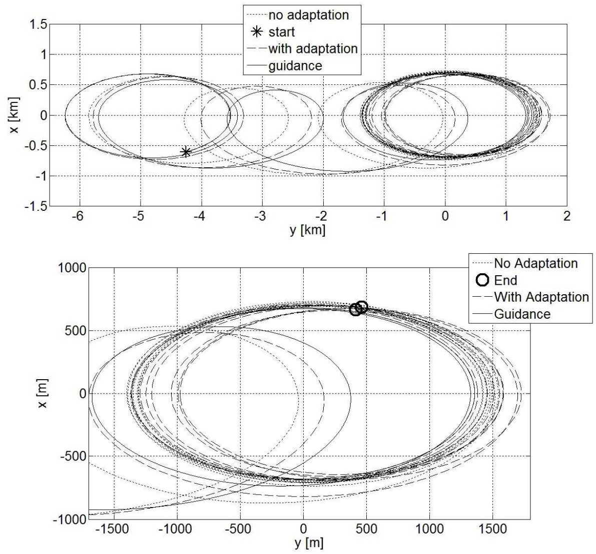

Figure 9: Nonlinear Simulation result (control update every 10 minutes): re-phasing to higher polar angle from equilibrium relative orbit initial condition and generation of a new closed relative motion around the target point. Guidance from Figure 4, with $\Delta t = 0.5T$. (TOP) full trajectory; (BOTTOM) zoom of last phase.

Figure 10 shows the same scenario as Figure 9, with an increased control frequency (from 10 to 5 minutes). In this case in the bottom image it is clear that both controllers provide good tracking. A preliminary interpretation of this behavior can be found in the nature of the maneuver. Since the spacecraft starts with a motion which is already oscillatory, the control action is only required to shift that motion and then stop the shift once the new desired location is reached. Roughly speaking, this maneuver is less challenging from the controllers point of view since the dynamics starts in a favorable initial condition.

In CASES 1 and 2 the spacecraft starts in a leader-follower state, thus requiring more effort from the input signal. In CASE 1, the controller is required to move the spacecraft away from its initial state, thus exciting the oscillations as well. These oscillations are controlled by choosing the correct Δt, and there is no need to drive them back to zero. In CASE 2, instead, the controller moves the spacecraft away from its leader-follower state, thus exciting oscillations, but it is also required to drive this motion to zero once the final desired location is approached. Once again, intuitively speaking, this implies more work for the controller. The above described differences in the maneuvers provide an interpretation for the fact that the benefits of adaptation are clearer in CASE 1 and 2 than in CASE 3.

Finally, Table 3 compares adaptive and non-adaptive simulations by showing the

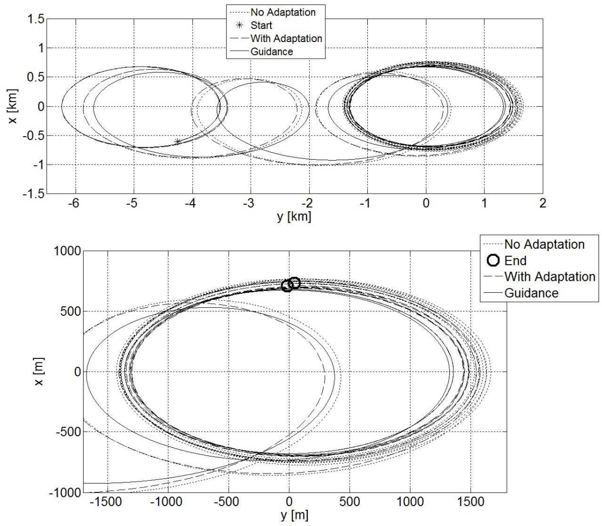

Figure 10: Nonlinear Simulation result (control update every 5 minutes): re-phasing to higher polar angle from equilibrium relative orbit initial condition and generation of a new closed relative motion around the target point. Guidance from Figure 4, with $\Delta t = 0.5T$. (TOP) full trajectory; (BOTTOM) zoom of last phase.

number of switches required (i.e. control effort, since electrical power would be required to actuate the devices), the average drag and critical drag, and the average control margin during the maneuvers, where the margin is calculated as the difference between real differential drag (it would not be known in real flight) and critical value. All the values in the table support the preference for the adaptation.

4.3 Results discussion

Both the closed-loop controllers require no numerical iterations, making them viable candidates for onboard implementation. The adaptive controller requires the implementation of the formulas for the derivatives ([12]) which is still analytical, but imposes more instructions on the spacecraft computer. Depending on the available memory, the designer may decide to only implement the non-adaptive controller. Overall, the adaptation provides better accuracy and less control effort (number of state switches for the drag surfaces), particularly allowing for better tracking of the guidance as the maneuver approaches the final stages. This is especially true for more demanding maneuvers in terms of guidance, where the dynamics may not be favorable with respect to the final desired state. For cases such as equilibrium-relative-orbit to equilibrium-relative-orbit, the non-adaptive controller may be equivalent to the adaptive in terms of control effort required, that is number of open/closed cycles.

		10 minutes control update			5 minutes control update		
		Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
Maneuver time (hr)		13.15	16.23	17.55	13.23	16.32	17.63
Non Adaptive	Control changes	41	68	64	80	133	127
	Mean critical value ($m/s^2 * 10^{-6}$)	-6.50	-4.30	-3.75	-5.90	-4.51	-3.40
	Mean actual drag ($m/s^2 * 10^{-5}$)	3.38	3.42	3.39	3.37	3.41	3.34
	Mean margin ($m/s^2 * 10^{-5}$)	4.03	3.85	3.77	3.96	3.86	3.68
Adaptive	Control changes	37	58	72	76	112	112
	Mean critical value ($m/s^2 * 10^{-6}$)	-7.23	-5.38	-3.81	-6.24	-4.87	-4.28
	Mean actual drag ($m/s^2 * 10^{-5}$)	3.38	3.44	3.39	3.40	3.41	3.33
	Mean margin ($m/s^2 * 10^{-5}$)	4.10	3.98	3.77	4.03	3.90	3.75

Table 3: Nonlinear simulations results (number of status switches for the drag devices, mean critical and real values of differential drag, and mean differential drag margin).

5 Conclusions

This paper introduced a novel framework combining previously presented analytical guidance and Lyapunov control solutions for propellant-less, drag-based spacecraft re-phasing relative maneuvers. The framework studied in this work, provides the groundwork for realistic finite magnitude and finite duration control, such as the control obtained via atmospheric differential drag. The analytical solutions can lead a spacecraft from an initial location along the orbit to a desired final location on the same course, as well as modify its path so that it will fly in an equilibrium fashion about a desired point ahead or behind its initial location. The guidance is graphically illustrated and employed within nonlinear models, where a closed-loop Lyapunov technique is used to track the guidance trajectory with satisfactory accuracy in the full nonlinear STK environment. The relative maneuvers are performed assuming differential drag control capability, which does not use any propellant. Observations derived from the results of the nonlinear simulations provide useful insights to spacecraft developers, and particularly to the mission designer who needs to implement the correct control law on the spacecraft onboard computer. Overall, the achieved results hold a promise for straightforward implementation onboard real spacecraft, particularly small spacecraft with limited computing capabilities.

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Integrable Time-Dependent Dynamical Systems: Generalized Ermakov-Pinney and Emden-Fowler Equations

Partha Guha [1*] and Anindya Ghose Choudhury [2]

[1] *S.N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata-700098, India.*
[2] *Department of Physics, Surendranath College, 24/2 Mahatma Gandhi Road, Calcutta-700009, India*

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Abstract: We consider the integrable time-dependent classical dynamics studied by Bartuccelli and Gentile (Phys Letts. **A307** (2003) 274–280; Appl. Math. Lett. **26** (2013) 1026–1030) and show its power to compute the first integrals of the (generalized) Ermakov-Pinney systems. A two component generalization of the Bartuccelli-Gentile equation is also given and its connection to Ermakov-Ray-Reid system and coupled Milne-Pinney equation has been illucidated. Finally, we demonstrate its application in other integrable ODEs, in particular, using the spirit of Bartuccelli-Gentile algorithm we compute the first integrals of the Emden-Fowler and describe the Lane-Emden type equations. A number of examples are given to illustrate the procedure.

Keywords: *time-dependent harmonic oscillator; Ermakov-Pinney equation; first integrals; Ermakov-Lewis invariant; Emden-Fowler equation.*

Mathematics Subject Classification (2010): 34A05, 34A34, 34C14.

* Corresponding author: mailto:partha@bose.res.in

1 Introduction

The study of nonlinear time-dependent ordinary differential equations (ODEs) has been going on for several years now. Since there are hardly any general methods for dealing with such equations one is often forced to look for interesting transformations which either enable us to simplify the equation or to map it to some linear or nonlinear equation whose features are already known. The linear harmonic oscillator has been a time honored favorite and has enhanced our understanding of several key areas of mathematics and physics. It has the added advantage of being a Hamiltonian system and serves as a first approximation for many nonlinear differential equations. In [4] Bartuccelli and Gentile made a beautiful observation regarding the equation of a linear harmonic oscillator,

$$\ddot{x} + \omega^2 x = 0. \tag{1.1}$$

Here the over dot represents differentiation with respect to the independent variable t. As is well known its solution is

$$x(t) = A\sin(\omega t + \phi), \tag{1.2}$$

where A and ϕ are arbitrary constants representing the amplitude and phase respectively. They observed that if (1.1), which may also be written as

$$\frac{d}{dt}\left(\frac{\dot{x}}{\omega}\right) + \omega x = 0, \tag{1.3}$$

one assumes that ω, instead of being a constant, is any arbitrary function of the independent variable t so that one actually has the following equation:

$$\frac{d}{dt}\left(\frac{\dot{x}}{\omega(t)}\right) + \omega(t) x = 0, \tag{1.4}$$

then its solution is similar in structure to (1.2) in the sense that

$$x(t) = A\sin(\int \omega(t)dt + \phi). \tag{1.5}$$

It was stressed in [4,5] that the equation in the form (1.4) is still quite interesting and can be generalized to various directions and gives new results. Our main aim is to explore all these directions in this paper.

It is obvious that (1.4) is not reducible to the equation of a time-dependent linear harmonic oscillator

$$\ddot{x} + \omega^2(t) x = 0. \tag{1.6}$$

Nevertheless the fact that the solution of (1.4) clearly reduces to that of the usual harmonic oscillator when ω is a constant is indeed remarkable. In fact the following generalization is also possible, namely we replace (1.4) by

$$\frac{d}{dt}\left(\frac{\dot{x}}{\omega(t)}\right) + \omega(t) F(x) = 0, \tag{1.7}$$

where $F(x)$ is some nonlinear C^1 function of x. Note that (1.7) may be written as the following system

$$\dot{x} = \omega(t) y, \quad \dot{y} = -\omega(t)\frac{dU(x)}{dx}, \tag{1.8}$$

with $F(x) = dU/dx$. In this paper equations (1.4) and (1.7) will be called the Bartuccelli-Gentile equations. In general for the linear equation

$$\ddot{x} + P(t)\dot{x} + Q(t)x = 0, \tag{1.9}$$

one can make use of Jacobi's Last Multiplier to derive a suitable Lagrangian. Indeed the last multiplier turns out to be the integrating factor of such an equation given by

$$M = \exp(\int^t P(s)ds).$$

The relationship between a last multiplier and the Lagrangian is given by

$$M = \frac{\partial^2 L}{\partial \dot{x}^2},$$

from which it follows that a Lagrangian for (2.12) is given by

$$L(t, x, \dot{x}) = e^{\int P(t)dt}\left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}Q(t)x^2\right). \tag{1.10}$$

By using the standard Legendre transformation it follows that the corresponding Hamiltonian is

$$H(t, x, p_x) = \frac{1}{2}\left(e^{-\int P(t)dt}p_x^2 + Q(t)e^{\int P(t)dt}x^2\right), \tag{1.11}$$

where the conjugate momentum is defined in the usual manner

$$p_x = \frac{\partial L}{\partial \dot{x}} = e^{\int P(t)dt}\dot{x}.$$

In case of (1.4) it is clear that $P(t) = -\dot{\omega(t)}/\omega(t)$ and $Q(t) = \omega^2(t)$, so that $M = \omega^{-1}(t)$, and the Hamiltonian therefore assumes the form

$$H = \frac{1}{2}\omega(t)\left(p_x^2 + x^2\right).$$

Note that (1.7) admits the following first integral

$$I(x, \dot{x}, t) = \frac{1}{2}\left(\frac{\dot{x}}{\omega(t)}\right)^2 + U(x), \tag{1.12}$$

where $U(x)$ is a primitive of $F(x)$, as is easy to verify. Clearly the level sets $I(x, \dot{x}, t) = E$ allow us to write

$$\int \frac{dx}{\sqrt{E - U(x)}} = \pm\sqrt{2}\int \omega(t)dt,$$

which in turn means that it is effectively a time-reparametrization of the usual time-independent case. The invariant (1.12) will be referred to as the Bartuccelli-Gentile invariant. The special case of $F(x) = x$ allows us to express this invariant as

$$I = \frac{1}{2}\left(p_x^2 + x^2\right),$$

where the definition of $p_x = \dot{x}/\omega$ has been used, and write

$$H(t, x, p_x) = I\omega.$$

Clearly the invariant I must have the dimension of action. It can be readily seen that $\frac{dH}{dt} \neq 0$, as is to be expected of a dissipative system. It is also necessary to mention that the expressions for the Lagrangian and Hamiltonian given in (2.32) and (2.36) reduce to those of Caldirola [9] and Kanai [22] when $P(t) = \gamma(t)$ and the case of $H = I\omega$ also appeared in connection with the derivation of Hannay's angle in [42].

The celebrated Ermakov-Pinney equation (see [21] for brief introduction) was introduced in the nineteenth century by V.P. Ermakov [15] to find the first integral for the time-dependent harmonic oscillator. In 1950 E. Pinney [34] found the solution of this equation. Ermakov systems have been extensively studied in physics as they often arise in the context of Bose-Einstein condensates, cosmological models, plasma confinements etc. Lewis [28,29] found independently an exact invariant for this system. Several methods have subsequently been devised for the derivation of the Lewis invariant, which was originally obtained in closed form through an application of the asymptotic theory of Kruskal [24]. Leach [26] has obtained the same result using a time-dependent canonical transformation. On the other hand Lutzky's [30] derivation was based on Noether's theorem. Moyo and Leach [31] used Noether symmetries to discuss the source of the Ermakov-Lewis invariant. Ray and Reid [37,38] by resurrecting Ermakov's original technique were able to obtain the existence of a Lewis-type invariant for the case of two coupled nonlinear equations. Grammaticos and Dorizzi [20] proposed a direct method to investigate the existence of an exact invariant for $2D$ time-dependent Hamiltonian systems. The construction of Bartuccelli and Gentile didn't consider the Ermakov issue. Although it is clear from their construction that there should be an explicit link between the Ermakov-Pinney equation and the Bartuccelli-Gentile equation.

The Emden-Fowler equation was first studied in an astrophysical context by Emden [14] and subsequently by Fowler who was instrumental in laying its mathematical foundation [16]. The celebrated Emden-Fowler equation appears in many areas in physics [33]. More recently Berry and Shukla [7] presented a class of models for particles moving under *curl forces* alone. They could not find closed-form solutions for general motions, but the dynamics can be reduced to the Emden-Fowler equation, for which a particular exact solution exists for a wide class of cases. In the study of stellar structure a star is usually considered as a gaseous sphere in thermodynamic and hydrostatic equilibrium described by a certain equation of state. In particular the polytropic equation of state yields the Lane-Emden equation, given by

$$xy'' + 2y' + xy^n = 0.$$

This was originally proposed by Jonathan Lane [25] and was analysed by R. Emden [14]. Several applications of the Emden-Fowler and Lane-Emden equations of various forms arising in astrophysics [11] and nonlinear dynamics have been reported. The Lane-Emden equation also arises in the study of fluid mechanics, relativistic mechanics, nuclear physics and in the study of chemical-reaction systems. A detailed account, though somewhat dated, can be found in the survey by Wong [43].

In recent years this equation has been generalized in many ways. For example, Goenner [17] studied a generalized class of the Lane-Emden equation

$$xy'' + k_1 y' + k_2 x^{\nu} y^n = 0, \quad \text{first kind,}$$

$$y'' + f(x)y' + g(x)y^n = 0, \quad \text{second kind .}$$

Kara and Mahomed [23] showed that when $n = -3$ the Lane-Emden equation,

$$y'' + (k/x)y' = \sigma x^w y^n, \qquad n \neq 0, 1, \ \sigma \neq 0,$$

generates the three-dimensional algebra $sl(2, R)$ in which case general solutions are known for $w = -2k$. Ranganathan [35, 36] has obtained solutions and first integrals for some classes of the Emden-Fowler equation.

1.1 Motivation, result and organization

In this paper we explore two important sets of integrable ODEs, namely, the Ermakov-Pinney systems and the Emden-Fowler systems. Many papers were devoted to the construction of the first integrals of these set of equations. We demonstrate in this survey that one can give a unified method to describe the first integrals of all these equations using Bartuccelli-Gentile's method.

At first we show how the Bartuccelli-Gentile invariant can be mapped to invariants of Ermakov type systems, then we present the two-component generalization of the Bartuccelli-Gentile construction. We extend their method to compute the first integrals of the Emden-Fowler equations and second first integrals for Lane-Emden type systems. It is true that the first integrals for many of these equations have already been found by means of a variety of different methods [6, 8, 19, 27, 35, 36, 40, 41]. In this paper we give an alternative and easy method to compute the first integrals of the Emden-Fowler class of equations.

This paper is organized as follows. In Section 2 we give an intimate connection between the Bartuccelli-Gentile construction and the Ermakov-Pinney equation, and extend this connection to coupled system also. We illustrate our construction through examples. Section 3 is devoted to Emden-Fowler type equations. We show just extending slightly the method of Bartuccelli-Gentile's construction one can easily obtain the first integrals of the Lane-Emden equations.

2 Ermakov-Pinney Equation and Bartuccelli-Gentile Construction

We begin by considering the equation of motion of a linear harmonic oscillator with time-dependent frequency, namely,

$$\ddot{x} + \omega^2(t)x = 0. \tag{2.1}$$

The problem of the time-dependent oscillator was first solved by Ermakov [15] who obtained an invariant for (2.1) by introducing the auxiliary equation

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3}. \tag{2.2}$$

Equation (2.2) is usually called the Ermakov-Pinney equation since Pinney provided the solution, some years after Ermakov's derivation of its first integral [34]. Ermakov obtained a first integral for the system of equations (2.1) and (2.2), by first of all eliminating $\omega^2(t)$ by multiplying (2.1) with ρ and (2.2) with x and subtracting the two and then finally by multiplying the resulting equation with the integrating factor $(\dot{x}\rho - x\dot{\rho})$. The resulting first integral is given by

$$I = \frac{1}{2}\big[(\rho\dot{x} - \dot{\rho}x)^2 + (x/\rho)^2\big], \tag{2.3}$$

and is called the Ermakov-Lewis invariant after Lewis independently *recalculated* it in 1966.

As mentioned in the previous section equation (1.7) which is explicitly given by

$$\ddot{x} - \frac{\dot{\omega}}{\omega}\dot{x} + \omega^2(t)F(x) = 0, \tag{2.4}$$

admits the first integral (1.12). Upon introducing the substitution

$$x = \frac{y}{\rho}, \tag{2.5}$$

into the first integral (1.12) the latter has the following appearance

$$I = \frac{1}{2}\left(\frac{\rho\dot{y} - y\dot{\rho}}{\omega(t)\rho^2}\right)^2 + U(y/\rho). \tag{2.6}$$

The transformation (2.5) is a particular case of a general transformation contained in Magnus and Winkler's book [32]. Moreover, under the above change of variables, (2.4) becomes

$$\frac{\rho\ddot{y} - y\ddot{\rho}}{\rho^2} - \left(\frac{\rho\dot{y} - y\dot{\rho}}{\rho^2}\right)\left(\frac{\dot{\omega}}{\omega} + 2\frac{\dot{\rho}}{\rho}\right) + \omega^2(t)F(y/\rho) = 0. \tag{2.7}$$

Setting

$$\frac{\dot{\omega}}{\omega} + 2\frac{\dot{\rho}}{\rho} = 0$$

so that

$$\omega(t)\rho^2 = c(>0) \text{ then leads to } \rho^2 = \frac{c}{\omega(t)}, \tag{2.8}$$

and causes (2.7) after partial elimination of the variable ρ, to reduce to the following equation (assuming $c = 1$),

$$\ddot{y} + \frac{1}{2}\left(\frac{\ddot{\omega}}{\omega} - \frac{3}{2}\left(\frac{\dot{\omega}}{\omega}\right)^2\right)y + \omega^2\rho F(y/\rho) = 0. \tag{2.9}$$

In view of (2.8) the first integral (2.6) therefore becomes

$$I = \frac{1}{2}\left(\frac{\rho\dot{y} - \dot{\rho}y}{c}\right)^2 + U(y/\rho).$$

Such a form of the first integral is suggestive of a deeper relation with the Ermakov system. Indeed if one assumes $F(x) = x$, then clearly (2.9) reduces to the time-dependent linear harmonic oscillator equation,

$$\ddot{y} + \Omega^2(t)y = 0, \tag{2.10}$$

with

$$\Omega^2(t) = \omega^2(t) + \frac{1}{2}\left(\frac{\ddot{\omega}}{\omega} - \frac{3}{2}\left(\frac{\dot{\omega}}{\omega}\right)^2\right). \tag{2.11}$$

On the other hand elimination of y from (2.7) leads to

$$\ddot{\rho} + (\Omega^2(t) - w^2(t))\rho = 0,$$

which in view of (2.8) is equivalent to the equation

$$\ddot{\rho} + \Omega^2(t)\rho = \rho^{-3}. \tag{2.12}$$

We are thus led to the following proposition.

Proposition 2.1 *Given the second-order linear time-dependent differential equation*

$$\frac{d}{dt}\left(\frac{\dot{x}}{\omega(t)}\right) + \omega(t)x(t) = 0, \tag{2.13}$$

then under the transformation $x = y/\rho$, the equation is equivalent to the coupled system

$$\ddot{y} + \Omega^2(t)y = 0, \qquad \ddot{\rho} + \Omega^2(t)\rho = \rho^{-3}, \tag{2.14}$$

provided $\rho^2\omega = 1$, where $\Omega^2(t)$ is defined by (2.11). The solution $x = \sin\big(\int \omega(t)\, dt\big)$ of the time-dependent equation (2.13) can also be mapped to the solution of the (y, ρ) pair of equations.

As to the proof of the latter part of the above proposition we note that the solution of the Bartuccelli-Gentile equation is $x = \sin\big(\int \omega(t)\, dt\big)$. Consequently substituting $x = y/\rho$ we obtain $y = \rho \sin\big(\int 1/\rho^2\, dt\big)$, which is a solution of

$$1 = \frac{\rho}{\sqrt{\rho^2 - y^2}}(\dot{y}\rho - \dot{\rho}y). \tag{2.15}$$

Differentiating (2.15) one can easily obtain the TDHO and the Ermakov-Pinney equations.□

2.1 Generalized Ermakov-Pinney equations

By an unbalanced Ermakov system [1] is meant a coupled second-order nonlinear system of the form

$$\ddot{x} + \omega_1^2(t)x = x^{-3}f(y/x), \qquad \ddot{y} + \omega_2^2(t)y = y^{-3}g(x/y), \tag{2.16}$$

where f and g are arbitrary functions of their arguments and where in general $\omega_1 \neq \omega_2$. When $\omega_1 = \omega_2$ the system is said to be balanced. Systems of the former type were studied by Ray and Reid [38] and as a result (2.16) is also known as the Ermakov-Ray-Reid system.

A crucial property of the balanced Ermakov system (i.e., when $\omega_1 = \omega_2 = \omega(t)$) is that it possesses an invariant, given by

$$I_{ERR} = \frac{1}{2}(x\dot{y} - \dot{x}y)^2 + \int^{y/x} [uf(u) - u^{-3}g(u)]du. \tag{2.17}$$

The invariance of I_{ERR} can be directly verified by checking that $dI_{ERR}/dt = 0$ along the trajectories of the Ermakov-Ray-Reid system.

The generalized Ermakov-Pinney equation is an Ermakov system in two-dimension given by a pair of coupled nonlinear second-order differential equations of the form

$$\ddot{x} + \omega^2(t)x = \frac{1}{yx^2}f(y/x), \qquad \ddot{y} + \omega^2(t)y = \frac{1}{xy^2}g(x/y), \tag{2.18}$$

where f and g are once again arbitrary functions of their arguments. This coupled system possesses the Lewis-Ray-Reid invariant

$$I_{GE} = \frac{1}{2}(x\dot{y} - \dot{x}y)^2 + U(y/x), \tag{2.19}$$

where $U(y/x) = \int^{y/x} f(u)du + \int^{x/y} g(u)du$.

