Nonlinear Dynamics and Systems Theory, **15** (2) (2015) 141–147

Global Stability of Phase Synchronization in Coupled Chaotic Systems

M. Boutefnouchet[1] H. Taghvafard[2] and G. H. Erjaee[2*]

[1] *Mathematics, Statistics and Physics Department, Qatar University, Doha, Qatar*
[2] *Mathematics Department, Shiraz University, Shiraz, Iran*

Received: June 25, 2014; Revised: April 5, 2015

Abstract: In analytical or numerical synchronizations studies of coupled chaotic systems, the phase synchronizations are less considered in the leading literatures. This paper is an attempt to find a sufficient analytical condition for the stability of phase synchronization in coupled chaotic systems. The method of nonlinear feedback function and the scheme of matrix measure have been used to justify this analytical stability, and tested numerically for the existence of the phase synchronization in some coupled chaotic systems.

Keywords: *chaos; phase synchronization; stability.*

Mathematics Subject Classification (2010): 37N35, 65P20, 65P99.

1 Introduction

Sensitivity to initial conditions is a generic feature of chaotic dynamical systems. Two chaotic systems starting from slightly different initial points in the state space separate away from each other with time. Therefore, how to control two chaotic systems to be synchronized has aroused a great deal of interest.

Recently, synchronization phenomena in coupled chaotic systems have received much attention [1–17]. Pecora and Carroll have shown [1–4] that in coupled chaotic systems a complete synchronization occurs if the difference between the various states of synchronized systems converges to zero. They have also shown that synchronization stability depends upon the signs of the conditional Lyapunov exponents: i.e., if all of the Lyapunov exponents of the response system under the action of the driver are negative, then there is a complete and stable synchronization between the drive and response systems.

* Corresponding author: mailto:erjaee@shirazu.ac.ir

Synchronization stability can also be verified using the Jacobian matrix in the linearized state difference between the drive and response chaotic systems [6]. Accordingly, despite the stability analysis in dynamical systems, if this Jacobian matrix is of full rank and all of its real parts of eigenvalues are negative, then the system will converge to zero, yielding complete synchronization.

The phenomenon of phase synchronization observed in systems of various nature [18, 19], including chemical, biological, and physiological systems, is today attracting much interest of researchers [19–21]. In this case, the Jacobian matrix has some zero eigenvalues and the difference between various states of synchronized systems may be not necessary converging to the zero, but will stay less than or equal to a constant. The main goal of this paper is to discuss the stability analysis of phase synchronization in coupled chaotic systems coupled by the nonlinear feedback function method [19]. Therefore, a brief discussion of the nonlinear coupling feedback function method is presented in Section 2, followed by the presentation of a criterion for the stability of synchronization in Section 3. In Section 4, we present some examples to corroborate our analytical assertion.

2 Description of the Method

There are different criteria for coupling two chaotic systems to be synchronized. In this paper, we apply the nonlinear coupling feedback function method introduced by Ali and Fang [19] to couple chaotic systems. Suppose $\dot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x}(t))$ is a chaotic system with $\mathbf{x}(t) \in \mathbb{R}^n$. Then decomposing vector-valued function $\mathbf{F}(t, \mathbf{x}(t))$ to a linear part, $\mathbf{L}(t, \mathbf{x}(t))$, and a nonlinear part, $\mathbf{N}(t, \mathbf{x}(t))$, yields

$$\mathbf{F}(t, \mathbf{x}(t)) = \mathbf{L}(t, \mathbf{x}(t)) + \mathbf{N}(t, \mathbf{x}(t)). \tag{1}$$

Now consider two chaotic systems, where their associated vector functions are decomposed as in (2) and coupled by using the nonlinear parts of their vector functions as follows:

$$\dot{\mathbf{x}}_1(t) = \mathbf{L}(t, \mathbf{x}_1(t)) - \mathbf{N}(t, \mathbf{x}_1(t)) + \alpha \left[\mathbf{N}(t, \mathbf{x}_1(t)) - \mathbf{N}(t, \mathbf{x}_2(t))\right], \tag{2}$$

$$\dot{\mathbf{x}}_2(t) = \mathbf{L}(t, \mathbf{x}_2(t)) - \mathbf{N}(t, \mathbf{x}_2(t)) + \alpha \left[\mathbf{N}(t, \mathbf{x}_2(t)) - \mathbf{N}(t, \mathbf{x}_1(t))\right]. \tag{3}$$

Here, systems (2) and (3) serve as drive and response systems, respectively, and α is the strength of their coupling. The synchronization stability of these two systems can be studied by using the evolutional equation of the difference between them, which is determined by the following linear approximation:

$$\dot{\mathbf{e}}(t) = \left[\mathbf{L}(t) + (2\alpha - 1)\frac{\partial \mathbf{N}(t, \mathbf{x}(t))}{\partial \mathbf{x}}\right]\mathbf{e}(t), \tag{4}$$

where $\mathbf{e}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$. Obviously, the stability type of the zero equilibrium in equation (4) shows the stability type of the synchronization between two chaotic systems. If $\mathbf{L}$ has full rank and $\alpha = 0.5$, we have

$$\dot{\mathbf{e}}(t) = \mathbf{L}(t)\