We can generalize this result to the time-dependent damped harmonic oscillator equation

$$\ddot{x} + P(t)\dot{x} + Q(t)x = 0, \tag{2.20}$$

in which case the invariant turns out to be

$$I_{dampedTD} = \frac{1}{2}\Big((x/\rho)^2 + (\dot{\rho}x - \rho\dot{x})^2 \exp\big(2\int_0^t P(t)\,dt\big)\Big) \tag{2.21}$$

with $\rho(t)$ satisfying the equation

$$\ddot{\rho} + P(t)\dot{\rho} + Q(t)\rho = \rho^{-3} \exp\big(-2\int_0^t P(t)\,dt\big). \tag{2.22}$$

The invariant $I_{dampedTD}$ of the damped time-dependent oscillator equation is called the Eliezer-Grey invariant.

Proposition 2.2 *The Eliezer-Grey invariant may be mapped to that of the time-dependent harmonic oscillator (TDHO) equation*

$$\frac{d}{dt}\left(\frac{\dot{x}}{\omega(t)}\right) + \omega(t)x(t) = 0$$

by setting $P = -\dot{w}/w$, $Q = w^2(t)$ *and* $\rho = 1$.

Proof. If we expand the time-dependent equation we can easily map it to damped TDHO provided $P = -\dot{\omega}/\omega$ and $Q = \omega^2(t)$. Hence we obtain

$$\exp\big(2\int_0^t P(t)\,dt\big) = \frac{1}{\omega^2}.$$

If we put $\rho = 1$, then from the Eliezer-Grey invariant we obtain the invariant

$$I = \frac{1}{2}\left(\frac{\dot{x}^2}{\omega^2(t)} + x^2\right). \quad \square$$

2.2 Ermakov-Ray-Reid system and Bartuccelli-Gentile construction

Let us consider the generalized time-dependent system

$$\frac{d}{dt}\left(\frac{\dot{x}}{\omega(t)}\right) + \omega(t)F(x) = 0,$$

where $F(x)$ is some nonlinear C^1 function of x, such that

$$F(x(t)) = \frac{1}{x^3}g(x^{-1}) + xf(x). \tag{2.23}$$

Proposition 2.3 *Given the second-order nonlinear time-dependent differential equation*

$$\frac{d}{dt}\left(\frac{\dot{x}}{\omega(t)}\right) + \omega(t)F(x) = 0,$$

if $x = y/\rho$, then this equation may be transformed to the coupled system:

$$\ddot{y} + \frac{1}{y^3}g\big(\frac{\rho}{y}\big) = 0, \qquad \ddot{\rho} + \frac{1}{\rho^3}f\big(\frac{y}{\rho}\big) = 0. \tag{2.24}$$

Proof. By direct calculation. □

Moreover, if we set $f = g = 1$ then an invariant can be readily found as

$$I = c_1\big(\frac{y}{\rho}\big)^2 + c_2\big(\frac{\rho}{y}\big)^2 + (y\dot{\rho} - \dot{y}\rho)^2.$$

Proposition 2.4 *Given the matrix second-order linear time-dependent differential equation*

$$\frac{d}{dt}\big(\Theta^{-1}\dot{X}\big) + \Theta X = 0, \tag{2.25}$$

where $\Theta = \Theta(t)$ is a differentiable function, such that its entries are all positive functions of time. This system has a first integral of motion given by

$$H = \frac{1}{2}\big(< \Theta^{-1}\dot{X}, \Theta^{-1}\dot{X} > + < X, X > \big) = \mathbb{E} = \textit{ constant }. \tag{2.26}$$

Proof. By explicit differentiation. □

Let

$$\Theta = \begin{pmatrix} \omega_1(t) & \omega_0(t) \\ \omega_0(t) & \omega_2(t) \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then the time-dependent matrix equation yields

$$\ddot{x} - \frac{1}{\Delta}\big((\dot{\omega_1}\omega_2 - \dot{\omega_0}\omega_0)\dot{x}) + (\dot{\omega_0}\omega_1 - \dot{\omega_1}\omega_0)\dot{y}\big) + (\omega_1^2 + \omega_2^2)x + \omega_0(\omega_1 + \omega_2)y = 0, \tag{2.27}$$

$$\ddot{y} - \frac{1}{\Delta}\big((\dot{\omega_2}\omega_1 - \dot{\omega_0}\omega_2)\dot{y}) + (\dot{\omega_0}\omega_2 - \dot{\omega_2}\omega_0)\dot{x}\big) + (\omega_2^2 + \omega_2^2)x + \omega_0(\omega_1 + \omega_2)x = 0, \tag{2.28}$$

where $\Delta(t) = \omega_1\omega_2 - \omega_0^2$. We consider now a special case.
Suppose $\omega_0 = 0$ and $\omega_1 \neq \omega_2$ then we obtain two decoupled equations of the form deduced earlier by Bartuccelli and Gentile, *viz*

$$\frac{d}{dt}\big(\frac{\dot{x}}{\omega_1}\big) + \omega_1 x = 0, \qquad \frac{d}{dt}\big(\frac{\dot{y}}{\omega_2}\big) + \omega_2 y = 0. \tag{2.29}$$

Finally if we define $\omega_2 = i\omega_1 \equiv \omega$ and $z = x + iy$, then equations (2.27) and (2.28) can be expressed as the following single complex differential equation

$$\frac{d}{dt}\big(\frac{\dot{z}}{\omega}\big) - i\omega z = 0. \tag{2.30}$$

Proposition 2.5 *The complex version of the Bartuccelli-Gentile equation has a first integral of motion given by*

$$I_{complex} = \frac{1}{2}\left(\frac{\dot{z}}{\omega}\right)^2 - iz^2. \tag{2.31}$$

Proof. By explicit differentiation we may obtain the desired first integral. □

2.3 Integrable coupled Milne-Pinney type dissipative systems

The study of coupled nonlinear ordinary differential equations of Ermakov-type originated in 1880 and in modern days the classical Ermakov-Pinney system was extended by Ray-Reid [37]. There is a class of Ermakov systems [2] given by

$$\ddot{q} + \omega^2(t)q = \frac{1}{q^3}f(q/p), \qquad \ddot{p} + \omega^2(t)p = \frac{1}{p^3}g(p/q), \tag{2.32}$$

where $\omega(t)$, f and g are essentially arbitrary functions of their arguments. In this case the Lewis-Ray-Reid invariant is

$$I = \frac{1}{2}(q\dot{p} - \dot{q}p)^2 - \int^{q/p} \big(u^{-3}f(u) - u\, g(u)\big)^2\, du. \tag{2.33}$$

We propose to study, in this section, the following time-dependent generalization of (2.32)

$$\frac{d}{dt}\big(\frac{\dot{q}}{\omega(t)}\big) + \omega(t)q = \frac{\omega(t)}{q^3}f(q/p), \qquad \frac{d}{dt}\big(\frac{\dot{p}}{\omega(t)}\big) + \omega(t)p = \frac{\omega(t)}{p^3}g(p/q). \tag{2.34}$$

In the following proposition an invariant of this system of coupled equation is provided.

Proposition 2.6 *The first integral of the coupled integrable Bartuccelli-Gentile equation of type (2.34) is*

$$I = \frac{1}{2}\frac{(\dot{q}p - q\dot{p})^2}{\omega(t)^2} + \int^{p/q} \big(uf(u^{-1}) - \frac{1}{u^3}g(u)\big)\, du, \tag{2.35}$$

where $\omega(t)$ is a differentiable function.

We can extend this result to a more general case. Consider the following generalized Ermakov system

$$\frac{d}{dt}\big(\frac{\dot{q}}{\omega(t)}\big) + \omega(t)q = \omega(t)q^m p^n f(q/p), \qquad \frac{d}{dt}\big(\frac{\dot{p}}{\omega(t)}\big) + \omega(t)p = \omega(t)q^n p^m g(p/q), \tag{2.36}$$

where $\omega(t)$ is a differentiable positive function.

Proposition 2.7 *The system (2.36) has a first integral of motion given by*

$$I = \frac{1}{2}\frac{(\dot{q}p - q\dot{p})^2}{\omega(t)^2} + \int^{p/q} \big(u^{n+1}f - \frac{1}{u^{n+3}}g\big)du, \tag{2.37}$$

where $m = -(n+3)$ and $u = p/q$.

2.3.1 Generalized Ince equation and coupled Bartuccelli-Gentile equation

Consider the class of second-order homogeneous differential equations

$$\frac{d^2p}{dt^2} + \frac{\alpha + \beta\cos 2t + \gamma\cos 4t}{(1 + a\cos 2t)^2}p = 0, \qquad \text{where } |a| < 1. \tag{2.38}$$

It is a four parameter family of Hill's equation which has been christened as the Ince equation by Magnus and Winkler [32]. A subclass of this system was studied by Athorne [3], and is given by

$$\frac{d^2p}{dt^2} + \big(1 + \frac{\alpha'}{(1 + a\cos 2t)^2}\big)p = 0. \tag{2.39}$$

One must note that $q(t) = B(1 + a\cos 2t)^{1/2}$ is a solution of the Ermakov-Pinney equation. It has been shown by Athorne that this equation can be replaced by the following coupled nonlinear equations of Ermakov type, namely

$$\ddot{p} + p = -\frac{\alpha' B^4}{q^4}p, \qquad \ddot{q} + q = \frac{\delta}{q^3}. \tag{2.40}$$

We propose to analyze a time-dependent generalization of (2.39) and consider the following generalization of the two-parameter version of the Ince equation

$$\frac{d}{dt}\big(\frac{\dot{p}}{\omega(t)}\big) + \big(1 + \frac{\alpha'}{(1 + acos\, 2t)^2}\big)\omega(t)p = 0. \tag{2.41}$$

This equation may also be replaced by the pair of equations:

$$\frac{d}{dt}\big(\frac{\dot{p}}{\omega(t)}\big) + \omega(t)p = -\frac{\alpha' B^4}{q^4}p, \qquad \frac{d}{dt}\big(\frac{\dot{q}}{\omega(t)}\big) + \omega(t)q = \frac{\omega(t)\delta}{q^3}, \tag{2.42}$$

and possesses a first integral which, in this case, is given by

$$I = \frac{1}{2}\Big[\frac{1}{\omega^2(t)}(q\dot{p} - \dot{q}p)^2 + \big(\frac{p}{q}\big)^2\Big], \tag{2.43}$$

as may easily be checked.

3 A Simple Algorithmic Method to Compute First Integrals of the Emden-Fowler Family

We can apply this straight forward scheme to compute the first integrals of the Lane-Emden equation. Consider the equation

$$y'' + p(x)y' = Ke^{-2F}y^n,$$

where $\int^x F\,dx = p(x)$. We can rewrite this equation as

$$(y'e^F)' = Ke^{-F}y^n,$$

from the prescription of Bartuccelli and Gentile one can immediately obtain the first integral

$$I = \frac{1}{2}(y'e^F)^2 - K/(n+1)y^{n+1},$$

where $\omega(x) = e^{-F}$.

We modify the preceding scheme to incorporate the Emden-Fowler equation. This will now be described.

Proposition 3.1 *The second-order ODE* $y'' + dx^r y^s = 0$ *with* $d > 0$ *and* $s \neq 1$ *admits a first integral of the form*

$$I = \frac{1}{2}(y'x - y)^2 + V(x,y),$$

where $V(x,y) = dx^{r+2}y^{s+1}/(s+1)$ *and* $r + s = -3$.

Proof. Setting $dI/dx = 0$ and using the given equation lead to

$$V_x = -dx^{r+1}y^{s+1} \text{ and } V_y = dx^{x+2}y^s,$$

respectively. The consistency of these partial derivatives then yields the condition $r+s = -3$ and $V(x,y)$ has the stated form. □

Remark 3.1 If we compare with the Bartucelli-Gentile construction we can readily see here $\omega(x) = x^{-1}$, furthermore there is a shift to define the first integral I of the Emden-Fowler equation. The nature of $\omega(x)$ is fixed for the entire family of the Emden-Fowler systems.

Proposition 3.2 *The second-order ODE* $y'' = \gamma_1^2 y + e^{-(2\gamma_1-\gamma_2)x}h(y)$ *admits a first integral of the form*

$$I = \frac{1}{2}(y' - \gamma_1 y)^2 e^{2\gamma_1 x} - e^{\gamma_2 x}\int^y h(u)du$$

provided $h(y) = y^{-(1+\gamma_2/\gamma_1)}$.

Proof. By an explicit calculation. □

Example 3.1 We can apply this scheme to compute the first integrals of the following Lane-Emden-Fowler equation [17]

$$y'' + \frac{k_1}{x}y' = \lambda x^{k_2}y^n. \tag{3.1}$$

This equation has been the subject of study by Rosenau [39] for its solution. It is worth mentioning here that from this equation one obtains immediately a generalization of Chandrasekhar's homology theorem. We can rewrite this equation as

$$(y'x + (k_1 - 1)x)' = x^{-1}\lambda x^{k_2+2}y^n$$

and from our prescription one can immediately obtain the first integral

$$I = \frac{1}{2}(y'x + (k_1 - 1)x)^2 - \frac{\lambda}{n+1}x^{k_2+2}y^{n+1},$$

where $(n+1)(k_1 - 1) = \lambda(k_2 + 2)$.

We present a slightly different method to compute the first integrals for the Emden-Fowler equation $y'' + dx^r y^s = 0$ for other sets of values of (r,s) than given in the previous section.

Proposition 3.3 *The Emden-Fowler equation* $y'' + dx^r y^s = 0$ *with* $d > 0$ *and* $r \neq 1$ *admits a first integral of the form* $I = y'(y'x - y) + V(x,y)$, *where* $V(x,y) = dx^{r+1}y^{s+1}/(r+1)$ *and* $2r + s = -3$.

Proof. It is clear that $(y'x - y)' = y''x$. We can recast the equation $y'' + dx^r y^s = 0$ as $(y'x - y)' + dx^{r+1}y^s = 0$. We compute $\frac{d}{dx}(y'(y'x - y))$ using the Emden-Fowler equation and equate it with the derivative of $V(x,y)$. This immediately yields the condition $2r + s = -3$. □

3.1 The (Generalized) Lane-Emden equation

Consider the Lane-Emden equation

$$y'' + 2\frac{y'}{x} + y^5 = 0.$$

One can rewrite this equation in either of the following two different forms, namely

$$(x^2y')' + x^2y^5 = 0, \qquad (y'x + y)' + xy^5 = 0.$$

Once again we use these two equations to compute $\big(x^2y'(y'x+y)\big)'$. Finally equating with a potential $V(x, y) = Kx^ny^m$ we obtain the first integral of the Lane-Emden equation

$$I = x^3(y')^2 + x^2yy' + \frac{1}{3}x^3y^6.$$

We can extend this scheme to more complicated systems. Let us compute the first integrals of the above Emden-Fowler equation for different values of (r, s). The generalized Lane-Emden equation as proposed by Goenner (3.1) in [17, 18] can be expressed either as

$$(y'x + (k_1 - 1)x)' = \lambda x^{k_2+1}y^n \quad \text{or} \quad (y'x^{k_1})' = \lambda x^{k_2+k_1}y^n.$$

Using these two forms we obtain

$$\frac{d}{dx}\big(y'x^{k_1}(y'x + (k_1 - 1)x)\big) = 2\lambda x^{k_1+k_2+1}y^ny' + \lambda(k_1 - 1)x^{k_1+k_2}y^{n+1}.$$

If we take $V = -2\lambda/(n+1)y^{n+1}x^{\beta}$ we obtain $\beta = k_1 + k_2 + 1 = (k_1 - 1)(n + 1)/2$. Thus we can get the first integral for equation (3.1)

$$I = y'x^{k_1}(y'x + (k_1 - 1)x) - 2\frac{\lambda}{n+1}y^{n+1}x^{(k_1-1)(n+1)/2}, \ n \neq -1.$$

Incidentally this first integral was first derived by Crespo Da Silva [12]. In this way we can find new first integrals for the Emden-Fowler type systems.

3.2 First integrals for other type of equations

One can extend the scheme to compute the first integral of more complicated equation with more terms, such as

$$y'' + \frac{k_1}{x}y' + \frac{k_3}{x^2}y = \lambda x^{k_2}y^n. \tag{3.2}$$

We then use our old trick to club the first two terms and express them either as

$$(y'x + (k_1 - 1)x)' = \lambda x^{k_2+1}y^n - \frac{k_3}{x}y \quad \text{or} \quad (y'x^{k_1})' = \lambda x^{k_2+k_1}y^n - k_3x^{k_1-1}y.$$

Once again we differentiate $(y'x_1^k)(y'x + (k_1 - 1)x)$ and equate it with the derivative of V and obtain the first integral of (3.2) in the form

$$I = y'x^{k_1}(y'x + (k_1 - 1)x) - 2\frac{\lambda}{n+1}y^{n+1}x^{(k_1-1)(n+1)/2} + k_3y^2x^{k_1-1}.$$

For an isothermal gaseous sphere, Emden studied also the equation

$$xy'' + 2y' + xe^{ny} = 0.$$

We can also compute the first integral from our method. It is easy to see that this equation can be rewritten either $(x^2y')' + x^2e^{ny} = 0$ or $(xy' + y)' + xe^{ny} = 0$. Again using our scheme we obtain the first integral

$$I = (x^2y'(xy' + y) + \frac{1}{3}x^3e^{ny}, \qquad \text{for} \quad n = 6.$$

Hence we have shown in this section how one can generalize the Barucelli-Gentile scheme to encompass various classes of Emden-Fowler systems.

4 Conclusion

In this paper we have examined the connection between a time-dependent second-order ODE due to Bartuccelli and Gentile which was derived by modifying the equation of a linear harmonic oscillator and the Ermakov-Pinney system of ODEs. It is interesting to note that though the system (1.8) can be generalized further to the following

$$\dot{x} = \omega(x, y, t)\frac{\partial G}{\partial y}, \quad \dot{y} = -\omega(x, y, t)\frac{\partial G}{\partial x},$$

with $G = G(x, y)$ and one can easily verify that $G(x, y)$ is an invariant, the solution of the above system is in general not known in closed form unlike that of (1.8) which can be obtained explicitly. This is the main reason for our interest in the Bartuccelli and Gentile construction. It is found that by a simple rational transformation of the dependent variable one can easily extract the well known Ermakov-Lewis invariant. Furthermore a matrix formulation is also considered and a decoupled version of the Bartuccelli-Gentile equation is obtained. Finally we present a simple scheme to compute the first integrals of several equations belonging to the Emden-Fowler and Lane-Emden class.

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Extremal Mild Solutions for Finite Delay Differential Equations of Fractional Order in Banach Spaces

Kamaljeet * and D. Bahuguna

Department of Mathematics & Statistics
Indian Institute of Technology Kanpur, Kanpur – 208016, India.

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Abstract: In this paper, we study the existence and uniqueness of extremal mild solutions for finite delay differential equations of fractional order in Banach spaces with the help of the monotone iterative technique based on lower and upper solutions. This technique uses the iterative procedure starting from a pair of ordered lower and upper solutions to obtain the extremal mild solutions. We also use the theory of fractional calculus, semigroup theory and measures of noncompactness to obtain the results. An example is presented to illustrate the main result.

Keywords: *fractional delay differential equations; semigroup theory; monotone iterative technique; Kuratowskii measures of noncompactness.*

Mathematics Subject Classification (2010): 34A08, 34G20, 34K30.

1 Introduction

In this paper, our aim is to study the existence of extremal mild solutions for the following finite delay differential equations of fractional order in an ordered Banach space X of the form:

$$\begin{cases} {}^cD^\alpha x(t) &= Ax(t) + f(t, x_t), \quad t \in J = [0, b], \\ x_0(\nu) &= \phi(\nu), \quad \nu \in [-a, 0], \end{cases} \tag{1}$$

where state $x(.)$ takes value in the Banach space X endowed with norm $\|.\|$; ${}^cD^\alpha$ is the Caputo fractional derivative of order α, $0 < \alpha < 1$; $A : D(A) \subset X \to X$ is a closed linear densely defined operator; A is an infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X. The function $f : J \times D \to X$ is given nonlinear function, here $D = C([-a, 0], X)$. If $x : [-a, b] \to X$ is a continuous function, then x_t denotes the

* Corresponding author: mailto:kamaljeetp2@gmail.com

function in D defined as $x_t(\nu) = x(t+\nu)$ for $\nu \in [-a, 0]$, here $x_t(.)$ represents the time history of the state from the time $t-a$ up to the present time t, and $\phi(.) \in D$.

Fractional calculus is generalization of ordinary differential equations and integration to arbitrary non integer orders. The subject is as old as differential calculus when it was invented by Newton and Leibnitz in the seventieth century. It has proved a valuable tool to describe many phenomena, arising in Engineering, Physics, Economics and Science. Indeed, we can find numerous applications in electrochemistry, control, porous media, electromagnetic, etc. (see [1–8]). Hence, in recent years, the researchers have paid more attention to fractional differential equations. In [9–19], the authors have discussed the existence of solutions of delay differential equations with or without fractional order.

This work is motivated by works [24, 26]. In this paper, we study the existence of extremal mild solutions of delay system (1) by using the monotone iterative technique. In the recent years, the monotonic iterative technique is also used to deal with fractional differential equations (see, for instance, [20–26] and references therein). The monotone iterative technique based on lower and upper solutions helps us to solve the differential equation with various kinds of boundary conditions. This technique uses the iterative procedure starting from a pair of ordered lower and upper solutions. The sequences of iterations uniformly converge to the extremal mild solutions between the lower and upper solutions. Further we prove the uniqueness of the solutions of the system. We also use the theory of fractional calculus, semigroup theory and measures of noncompactness to obtain the results. To the best of our knowledge, up to now, no work has been reported on finite delay differential equations of fractional order by using the monotone iterative technique.

The rest of paper is organized as follows. In the next Section we give some basic definitions and notations. In Section 3, we study the existence of extremal mild solution of delay system (1) and uniqueness of solutions of the system. Finally, in Section 4, we present an example to illustrate our results.

2 Preliminaries

In this section, we introduce some basic definitions and notations which are used throughout this paper. We denote by X a Banach space with the norm $\|.\|$ and $A : D(A) \to X$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$. This means that there exists $M \geq 1$ such that $\sup_{t\in J} \|T(t)\| \leq M$.

Definition 2.1 (see [8]) The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function f is given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0,$$

where Γ is the gamma function, and $f \in L^1([0, b], X)$.

Definition 2.2 (see [8]) The fractional derivative of order $0 \leq n-1 < \alpha < n$ in the Caputo sense is defined as

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0,$$

where f is an n-times continuous differentiable function and Γ is a gamma function.

If f is an abstract function with values in a Banach space X, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Let $P = \{y \in X : y \geq \theta\}$ (θ is a zero element of X) be positive cone in X which defines a partial ordering in X by $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$ we write $x < y$. The cone P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$ and P is said to be fully regular if $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots,\quad \sup_n \|x_n\| < \infty$ implies $\|x_n - x\| \to 0$ as $n \to \infty$ for some $x \in X$. Clearly full regularity of P implies the normality of P.

Since $C([-a,b],X)$ is the Banach space of all continuous X-valued functions on interval $[-a,b]$ with norm $\|.\|_C = \sup_{t\in[-a,b]} \|x(t)\|$. Then $C([-a,b],X)$ is an ordered Banach space whose partial ordering $\leq$ reduced by positive cone $P_C = \{x \in C([-a,b],X) \mid x(t) \geq \theta,\ t \in [-a,b]\}$. Similarly D is also an ordered Banach space with norm $\|.\|_D = \sup_{t\in[-a,0]} \|x(t)\|$ and partial ordering $\leq$ reduced by $P_D = \{x \in C([-a,0],X) \mid x(t) \geq \theta, t \in [-a,0]\}$. P_C and P_D are also normal cones with the same normal constant N. For $x,y \in C(I,X)$ with $x \leq y$, denote the ordered interval $[x,y] = \{z \in C(I,X),\ x \leq z \leq y\}$ in $C(I,X)$, and $[x(t),y(t)] = \{u \in X | x(t) \leq u \leq y(t)\}$ $(t \in I)$ in X, here $I = [-a,b]$ or $I = [-a,0]$.

Let $C^\alpha([-a,b],X) = \{u \in C([-a,b],X) :\ {}^cD^\alpha u$ exists on $[0,b]$, ${}^cD^\alpha u|_{[0,b]} \in C([0,b],X)$ and $u(t) \in D(A)$ for $t \geq 0\}$. An abstract function $u \in C^\alpha([-a,b],X)$ is called a solution of (1) if $u(t)$ satisfies equation (1).

Definition 2.3 (see [26]) The function $y \in C^\alpha([-a,b],X)$ is called a lower solution of the problem (1) if it satisfies the following inequalities

$$\begin{cases} {}^cD^\alpha y(t) \leq Ay(t) + f(t,y_t), \quad t \in I = [0,b], \\ y_0(\nu) \leq \phi(\nu), \quad \nu \in [-a,0]. \end{cases} \tag{2}$$

If all inequalities of (2) are reversed, we call $y(\cdot)$ an upper solution of the problem (1).

Lemma 2.1 *If h satisfies a uniform Hölder condition, with exponent $\beta \in (0,1]$, then the unique solution of the linear initial value problem*

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + h(t), \quad t \in J, \\ x(0) = x_0 \in X, \end{cases} \tag{3}$$

is given by

$$x(t) = U(t)x_0 + \int_0^t (t-s)^{\alpha-1} V(t-s)h(s))ds, \quad t \in J, \tag{4}$$

where

$$U(t) = \int_0^\infty \psi_\alpha(\vartheta)T(t^\alpha\vartheta)d\vartheta, \quad V(t) = \alpha\int_0^\infty \vartheta\psi_\alpha(\vartheta)T(t^\alpha\vartheta)d\vartheta, \tag{5}$$

$$\psi_\alpha(\vartheta) = \frac{1}{\alpha}\vartheta^{-1-1/\alpha}\rho_\alpha(\vartheta^{-1/\alpha}).$$

Note that $\psi_\alpha(\vartheta)$ satisfies the condition of a probability density function defined on $(0,\infty)$, that is $\psi_\alpha(\vartheta) \geq 0$, $\int_0^\infty \psi_\alpha(\vartheta)d\vartheta = 1$ and $\int_0^\infty \vartheta\psi_\alpha(\vartheta) = \frac{1}{\Gamma(1+\alpha)}$. Also the term $\rho_\alpha(\vartheta)$ is defined as

$$\rho_\alpha(\vartheta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\vartheta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \vartheta \in (0,\infty).$$

Definition 2.4 A function $x(.) \in C([-a,b],X)$ is said to be a mild solution of the system (1) if $x(t) = \phi(t)$ on $[-a,0]$ and the following integral equation is satisfied:

$$x(t) = U(t)\phi(0) + \int_0^t (t-s)^{\alpha-1} V(t-s) f(s, x_s) ds, \quad t \in J, \tag{6}$$

where $U(t)$ and $V(t)$ are defined by (5).

Lemma 2.2 *The following properties are valid:*

(i) for fixed $t \geq 0$ and any $x \in X$, we have

$$\|U(t)x\| \leq M\|x\|, \quad \|V(t)x\| \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|x\| = \frac{M}{\Gamma(\alpha)}\|x\|.$$

(ii) The operators are $U(t)$ and $V(t)$ are strongly continuous for all $t \geq 0$.

(iii) If $S(t)(t > 0)$ is a compact semigroup in X, then $U(t)$ and $V(t)$ are norm-continuous in X for $t > 0$.

(iv) If $S(t)(t > 0)$ is a compact semigroup in X, then $U(t)$ and $V(t)$ are compact operators in X for $t > 0$.

Definition 2.5 A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called a positive semigroup, if $T(t)x \geq \theta$ for all $x \geq \theta$ and $t \geq 0$.

Now we recall the definition of Kuratowski's measure of noncompactness, which is used in the next section to study the existence of extremal mild solutions for finite delay differential equation of fractional order.

Definition 2.6 (see [27,28]) Let X be a Banach space and $\mathcal{B}(X)$ be family of bounded subset of X. Then $\mu : \mathcal{B}(X) \to \mathbb{R}^+$, defined by

$$\mu(S) = \inf\{\delta > 0 \ : S \text{ admits a finite cover by sets of diameter } \leq \delta \ \},$$

where $S \in \mathcal{B}(X)$, is called the Kuratowski measure of noncompactness.
Clearly $0 \leq \mu(S) < \infty$.

We need to use the following basic properties of the μ measure.

Lemma 2.3 (see [27, 28]) *Let S, S_1 and S_2 be bounded sets of a Banach space X. Then:*

(i) $\mu(S) = 0$ if and only if S is relatively compact set in X;

(ii) $\mu(S_1) \leq \mu(S_2)$ if $S_1 \subset S_2$;

(iii) $\mu(S_1 + S_2) \leq \mu(S_1) + \mu(S_2)$;

(iv) $\mu(\lambda S) \leq |\lambda|\mu(S)$ for any $\lambda \in \mathbb{R}$.

Lemma 2.4 (see [27,28]) *If $W \subset C([a,b],X)$ is bounded and equicontinuous on $[a,b]$, then $\mu(W(t))$ is continuous for $t \in [a,b]$ and*

$$\mu(W) = \sup\{\mu(W(t)), t \in [a,b]\}, \text{ where } W(t) = \{x(t) : x \in W\} \subseteq X.$$

Remark 2.1 (see [27,28]) If B is a bounded set in $C([a,b],X)$, then $B(t)$ is bounded in X, and $\mu(B(t)) \leq \mu(B)$.

Lemma 2.5 (see [27,28]) *Let $B = \{u_n\} \subset C(I,X)(n = 1,2,\dots)$ be a bounded and countable set. Then $\mu(B(t))$ is Lebesgue integrable on I, and*

$$\mu\left(\left\{\int_I u_n(t)dt \mid n = 1,2,\dots\right\}\right) \leq 2\int_I \mu(B(t))dt, \text{ here } I = [a,b]. \tag{7}$$

3 Main Result

In this section, we prove the existence of extremal mild solutions of the problem (1) and then prove the uniqueness in the next theorem.

Theorem 3.1 *Let X be an ordered Banach space, whose positive cone P is normal with normal constant N and $T(t)(t \geq 0)$ be a positive operator. Also assume that the Cauchy delay problem (1) has a lower solution $x^{(0)} \in C([-a,b],X)$ and an upper solution $y^{(0)} \in C([-a,b],X)$ with $x^{(0)} \leq y^{(0)}$. The system (1) has minimal and maximal mild solutions between $x^{(0)}$ and $y^{(0)}$ if the following assumptions (H1)-(H4) are satisfied:*

(H1) The function $f : J \times D \to X$ is such that for $t \in J$, the function $f(t,.) : D \times X \to X$ is continuous and for all $\varphi \in D$, the function $f(.,\varphi)$ is strongly measurable.

(H2) For any $t \in [0,b]$, the function $f(t,.) : D \to X$ satisfies the following

$$f(t,\varphi_1) \leq f(t,\varphi_2),$$

where $\varphi_1, \varphi_2 \in D$ with $x_t^0 \leq \varphi_1 \leq \varphi_2 \leq y_t^0$.

(H3) There exists a constant $L \geq 0$ such that

$$\mu(f(t,E)) \leq L\left[\sup_{-a\leq\nu\leq 0} \mu(E(\nu))\right],$$

for a.e. $t \in J$ and $E \subset D$, where $E(\nu) = \{\varphi(\nu) : \varphi \in E\}$.

(H4) $K = \frac{2MLb^\alpha}{\Gamma(\alpha+1)} < 1$,

Proof. Let $B = [x^{(0)}, y^{(0)}] = \{x \in C([-a,b],X) \mid x^{(0)} \leq x \leq y^{(0)}\}$. We define a map $Q : B \to C([-a,b],X)$ by

$$Qx(t) = \begin{cases} U(t)\phi(0) + \int_0^t (t-s)^{\alpha-1} V(t-s) f(s,x_s)ds, & t \in [0,b], \\ \phi(t), & t \in [-a,0]. \end{cases} \tag{8}$$

By (H2) and for any $x \in B$, we have that

$$f(t,x_t^{(0)}) \leq f(t,x_t) \leq f(t,y_t^{(0)}).$$

By the normality of the positive cone P, there exists a constant $k > 0$ such that

$$\|f(t,x_t)\| \leq k, \quad x \in B. \tag{9}$$

Clearly $Q : B \to C([-a, b], X)$ is continuous. Let $x, y \in B$ and $x \leq y$, then $x(t) \leq y(t),\ t \in [-a, b]$. Therefore, for any $t \in [0, b]$, $x_t \leq y_t$ in the ordered Banach space D. Now by positivity of operators $U(t)$ and $V(t)$, (H2), we have

$$Qx \leq Qy. \tag{10}$$

For showing $x^{(0)} \leq Qx^{(0)}$ and $Qy^{(0)} \leq y^{(0)}$, we let ${}^cD^\alpha x^{(0)}(t) = Ax^{(0)}(t) + \xi(t),\ \ t \in J$, then by Definition 2.3, Lemma 2.1 and the positivity of $U(t)$ and $V(t)$ for $t \in J$, we get that

$$\begin{aligned} x^{(0)}(t) =& U(t)x^{(0)}(0) + \int_0^t (t-s)^{\alpha-1} V(t-s)\xi(s)ds \\ &\leq U(t)\phi(0) + \int_0^t (t-s)^{\alpha-1} V(t-s) f(s, x_s^{(0)})ds, \quad t \in J \end{aligned}$$

and also $x^{(0)}(t) \leq \phi(t) = Qx^{(0)}(t),\ t \in [-a, 0]$. Thus $x^{(0)}(t) \leq Qx^{(0)}(t),\ t \in [-a, b]$. Similarly we can prove that $Qy^{(0)}(t) \leq y^{(0)}(t),\ t \in [-a, b]$. Thus $Q : B \to B$ is an increasing monotonic operator. Now we define the sequences as

$$x^{(n)} = Qx^{(n-1)} \text{ and } y^{(n)} = Qy^{(n-1)}, \quad n = 1, 2, \ldots, \tag{11}$$

and from (10), we have

$$x^{(0)} \leq x^{(1)} \leq \ldots x^{(n)} \leq \ldots \leq y^{(n)} \leq \ldots \leq y^{(1)} \leq y^{(0)}. \tag{12}$$

Now we show that Q is equicontinuous on $[-a, b]$. For this, we let any $x \in B$ and $t_1, t_2 \in [-a, b]$ with $t_1 \leq t_2$. First we take $t_1,\ t_2 \in [-a, 0]$, then $\|Qx(t_2) - Qx(t_1)\| = \|\phi(t_2) - \phi(t_1)\| \to 0$ as $\phi(.)$ is continuous and $t_1 \to t_2$ independent of $x \in B$. Further, if $t_1, t_2 \in J$ with $t_1 \leq t_2$ and by (9), then we have that

$$\begin{aligned} \|Qx(t_2) - Qx(t_1)\| \leq& \|U(t_2)\phi(0) - U(t_1)\phi(0)\| \\ &+ \| \int_0^{t_1} (t_2-s)^{\alpha-1} \left[V(t_2-s) - V(t_1-s)\right] f(s, x_s)ds\| \\ &+ \| \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] V(t_1-s) f(s, x_s)ds\| \\ &+ \int_{t_1}^{t_2} (t-s)^{\alpha-1} V(t_2-s) f(s, x_s)ds \\ \leq& \|U(t_2)\phi(0) - U(t_1)\phi(0)\| \\ &+ k \int_0^{t_1} (t_2-s)^{\alpha-1} \|V(t_2-s) - V(t_1-s)\|ds \\ &+ \frac{Mk}{\Gamma(\alpha)} \int_0^{t_1} |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|ds \\ &+ \frac{Mk}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t-s)^{\alpha-1}ds \\ =& I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{13}$$

where

$$
\begin{aligned}
I_1 =&\|U(t_2)\phi(0) - U(t_1)\phi(0)\|, \\
I_2 =&k\int_0^{t_1}(t_2-s)^{\alpha-1}\|V(t_2-s)-V(t_1-s)\|\,ds, \\
I_3 =&\frac{Mk}{\Gamma(\alpha)}\int_0^{t_1}|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|\,ds, \\
I_4 =&\frac{Mk}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t-s)^{\alpha-1}ds.
\end{aligned}
$$

For any $\epsilon \in (0, t_1)$, we have

$$
\begin{aligned}
I_2 \leq& k\int_0^{t_1-\epsilon}(t_2-s)^{\alpha-1}\|V(t_2-s)-V(t_1-s)\|ds \\
&+ k\int_{t_1-\epsilon}^{t_1}(t_2-s)^{\alpha-1}\|V(t_2-s)-V(t_1-s)\|ds \\
\leq& k\int_0^{t_1-\epsilon}(t_2-s)^{\alpha-1}ds. \sup_{s\in[0,t_1-\epsilon]}\|V(t_2-s)-V(t_1-s)\| \\
&+ \frac{2Mk}{\Gamma(\alpha)}\int_{t_1-\epsilon}^{t_1}(t_2-s)^{\alpha-1}ds.
\end{aligned} \tag{14}
$$

By Lemma 2.2, we get that $I_2 \to 0$ as $t_1 \to t_2$ and $\epsilon \to 0$ independent of $x \in B$. From expression of I_1, I_3 and I_4, we can easily show that $I_2 \to 0$, $I_3 \to 0$ and $I_4 \to 0$ as $t_2 \to t_1$ independent of $x \in B$. Therefore $\|Qx(t_2) - Qx(t_1)\| \to 0$ as $t_1 \to t_2$ independent of $x \in B$. Thus for $t_1, t_2 \in [-a, b]$ with $t_1 \leq t_2$, we have that $\|Qx(t_2) - Qx(t_1)\| \to 0$ as $t_1 \to t_2$ independent of $x \in B$. Therefore $Q(B)$ is equicontinuous on $[-a, b]$.

From (8), we must have $x^{(n)}(t) = y^{(n)}(t) = \phi(t)$, $\quad n = 1, 2, \ldots,\ t \in [-a, 0]$. So $x^{(n)} \to \phi$ and $y^{(n)} \to \phi$ on $[-a, 0]$. Let $S = \{x^{(n)}\}_{n=1}^{\infty}$. The normality of positive cone P and (12) imply that S is bounded. Note that $\mu(S(t)) = 0$, for any $t \in [-a, 0]$. Since $S(t) = \{x^{(1)}(t)\} \cup \{Q(S)(t)\}$ for any $t \in J$, then $\mu(S(t)) = \mu(Q(S)(t))$ for any $t \in J$. By using (H3), (8), (11) and for $t \in J$, we have that

$$
\begin{aligned}
\mu(S(t)) =&\mu\left(\left\{U(t)\phi(0) + \int_0^t(t-s)^{\alpha-1}V(t-s)f(s,x_s^{(n)})ds\right\}_{n=1}^{\infty}\right) \\
\leq&\mu\left(\left\{\int_0^t(t-s)^{\alpha-1}V(t-s)f(s,x_s^{(n)})ds\right\}_{n=1}^{\infty}\right) \\
\leq&\frac{2M}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mu\left(\left\{f(s,x_s^{(n)})\right\}_{n=1}^{\infty}\right)ds \\
\leq&\frac{2M}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}L\sup_{-a\leq\nu\leq 0}\mu\left(\left\{x^{(n)}(s+\nu)\right\}_{n=1}^{\infty}\right)ds
\end{aligned}
$$

$$\leq \frac{2ML}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{0\leq\tau\leq s} \mu\left(\left\{x^{(n)}(\tau)\right\}_{n=1}^{\infty}\right) ds$$
$$\leq \frac{2ML}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds. \sup_{0\leq\tau\leq b} \mu\left(\left\{x^{(n)}(\tau)\right\}_{n=1}^{\infty}\right)$$
$$\leq \frac{2MLb^{\alpha}}{\Gamma(\alpha+1)} \sup_{0\leq\tau\leq b} \mu\left(\left\{x^{(n)}(\tau)\right\}_{n=1}^{\infty}\right). \tag{15}$$

Since $\left\{Qx^{(n)}\right\}_{n=0}^{\infty}$, i.e. $\left\{x^{(n)}\right\}_{n=1}^{\infty}$, are equicontinuous on $[-a,b]$ and $\mu(S(t))=0$, for any $t\in[-a,0]$, then Lemma 2.4 and inequality (15) imply that

$$\mu(S) \leq \frac{2MLb^{\alpha}}{\Gamma(\alpha+1)}\mu\left(\left\{x^{(n)}\right\}_{n=1}^{\infty}\right) = K\mu(S). \tag{16}$$

Since $K<1$ as given in (H4), this implies that $\mu(S)=0$, i.e. $\mu(\{x^{(n)}\}_{n=1}^{\infty})=0$. Thus the set $\{x^{(n)} : n\geq 1\}$ is relatively compact in B. So we have that the sequence $\{x^{(n)}\}$ has a convergent subsequence in B. In view of (12), we can easily show that $\{x^{(n)}\}$ itself is convergent in B. So there exists $\underline{x}\in B$ such that $x^{(n)}\to\underline{x}$ as $n\to\infty$. By (8)and (11), we have that

$$x^{(n)}(t) = \begin{cases} U(t)\phi(0) + \int_0^t (t-s)^{\alpha-1} V(t-s) f(s, x_s^{(n-1)}) ds, & t\in[0,b], \\ \phi(t), & t\in[-a,0]. \end{cases} \tag{17}$$

Taking $n\to\infty$ and Lebesgue dominated convergence theorem, we have that

$$\underline{x}(t) = \begin{cases} U(t)\phi(0) + \int_0^t (t-s)^{\alpha-1} V(t-s) f(s, \underline{x}_s) ds, & t\in[0,b], \\ \phi(t), & t\in[-a,0]. \end{cases} \tag{18}$$

Then $\underline{x}\in C([-a,b],X)$ and $\underline{x}=Q\underline{x}$. Thus $\underline{x}$ is a fixed point of Q, hence $\underline{x}$ becomes a mild solution of (1). Similarly we can prove that there exists $\overline{x}\in C([-a,b],X)$ such that $y^{(n)}\to\overline{x}$ as $n\to\infty$ and $\overline{x}=Q\overline{x}$. Let $x\in B$ be any fixed point of Q, then by (10), $x^{(1)}=Qx^{(0)}\leq Qx=x\leq Qy^{(0)}=y^{(1)}$. By induction, $x^{(n)}\leq x\leq y^{(n)}$. Using (12) and taking the limit as $n\to\infty$ we conclude that $x^{(0)}\leq\underline{x}\leq x\leq\overline{x}\leq y^{(0)}$. Hence $\underline{x}$, $\overline{x}$ are the minimal and maximal mild solutions of the finite delay differential equations of fractional order (1) on $[x^{(0)},y^{(0)}]$ respectively. □

In the next theorem, we shall prove the uniqueness of the solution of system (1) by using monotone iterative procedure. For this we make the the following additional assumption:

(H5) $f: J\times D\to X$ is a continuous function and there exists a constant $\eta\geq 0$ such that

$$f(t,\varphi_2)-f(t,\varphi_1)\leq\eta(\varphi_2(\nu)-\varphi_1(\nu)), \quad \text{for some } \nu\in[-a,0]$$

for any $t\in J$ and $x_t^{(0)}\leq\varphi_1\leq\varphi_2\leq y_t^{(0)}$.

Theorem 3.2 *Let X be an ordered Banach space, whose positive cone P is normal with normal constant N and $T(t)(t\geq 0)$ be a positive operator. Also assume that the Cauchy delay problem (1) has a lower solution $x^{(0)}\in C([-a,b],X)$ and an upper solution $y^{(0)}\in C([-a,b],X)$ with $x^{(0)}\leq y^{(0)}$. If the assumptions (H2) and (H5) hold and $K=\frac{2MN\eta b^{\alpha}}{\Gamma(\alpha+1)}<1$, then the Cauchy delay problem (1) has a unique mild solution between $x^{(0)}$ and $y^{(0)}$.*

Proof. Let $\{x^{(n)}\} \subset B$ be monotone increasing sequence. For any $m, n = 1, 2, \ldots$, with $m > n$, by H(4) and H(6), we have that

$$\theta \le f(t, x_t^{(m)}) - f(t, x_t^{(n)}) \le \eta(x_t^{(m)}(\nu) - x_t^{(n)}(\nu)).$$

Using the normality of the positive cone P, we get

$$\|f(t, x_t^{(m)}) - f(t, x_t^{(n)})\| \le N\eta\|x_t^{(m)}(\nu) - x_t^{(n)}(\nu)\|. \tag{19}$$

From the definition of measure of noncompactness and (19), we get

$$\mu\left(\left\{f\left(s, x_t^{(n)}\right)\right\}\right) \le N\eta \sup_{-a\le\nu\le 0} \mu\left(\left\{x_t^{(n)}(\nu)\right\}\right). \tag{20}$$

From (19), f is a Lipschitz continuous for second variable. So f satisfies the assumptions (H1) and (H3) with $L = N\eta$. Thus all the conditions of Theorem 3.1 are satisfied, the Cauchy delay problem (1) has maximal and minimal solutions on the ordered interval $B = [x^{(0)}, y^{(0)}]$.

Let $\underline{x}(t)$ and $\overline{x}(t)$ be the minimal solution and maximal solution of Cauchy delay problem (1) respectively on the ordered interval $B = [x^{(0)}, y^{(0)}]$. Since $\underline{x}(t) \equiv \overline{x}(t)$ for $t \in [-a, 0]$, then we have to prove that $\overline{x}(t) \equiv \underline{x}(t)$ on J for the uniqueness. By (8), (H5) and the positivity of operator $U(t)$ and $V(t)$ and take $t \in J$, we get

$$\begin{aligned}
\theta &\le \overline{x}(t) - \underline{x}(t) = Q\overline{x}(t) - Q\underline{x}(t)\\
&= \int_0^t (t-s)^{\alpha-1} V(t-s)\left[f(s, \overline{x}_s) - f(s, \underline{x}_s)\right] ds\\
&\le \eta \int_0^t (t-s)^{\alpha-1} V(t-s)(\overline{x}_s(\nu) - \underline{x}_s(\nu)) ds, \qquad \text{for some } \nu \in [-a, 0].
\end{aligned}$$

By applying the normality of the positive cone P, we get

$$\begin{aligned}
\|\overline{x}(t) - \underline{x}(t)\| &\le N\eta\| \int_0^t (t-s)^{\alpha-1} V(t-s)(\overline{x}_s(\nu) - \underline{x}_s(\nu)) ds\|\\
&\le \frac{MN\eta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\overline{x}_s(\nu) - \underline{x}_s(\nu)\| ds\\
&= \frac{MN\eta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\overline{x}(s+\nu) - \underline{x}(s+\nu)\| ds\\
&\le \frac{MN\eta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\overline{x} - \underline{x}\| ds\\
&\le \frac{MN\eta b^\alpha}{\Gamma(\alpha+1)} \|\overline{x} - \underline{x}\|.
\end{aligned} \tag{21}$$

Inequality implies that $\|\overline{x} - \underline{x}\| \le K\|\overline{x} - \underline{x}\|$. Since $K < \frac{1}{2}$, then $\|\overline{x} - \underline{x}\| = 0$, i.e. $\overline{x} = \underline{x}$ on $[-a, b]$. Hence $\overline{x} = \underline{x}$ is the unique mild solution of the Cauchy delay problem (1) between $x^{(0)}$ and $y^{(0)}$. $\square$

4 Example

Let $X = L^2([0,\pi],\mathbb{R})$. Consider the following finite delay patial differential equation of fractional order:

$$\begin{cases} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} z(t,y) = \frac{\partial^2}{\partial y^2} z(t,y) + 2\eta \sin(\frac{z(t-1,y)}{2}), \quad (t,y) \in [0,\frac{\pi}{2}] \times [0,\pi], \\ z(t,0) = z(t,\pi) = 0, \quad t \in [0,\frac{\pi}{2}], \\ z(\nu,y) = \phi(\nu,y) \quad \nu \in [-1,0], \end{cases} \tag{22}$$

where $\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}$ is the Caputo fractional partial derivative, $0 \le \eta \le \min\{\frac{2}{\sqrt{\pi}}, \frac{\sqrt{\pi}}{4M}\}$, $f : J \times D \to X$ is a nonlinear functions, here $D = C([-1,0] \times [0,\pi], X)$ and $\phi(\nu,y) \in D$.

Let $P = \{\phi \in X | \phi(y) \ge 0 \text{ a.e. } y \in [0,\pi]\}$. Then P is a normal cone in Banach space X and its normal constant is 1, i.e. $N = 1$. We define an operator $A : X \to X$ by $Av = v''$ with domain

$$D(A) = \{v \in X : v, v' \text{ is absolutely continuous } v'' \in X, v(0) = v(\pi) = 0\}.$$

It is well known that A is an infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operator $\{T(t), t \ge 0\}$ in X. Now we define $x(t)(y) = z(t,y)$, ${}^cD_t^{\frac{1}{2}} x(t)(y) = \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} z(t,y)$, $f(t,x_t)(y) = 2\eta \sin(\frac{z(t-1,y)}{2})$, $x(\nu)(y) = \phi(\nu)(y) = \phi(\nu,y)$. Therefore, the above impulsive fractional differential equation (22) can be written as the abstract form (1).

The continuous function ϕ is such that $0 \le \phi(\nu,y) \le -\nu y(\pi - y)$, $(\nu,y) \in [-1,0] \times [0,\pi]$. Let $v(t,y) = 0$, $(t,y) \in [-1,\frac{\pi}{2}] \times [0,\pi]$. Then $f(t, v_t(\nu,y)) = 0$ for $t \in [0,\frac{\pi}{2}]$ and $\phi(\nu,y) \ge v(\nu,y)$ for $\nu \in [-1,0]$. Thus v becomes a lower solution of the problem (1). Now we take $w(t,y)$ such that

$$w(t,x) = \begin{cases} ty(\pi - y), \quad (t,y) \in [0,\frac{\pi}{2}] \times [0,\pi], \\ -ty(\pi - y), \quad (t,y) \in [-1,0] \times [0,\pi]. \end{cases}$$

Note that $\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} w(t,y) = \frac{2t^{\frac{1}{2}} y(\pi-y)}{\sqrt{\pi}}$ and $\frac{\partial^2}{\partial y^2} w(t,y) = -2t$. Since $\frac{t^{\frac{1}{2}} y(\pi-y)}{2} \ge \frac{ty(\pi-y)}{2}$ for $0 \le t \le 1$, the function $\sin(.)$ is increasing for interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\frac{4}{\sqrt{\pi}} \ge 2\eta$, these imply that

$$\frac{2t^{\frac{1}{2}} y(\pi - y)}{\sqrt{\pi}} \ge 2\eta \sin(\frac{ty(\pi - y)}{2}) \ge 2\eta \sin(\frac{(t-1)y(\pi - y)}{2}).$$

Thus

$$\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} w(t,y) \ge \frac{\partial^2}{\partial y^2} w(t,y) + 2\eta \sin(\frac{w(t-1,y)}{2}),$$

and $w(\nu,y) \ge \phi(\nu,y)$ for $\nu \in [-1,0]$. So w is an upper solution of the problem (1). Clearly the function $f(t,\varphi)$ is increasing in φ for $v \le \varphi \le w$, so the assumptions (H2) is satisfied. Since the function $\sin(.)$ is Lipschitz function and is increasing for interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. So the function f satisfies the following condition:

$$0 \le f(t, z^{(2)}(t-1,y)) - f(t, z^{(1)}(t-1,y)) \le \eta(z^{(2)}(t-1,y) - z^{(1)}(t-1,y)), \quad \nu \in [-1,0]$$

for any $v(t,y) \leq z^{(1)}(t,y) \leq z^{(2)}(t,y) \leq w(t,y)$, $(t,y) \in [-1, \frac{\pi}{2}] \times [0,\pi]$. This means

$$\theta(y) \leq f(t, x_t^{(2)})(y) - f(t, x_t^{(1)})(y) \leq \eta(x_t^{(2)}(-1)(y) - x_t^{(1)}(-1)(y))$$

for any $v \leq x^{(1)} \leq x^{(2)} \leq w$. Thus the assumption (H5) is also satisfied. At last $K = \frac{2MN\eta}{\Gamma(1+\frac{1}{2})} = \frac{4M\eta}{\sqrt{\pi}} < 1$. All the conditions of the Theorem 3.2 are satisfied, hence the system (22) has a unique solution. □

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Pullback Attractors of Nonautonomous Boundary Cauchy Problems

M. Moussi *

Department of Mathematics and Informatics, Faculty of sciences, University Mohamed I 60000 Oujda, Morocco

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Abstract: In this work, we establish the existence of pullback attractors for nonautonomous nonlinear boundary Cauchy problems. We apply our result to a reaction-diffusion equation.

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Mathematics Subject Classification (2010): 47J35, 45Exx, 34D45, 35K57.

1 Introduction

Consider the nonlinear boundary Cauchy problem for arbitrary $s \in \mathbb{R}$

$$\begin{cases} \frac{d}{dt}u(t) = A_{\max}(t)u(t), \quad t \in [s, \infty), \\ L(t)u(t) = f(t, u(t)), \quad t \in [s, \infty), \\ u(s) = x, \end{cases} \tag{1}$$

where $A_{\max}(t)$ is a closed operator on a Banach space X endowed with a maximal domain $D(A_{\max}(t))$, and $L(t) : D(A_{\max}(t)) \to \partial X$, with a 'boundary space' ∂X and a function $f : \mathbb{R} \times X \to \partial X$, the solution $u : [s, \infty) \to X$ takes the initial value $x \in X$ at time s. Moreover, the restriction $A(t) := A_{\max}(t)|_{ker(L(t))}$ is assumed to generate an evolution family $(U(t, s))_{t \geq s}$, on the state space X. That is $U(t, s)x$ is a solution of the corresponding linear boundary Cauchy problem of (1) given by

$$\begin{cases} \frac{d}{dt}u(t) = A_{\max}(t)u(t), \quad t \in [s, \infty), \\ L(t)u(t) = 0, \quad t \in [s, \infty), \\ u(s) = x. \end{cases} \tag{2}$$

* Corresponding author: mailto:mohmoussi@hotmail.com

This type of equations has recently been suggested and investigated as a model class with various applications like population equations, retarded differential (difference) equations, dynamical population equations and boundary control problems (see e.g. [2, 3, 7] and the references therein).

A crucial question concerning nonautonomous boundary equations is the existence of solutions. Recently, in [3, 9], the existence and uniqueness of classical solutions for (1) in the case that $f(t,x(t)) \equiv f(t)$ was proved. Moreover, it was shown that these solutions are given by a variation of constants formula which can be easily extended, using the contraction fixed point theorem, to the following variation of constants formula solution of (1):

$$x(t,s) = U(t,s)x_0 + \lim_{\lambda \to +\infty} \int_s^t U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma, x(\sigma))\, d\sigma, \quad t \geq s. \tag{3}$$

Here $L_{\lambda,t}$ is the inverse of $L(t)|_{ker(\lambda - A_{\max}(t))}$.

The study of the regularity properties and the long-time behavior of infinite dimensional dynamical systems is one of the most important problems of modern mathematical physics. In this direction, some studies have been done for the problem (1), we cite for example the compactness of solutions [3], the study of controllability [2], the almost periodicity and automorphicity of solutions [1].

Another important question concerning the long-time behavior is the existence of invariant manifolds. This question was recently studied in [7].

The long-time behavior of the above systems can be also expressed by the term of attractors. To the best of our knowledge, the existence of attractors for nonautonomous dynamical systems is not as well developed as for the autonomous case. There exist several non equivalent definitions for nonautonomous attractors, e.g. forward and pullback attractors describing, respectively, the future and the past of nonautonomous equations (see e.g. [6, 15] and the references therein).

Recently, in [5], the authors showed the existence of pullback attractors for evolution processes. Inspired by the ideas in [5], we are concerned in the present work with the study of the existence of pullback attractors for the boundary evolution equation (1), our main tool is the variation of constants formula (3).

Roughly speaking, our goal is to establish sufficient conditions for guaranteeing the existence of a pullback attractor which is a family of compact invariant subsets pullback attracting bounded subsets. More precisely, by assuming some regularity conditions on $(U(t,s))_{t\geq s}$, we will prove that the solution x given in (3) is both pullback strongly bounded dissipative and pullback asymptotically compact.

Finally, to illustrate our general assumptions we give an application to the following reaction diffusion equation:

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,x) = \dfrac{\partial^2 v}{\partial x^2}(t,x) - \beta(t)v(t,x), \quad t \geq 0, x \in [0,1], \\ \dfrac{\partial}{\partial x}v(t,0) = g_1(t,v);\ \dfrac{\partial}{\partial x}v(t,1) = g_2(t,v), \quad t \geq 0, \\ v(0,x) = v_0(x), \quad x \in [0,1]. \end{cases} \tag{4}$$

The structure of the paper is as follows. In Section 2 we list natural assumptions for well-posedness of equation (1) and the concepts of mild solution. Section 3 is devoted to a pullback attractors theorem for (1) which yields sufficient conditions for the existence of pullback attractors. Section 4 is devoted to an application of the reaction diffusion equation (4).

2 Preliminaries

In this section we recall some definitions and results and formulate assumptions.

2.1 Linear nonautonomous boundary Cauchy problems

A family of linear (unbounded) operators $(A(t))_{t\geq 0}$ defined on a Banach space X is called a *stable family* if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A(t))$ for all $t \geq 0$ and

$$\Big\| \prod_{i=1}^{k} R(\lambda, A(t_i)) \Big\| \leq M(\lambda - \omega)^{-k}$$

for $\lambda > \omega$ and any finite sequence $0 \leq t_1 \leq \cdots \leq t_k$, where

$$\rho(A(t)) := \{\lambda \in \mathbb{C} \mid \lambda\, \mathrm{id}_X - A(t) : D(A(t)) \to X \text{ is bijective}\}$$

denotes the *resolvent set* of $A(t)$. For $\lambda \in \rho(A(t))$, the inverse $R(\lambda, A(t)) := (\lambda\, \mathrm{id}_X - A(t))^{-1}$ is called the *resolvent* of $A(t)$.

Remark 2.1 If there exists a constant $\omega \in \mathbb{R}$ such that

$$\|R(\lambda, A(t))\| \leq \frac{1}{\lambda - \omega},$$

for all $\lambda > \omega$ and $t \geq 0$, then $(A(t))_{t\geq 0}$ is a stable family.

Definition 2.1 A family of linear bounded operators $(U(t,s))_{t\geq s\in J}$, $J := \mathbb{R}_+$ or $\mathbb{R}$, on a Banach space X is called *evolution family* if

(1) $U(t,s) = U(t,r)U(r,s)$ and $U(s,s) = \mathrm{id}_X$ for all $t \geq r \geq s \in J$,

(2) the mapping $\{(t,s) \in J \times J : t \geq s\} \ni (t,s) \mapsto U(t,s) \in \mathcal{L}(X)$ is strongly continuous.

The *growth bound* of $(U(t,s))_{t\geq s}$ is defined by

$$\omega(U) := inf \Big\{\omega \in \mathbb{R} : \exists\, M_\omega \geq 1 \text{ with } \|U(t,s)\| \leq M_\omega e^{\omega(t-s)}\ \forall\, t \geq s \in J\Big\}.$$

The evolution family $(U(t,s))_{t\geq s}$ is called *exponentially bounded* provided that $\omega(U) < \infty$ and *exponentially stable* provided that $\omega(U) < 0$.

Let $X, D, \partial X$ be Banach spaces such that D is dense and continuously embedded in X. On these spaces, the operators $A_{\max}(t) \in \mathcal{L}(D, X), L(t) \in \mathcal{L}(D, \partial X)$, for $t \in \mathbb{R}$, are supposed to satisfy the following hypotheses:

(H1) There are positive constants C_1, C_2 such that

$$C_1\|x\|_D \leq \|x\| + \|A_{\max}(t)x\| \leq C_2\|x\|_D$$

for all $x \in D$ and $t \in \mathbb{R}$;

(H2) for each $x \in D$ the mapping $\mathbb{R} \ni t \mapsto A_{\max}(t)x \in X$ is continuously differentiable;

(H3) the operators $L(t) : D \to \partial X, t \in \mathbb{R}$, are surjective;

(H4) for each $x \in D$ the mapping $\mathbb{R} \ni t \mapsto L(t)x \in \partial X$ is continuously differentiable;

(H5) there exist constants $\gamma > 0$ and $\omega \in \mathbb{R}$ such that

$$||L(t)x||_{\partial X} \geq \gamma^{-1}(\lambda - \omega)||x||_X,$$

for $x \in \ker(\lambda\,\mathrm{id}_X - A_{\max}(t)), \lambda > \omega$ and $t \in \mathbb{R}$;

(H6) the family of operators $(A(t))_{t\in\mathbb{R}}$, $A(t) := A_{\max}(t)|_{\ker L(t)}$, is stable.

In the following lemma, we cite consequences of the above assumptions from [10, Lemma 1.2] which will be needed below.

Lemma 2.1 *The restriction* $L(t)|_{\ker(\lambda\,\mathrm{id}_X - A_{\max}(t))}$ *is an isomorphism from* $\ker(\lambda\,\mathrm{id}_X - A_{\max}(t))$ *into* ∂X *and its inverse* $L_{\lambda,t} := \left[L(t)|_{\ker(\lambda\,\mathrm{id}_X - A_{\max}(t))}\right]^{-1} : \partial X \to \ker(\lambda\,\mathrm{id}_X - A_{\max}(t))$ *satisfies*

$$\|L_{\lambda,t}\| \leq \gamma(\lambda - \omega)^{-1} \quad \textit{for } \lambda > \omega,\ t \in \mathbb{R}.$$

Under assumptions (H1)-(H6), it was shown that the linear boundary Cauchy problem (2) is well-posed. More precisely, there exists an evolution family $(U(t,s))_{t\geq s}$ generated by the family of operators $(A(t))_{t\in\mathbb{R}}$. See [12,13].

2.2 Nonlinear boundary Cauchy problems

In case $f \equiv 0$ the boundary Cauchy problem (1) reduces to the linear boundary Cauchy problem (2) which was studied in the last subsection under the assumptions (H1)-(H6). In particular, let $(U(t,s))_{t\geq s}$ denote the evolution family solution to the problem (2). We want to study nonlinear perturbations (1) of (2) and therefore assume that the nonlinearity f satisfies:

(H7) The nonlinear part $f : \mathbb{R} \times X \to \partial X$ is assumed to be continuous and there exists a positive constant ℓ such that one has the global Lipschitz estimate

$$\|f(t,x) - f(t,\bar{x})\| \leq \ell\|x - \bar{x}\| \quad \text{for all } x, \bar{x} \in X, t \in \mathbb{R}.$$

Under the assumptions (H1)-(H7) the semilinear boundary Cauchy problem (1) admits a unique mild solution. For $s \in \mathbb{R}$, $x \in X$, a function $u = u(\cdot, s, x) : [s, \infty) \to X$ is called *mild solution of* (1) if it satisfies the integral equation

$$u(t,s,x) = U(t,s)x + \lim_{\lambda\to\infty} \int_s^\infty U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma, u(\sigma,s,x))\, d\sigma, \quad t \geq s. \tag{5}$$

The unique existence follows with the usual contraction arguments (see e.g. [2,11,14]) and uses the *variation of constants formula* from [3] for solutions $v : [s,\infty) \to X$ of inhomogeneous boundary Cauchy problems, i.e. systems (1) with $f(t,u(t)) \equiv g(t)$ independent of $u(t)$

$$v(t) = U(t,s)x + \lim_{\lambda\to\infty} \int_s^\infty U(t,\sigma)\lambda L_{\lambda,\sigma} g(\sigma)\, d\sigma, \quad t \geq s.$$

Let us define on X the family of operators:

$$V(t,s)x := u(t,s,x) \text{ for } x \in X \text{ and } t \geq s. \tag{6}$$

Our goal in the next section is to study the existence of pullback attractors for the family of operators $(V(t,s))_{t\geq s}$.

3 Pullback Attractors of Nonlinear Boundary Cauchy Problems

In this section, we consider the following system

$$\begin{cases} \frac{d}{dt}u(t) = A_{\max}(t)u(t), & t \in \mathbb{R}, \\ L(t)u(t) = f(t,u(t)), & t \in \mathbb{R}, \end{cases} \tag{7}$$

where $A_{\max}(t), L(t), f(t,x)$ are assumed to satisfy assumptions (H1)-(H7). We want to study the existence of pullback attractors of the nonlinear problem (7), therefore the evolution family $(U(t,s))_{t\geq s}$ associated with the linear problem (2) is assumed to satisfy the following:

(H8) $(U(t,s))_{t\geq s}$ is exponentially stable, that is, there exist constants $\alpha > 0$ and $M_1 \geq 1$ such that

$$\|U(t,s)\| \leq M_1 e^{-\alpha(t-s)},\ t \geq s;$$

(H9) for all $t > s$, $U(t,s)$ is a compact operator on X.

To get our aim, we will use the following sufficient condition result shown in [5, Theorem 2.3].

Theorem 3.1 *If $(V(t,s))_{t\geq s}$ is pullback strongly bounded dissipative and pullback asymptotically compact, then it has a pullback attractor $(\mathcal{A}(t))_{t\in\mathbb{R}}$ with the property that $\bigcup_{s\leq t} \mathcal{A}(s)$ is bounded for each $t \in \mathbb{R}$.*

The concepts of pullback strongly bounded dissipative and pullback asymptotically compact are given in the following definitions.

Definition 3.1 We say that $(V(t,s))_{t\geq s}$ is *pullback strongly bounded dissipative* if, for each $t \in \mathbb{R}$, there is a bounded subset $B(t)$ of X which pullback attracts bounded subsets of X at time t, that is, given a bounded subset $B \subset X$ and $t \in \mathbb{R}$, there exists $s(t,B) \leq t$ such that $V(t,s)B \subset B(t)$ for all $s \leq s(t,B)$.

Definition 3.2 We say that $(V(t,s))_{t\geq s}$ is *pullback asymptotically compact* if, for each $t \in \mathbb{R}$, sequence $(s_k)_{k\in\mathbb{N}}$ in $(-\infty,t]$ and bounded sequence $(x_k)_{k\in\mathbb{N}}$ in X such that $s_k \longrightarrow -\infty$ as $k \to +\infty$ and $\{V(t,s_k)x_k : k \in \mathbb{N}\}$ is bounded, the sequence $(V(t,s_k)x_k)_{k\in\mathbb{N}}$ has a convergent subsequence.

We first show the following lemma.

Lemma 3.1 *The family of operators $(V(t,s))_{t\geq s}$ is pullback strongly bounded dissipative provided that $M_1\gamma\ell - \alpha < 0$.*

Proof. Let $x \in X$ and $t \geq s$. From (H7) we obtain

$$\|f(t,x)\| \quad \leq \quad \|f(t,0)\| + \|f(t,x) - f(t,0)\| \leq \|f(t,0)\| + \ell\|x\|.$$

We put $C := \sup_{t\in\mathbb{R}} \|f(t,0)\|$. Using (H8) and Lemma 2.1 we obtain

$$\begin{aligned}
\|V(t,s)x\| &\leq M_1 e^{-\alpha(t-s)}\|x\| + \lim_{\lambda\to\infty} \int_s^t M_1 e^{-\alpha(t-\sigma)} \frac{\lambda\gamma}{\lambda-\omega} \|f(\sigma, V(\sigma,s)x)\| d\sigma \\
&\leq M_1 e^{-\alpha(t-s)}\|x\| + M_1\gamma \int_s^t e^{-\alpha(t-\sigma)} (C + \ell\|V(\sigma,s)x\|) d\sigma \\
&\leq M_1 e^{-\alpha(t-s)}\|x\| + M_1\gamma C \int_s^t e^{-\alpha(t-\sigma)}\, d\sigma \\
&\quad + M_1\gamma\ell \int_s^t e^{-\alpha(t-\sigma)} \|V(\sigma,s)x\|\, d\sigma
\end{aligned}$$

then we get

$$\begin{aligned}
e^{\alpha t}\|V(t,s)x\| &\leq M_1 e^{\alpha s}\|x\| + M_1\gamma C \int_s^t e^{\alpha\sigma}\, d\sigma + M_1\gamma\ell \int_s^t e^{\alpha\sigma}\|V(\sigma,s)x\|\, d\sigma \\
&= M_1 e^{\alpha s}\|x\| + \frac{M_1\gamma C}{\alpha}(e^{\alpha t} - e^{\alpha s}) + M_1\gamma\ell \int_s^t e^{\alpha\sigma}\|V(\sigma,s)x\|\, d\sigma
\end{aligned}$$

Using the generalized Gronwall's lemma we obtain

$$\begin{aligned}
e^{\alpha t}\|V(t,s)x\| &\leq M_1 e^{\alpha s}\|x\| + \frac{M_1\gamma C}{\alpha}(e^{\alpha t} - e^{\alpha s}) \\
&\quad + \int_s^t \left[M_1 e^{\alpha s}\|x\| + \frac{M_1\gamma C}{\alpha}(e^{\alpha\sigma} - e^{\alpha s}) \right] M_1\gamma\ell e^{\int_\sigma^t M_1\gamma\ell\, du}\, d\sigma \\
&= \frac{M_1\gamma C}{\alpha} e^{\alpha t} + M_1 e^{\alpha s}\|x\| e^{M_1\gamma\ell(t-s)} + \frac{M_1\gamma\ell M_1\gamma C}{\alpha(\alpha - M_1\gamma\ell)} e^{\alpha t} \\
&\quad - \frac{M_1\gamma\ell M_1\gamma C}{\alpha(\alpha - M_1\gamma\ell)} e^{M_1\gamma\ell t} e^{(\alpha - M_1\gamma\ell)s} - \frac{M_1\gamma C}{\alpha} e^{\alpha s} e^{M_1\gamma\ell(t-s)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|V(t,s)x\| &\leq \frac{M_1\gamma C}{\alpha} + M_1 e^{-\alpha(t-s)}\|x\| e^{M_1\gamma\ell(t-s)} + \frac{M_1\gamma\ell M_1\gamma C}{\alpha(\alpha - M_1\gamma\ell)} \\
&\quad - \frac{M_1\gamma\ell M_1\gamma C}{\alpha(\alpha - M_1\gamma\ell)} e^{M_1\gamma\ell(t-s)} e^{(-\alpha(t-s)} - \frac{M_1\gamma C}{\alpha} e^{-\alpha(t-s)} e^{M_1\gamma\ell(t-s)} \\
&= \frac{M_1\gamma C}{\alpha} + \frac{M_1\gamma\ell M_1\gamma C}{\alpha(\alpha - M_1\gamma\ell)} \\
&\quad + e^{(M_1\gamma\ell - \alpha)(t-s)} \left[M_1\|x\| - \frac{M_1\gamma\ell M_1\gamma C}{\alpha(\alpha - M_1\gamma\ell)} - \frac{M_1\gamma C}{\alpha} \right] \\
&= \frac{M_1\gamma C}{\alpha - M_1\gamma\ell} + \left(M_1\|x\| - \frac{M_1\gamma C}{\alpha - M_1\gamma\ell} \right) e^{(M_1\gamma\ell - \alpha)(t-s)}.
\end{aligned}$$

We have then

$$\|V(t,s)x\| \leq K + (M_1\|x\| - K) e^{-\beta s} e^{\beta t}$$

with $K := \dfrac{M_1\gamma C}{\alpha - M_1\gamma\ell}$ and $\beta := M_1\gamma\ell - \alpha$. By hypothesis, $\beta < 0$.
Since $(M_1\|x\| - K)e^{-\beta s} \longrightarrow 0$ as $s \longrightarrow -\infty$. Then for fixed $t \in \mathbb{R}$ and $x \in B$ bounded, there exists $s_0(t, B)$ such that $(M_1\|x\| - K)e^{-\beta s} < 1$ for all $s \leq s_0(t, B)$. This implies

$$\|V(t, s)x\| \leq K + e^{\beta t}.$$

We take $B(t) = B(0, K + e^{\beta t})$ the ball with center 0 and radius $K + e^{\beta t}$. Then the dissipativity of the family $(V(t, s))_{t \geq s}$ holds.

To get the main result, it remains to show that $V(t, s)$, $t \geq s$, is pullback asymptotically compact. To do that, from [5, Theorem 2.4], it is sufficient to prove the following lemma.

Lemma 3.2 *There exist $(T(t, s))_{t\geq s}$ and $(R(t, s))_{t\geq s}$ such that $V(t, s) = T(t, s) + R(t, s)$, where*

(i) $R(t, s)$, $t > s$, is compact,

(ii) there exists a non-increasing function $k : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ with $k(\sigma, r) \longrightarrow 0$ when $\sigma \to \infty$, and for all $s \leq t$ and $x \in X$ with $\|x\| \leq r$, $\|T(t, s)\| \leq k(t - s, r)$.

Proof. Define the families of operators $R(t, s) := U(t, s)$ and

$$T(t, s) := \lim_{\lambda \to +\infty} \int_s^t U(t, \sigma)\lambda L_{\lambda,\sigma} f(\sigma, V(\sigma, s)\cdot))\, d\sigma. \tag{8}$$

The assertion (i) is satisfied by hypothesis (H9).

To prove (ii), we assume that $M_1\gamma l < \alpha$ and we will show that

$$\begin{aligned} \|T(t, s)x\| \quad &\leq \quad M_1\|x\|e^{(M_1\gamma l - \alpha)(t-s)} - \frac{M_1\gamma C}{\alpha - M_1\gamma l} e^{(M_1 M l - \alpha)(t-s)} \\ &\qquad -M_1\|x\|e^{-\alpha(t-s)} + \frac{M_1\gamma C}{\alpha - M_1 M l}. \end{aligned}$$

In fact, we have

$$\begin{aligned} \|T(t, s)x\| \quad &\leq \quad \lim_{\lambda\to\infty} \int_s^t M_1 e^{-\alpha(t-\sigma)} \frac{\lambda\gamma}{\lambda - \omega} \|f(\sigma, V(\sigma, s)x)\| d\sigma \\ &\leq \quad M_1\gamma \int_s^t e^{-\alpha(t-\sigma)} (C + \ell\|V(\sigma, s)x\|) d\sigma \\ &\leq \quad M_1\gamma C \int_s^t e^{-\alpha(t-\sigma)}\, d\sigma \\ &\qquad + M_1\gamma\ell \int_s^t e^{-\alpha(t-\sigma)} (\|U(\sigma, s)x\| + \|T(\sigma, s)x\|)\, d\sigma. \end{aligned}$$

Then we get

$$\begin{aligned}
e^{\alpha t}\|T(t,s)x\| &\leq M_1\gamma C\int_s^t e^{\alpha\sigma}\,d\sigma + M_1\gamma\ell\int_s^t e^{\alpha\sigma}\|U(\sigma,s)x\|\,d\sigma \\
&\quad +M_1\gamma\ell\int_s^t e^{\alpha\sigma}\|T(\sigma,s)x\|\,d\sigma \\
&\leq M_1\gamma C\int_s^t e^{\alpha\sigma}\,d\sigma + M_1M_1\gamma\ell\int_s^t e^{\alpha s}\|x\|\,d\sigma \\
&\quad +M_1\gamma\ell\int_s^t e^{\alpha\sigma}\|T(\sigma,s)x\|\,d\sigma \\
&= \frac{M_1\gamma C}{\alpha}(e^{\alpha t}-e^{\alpha s}) + M_1M_1\gamma\ell e^{\alpha s}(t-s)\|x\| \\
&\quad +M_1\gamma\ell\int_s^t e^{\alpha\sigma}\|T(\sigma,s)x\|\,d\sigma.
\end{aligned}$$

Using the generalized Gronwall's lemma we obtain

$$\begin{aligned}
e^{\alpha t}\|T(t,s)x\| &\leq \frac{M_1\gamma C}{\alpha}(e^{\alpha t}-e^{\alpha s}) + M_1M_1\gamma\ell e^{\alpha s}(t-s)\|x\| \\
&\quad +\int_s^t\left[\frac{M_1\gamma C}{\alpha}(e^{\alpha\sigma}-e^{\alpha s}) + M_1M_1\gamma\ell e^{\alpha s}(\sigma-s)\|x\|\right] \\
&\qquad\qquad \times M_1\gamma\ell e^{\int_\sigma^t M_1\gamma\ell\,du}\,d\sigma \\
&= \frac{M_1\gamma C}{\alpha}e^{\alpha t} - \frac{M_1\gamma C}{\alpha}e^{\alpha s} + M_1M_1\gamma\ell e^{\alpha s}(t-s)\|x\| \\
&\quad +\int_s^t \frac{M_1\gamma C}{\alpha}e^{\alpha\sigma}M_1\gamma\ell e^{M_1\gamma\ell(t-\sigma)}\,d\sigma \\
&\quad -\int_s^t \frac{M_1\gamma C}{\alpha}e^{\alpha s}M_1\gamma\ell e^{M_1\gamma\ell(t-\sigma)}\,d\sigma \\
&\quad +\int_s^t M_1M_1\gamma_1\gamma\ell e^{\alpha s}(\sigma-s)\|x\|M_1\gamma\ell e^{M_1\gamma\ell(t-\sigma)}\,d\sigma \\
&= \frac{M_1\gamma C}{\alpha}e^{\alpha t} - \frac{M_1\gamma C}{\alpha}e^{\alpha s} + M_1M_1\gamma\ell e^{\alpha s}(t-s)\|x\| \\
&\quad +\int_s^t \frac{M_1\gamma C}{\alpha}M_1\gamma\ell e^{M_1\gamma\ell t}e^{(\alpha-M_1\gamma\ell)\sigma}\,d\sigma \\
&\quad -\int_s^t \frac{M_1\gamma C}{\alpha}e^{\alpha s}M_1\gamma\ell e^{M_1\gamma\ell(t-\sigma)}\,d\sigma \\
&\quad +M_1M_1\gamma\ell\|x\|M_1\gamma\ell e^{\alpha s}\int_s^t(\sigma-s)e^{M_1\gamma\ell(t-\sigma)}\,d\sigma
\end{aligned}$$

$$
\begin{aligned}
&= \frac{M_1\gamma C}{\alpha}e^{\alpha t} - \frac{M_1\gamma C}{\alpha}e^{\alpha s} + M_1M_1\gamma\ell e^{\alpha s}(t-s)\|x\| \\
&\quad + \frac{M_1\gamma C}{\alpha}M_1\gamma\ell e^{M_1\gamma\ell t}\frac{1}{\alpha - M_1\gamma\ell}\left(e^{(\alpha-M_1\gamma\ell)t} - e^{(\alpha-M_1\gamma\ell)s}\right) \\
&\quad - \frac{M_1\gamma C}{\alpha}e^{\alpha s}\left(-1 + e^{M_1\gamma\ell(t-s)}\right) \\
&\quad + M_1M_1\gamma\ell\|x\|M_1\gamma\ell e^{\alpha s}\int_s^t (\sigma - s)e^{M_1\gamma\ell(t-\sigma)}\,d\sigma \\
&= \frac{M_1\gamma C}{\alpha}e^{\alpha t} - \frac{M_1\gamma C}{\alpha}e^{\alpha s} + M_1M_1\gamma\ell e^{\alpha s}(t-s)\|x\| \\
&\quad + \frac{M_1\gamma C}{\alpha}\frac{M_1\gamma\ell}{\alpha - M_1\gamma\ell}e^{M_1\gamma\ell t}\left(e^{(\alpha-M_1\gamma\ell)t} - e^{(\alpha-M_1\gamma\ell)s}\right) \\
&\quad + \frac{M_1\gamma C}{\alpha}e^{\alpha s} - \frac{M_1\gamma C}{\alpha}e^{\alpha s}e^{M_1\gamma\ell(t-s)} \\
&\quad + M_1M_1\gamma\ell\|x\|M_1\gamma\ell\left[-\frac{(t-s)}{M_1\gamma\ell} - \frac{1}{(M_1\gamma\ell)^2}(1 - e^{M_1\gamma\ell(t-s)})\right] \\
&= \frac{M_1\gamma C}{\alpha}e^{\alpha t} + \frac{M_1\gamma C}{\alpha}\frac{M_1\gamma\ell}{\alpha - M_1\gamma\ell}e^{\alpha t} - \frac{M_1\gamma C}{\alpha}\frac{M_1\gamma\ell}{\alpha - M_1\gamma\ell}e^{\alpha s}e^{M_1\gamma\ell(t-s)} \\
&\quad - \frac{M_1\gamma C}{\alpha}e^{\alpha s}e^{M_1\gamma\ell(t-s)} - \frac{M_1M_1\gamma\ell}{M_1\gamma\ell}e^{\alpha s}\|x\| + \frac{M_1M_1\gamma\ell}{M_1\gamma\ell}e^{\alpha s}\|x\|e^{M_1\gamma\ell(t-s)}.
\end{aligned}
$$

Multiplying both sides by $e^{-\alpha t}$, we get

$$
\begin{aligned}
\|T(t,s)x\| &\leq \frac{M_1\gamma C}{\alpha} + \frac{M_1\gamma C}{\alpha}\frac{M_1\gamma\ell}{\alpha - M_1\gamma\ell} - \frac{M_1\gamma C}{\alpha}\frac{M_1\gamma\ell}{\alpha - M_1\gamma\ell}e^{(M_1\gamma\ell-\alpha)(t-s)} \\
&\quad - \frac{M_1\gamma C}{\alpha}e^{(M_1\gamma\ell-\alpha)(t-s)} - \frac{M_1M_1\gamma\ell}{M_1\gamma\ell}e^{-\alpha(t-s)}\|x\| \\
&\quad + \frac{M_1M_1\gamma\ell}{M_1\gamma\ell}\|x\|e^{(M_1\gamma\ell-\alpha)(t-s)} \\
&= M_1\|x\|e^{(M_1\gamma l-\alpha)(t-s)} - \frac{M_1\gamma C}{\alpha - M_1\gamma l}e^{(M_1\gamma l-\alpha)(t-s)} \\
&\quad - M_1\|x\|e^{-\alpha(t-s)} + \frac{M_1\gamma C}{\alpha - M_1\gamma l}.
\end{aligned}
$$

To end the proof, we take the function $k(\cdot,\cdot)$ as follows

$$k(\sigma, r) = M_1re^{\beta\sigma} + \frac{M_1\gamma C}{\beta}e^{\beta\sigma} - M_1re^{-\alpha\sigma} - \frac{M_1\gamma C}{\beta}$$

with $\beta := M_1\gamma l - \alpha$. Since, by hypothesis, $\beta < 0$, it is clear that $k(t,s)$ satisfies assertion (ii). Then the proof is achieved.

From the previous lemmas, we are now ready to state our main result.

Theorem 3.2 *Assume that* (7) *satisfies the assumptions (H1)-(H9) with* $M_1\gamma l < \alpha$. *Then the family of operators* $(V(t,s))_{t\geq s}$ *has a pullback attractor* $(\mathcal{A}(t))_{t\in\mathbb{R}}$ *with the property that* $\bigcup_{s\leq t}\mathcal{A}(s)$ *is bounded for each* $t \in \mathbb{R}$.

4 Application

consider the following reaction diffusion equation

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,x) = \dfrac{\partial^2 v}{\partial x^2}(t,x) - \beta(t)v(t,x), t \geq 0, x \in [0,1], \\ \dfrac{\partial}{\partial x}v(t,0) = g_1(t,v);\ \dfrac{\partial}{\partial x}v(t,1) = g_2(t,v) \quad t \geq 0, \\ v(0,x) = v_0(x), \quad x \in [0,1]. \end{cases} \tag{9}$$

Here $\beta(\cdot)$ is a continuously differentiable positive function. Moreover, we assume that

(i) There exist positive constants $\overline{\beta}$ and $\underline{\beta}$ such that $\underline{\beta} \leq \beta(t) \leq \overline{\beta}$ for all $t \geq 0$.

(ii) $g_1 : \mathbb{R}^+ \times L^1[0,1] \longrightarrow \mathbb{R}$ and $g_2 : \mathbb{R}^+ \times L^1[0,1] \longrightarrow \mathbb{R}$ are continuous functions and globally Lipschitz with respect to the second variable uniformly to the first one.

Our aim is to write equation (9) as a boundary Cauchy problem of the form (7) satisfying the assumptions (H1)-(H9). For this purpose, we define the Banach spaces

$$\partial X := \mathbb{R}^2,\ X := L^1[0,1] \text{ and } D := W^{2,1}[0,1],$$

with

$$W^{2,1}[0,1] = \left\{ u \in L^1[0,1] \mid u^{'}, u^{''} \in L^1[0,1] \right\}$$

endowed with the norm

$$\|u\|_D := \|u\|_1 + \|u'\|_1 + \|u''\|_1 \text{ for } u \in W^{2,1}[0,1].$$

Here $\|u\|_1$ denotes the norm of $L^1[0,1]$.

$(D, \|\cdot\|_D)$ is a Banach space dense and continuously embedded in X.

For each $t \geq 0$ the operator $A_{\max}(t) : X \to X$ is defined by $D(A_{\max}(t)) = D$ and

$$(A_{\max}(t)\varphi)(a) = \varphi^{''} - \beta(t)\varphi \tag{10}$$

for all $\varphi \in D$.

For each $t \geq 0$, we define $L(t) : D \longrightarrow \partial X$ by

$$L(t)\varphi = (\varphi^{'}(0), \varphi^{'}(1))^T \quad \text{ for all } \varphi \in D. \tag{11}$$

We show now that the hypotheses (H1)–(H9) are satisfied.

Verification of (H1): since, from [4, Remarque 11], the norms $\|\varphi\|_D$ and $\|\varphi\|_1 + \|\varphi''\|_1$ are equivalent in D, then (H1) holds.

Verification of (H2): holds from assumptions on $t \to \beta(t)$.

Verification of (H3): to show the surjectivity of $L(t)$, let $(a,b) \in \mathbb{R}^2$ be arbitrary. Define

$$u(x) = bx + a(1-x) \text{ for all } x \in [0,1].$$

We have $u \in D$ and $L(t)u = (a,b)$. Therefore $L(t)$ is surjective.

Verification of (H4): is obvious since $L(t)$ is independent of t.

Verification of (H5): let $u \in ker(\lambda - A_{\max}(t))$ for $\lambda > \overline{\beta}$, then there exists $(a,b) \in \mathbb{R}^2$ such that $u(x) = ae^{\alpha(t)} + be^{-\alpha(t)}$ for $x \in [0,1]$ with $\alpha(t) := \sqrt{\lambda - \beta(t)}$. We have

$$|u(x)| = |ae^{\alpha(t)x} + be^{-\alpha(t)x}| \leq |a|e^{\alpha(t)x} + |b|e^{-\alpha(t)x} = \frac{1}{\alpha(t)}\left[|a|\frac{d}{dx}e^{\alpha(t)x} - |b|\frac{d}{dx}e^{-\alpha(t)x}\right].$$

Integrating both sides on x, one can have

$$\begin{aligned}\int_0^1 |u(x)|\,dx &\leq \frac{1}{\alpha(t)}\int_0^1 |a|\frac{d}{dx}e^{\alpha(t)x} - |b|\frac{d}{dx}e^{-\alpha(t)x}\,dx\\ &= \frac{1}{\alpha(t)}\left[|a|e^{\alpha(t)} - |b|e^{-\alpha(t)} - |a| + |b|\right]\\ &\leq \frac{1}{\alpha(t)}\left[|ae^{\alpha(t)} - be^{-\alpha(t)}| + |a-b|\right]\\ &= \frac{1}{\alpha^2(t)}(|u^{'}(0)| + |u^{'}(1)|).\end{aligned}$$

We obtain then $\|Lu\|_{\mathbb{R}^2} \geq (\lambda - \overline{\beta})\|u\|_1$. This shows (H5) with $\gamma = 1$ and $\omega = \overline{\beta}$.

Verification of (H6): Define the operator

$$Au := \Delta u, \quad D(A) = \left\{u \in W^{2,1}[0,1] \mid u^{'}(0) = u^{'}(1) = 0\right\}.$$

It is known that A generates an immediately compact analytic semigroup $(T(t))_{t\geq 0}$ of contraction on the Banach space $L^1[0,1]$, that is $T(t)$ is compact for all $t > 0$ and

$$\|T(t)u\| \leq 1 \text{ for } t \geq 0 \text{ and } u \in L^1[0,1]. \tag{12}$$

See, for example, [8]. Then from Hille-Yosida theorem (see [8, Theorem II.3.8]), $\forall \lambda > 0$ one has $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}.$$

Then, for every $\lambda + \underline{\beta} > 0$, we have $\lambda \in \rho(A(t))$. Moreover,

$$R(\lambda, A(t)) = R(\lambda + \beta(t), A).$$

Therefore,

$$\|R(\lambda, A(t))\| \leq \frac{1}{\lambda + \underline{\beta}} = \frac{1}{\lambda - (-\underline{\beta})},$$

by Remark 2.1, it follows that

$$\Big\|\prod_{i=1}^{n} R(\lambda, A(t_i))\Big\| \leq \frac{1}{(\lambda - (-\underline{\beta}))^n}$$

for $\lambda > -\underline{\beta}$ and any finite sequence $0 \leq t_1 \leq \cdots \leq t_n$. Hence (H6) is satisfied.

Verification of (H7): Follows from assumptions on the functions g_1 and g_2.

Verification of (H8): We note that the evolution family $(U(t,s))_{t\geq s}$ generated by $(A(t))_{t\geq 0}$ is given by

$$U(t,s) = exp\left(\int_s^t -\beta(\sigma)\,d\sigma\right)T(t-s),\ t \geq s \geq 0.$$

Then, from (12) one can see that $\|U(t,s)\| \leq e^{-\underline{\beta}(t-s)},\ t \geq s$. Hence $(U(t,s))_{t\geq s}$ is exponentially stable and (H8) holds.

Verification of (H9): Is obvious from the fact that the semigroup $T(t)$ is compact for all $t > 0$.

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Chaos Synchronization Approach Based on New Criterion of Stability

Adel Ouannas*

LAMIS Laboratory, Department of Mathematics and Computer Science, University of Tebessa, Algeria.

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Abstract: In this paper, we propose a simple method for chaos synchronization in continuous-time based on a new criterion for stability. This criterion implies the Lyapunov stabilization criterion, and is applicable to some typical chaotic systems. Numerical simulations in 3D and 4D are presented to demonstrate the effectiveness of the synchronization results derived in this paper.

Keywords: *chaos synchronization; new criterion; dynamical system; continuous-time; Lyapunov stability.*

Mathematics Subject Classification (2010): 37B25, 37B55, 37C75.

1 Introduction

During the last decade, chaos synchronization has become an active research area, due to its potential applications in information processing such as secure communication [1, 2]. Many types of synchronization have been presented [3–6] and various methods have been developed for synchronization of chaotic systems such as active and adaptive control method [7, 8], backstepping design technique [9], sliding mode control [10], generalized Hamiltonian systems approach [11, 12], and so on. Most of synchronization methods are based on Lyapunov stability theory to guarantee zero stability of errors dynamical system between master and slave chaotic systems.

In this paper, based on some lemma derived from Halanay inequality, we introduce a new and simple stability criterion to synchronize chaotic dynamical systems in continuous-time. In [13], authors derived an important result using Halanay inequality, we give it in the following lemma:

* Corresponding author: mailto:ouannas_adel@yahoo.fr

Lemma 1.1 *Suppose that the continuous functional h satisfies:*

$$|h\left(t,z_{\tau}\right)| \leq \max|z_{\tau}| \tag{1}$$

and

$$ess\sup|\beta(t)| = \beta \leq \alpha. \tag{2}$$

Then every solution $z\left(t\right)$ of

$$\dot{z}\left(t\right) = -\alpha z\left(t\right) + \beta(t)h\left(t,z_{\tau}\right) \tag{3}$$

converges to zero.

This lemma allows us to achieve synchronization without using Lyapunov theory. This paper is organized as follows: In Section 2, a new synchronization criterion for different chaotic systems is proposed. In Section 3, the case of two identical chaotic systems is investigated. In Section 4, numerical examples of 3D chaotic systems and 4D hyperchaotic systems are discussed and numerical simulations are given. In Section 5, conclusion follows.

2 Synchronization Criterion for Different Systems

Consider the chaotic system described by

$$\dot{X}(t) = f(X(t)), \tag{4}$$

where $X(t) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. We consider the system (4) as the master system, as the slave system we consider the following chaotic system described by

$$\dot{Y}(t) = BY(t) + g(Y(t)) + U, \tag{5}$$

where $Y(t) \in \mathbb{R}^n$, B is the $n \times n$ matrix of parameters system, $g : \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear part of the system (5) and U is the vector controller. The synchronization problem is to find a controller U, which stabilizes the error system

$$e(t) = Y(t) - X(t), \tag{6}$$

then the aim of synchronization is to make $\lim_{\ t\longrightarrow+\infty} \|e(t)\| = 0$, where $\|.\|$ is the Euclidean norm.

Remark 2.1 Most of chaotic systems, including all Lur'e nonlinear systems and Lipschitz nonlinear systems, can be described by form of (5) without the function controller U.

The error dynamical system between the master system (4) and the slave system (5), can be derived as follows

$$\dot{e}(t) = Be(k) + BX(t) + g(Y(t)) - f(X(t)) + U. \tag{7}$$

To achieve synchronization between the master system (4) and the slave system (5), we can choose the vector controller U as follows

$$U = \left(C-K\right)Y(t) + \left(K-C-B\right)X(t) + f(X(t)) - g(Y(t)), \tag{8}$$

where $C = (c_{ij}) \in \mathbb{R}^{n \times n}$ such that

$$c_{ij} = \begin{cases} -b_{ij}, & if \quad i \neq j, \\ 0, & if \quad i = j, \end{cases} \tag{9}$$

and $K = diag\,(k_1, k_2, ..., k_n)$ is unknown control diagonal matrix to be determined.

By substituting Eq.(8) into Eq.(7), the error system can be written as

$$\dot{e}(t) = -e(t) + (B + C - K + I)\, e(t). \tag{10}$$

Theorem 2.1 *If the control constants $(k_i)_{1 \leq i \leq n}$ are chosen such that*

$$b_{ii} < k_i < 2 + b_{ii}, \quad 1 \leq i \leq n, \tag{11}$$

then the two systems (4) and (5) are globally synchronized.

Proof. The Eq. (10) allows us to get the following scalar systems

$$\dot{e}_i(t) = -e_i(t) + (b_{ii} - k_i + 1)\, e_i(t), \quad 1 \leq i \leq n. \tag{12}$$

If we put $\tau = 0$ in Eq. (3), we can see that the Eq.(14) is the same as Eq.(3) in Lemma 1.1 with: $z\,(t) = e_i(t)$, $\alpha = 1$, $\beta\,(t) = 1$ and $h\,(t, z\,(t)) = (b_{ii} - k_i + 1)\, e_i(t)$. Now, by using condition (11), we can verify conditions of Lemma 1.1 to (12)

$$|(b_{ii} - k_i + 1)\, e_i(t)| = |b_{ii} - k_i + 1|\, |e_i(t)| \leq \max |e_i(t)| \tag{13}$$

and

$$ess \sup |\beta(t)| = \beta = 1 \leq \alpha, \tag{14}$$

hence

$$\lim_{t \to \infty} e_i(t) = 0, \ (1 \leq i \leq n)\,, \tag{15}$$

and from the fact $\lim_{t \to \infty} \|e(t)\| = 0$, we conclude that systems (4) and (5) are globally synchronized.

Proposition 2.1 *The stability criterion of Theorem 2.1 implies Lyapunov stabilization criterion.*

Proof. Assume that systems (4) and (5) are globally synchronized with the criterion of Theorem 2.1, and we consider the following Lyapunov function:

$$V\,(e(t)) = \sum_{i=1}^{n} \frac{1}{2} e_i^2(t), \tag{16}$$

we get

$$\begin{aligned} \dot{V}\,(e(t)) &= \textstyle\sum_{i=1}^{n} \dot{e}_i(t) e_i(t) \\ &= \textstyle\sum_{i=1}^{n} \left(-e_i(t) + (b_{ii} - k_i + 1)\, e_i(t)\right) e_i(t) \\ &= \textstyle\sum_{i=1}^{n} (b_{ii} - k_i)\, e_i^2(t) \end{aligned}$$

and by Theorem 2.1, we have

$$-2 < b_{ii} - k_i < 0, \quad 1 \leq i \leq n, \tag{17}$$

then $\dot{V}\,(e(t)) < 0$, and the implication is verified.

3 Synchronization Criterion for Identical Systems

Now, we consider the master system and the slave system in the following forms

$$X(t) = AX(t) + f(X(t)), \tag{18}$$

$$Y(t) = AY(t) + f(Y(t)) + U, \tag{19}$$

where $X(t) \in \mathbb{R}^n$, $Y(t) \in \mathbb{R}^n$ are the state vectors of the master system and the slave system, respectively, A is the $n \times n$ matrix of parameters of system, $f = (f_i(X(t)))_{1 \le i \le n}$, such that f_i are continuous nonlinear scalar functions and verifying the following condition

$$|f_i(Y(t)) - f_i(X(t))| \le \rho_i |y_i(t)) - x_i(t)|, \quad 1 \le i \le n, \tag{20}$$

where $0 < \rho_i < 1$ and U is the vector controller to be determined.

The error system between the master system (18) and the slave system (19), can be derived as folow:

$$\dot{e}(t) = Ae(t) + f(Y(t)) - f(X(t)) + U, \tag{21}$$

To achieve synchronization between systems (18) and (19), we choose the vector controller as

$$U = (C - K)\, e(t), \tag{22}$$

where $C = (c_{ij}) \in \mathbb{R}^{n \times n}$, such that:

$$c_{ij} = \begin{cases} -a_{ij}, & if \quad i \neq j, \\ 0, & if \quad i = j, \end{cases} \tag{23}$$

and $K = diag\,(k_1, k_2, ..., k_n) \in \mathbb{R}^{n \times n}$ is unknown control diagonal matrix to be designed later.

By substituting Eq.(22) into Eq.(21), one can obtain the following formula for the error system:

$$\dot{e}(t) = (A + C - K)\, e(t) + f(Y(t)) - f(X(t)). \tag{24}$$

Theorem 3.1 *If the control constants $(k_i)_{1 \le i \le n}$ are chosen such that*

$$a_{ii} + \rho_i < k_i < 2 + a_{ii} - \rho_i, \quad 1 \le i \le n, \tag{25}$$

then the two systems (18) and (19) are globally synchronized.

Proof. According to the same procedure as in the proof of Theorem 2.1, the Eq. (24) provides the following scalar equations

$$\dot{e}_i(t) = -e_i(t) + (a_{ii} - k_i + 1)\, e_i(t) + f_i(Y(t)) - f_i(X(t)), \quad 1 \le i \le n, \tag{26}$$

and we can see that Eq.(26) is the same as Eq.(3) with: $z\,(t) = e_i(t)$, $\alpha = 1$, $\beta\,(t) = 1$ and $h\,(t, z\,(t)) = (a_{ii} - k_i + 1)\, e_i(t) + f_i(Y\,(t)) - f_i(X\,(t)$.

Thus, we apply Lemma 1.1 to Eq.(26) and by using Theorem 3.1, we obtain

$$|h\,(t, z\,(t))| \quad \le (|a_{ii} - k_i + 1| + \rho_i)\, |e_i\,(t)| \quad \le \max\left(|e_i\,(t)|\right), \tag{27}$$

and we have also

$$ess \sup |\beta(t)| = 1 \le \alpha, \tag{28}$$

Hence

$$\lim_{t \to \infty} e_i\,(t) = 0, \quad 1 \le i \le n, \tag{29}$$

implying $\lim_{t \to \infty} \|e\,(t)\| = 0$, i.e., systems (18) and (19) are globally synchronized.

4 Numerical Examples and Simulations

In this section, to demonstrate the use of chaos synchronization criterion proposed herein, two numerical examples are considered.

4.1 Example 1

Here, as the master system we consider the Chen system [14] described by

$$\begin{cases} \dot{x}_1 = a\left(x_2 - x_1\right), \\ \dot{x}_2 = \left(c - a\right)x_1 + cx_2 - x_1x_3, \\ \dot{x}_3 = -bx_3 + x_1x_2, \end{cases} \tag{30}$$

when $a = 35$, $b = 3$ and $c = 28$, the Chen system has chaotic attractor.

As the slave system, we consider the controlled Lü system [15] described by

$$\begin{cases} \dot{y}_1 = \alpha(y_2 - y_1) + u_1, \\ \dot{y}_2 = \beta y_2 - y_1y_3 + u_2, \\ \dot{y}_3 = -\gamma y_3 + y_1y_2 + u_3, \end{cases} \tag{31}$$

where u_1, u_2, u_3 are synchronization controllers and when $\alpha = 36$, $\beta = 3$, $\gamma = 20$, the Lü system is chaotic.

Corollary 4.1 *For the two coupled Chen system and Lü system, if $(k_i)_{1\leq i\leq 3}$ are chosen such that the inequalities: $-36 < k_1 < -34$, $3 < k_2 < 5$ and $-20 < k_3 < -18$, holds. Then they are globally synchronized.*

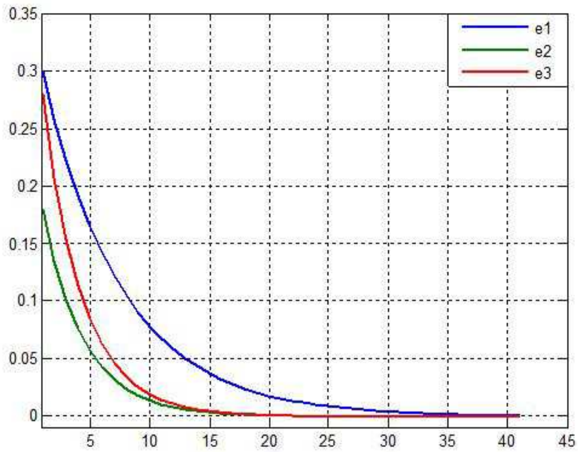

Figure 1: Time evolution of synchronization errors between the master system (30) and the slave system (31).

4.2 Example 2

Now, as the master system we consider the hyperchaotic Lü system [16], described by

$$\begin{cases} \dot{x}_1 = a\left(x_2 - x_1\right) + x_4, \\ \dot{x}_2 = cx_2 - x_1 x_3, \\ \dot{x}_3 = -bx_3 + x_1 x_2, \\ \dot{x}_4 = x_1 x_3 + dx_4, \end{cases} \tag{32}$$

when $a = 36$, $b = 3$, $c = 20$ and $-0.35 < d \leq 1.3$, the 4D Lü system has hyperchaotic attractor.

As the slave system, we consider the controlled hyperchaotic Chen system [17] described by

$$\begin{cases} \dot{y}_1 = b_1\left(y_2 - y_1\right) + u_1, \\ \dot{y}_2 = 4\left(y_1 + y_4\right) + b_2 y_2 - 10 y_1 y_3 + u_2, \\ \dot{y}_3 = -b_3 y_3 + y_2^2 + u_3, \\ \dot{y}_4 = -b_4 y_1 + u_4, \end{cases} \tag{33}$$

where u_1, u_2, u_3 and u_4.are synchronization controllers. The 4D Chen system is hyperchaotic when the parameter values are taken as $b_1 = 35$, $b_2 = 21$, $b_3 = 3$, $b_4 = 2$.

Corollary 4.2 *For the two coupled, hyperchaotic Lü system and hyperchaotic Chen system, if $(k_i)_{1\leq i\leq 4}$ are chosen such that the inequalities: $-35 < k_1 < -33$, $21 < k_2 < 23$, $-3 < k_3 < -1$ and $0 < k_4 < 2$, hold. Then they are globally synchronized.*

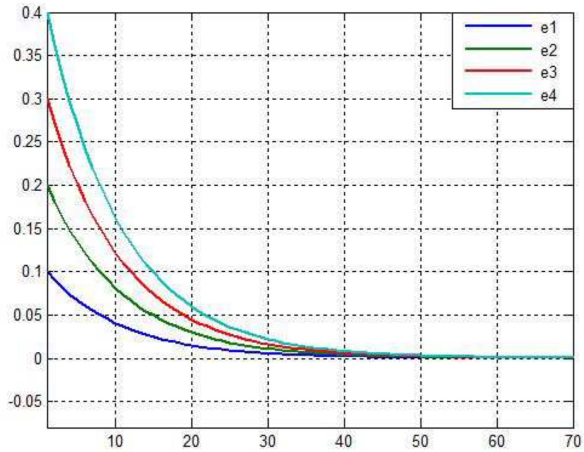

Figure 2: Time evolution of synchronization errors between the master system (32) and the slave system (33).

5 Conclusion

In this paper, using Lemma 1.1, a new criterion was derived. It was also demonstrated that this criterion can be applied to some chaotic and hyperchaotic systems. Finally, we remark that in the first case when the chaotic systems are different the controller is taken in a nonlinear form, but in the case of identical systems the controller is linear.

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Using Dynamic Vibration Absorber for Stabilization of a Double Pendulum Oscillations

V.E. Puzyrev and N.V. Savchenko *

Institute of Applied Mathematics and Mechanics of NAS of Ukraine, Roza Luksemburg st., 74, Donetsk, 83114, Ukraine

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Abstract: In this paper a stability problem for double pendulum is discussed. A damping device of passive type is used to stabilize small free oscillations of perturbed system. The simplified approach is suggested to prove the asymptotic stability of equilibrium.

Keywords: *double pendulum; dynamic vibration absorber; asymptotic stability; Lagrange function.*

Mathematics Subject Classification (2010): 34C46, 34D20, 70E50, 70E55, 70K20.

1 Introduction

The double pendulum may be considered as a simplified model of the coupled rigid bodies and finds wide use in engineering and technology. Both mathematical and physical interest to this model arises from the phenomena of its motion. Although this motion is described by rather simple ODE system, the pendulum exhibits the dynamical behavior which may be complex and unpredictable [1,2]. In particular, the motion of the double pendulum has the ability of ***beats*** and is strongly sensitive to the initial perturbations. These perturbations may provoke an increased amplitude of the second limb oscillations and, as a result, the switch from regular regime to chaotic one [3,4].

The problem of elimination or reduction of the undesired vibration in various technical systems has a long history and great achievements [5], mostly during the last century. For this purpose the damping devices are used, which may be divided into active and passive dampers. The classical example of passive damper is a dynamic vibration absorber (DVA) [6,7] or vibration neutralizer. It represents the mechanical appendage comprising inertia, stiffness, and damping elements and is connected to a given structure, named herein the primary [5] or original [8] system, with the aim to absorb the excessive vibratory energy. A DVA may be used both in cases of free oscillations and vibrations caused by harmonic excitations. For the case of a simple pendulum, DVA was used in papers [8,9].

* Corresponding author: mailto:nina_savchenko@hotmail.com

2 Description of the Model

Consider the double pendulum with distributed mass (Fig. 1) which has a fixed point O and is in a gravitational field. Assume that the mass center of the first limb is located at C_1. At the point O_1 located on the axis OC_1 a second limb is pivotally attached. The point C_2 is mass center of the second link. The first limb (configuration A) is attached with a dynamic absorber with stiffness k and damping coefficient h. The absorber oscillates along the axis O_2x', which is orthogonal to the line OO_1 and intersects it at the point O_2. Hinges at the points $O,\ O_1$ are supposed frictionless.

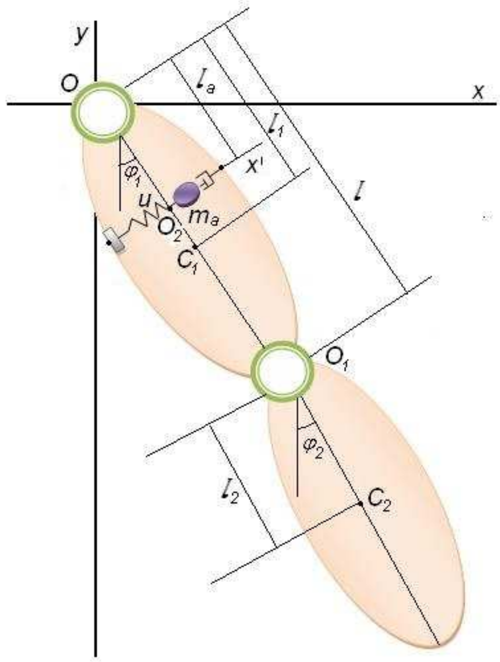

Figure 1: Double pendulum with dynamic vibration absorber in first limb.

Let us write the Lagrange function for the described mechanical system. One can get the kinetic energy K of the system in the form

$$K = K_p + K_a,$$

where K_p, K_a are the kinetic energies of the primary system (pendulum without absorber) and vibration absorber, respectively, calculated by the formulas

$$K_p = \frac{1}{2}[J_1\dot{\varphi}_1^2 + J_2\dot{\varphi}_2^2 + m_2l^2\dot{\varphi}_1^2 + 2m_2ll_2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2)],$$

$$K_a = \frac{1}{2}m_a[\dot{\varphi}_1^2(l_a^2 + u^2) + 2l_a\dot{\varphi}_1\dot{u} + \dot{u}^2].$$

Here J_1, J_2 are the moments of inertia of the first and second limbs of pendulum with respect to poles $O,\ O_1$ respectively, m_1, m_2, m_a are the masses of the first and second links, and absorber respectively, u is the extension of the spring, φ_1, φ_2 are the angles of deflection of the pendulum limbs about a vertical axis, l is the length of the first limb, l_1, l_2 are the distances from the suspension points of each of the links to its mass center, l_a is the distance OO_2.

The potential energy can be written as

$$\Pi = -g\cos\varphi_1(m_a l_a + m_1 l_1 + m_2 l) - m_2 l_2 g\cos\varphi_2 + m_a g u\sin\varphi_1 + \frac{1}{2}ku^2.$$

The equations of motion can be written in the form of Lagrange

$$\begin{gathered}(J_1 + m_2 l^2 + m_a l_a^2)\ddot{\varphi}_1 + m_2 l l_2 \ddot{\varphi}_2\cos(\varphi 1 - \varphi 2) + m_2 l l_2 \dot{\varphi}_2(\dot{\varphi}_1 - \dot{\varphi}_2)\sin(\varphi_1 - \varphi_2) + \\ + m_a l_a \ddot{u} + g\sin\varphi_1(m_1 l_1 + m_2 l + m_a l_a) + m_a g u\cos\varphi_1 = 0, \\ J_2\ddot{\varphi}_2 + m_2 l l_2 \ddot{\varphi}_1\cos(\varphi_1 - \varphi_2) - m_2 l l_2 \dot{\varphi}_1(\dot{\varphi}_1 + \dot{\varphi}_2)\sin(\varphi_1 - \varphi_2) + m_2 g l_2\sin\varphi_2 = 0, \\ m_a l_a \ddot{\varphi}_1 + m_a \ddot{u} + m_a g\sin\varphi_1 + ku = -h\dot{u}.\end{gathered} \tag{2.1}$$

Let us define the conditions of stability of motion of the system (2.1) when the pendulum is in the lower position of equilibrium, i.e. solution

$$\varphi_1 = 0, \varphi_2 = 0, u = 0, \dot{\varphi}_1 = 0, \dot{\varphi}_2 = 0, \dot{u} = 0. \tag{2.2}$$

3 Stabilization Conditions

Firstly, we write the linear approximation of the system (2.1)

$$\begin{gathered}(J_1 + m_2 l^2 + m_a l_a^2)\ddot{\varphi}_1 + m_2 l l_2 \ddot{\varphi}_2 + m_a l_a \ddot{u} + g(m_1 l_1 + m_2 l + m_a l_a)\varphi_1 + m_a g u = 0, \\ J_2\ddot{\varphi}_2 + m_2 l l_2 \ddot{\varphi}_1 + m_2 g l_2 \varphi_2 = 0, \\ m_a l_a \ddot{\varphi}_1 + m_a \ddot{u} + m_a g \varphi_1 + ku = -h\dot{u}.\end{gathered} \tag{3.1}$$

We introduce the dimensionless parameters by the formulas

$$\begin{gathered}\widetilde{m}_a = \frac{m_a}{m_1},\ \widetilde{m}_2 = \frac{m_2}{m_1},\ \widetilde{l}_a = \frac{l_a}{l_1},\ \widetilde{l} = \frac{l}{l_1},\ \widetilde{l}_2 = \frac{l_2}{l_1},\ \tau = \sqrt{\frac{g}{l_1}}t, \\ \widetilde{k} = \frac{kl_1}{m_1 g},\ \widetilde{h} = \frac{h}{m_1}, \widetilde{u} = \frac{u}{l_1}.\end{gathered} \tag{3.2}$$

The system (3.1) can be rewritten as

$$\begin{gathered}(J_1 + \widetilde{m}_2\widetilde{l}^2 + \widetilde{m}_a\widetilde{l}_a^2)\widetilde{\varphi}_1'' + \widetilde{m}_2\widetilde{l}\widetilde{l}_2\widetilde{\varphi}_2'' + \widetilde{m}_a\widetilde{l}_a\widetilde{u}'' + (1 + \widetilde{m}_2\widetilde{l} + \widetilde{m}_a\widetilde{l}_a)\widetilde{\varphi}_1 + \widetilde{m}_a\widetilde{u} = 0, \\ \widetilde{J}_2\widetilde{\varphi}_2'' + \widetilde{m}_2\widetilde{l}\widetilde{l}_2\widetilde{\varphi}_1'' + \widetilde{m}_2\widetilde{l}_2\widetilde{\varphi}_2 = 0, \\ \widetilde{m}_a\widetilde{l}_a\widetilde{\varphi}_1'' + \widetilde{m}_a\widetilde{u}'' + \widetilde{m}_a\widetilde{\varphi}_1 + \widetilde{k}\widetilde{u} = -\widetilde{h}\widetilde{u}'.\end{gathered} \tag{3.3}$$

For simplicity, we omit the symbol $\sim$ in what follows.

To investigate the problem of the stability of motion (2.2) we will use the results from [10] below.

Suppose that the motion equations of a mechanical system are described by the following system of differential equations

$$\boldsymbol{A}\ddot{\boldsymbol{q}} + \boldsymbol{B}\dot{\boldsymbol{q}} + \boldsymbol{C}\boldsymbol{q} = \boldsymbol{F}(t, \dot{\boldsymbol{q}}, \boldsymbol{q})\dot{\boldsymbol{q}}_1 + \boldsymbol{N}(t, \dot{\boldsymbol{q}}, \boldsymbol{q}), \tag{3.4}$$

where square matrices $\boldsymbol{A}$, $\boldsymbol{C}$ of order $m+n$, and $\boldsymbol{F}(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ of order m are symmetric, square matrix $\boldsymbol{B}$ is skew-symmetric, $\boldsymbol{q} = (\boldsymbol{q}_1, \boldsymbol{q}_2)^T$, i.e. vector $\boldsymbol{q}$ is divided into sub-vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ with orders m, n respectively. Denotation "T" means transposition, vector $N(t,\dot{\boldsymbol{q}},\boldsymbol{q})$ represents a set of arbitrary nonlinear terms. Dependence on t is periodic or quasi-periodic.

We assume that the system provides steady motion:

$$\boldsymbol{q} = 0, \;\; \dot{\boldsymbol{q}} = 0. \tag{3.5}$$

It is supposed that the matrix $\boldsymbol{F}_0 = \boldsymbol{F}(t,0,0)$ is positive definite for $t \geq 0$. Denote by $\boldsymbol{d}, \boldsymbol{d}_{22}$ the linear differential operators

$$\boldsymbol{d} = \boldsymbol{A}\frac{d^2}{dt^2} + (\boldsymbol{B}+\boldsymbol{F}_0)\frac{d}{dt} + \boldsymbol{C}, \;\; \boldsymbol{d}_{22} = \boldsymbol{A}_{22}\frac{d^2}{dt^2} + \boldsymbol{B}_{22}\frac{d}{dt} + \boldsymbol{C}_{22},$$

and $\boldsymbol{D}(\lambda), \boldsymbol{D}_{22}(\lambda)$ are the corresponding λ-matrices:

$$\boldsymbol{D}(\lambda) = \boldsymbol{A}\lambda^2 + (\boldsymbol{B}+\boldsymbol{F}_0)\lambda + \boldsymbol{C}, \;\; \boldsymbol{D}_{22}(\lambda) = \boldsymbol{A}_{22}\lambda^2 + \boldsymbol{B}_{22}\lambda + \boldsymbol{C}_{22}.$$

Let λ_0 be an eigenvalue of $\boldsymbol{d}_{22}$, and $\boldsymbol{\gamma}_{20}$ be the corresponding eigenvector. Introduce the equality

$$\boldsymbol{D}_{12}(\lambda_0)\boldsymbol{\gamma}_{20} = 0. \tag{3.6}$$

Theorem 3.1 *Let us consider a mechanical system whose motion equations are discribed by (3.4) and suppose that none of the eigenvectors of operator $\boldsymbol{d}_{22}$ satisfies condition (3.6). Then adding to system an arbitrary dissipative force, which provides full dissipation (by linear terms) on $\dot{\boldsymbol{q}}_1$ leads to the following results:*

I) If all eigenvalues of matrix $\boldsymbol{C}$ are positive, then equilibrium (3.5) becomes asymptotically stable. Stability is exponential and uniform.

II) If matrix $\boldsymbol{C}$ has some negative eigenvalues, then equilibrium (3.5) is unstable, even if it was stabilized before by gyroscopic forces. Among particular solutions of the system at least one has negative Liapunov characteristic number.

According to the above statements, matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ ($\boldsymbol{B} = \boldsymbol{0}$) for the system (3.3) take the following form

$$\boldsymbol{A} = \begin{pmatrix} J_1 + m_2 l^2 + m_a l_a^2 & m_2 l l_2 & m_a l_a \\ m_2 l l_2 & J_2 & 0 \\ m_a l_a & 0 & m_a \end{pmatrix},$$

$$\boldsymbol{C} = \begin{pmatrix} 1 + m_2 l + m_a l_a & 0 & m_a \\ 0 & m_2 l_2 & 0 \\ m_a & 0 & k \end{pmatrix}.$$

To verify condition (3.6) one may investigate the compatibility of the following system

$$\begin{gathered} [\lambda^2(J_1 + m_2 l^2 + m_a l_a^2) + 1 + m_2 l + m_a l_a]\gamma_1 + \lambda^2 m_2 l l_2 \gamma_2 = 0, \\ \lambda^2 m_2 l l_2 \gamma_1 + (\lambda^2 J_2 + m_2 l_2)\gamma_2 = 0, \\ (\lambda^2 l_a + 1)\gamma_1 = 0. \end{gathered} \tag{3.7}$$

The third equation of (3.7) implies that $\lambda^2 = -1/l_a$. Then the condition of compatibility of the system (3.7) takes the form

$$\delta_1 = (m_2 l_2 + m_2^2 l l_2) l_a^2 - (J_1 m_2 l_2 + m_2^2 l^2 l_2 + m_2 l J_2 + J_2) l_a + \\ + J_1 J_2 + m_2 l^2 J_2 - m_2^2 l^2 l_2^2 = 0. \quad (3.8)$$

Choosing an arbitrary l_a ($l_a \leq l$), excluding the value which transforms (3.8) into true equality, we obtain an inconsistent system (3.7). Consequently, the conditions of the theorem are satisfied and we have asymptotic stability of the studied solution.

For more clarity let us compare the results obtained with the standart procedure based on the Routh–Hurwitz criterion [11].

Characteristic equation of system (3.3) is written in the form

$$a_0\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0,$$

where the coefficients are given by the formulas

$$a_0 = m_a[J_1J_2 + m_2 l^2(J_2 - m_2 l_2^2)],\ a_1 = h[m_2 l^2(J_2 - m_2 l_2^2) + J_2(J_1 + l_a^2 m_a)],$$
$$a_2 = m_2 l^2 k(J_2 - m_2 l_2^2) + J_1 J_2 k + [m_2 l_2(J_1 + m_2 l^2) + J_2(1 + m_2 l)]m_a +$$
$$+ J_2 m_a l_a(k l_a - m_a),\ a_3 = h[J_2 + J_1 m_2 l_2 + m_2 l(J_2 + m_2 l l_2) + m_a l_a(J_2 + l_a m_2 l_2)],$$
$$a_4 = k[m_2 l_2(J_1 + m_2 l^2) + J_2(1 + m_2 l)] + (1 + m_2 l)m_a m_2 l_2 - J_2 m_a^2 +$$
$$+(J_2 k - m_a m_2 l_2) l_a m_a + m_2 l_2 k m_a l_a^2,\ a_5 = m_2 l_2 h(1 + m_2 l + m_a l_a),$$
$$a_6 = m_2 l_2 k(1 + m_2 l) + m_a m_2 l_2(k l_a - m_a).$$

The solution of the system will be asymptotically stable if and only if the following conditions hold

$$a_0 > 0,\ a_3 > 0,\ a_5 > 0,\ a_6 > 0,\ \Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0,$$

$$\Delta_5 = \begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 \\ 0 & a_6 & a_5 & a_4 & a_3 \\ 0 & 0 & 0 & a_6 & a_5 \end{vmatrix} > 0. \quad (3.9)$$

It is not hard to see that $a_0,\ a_3,\ a_5,\ a_6$ are positive.

$$\Delta_3 = h^2 m_a^2 \Delta_{30} = h^2 m_a^2 (p_0 - 2p_1 l_a + p_2 l_a^2 + m_a l_a^4 m_2^4 l_2^4 l^2),$$

where

$$p_0 = J_2[l^2 m_2(J_2 - m_2 l_2^2) + J_1 J_2]^2,$$
$$p_1 = (m_2^3 l_2^3 l^2 + m_2 l J_2^2 + J_2^2)[l^2 m_2(J_2 - m_2 l_2^2) + J_1 J_2],$$
$$p_2 = J_2(1 + m_2 l)[J_2^2(1 + m_2 l) + 2m_2^3 l_2^3 l^2] + m_2^4 l^2 l_2^4(m_2 l^2 + J_1).$$

Let us transform the expression for Δ_{30} to the following form

$$\Delta_{30} = p_0(l_a - \frac{p_1}{p_2})^2 + m_2^4 l_2^4 l^2 (J_1 J_2 + m_2 l^2 J_2 - m_2^2 l^2 l_2^2)^3 + m_a m_2^4 l_2^4 l^2 l_a^4.$$

So, it is obviously positive, because of $p_0 > 0$, $J_2 \geq m_2 l_2^2$.

The determinant Δ_5 can be represented as $\Delta_5 = m_2^4 l^2 l_2^4 h^3 m_a^4 \delta_1^2$.

Obviously, the conditions of criterion Routh–Hurwitz for system (3.3) are always satisfied, except for $\delta_1 = 0$.

Therefore, $\delta_1 \neq 0$ is a necessary and sufficient condition for asymptotic stability of motion of the system (2.1). That is, selecting a value of parameter l_a that does not satisfy (3.7), we can achieve the exponential stability of a double pendulum motion with additionally introduced mass.

Consider the case where the vibration absorber is located in the second link of the pendulum (Fig. 2).

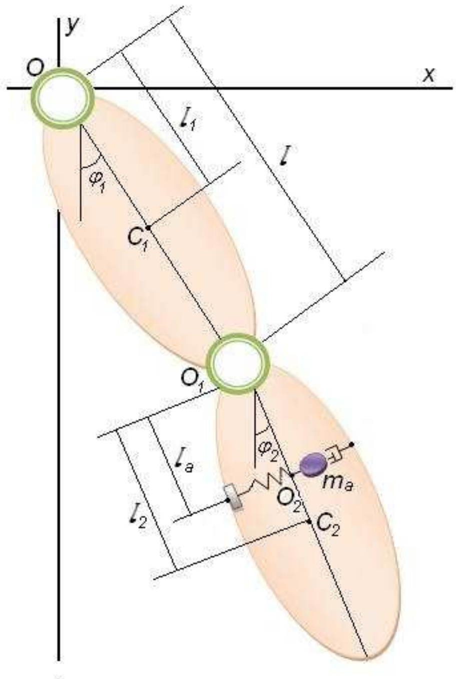

Figure 2: Double pendulum with dynamic vibration absorber in second limb.

In this situation, the choice of dimensionless parameters should be replaced by m_1 to m_2 and l_1 to l_2. Then the matrices take the form

$$\boldsymbol{A} = \begin{pmatrix} J_1 + l^2 + m_a l^2 & l + m_a l l_a & m_a l \\ l + m_a l l_a & J_2 + m_a l_a^2 & m_a l_a \\ m_a l & m_a l_a & m_a \end{pmatrix},$$

$$\boldsymbol{C} = \begin{pmatrix} m_1 l_1 + l + m_a l & 0 & 0 \\ 0 & 1 + m_a l_a & m_a \\ 0 & m_a & k \end{pmatrix}.$$

Obtain a system of conditions

$$[\lambda^2(J_1 + l^2 + m_a l^2) + m_1 l_1 + l + m_a l]\gamma_1 + \lambda^2(l + m_a l l_a)\gamma_2 = 0,$$

$$\lambda^2(l+m_a l l_a)\gamma_1+[\lambda^2(J_2+m_a l_a^2)+1+m_a l_a]\gamma_2=0, \tag{3.10}$$
$$\lambda^2 l\gamma_1+\gamma_2\lambda^2 l_a+\gamma_2=0.$$

To check the consistency of the system express from the second equation (3.10)

$$\gamma_2=-\frac{\lambda^2 l(1+m_a l_a)\gamma_1}{\lambda^2 J_2+\lambda^2 m_a l_a^2+1+m_a l_a}.$$

Upon substituting this expression into the third equation (3.10) we obtain

$$\frac{\lambda^2 l\gamma_1(\lambda^2 J_2+2+2m_a l_a-\lambda^2 l_a)}{\lambda^2 J_2+\lambda^2 m_a l_a^2+1+m_a l_a}=0,$$

whence $\lambda^2=-2(1+m_a l_a)/(J_2-l_a)$.

The condition of compatibility of the system of (3.10) can be represented in the form

$$\delta_2=l_a^3 m_a[(4l^2+4J_1+2l)m_a+2l+2m_1l_1]-l_a^2[(6l^2+2lJ_2)m_a^2-(l-2lJ_2-2m_1l_1J_2+$$
$$+6J_1+6l^2)m_a-l-m_1l_1]+l_a[(2l^2m_a^2+2J_1m_a+2m_al^2)J_2-10m_al^2+2J_1+$$
$$+2l^2]-(m_al+m_1l_1+l)J_2^2+(2J_1+2m_al^2+2l^2)J_2-4l^2=0. \tag{3.11}$$

Similarly to the first case by selecting l_a that does not satisfy equality (3.11) asymptotic stability of the studied solutions can be obtained.

It is possible to verify that the conditions of asymptotic stability obtained by using the Routh–Hurwitz criterion, are also satisfied for $\delta_2 \neq 0$.

Remark 3.1 In the case when equality (3.7) or (3.11) holds, this fact does not prevent the asymptotic stability of equilibrium. The linear approximation has a pair of pure imaginary roots, and we get the critical case in Liapunov sense. To prove the asymptotic stability, the Liapunov function may be constructed [8]. This function is a sum of positively defined quadratic form and form of fourth order and has negative derivative on time. Basically, this procedure is not difficult, but it leads to extremely huge analytical expressions for coefficients of the function (and its derivative) and cannot be given here.

Remark 3.2 The approach employed to prove the asymptotic stability of the motion is relatively simple and much more easier than the use of determinants or innors technique. However, it does not provide the estimation of the damping rate for perturbed oscillations of primary system. For this purpose our approach can be modified, or added by some special evaluating procedure. Obviously, in exchange for this gain, it (approach) will lose a part of simplicity.

We don't discuss now the problem about choice of absorbers parameters with the aim to optimize the decaying rate. For arbitrary set of the pendulum parameters this problem leads to extrema problem for function of high order and, probably, has no explicit finite solution. However, if the pendulum mass distribution is given, numerical calculations may help. Our simulations witness, that configuration B with small distance l_a is a bet, and values k, h strongly depend on primary system parameters. For example, with $m_1=m_2=m$, $l_1=l_2=l, J_1=J_2=ml^2$, $\widetilde{m}_a=2m/5$, for configuration A one gets $\widetilde{l}_a=0.552$, $\widetilde{k}=0.45, \widetilde{h}=0.463$, and $\sigma=max\{Re\lambda_j\}\approx -0.0140$. For configuration B corresponding values are $\widetilde{l}_a=0.05, \widetilde{k}=0.486, \widetilde{h}=0.234$, and $\sigma=max\{Re\lambda_j\}\approx$ ≈ -0.0943.

4 Conclusion

In the paper we prove that attaching a DVA to double pendulum stabilizes its equilibrium i.e. provides the exponential stability. Special simple procedure to verify the conditions of stabilization is applied. Some aspects of the optimal absorber's configuration are discussed.

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Time-Fractional Generalized Equal Width Wave Equations: Formulation and Solution via Variational Methods

Youwei Zhang *

Department of Mathematics, Hexi University, Gansu 734000, China

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Abstract: This paper presents the formulation of the time-fractional generalized Equal Width Wave (EWW) equation and generalized Equal Width Wave-Burgers (EWW-Burgers) equation using the Euler-Lagrange variational technique in the Riemann-Liouville derivative sense, and derive respectively an approximate solitary wave solution. Our results witness that He's variational-iteration method was very efficient and powerful technique in finding the solution of the proposed equation.

Keywords: *Riemann-Liouville fractional operator; Euler-Lagrange equation; fractional EWW equation; He's variational-iteration method; solitary wave.*

Mathematics Subject Classification (2010): 35R11, 35G20.

1 Introduction

The generalized EWW equation has been used to describe approximately the unidirectional propagation of the regularized long wave in certain nonlinear dispersive systems [1], and has been proposed by Benjamin, Bona and Mahony as a model for small-amplitude long waves on the surface of water in a channel [2]. In physical situations one has unidirectional waves propagating in a water channel, long-crested waves in near-shore zones and many others. This equation also serves as an alternative model to the generalized regularised long wave equation and generalized Korteweg-de Vries equation (KdV) [3–5].

During the past three decades or so, fractional calculus has gained considerable popularity and importance as generalizations of integer-order evolution equations, and is applied to model problems in neurons, hydrology, viscoelasticity and rheology, image processing, mechanics, mechatronics, physics, finance and control theory, see [6–11]. If

* Corresponding author: mailto:ywzhang0288@163.com

the Lagrangian of conservative system is constructed using fractional derivatives, the resulting equations of motion can be nonconservative. Therefore, in many cases, the real physical processes could be modeled in a reliable manner using fractional-order differential equations rather than integer-order equations [12]. In [13], the semi-inverse method has been used to derive the Lagrangian of the KdV equation, the time operator of the Lagrangian of the KdV equation has been transformed into fractional domain in terms of the left-Riemann-Liouville fractional differential operator, the variational of the functional of this Lagrangian leads neatly to Euler-Lagrange equation. Based on the stochastic embedding theory, Cresson [14] defined the fractional embedding of differential operators and provided a fractional Euler-Lagrange equation for Lagrangian systems, then investigated a fractional Noether theorem and a fractional Hamiltonian formulation of fractional Lagrangian systems. Herzallah and Baleanu [15] presented the necessary and sufficient optimality conditions for the Euler-Lagrange fractional equations of fractional variational problems with determining in which spaces the functional must exist. Malinowska [16] proposed the Euler-Lagrange equations for fractional variational problems with multiple integrals and proved the fractional Noether-type theorem for conservative and nonconservative generalized physical systems. Riewe [17] formulated a version of the Euler-Lagrange equation for problems of calculus of variation with fractional derivatives. Wu and Baleanu [18] developed some new variational-iteration formulae to find approximate solutions of fractional differential equations and determined the Lagrange multiplier in a more accurate way. For generalized fractional Euler-Lagrange equations we can refer to the works by Odzijewicz [19, 20]. Other known results can be found in Agrawal [21–23], Baleanu et al [24], Inokuti et al [25] and Zhang [26]. In view of the fact that most of physical phenomena may be considered as nonconservative, they can be described using fractional-order differential equations. Recently, several methods have been used to solve nonlinear fractional evolution equation using techniques of nonlinear analysis, such as Adomian decomposition method [27], homotopy analysis method [28,29] and homotopy perturbation method [30]. It was mentioned that the variational-iteration method has been used successfully to solve different types of integer and fractional nonlinear evolution equations. Making use of the variational-iteration method, this work's main motivation is to formulate the time-fractional generalized EWW equation and generalized EWW-Burgers equation and to derive an approximate solitary wave solution, respectively.

This paper is organized as follows: Section 2 states some background material from fractional calculus. Section 3 presents the principle of He's variational-iteration method. Sections 4 and 5 are devoted to describing the formulation of the time-fractional generalized EWW equation and generalized EWW-Burgers equation using the Euler-Lagrange variational technique and to deriving an approximate solitary wave solution, respectively. Section 6 makes some analysis for the obtained graphs and figures and discusses the present work.

2 Preliminaries

We recall the necessary definitions for the fractional calculus (see, e.g. [31–33]) which is used throughout the remaining sections of this paper.

Definition 2.1 A real multivariable function $\varphi(x, t)$, $t > 0$ is said to be in the space C_γ, $\gamma \in \mathbb{R}$, with respect to t if there exists a real number $p > \gamma$, such that

$\varphi(x,t) = t^p\varphi_1(x,t)$, where $\varphi_1 \in C(\Omega \times T)$, $\Omega \subseteq \mathbb{R}$ and $T = [0,t_0](t_0 > 0)$. Obviously, $C_\gamma \subset C_\delta$ if $\delta \leq \gamma$.

Definition 2.2 The left-hand side Riemann-Liouville fractional integral of a function $\varphi \in C_\gamma$, $(\gamma \geq -1)$ is defined by

$$
\begin{aligned}
{}_0I_t^\alpha \varphi(x,t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\varphi(x,\tau)d\tau, \quad \alpha > 0, \quad t \in T,\\
{}_0I_t^0 \varphi(x,t) &= \varphi(x,t).
\end{aligned}
$$

Definition 2.3 The Riemann-Liouville fractional derivatives of the order $n-1 \leq \alpha < n$ of a function $\varphi \in C_\gamma$, $(\gamma \geq -1)$ are defined as

$$
\begin{aligned}
{}_0D_t^\alpha \varphi(x,t) &= \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_0^t (t-\tau)^{n-\alpha-1}\varphi(x,\tau)d\tau,\\
{}_tD_{t_0}^\alpha \varphi(x,t) &= \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_t^{t_0} (\tau-t)^{n-\alpha-1}\varphi(x,\tau)d\tau, \quad t \in T.
\end{aligned}
$$

Lemma 2.1 *The integration of Riemann-Liouville fractional derivative of the order $0 < \alpha < 1$ of the functions φ, ϕ, ${}_tD_{t_0}^\alpha\varphi(x,t)$ and ${}_0D_t^\alpha\phi(x,t) \in C(\Omega \times T)$ by parts are given by the rule*

$$
\int_T \varphi(x,t){}_0D_t^\alpha\phi(x,t)dt = \int_T \phi(x,t){}_tD_{t_0}^\alpha\varphi(x,t)dt.
$$

Definition 2.4 The Riesz fractional integral of the order $n-1 \leq \alpha < n$ of a function $\varphi \in C_\gamma$, $(\gamma \geq -1)$ is defined as

$$
{}_0^RI_t^\alpha\varphi(x,t) = \frac{1}{2}\big({}_0I_t^\alpha\varphi(x,t) + {}_tI_{t_0}^\alpha\varphi(x,t)\big) = \frac{1}{2\Gamma(\alpha)}\int_0^{t_0} |t-\tau|^{\alpha-1}\varphi(x,\tau)d\tau,
$$

where ${}_0I_t^\alpha$ and ${}_tI_{t_0}^\alpha$ are respectively the left- and right-hand side Riemann-Liouville fractional integral operators.

Definition 2.5 The Riesz fractional derivative of the order $n-1 \leq \alpha < n$ of a function $\varphi \in C_\gamma$, $(\gamma \geq -1)$ is defined by

$$
\begin{aligned}
{}_0^RD_t^\alpha\varphi(x,t) &= \frac{1}{2}\big({}_0D_t^\alpha\varphi(x,t) + (-1)^n{}_tD_{t_0}^\alpha\varphi(x,t)\big)\\
&= \frac{1}{2\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^{t_0} |t-\tau|^{n-\alpha-1}\varphi(x,\tau)d\tau,
\end{aligned}
$$

where ${}_0D_t^\alpha$ and ${}_tD_{t_0}^\alpha$ are respectively the left- and right-hand side Riemann-Liouville fractional differential operators.

Lemma 2.2 *Let $\alpha > 0$ and $\beta > 0$ be such that $n-1 < \alpha < n$, $m-1 < \beta < m$ and $\alpha+\beta < n$, and let $\varphi \in L_1(\Omega \times T)$ and ${}_0I_t^{m-\alpha}\varphi \in AC^m(\Omega \times T)$. Then we have the following index rule:*

$$
{}_0^RD_t^\alpha\big({}_0^RD_t^\beta\varphi(x,t)\big) = {}_0^RD_t^{\alpha+\beta}\varphi(x,t) - \sum_{i=1}^m {}_0^RD_t^{\beta-i}\varphi(x,t)|_{t=0}\frac{t^{-\alpha-i}}{\Gamma(1-\alpha-i)}.
$$

Remark 2.1 One can express the Riesz fractional differential operator ${}_0^RD_t^{\alpha-1}$ of the order $0 < \alpha < 1$ as the Riesz fractional integral operator ${}_0^RI_\tau^{1-\alpha}$, i.e.

$$
{}_0^RD_t^{\alpha-1}\varphi(x,t) = {}_0^RI_t^{1-\alpha}\varphi(x,t), \quad t \in T.
$$

3 Variational-iteration Method

The variational-iteration method [34–36] provides an effective procedure for explicit and solitary wave solutions of a wide and general class of differential systems representing real physical problems. Moreover, the variational-iteration method can overcome the foregoing restrictions and limitations of approximate techniques so that it provides us with a possibility to analyze strongly nonlinear evolution equations. Therefore, we extend this method to solve the time-fractional generalized EWW equation. The basic features of the variational-iteration method are outlined as follows.

Considering a nonlinear evolution equation consists of a linear part $\mathcal{L}u$, nonlinear part $\mathcal{N}u$, and a free term $f = f(x,t)$ represented as

$$\mathcal{L}u + \mathcal{N}u = f. \tag{1}$$

According to the variational-iteration method, the $n+1$-th approximate solution of (1) can be read using iteration correction functional as

$$u_{n+1} = u_n + \int_0^t \lambda(\tau)\big(\mathcal{L}\tilde{u} + \mathcal{N}\tilde{u} - f\big)d\tau, \tag{2}$$

where $\lambda(\tau)$ is a general Lagrange's multiplier, which can be identified via the variational theory and $\tilde{u}$ is considered as a restricted variation function which means $\delta\tilde{u} = 0$. Extreming the variation of the correction functional (2) leads to the Lagrangian multiplier $\lambda(\tau)$. The initial iteration u_0 can be used as the initial value $u(x,0)$, as n tends to infinity, the iteration leads to the solitary wave solution of (1), i.e.

$$u = \lim_{n\to\infty} u_n.$$

4 Time-fractional Generalized EWW Equation

In this section, He's variational-iteration method is applied to solve time-fractional generalized EWW equation

$${}_0^R D_t^\alpha u + a u^p u_x - \mu u_{xxt} = 0,$$

where $a \neq 0$, p and $\mu > 0$, $u = u(x,t)$ is a field variable, $x \in \Omega \subseteq \mathbb{R}$ is a space coordinate in the propagation direction of the field and $t \in T = [0, t_0](t_0 > 0)$ is the time, the subscripts denote the partial differentiation of the function u with respect to the parameter x and t, ${}_0^R D_t^\alpha$ is the Riesz fractional derivative.

The generalized EWW equation in (1+1) dimensions is given as

$$u_t + a u^p u_x - \mu u_{xxt} = 0. \tag{3}$$

Employing a potential function v on the field variable, set $u = v_x$ yields the potential equation of the generalized EWW equation (3) in the form,

$$v_{xt} + a v_x^p v_{xx} - \mu v_{xxxt} = 0. \tag{4}$$

The Lagrangian of this generalized EWW equation (3) can be defined using the semi-inverse method [37, 38] as follows. The functional of the potential equation (4) can be represented as

$$J(v) = \int_\Omega dx \int_T \Big(v\big(c_1 v_{xt} + a c_2 v_x^p v_{xx} - \mu c_3 v_{xxxt}\big)\Big)dt, \tag{5}$$

with c_i $(i = 1, 2, 3)$ as an unknown constant to be determined later. Integrating (5) by parts and taking $v_x|_{\partial\Omega} = v_x|_{\partial T} = v_{xxt}|_{\partial\Omega} = 0$ yield

$$J(v) = \int_\Omega dx \int_T \Big(- c_1 v_t v_x - \frac{ac_2}{p+1} v_x^{p+2} + \mu c_3 v_{xxt} v_x \Big) dt. \tag{6}$$

The constants c_i $(i = 1, 2, 3)$ can be determined taking the variation of the functional (6) to make it optimal. By applying the variation of the functional, integrating each term by parts, and making use of the variation optimum condition of the functional $J(v)$, it yields the following representation

$$2c_1 v_{tx} + (p + 2)ac_2 v_x^p v_{xx} - 2\mu c_3 v_{xxxt} = 0. \tag{7}$$

Note that the obtained result (7) is equivalent to (4), so one has that the constants c_i $(i = 1, 2, 3)$ are respectively

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{p+2}, \quad c_3 = \frac{1}{2}.$$

In addition, the functional representation given by (6) obtains directly the Lagrangian form of the generalized EWW equation,

$$L(v_t, v_x, v_{xxt}) = -\frac{1}{2} v_t v_x - \frac{a}{(p+1)(p+2)} v_x^{p+2} + \frac{\mu}{2} v_{xxt} v_x.$$

Similarly, the Lagrangian of the time-fractional version of the generalized EWW equation could be read as

$$F({}_0D_t^\alpha v, v_x, v_{xxt}) = -\frac{1}{2} {}_0D_t^\alpha v v_x - \frac{a}{(p+1)(p+2)} v_x^{p+2} + \frac{\mu}{2} v_{xxt} v_x, \quad \alpha \in]0, 1]. \tag{8}$$

Then the functional of the time-fractional generalized EWW equation will take the representation

$$J(v) = \int_\Omega dx \int_T F({}_0D_t^\alpha v_t, v_x, v_{xxt}) dt, \tag{9}$$

where the time-fractional Lagrangian $F({}_0D_t^\alpha v_t, v_x, v_{xx}, v_{xxt}, v_{xxx})$ is given by (8). Following Agrawal's method [21–23], the variation of functional (9) with respect to v leads to

$$\delta J(v) = \int_\Omega dx \int_T \Big(\frac{\partial F}{\partial {}_0D_t^\alpha v} \delta({}_0D_t^\alpha v) + \frac{\partial F}{\partial v_x} \delta v_x + \frac{\partial F}{\partial v_{xxt}} \delta v_{xxt} \Big) dt. \tag{10}$$

By Lemma 2.1, upon integrating the right-hand side of (10), one has

$$\delta J(v) = \int_\Omega dx \int_T \Big({}_tD_T^\alpha \big(\frac{\partial F}{\partial {}_0D_t^\alpha v} \big) - \frac{\partial}{\partial x} \big(\frac{\partial F}{\partial v_x} \big) - \frac{\partial^3}{\partial x^2 \partial t} \big(\frac{\partial F}{\partial v_{xxt}} \big) \Big) \delta v dt,$$

noting that $\delta v|_{\partial T} = \delta v|_{\partial\Omega} = \delta v_x|_{\partial\Omega} = \delta v_{xx}|_{\partial T} = 0$.

Obviously, optimizing the variation of the functional $J(v)$, i.e., $\delta J(v) = 0$, yields the Euler-Lagrange equation for time-fractional generalized EWW equation in the following representation

$${}_tD_T^\alpha \big(\frac{\partial F}{\partial {}_0D_t^\alpha v} \big) - \frac{\partial}{\partial x} \big(\frac{\partial F}{\partial v_x} \big) - \frac{\partial^3}{\partial x^2 \partial t} \big(\frac{\partial F}{\partial v_{xxt}} \big) = 0. \tag{11}$$

Substituting the Lagrangian of the time-fractional generalized EWW equation (8) into Euler-Lagrange formula (11) gives

$$-\frac{1}{2}{}_tD_{T_0}^{\alpha}v_x + \frac{1}{2}{}_0D_t^{\alpha}v_x + av_x^p v_{xx} - \mu v_{xxxt} = 0.$$

Once again, substituting the potential function v_x for u, yields the time-fractional generalized EWW equation for the state function u as

$$\frac{1}{2}\left({}_0D_t^{\alpha}u - {}_tD_{T_0}^{\alpha}u\right) + au^p u_x - \mu u_{xxt} = 0. \tag{12}$$

According to the Riesz fractional derivative ${}_0^RD_t^{\alpha}u$, the time-fractional generalized EWW equation represented in (12) can be written as

$${}_0^RD_t^{\alpha}u + au^p u_x - \mu u_{xxt} = 0. \tag{13}$$

Acting from the left-hand side by the Riesz fractional operator ${}_0^RD_t^{1-\alpha}$ on (13) leads to

$$\frac{\partial}{\partial t}u - {}_0^RD_t^{\alpha-1}u|_{t=0}\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + {}_0^RD_t^{1-\alpha}\Big(au^p u_x - \mu u_{xxt}\Big) = 0, \tag{14}$$

from Lemma 2.2. In view of the variational-iteration method, combining with (14), the $n+1$-th approximate solution of (13) can be read using iteration correction functional as

$$\begin{aligned} u_{n+1} = u_n + \int_0^t \lambda(\tau)\bigg(&\frac{\partial}{\partial\tau}u_n - {}_0^RD_\tau^{\alpha-1}u_n|_{\tau=0}\frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} \\ &+ {}_0^RD_\tau^{1-\alpha}\Big(a\tilde{u}_n^p\frac{\partial}{\partial x}\tilde{u}_n - \mu\frac{\partial^3}{\partial x^2\partial t}\tilde{u}_n\Big)\bigg)d\tau, \end{aligned} \tag{15}$$

where the function $\tilde{u}_n$ is considered as a restricted variation function, i.e., $\delta\tilde{u}_n = 0$. The extreme of the variation of (15) subject to the restricted variation function straightforwardly yields

$$\delta u_{n+1} = \delta u_n + \int_0^t \lambda(\tau)\delta\frac{\partial}{\partial\tau}u_n d\tau = \delta u_n + \lambda(\tau)\delta u_n|_{\tau=t} - \int_0^t \frac{\partial}{\partial\tau}\lambda(\tau)\delta u_n d\tau = 0.$$

This representation reduces the following stationary conditions

$$\frac{\partial}{\partial\tau}\lambda(\tau) = 0, \quad 1 + \lambda(\tau) = 0,$$

which converted to the Lagrangian multiplier at $\lambda(\tau) = -1$. Therefore, the correction functional (15) takes the following form

$$u_{n+1} = u_n - \int_0^t\bigg(\frac{\partial}{\partial\tau}u_n - {}_0^RI_\tau^{1-\alpha}u_n|_{\tau=0}\frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} + {}_0^RD_\tau^{1-\alpha}\Big(au_n^p\frac{\partial}{\partial x}u_n - \mu\frac{\partial^3}{\partial x^2\partial t}u_n\Big)\bigg)d\tau, \tag{16}$$

since $\alpha - 1 < 0$, the fractional derivative operator ${}_0^RD_t^{\alpha-1}$ reduces to fractional integral operator ${}_0^RI_t^{1-\alpha}$ by Remark 2.1.

In view of the right-hand side Riemann-Liouville fractional derivative is interpreted as a future state of the process in physics. For this reason, the right-derivative is usually neglected in applications, when the present state of the process does not depend on the results of the future development, and so the right-derivative is used equal to zero in the following calculations. The zero order solitary wave solution can be taken as the initial value of the state variable, which is taken in this case as

$$u_0(x,t) = u(x,0) = \Big(\frac{(p+1)(p+2)c}{2a}\mathrm{sech}^2\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\Big)^{\frac{1}{p}},$$

where c and x_0 are constants.

Substituting this zero order solitary wave solution into (16) and using the Definition 2.5 lead to the first order solitary wave solution

$$\begin{aligned} u_1(x,t) = {} & \Big(\frac{(p+1)(p+2)c}{2a}\mathrm{sech}^2\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\Big)^{\frac{1}{p}} + \frac{t^\alpha}{\Gamma(\alpha+1)}\frac{a}{\sqrt{\mu}}\Big(\frac{(p+1)(p+2)c}{2a}\Big)^{\frac{1+p}{p}} \\ & \times \sinh\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\mathrm{sech}^{\frac{2+3p}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big). \end{aligned}$$

Substituting first order solitary wave solution into (16) and using the Definition 2.5 then lead to the second order solitary wave solution in the following form

$$\begin{aligned} u_2(x,t) = {} & \Big(\frac{(p+1)(p+2)c}{2a}\mathrm{sech}^2\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\Big)^{\frac{1}{p}} \\ & + \frac{t^\alpha}{\Gamma(\alpha+1)}\frac{(p+1)^2(p+2)^2c^2}{2ap}\Big(\frac{(p+1)(p+2)c}{2a}\mathrm{sech}^2\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\Big)^{\frac{1-p}{p}} \\ & \times \sinh\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\mathrm{sech}^5\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big) \\ & - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big(\frac{(p+1)(p+2)c}{2a}\Big)^{\frac{1+p}{p}}\Bigg(\frac{acp(p+1)(p+2)}{4\mu}\mathrm{sech}^{\frac{2+4p}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big) \\ & - \frac{ac(p+1)(p+2)(2+3p)}{4\mu}\sinh^2\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\mathrm{sech}^{\frac{2+6p}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big) \\ & - \frac{a^2c(p+1)(p+2)}{\sqrt{\mu}}\sinh^2\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\mathrm{sech}^{\frac{2+6p}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\Bigg) \\ & - \frac{t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)}\frac{ap(p+1)^2(p+2)^2c^2}{8\mu\sqrt{\mu}}\Big(\frac{(p+1)(p+2)c}{2a}\Big)^{\frac{1+p}{p}} \\ & \times \Bigg(p\mathrm{sech}^{\frac{2}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big) - (2+3p)\sinh^2\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big) \\ & \times \mathrm{sech}^{\frac{2+2p}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\Bigg)\sinh\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\mathrm{sech}^7\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big) \end{aligned}$$

$$
\begin{aligned}
&+\frac{t^{2\alpha-1}a}{2\sqrt{\mu}\Gamma(2\alpha)}\Big(\frac{(p+1)(p+2)c}{2a}\Big)^{\frac{1+p}{p}}\Bigg((-3p-4p^2)\sinh\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\\
&\times\operatorname{sech}^{\frac{2+3p}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)+(1+2p)(2+3p)\sinh^3\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\\
&\times\operatorname{sech}^{\frac{2+5p}{p}}\big(\frac{p}{2\sqrt{\mu}}(x-x_0)\big)\Bigg).
\end{aligned}
$$

Making use of Definition 2.5 and the Maple or Mathematics and substituting $n-1$ order solitary wave solution into (16), lead to the solitary wave solution u_3, u_4, $\ldots$, u_n, $\ldots$. As n tends to infinity, the iteration leads to the solitary wave solution of the time-fractional generalized EWW equation

$$
u(x,t)=\lim_{n\to\infty}u_n=\Big(\frac{(p+1)(p+2)c}{2a}\operatorname{sech}^2\big(\frac{p}{2\sqrt{\mu}}(x-ct-x_0)\big)\Big)^{\frac{1}{p}}.
$$

Selecting the appropriate values of p, a, μ, c and x_0, we can present the distribution function u as a 3-dimensions graph and 2-dimensions graph to the approximate solitary wave solution.

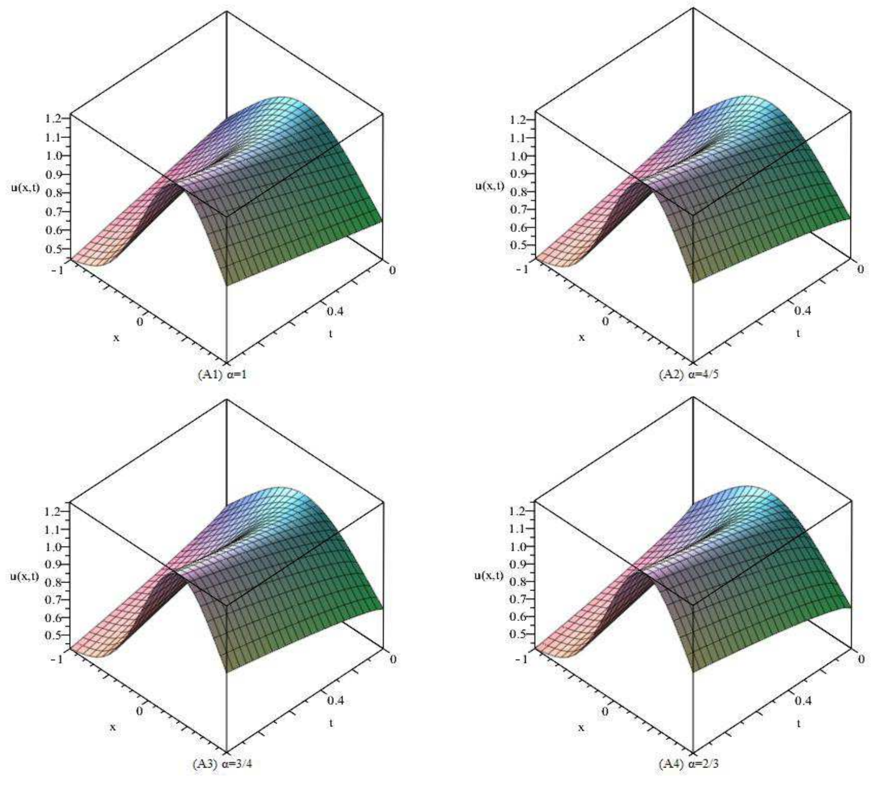


Figure 1: The distribution function u as a 3-dimensions graph for different order α.

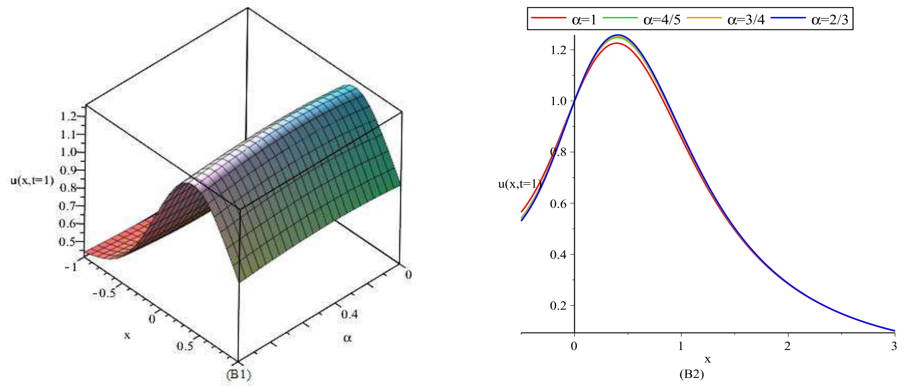


Figure 2: The distribution function u as a function of space x at time $t = 1$ for different order α: (B1) 3-dimensions graph, (B2) 2-dimensions graph.

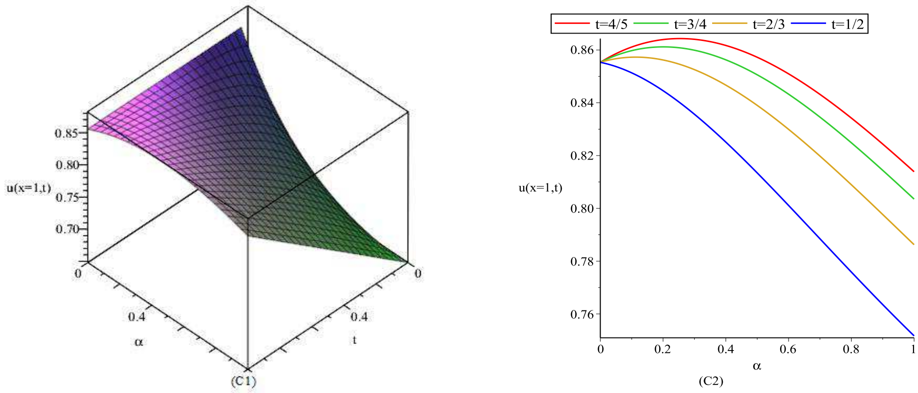


Figure 3: The distribution function u as a function of time t at space $x = 1$ of the different order α: (C1) 3-dimensions graph, (C2) 2-dimensions graph.

5 Time-fractional Generalized EWW-Burgers Equation

In this section, He's variational-iteration method is applied to solve time-fractional generalized EWW-Burgers equation

$$ {}_0^R D_t^{\alpha} u + a u^p u_x - \lambda u_{xx} - \mu u_{xxt} = 0. $$

The generalized EWW-Burgers equation in (1+1) dimensions is given as

$$ u_t + a u^p u_x - \lambda u_{xx} - \mu u_{xxt} = 0. \tag{17} $$

Employing a potential function v on the field variable, and setting $u = v_x$ yield the potential equation of the generalized EWW-Burgers equation (17) in the form,

$$ v_{xt} + a v_x^p v_{xx} - \lambda v_{xxx} - \mu v_{xxxt} = 0. \tag{18} $$

The Lagrangian of this generalized EWW-Burgers equation (17) can be defined using the semi-inverse method as follows. The functional of the potential equation (18) can be represented as

$$J(v) = \int_\Omega dx \int_T \Big(v\big(d_1 v_{xt} + a d_2 v_x^p v_{xx} - \lambda d_3 v_{xxx} - \mu d_4 v_{xxxt}\big)\Big) dt, \tag{19}$$

with d_i $(i = 1, 2, 3, 4)$ as an unknown constant to be determined later. Integrating (19) by parts and taking $v_x|_{\partial\Omega} = v_x|_{\partial T} = v_{xxt}|_{\partial\Omega} = 0$ yield

$$J(v) = \int_\Omega dx \int_T \Big(- d_1 v_t v_x - \frac{a d_2}{p+1} v_x^{p+2} + \lambda d_3 v_{xx} v_x + \mu d_4 v_{xxt} v_x \Big) dt. \tag{20}$$

The constants d_i $(i = 1, 2, 3, 4)$ can be determined taking the variation of the functional (20) to make it optimal. By applying the variation of the functional, integrating each term by parts, and making use of the variation optimum condition of the functional $J(v)$, yield the following representation

$$2d_1 v_{tx} + (p+2) a d_2 v_x^p v_{xx} - 2\lambda d_3 v_{xxx} - 2\mu d_4 v_{xxxt} = 0. \tag{21}$$

Notice that the obtained result (21) is equivalent to (18), so one has that the constants d_i $(i = 1, 2, 3, 4)$ are respectively

$$d_1 = \frac{1}{2}, \quad d_2 = \frac{1}{p+2}, \quad d_3 = d_4 = \frac{1}{2}.$$

In addition, the functional representation given by (20) obtains directly the Lagrangian form of the generalized EWW-Burgers equation,

$$L(v_t, v_x, v_{xx}, v_{xxt}) = -\frac{1}{2} v_t v_x - \frac{a}{(p+1)(p+2)} v_x^{p+2} + \frac{\lambda}{2} v_{xx} v_x + \frac{\mu}{2} v_{xxt} v_x.$$

Similarly, the Lagrangian of the time-fractional version of the generalized EWW-Burgers equation could be read as

$$\begin{aligned} F({}_0D_t^\alpha v, v_x, v_{xx}, v_{xxt}) = -\frac{1}{2} {}_0D_t^\alpha v v_x - \frac{a}{(p+1)(p+2)} v_x^{p+2} + \frac{\lambda}{2} v_{xx} v_x + \frac{\mu}{2} v_{xxt} v_x, \\ \alpha \in]0, 1]. \end{aligned} \tag{22}$$

Then the functional of the time-fractional generalized EWW-Burgers equation will take the form

$$J(v) = \int_\Omega dx \int_T F({}_0D_t^\alpha v_t, v_x, v_{xx}, v_{xxt}) dt, \tag{23}$$

where the time-fractional Lagrangian $F({}_0D_t^\alpha v_t, v_x, v_{xx}, v_{xxt}, v_{xxx})$ is given by (22). Following Agrawal's method, the variation of functional (23) with respect to v leads to

$$\delta J(v) = \int_\Omega dx \int_T \Big(\frac{\partial F}{\partial {}_0D_t^\alpha v} \delta({}_0D_t^\alpha v) + \frac{\partial F}{\partial v_x} \delta v_x + \frac{\partial F}{\partial v_{xx}} \delta v_{xx} + \frac{\partial F}{\partial v_{xxt}} \delta v_{xxt} \Big) dt. \tag{24}$$

By Lemma 2.1, upon integrating the right-hand side of (24), one has

$$\delta J(v) = \int_\Omega dx \int_T \Big({}_tD_T^\alpha \big(\frac{\partial F}{\partial {}_0D_t^\alpha v}\big) - \frac{\partial}{\partial x}\big(\frac{\partial F}{\partial v_x}\big) + \frac{\partial^2}{\partial x^2}\big(\frac{\partial F}{\partial v_{xx}}\big) - \frac{\partial^3}{\partial x^2 \partial t}\big(\frac{\partial F}{\partial v_{xxt}}\big) \Big) \delta v dt,$$

noting that $\delta v|_{\partial T} = \delta v|_{\partial\Omega} = \delta v_x|_{\partial\Omega} = \delta v_{xx}|_{\partial T} = 0$.

Obviously, optimizing the variation of the functional $J(v)$, i.e., $\delta J(v) = 0$, yields the Euler-Lagrange equation for time-fractional generalized EWW-Burgers equation in the following form

$$ {}_tD_T^\alpha\Big(\frac{\partial F}{\partial_0 D_t^\alpha v}\Big) - \frac{\partial}{\partial x}\Big(\frac{\partial F}{\partial v_x}\Big) + \frac{\partial^2}{\partial x^2}\Big(\frac{\partial F}{\partial v_{xx}}\Big) - \frac{\partial^3}{\partial x^2 \partial t}\Big(\frac{\partial F}{\partial v_{xxt}}\Big) = 0. \tag{25} $$

Substituting the Lagrangian of the time-fractional generalized EWW-Burgers equation (22) into Euler-Lagrange formula (25) one obtains

$$ -\frac{1}{2}{}_tD_{T_0}^\alpha v_x + \frac{1}{2}{}_0D_t^\alpha v_x + av_x^p v_{xx} - \lambda v_{xxx} - \mu v_{xxxt} = 0. $$

Once again, substituting the potential function v_x for u, yields the time-fractional generalized EWW-Burgers equation for the state function u as

$$ \frac{1}{2}\big({}_0D_t^\alpha u - {}_tD_{T_0}^\alpha u\big) + au^p u_x - \lambda u_{xx} - \mu u_{xxt} = 0. \tag{26} $$

According to the Riesz fractional derivative ${}_0^RD_t^\alpha u$, the time-fractional generalized EWW-Burgers equation represented in (26) can be written as

$$ {}_0^RD_t^\alpha u + au^p u_x - \lambda u_{xx} - \mu u_{xxt} = 0. \tag{27} $$

Acting from the left-hand side by the Riesz fractional operator ${}_0^RD_t^{1-\alpha}$ on (27) leads to

$$ \frac{\partial}{\partial t}u - {}_0^RD_t^{\alpha-1}u|_{t=0}\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + {}_0^RD_t^{1-\alpha}\Big(au^p u_x - \lambda u_{xx} - \mu u_{xxt}\Big) = 0, \tag{28} $$

from Lemma 2.2. In view of the variational-iteration method, combining with (28), the $n+1$-th approximate solution of (27) can be read using iteration correction functional as

$$ \begin{aligned} u_{n+1} = u_n + \int_0^t \lambda(\tau)\Big(&\frac{\partial}{\partial \tau}u_n - {}_0^RD_\tau^{\alpha-1}u_n|_{\tau=0}\frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} \\ &+ {}_0^RD_\tau^{1-\alpha}\Big(a\tilde{u}_n^p\frac{\partial}{\partial x}\tilde{u}_n - \lambda\frac{\partial^2}{\partial x^2}\tilde{u}_n - \mu\frac{\partial^3}{\partial x^2\partial t}\tilde{u}_n\Big)\Big)d\tau, \end{aligned} \tag{29} $$

where the function $\tilde{u}_n$ is considered as a restricted variation function, i.e., $\delta\tilde{u}_n = 0$. By the same argument as in Section 4, it is converted to the Lagrangian multiplier at $\lambda(\tau) = -1$. Therefore, the correction functional (29) takes the following form

$$ \begin{aligned} u_{n+1} = u_n - \int_0^t \Big(&\frac{\partial}{\partial \tau}u_n - {}_0^RI_\tau^{1-\alpha}u_n|_{\tau=0}\frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} \\ &+ {}_0^RD_\tau^{1-\alpha}\Big(au_n^p\frac{\partial}{\partial x}u_n - \lambda\frac{\partial^2}{\partial x^2}u_n - \mu\frac{\partial^3}{\partial x^2\partial t}u_n\Big)\Big)d\tau, \end{aligned} \tag{30} $$

since $\alpha - 1 < 0$, the fractional derivative operator ${}_0^RD_t^{\alpha-1}$ reduces to fractional integral operator ${}_0^RI_t^{1-\alpha}$ by Remark 2.1.

The zero order solitary wave solution can be taken as the initial value of the state variable, which is taken in this case as

$$u_0(x,t) = \Big(A - A\tanh\kappa(x-x_0) - \frac{A}{2}\mathrm{sech}^2\kappa(x-x_0)\Big)^{\frac{1}{p}}.$$

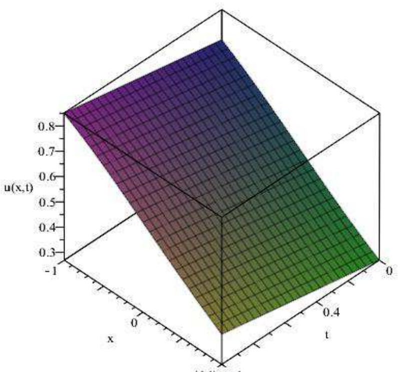
Substituting zero order solitary wave solution into (30) and using the Definition 2.5 lead to the first order solitary wave solution

$$\begin{aligned} u_1(x,t) = &\Big(A - A\tanh\kappa(x-x_0) - \frac{A}{2}\mathrm{sech}^2\kappa(x-x_0)\Big)^{\frac{1}{p}} \\ &- \frac{t^\alpha}{\Gamma(\alpha+1)}\Bigg[\frac{a}{p}\Big(A - A\tanh\kappa(x-x_0) - \frac{A}{2}\mathrm{sech}^2\kappa(x-x_0)\Big)^{\frac{1}{p}} \\ &\times\Big(A\kappa\mathrm{sech}^2\kappa(x-x_0)\tanh\kappa(x-x_0) - A\kappa\mathrm{sech}^2\kappa(x-x_0)\Big) \\ &- \frac{\lambda(1-p)}{p^2}\Big(A - A\tanh\kappa(x-x_0) - \frac{A}{2}\mathrm{sech}^2\kappa(x-x_0)\Big)^{\frac{1-2p}{p}} \\ &\times\Big(A\kappa\mathrm{sech}^2\kappa(x-x_0)\tanh\kappa(x-x_0) - A\kappa\mathrm{sech}^2\kappa(x-x_0)\Big)^2 \\ &- \frac{\lambda}{p}\Big(A - A\tanh\kappa(x-x_0) - \frac{A}{2}\mathrm{sech}^2\kappa(x-x_0)\Big)^{\frac{1-p}{p}} \\ &\times\Big(2A\kappa^2\mathrm{sech}^2\kappa(x-x_0)\tanh\kappa(x-x_0) - 2A\kappa^2\mathrm{sech}^2\kappa(x-x_0)\tanh^2\kappa(x-x_0) \\ &+ A\kappa^2\mathrm{sech}^4\kappa(x-x_0)\Big)\Bigg]. \end{aligned}$$

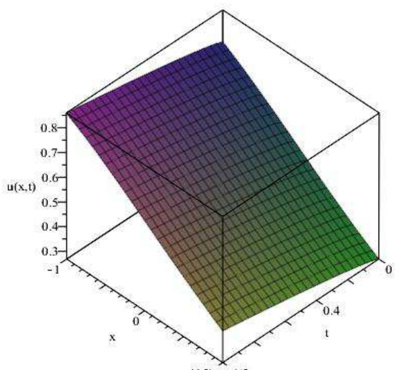
Substituting first order solitary wave solution into (30) and using the Definition 2.5 then lead to the solitary wave solution u_2, u_3, …, u_n, …. As n tends to infinity, the iteration leads to the solitary wave solution of the time-fractional generalized EWW-Burgers equation

$$u(x,t) = \lim_{n\to\infty} u_n = \Big(A - A\tanh\kappa(x-ct-x_0) - \frac{A}{2}\mathrm{sech}^2\kappa(x-ct-x_0)\Big)^{\frac{1}{p}}.$$

Selecting the appropriate values of $p, a, \lambda, \mu, A, \kappa, c$ and x_0, we can present the distribution function u as a 3-dimensions graph and 2-dimensions graph to the approximate solitary wave solution.



(A1) α=1



(A2) α=4/5

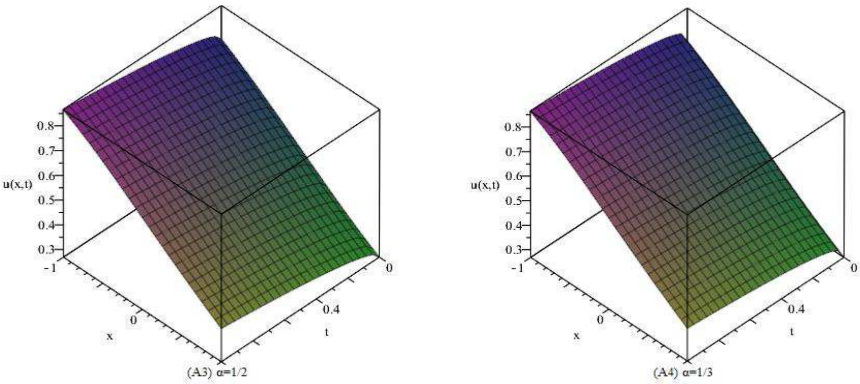


Figure 4: The distribution function u as a 3-dimensions graph for different order α.

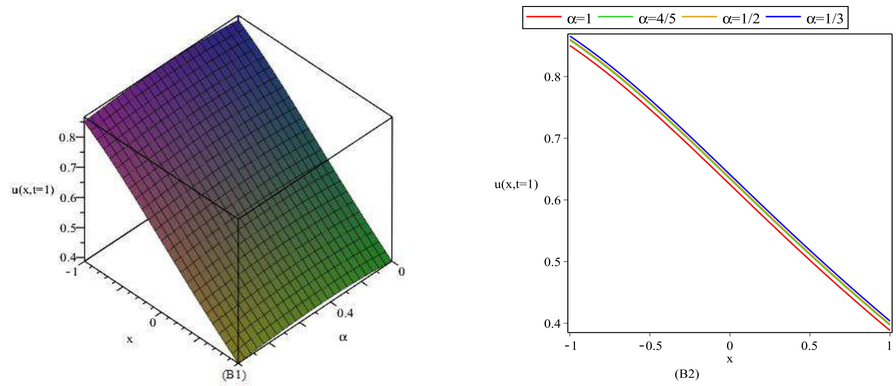


Figure 5: The distribution function u as a function of space x at time $t = 1$ for different order α: (B1) 3-dimensions graph, (B2) 2-dimensions graph.

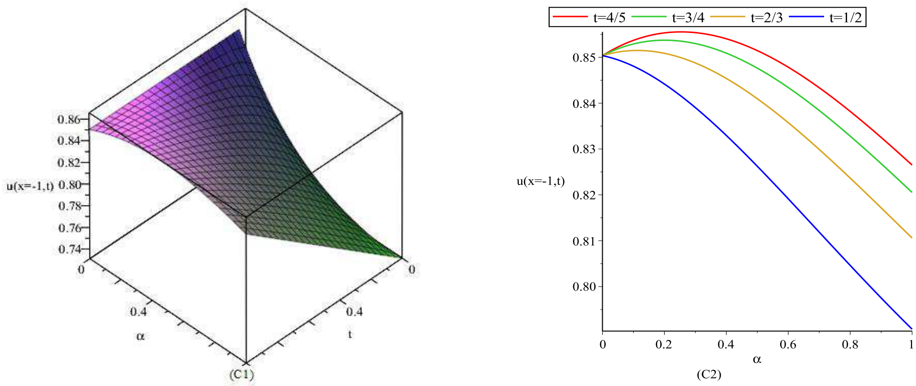


Figure 6: The distribution function u as a function of time t at space $x = -1$ for different order α: (C1) 3-dimensions graph, (C2) 2-dimensions graph.

6 Discussion

The purpose of the present work is to explore the effect of the fractional order derivative on the structure and propagation of the resulting solitary waves obtained from time-fractional generalized EWW equation. We derive the Lagrangian of the generalized EWW equation by the semi-inverse method, then take a similar form of Lagrangian to the time-fractional generalized EWW equation. Using the Euler-Lagrange variational technique, we continue our calculations until the high-order iteration. During this period, our approximate calculations are carried out concerning the solution of the time-fractional generalized EWW equation as well as generalized EWW-Burgers equation. The results of approximate solitary wave solution of time-fractional generalized EWW equation and generalized EWW-Burgers equation are obtained. In addition, 3-dimensional representation of the solution u for the time-fractional generalized EWW equation and generalized EWW-Burgers equation with space x and time t for different values of the order α are presented respectively in Figures 1 and 4, the solution u is still a single soliton wave solution for all values of the order α. It shows that the balancing scenario between nonlinearity and dispersion is still valid. Figures 2 and 5 present respectively the change of amplitude and width of the soliton due to the variation of the order α, 2- and 3-dimensional graphs depicted the behavior of the solution u at time $t = 1$ corresponding to different values of the order α. This behavior indicates that the increases of the value α increasing both the height and the width of the solitary wave solution. That is, the order α can be used to modify the shape of the solitary wave without change of the nonlinearity and the dispersion effects in the medium. Figures 3 and 6 are respectively devoted to studying the representation between the amplitude of the soliton and the fractional order at different time values, these figures show that at the same time, the increasing of the fractional α decreases the amplitude of the solitary wave to some value of α.

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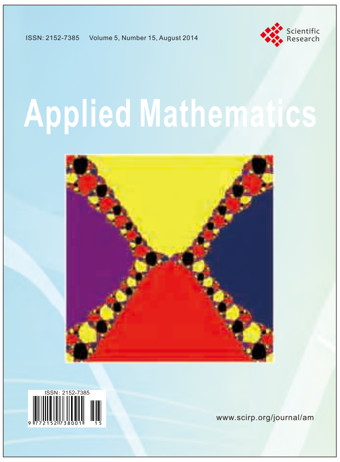

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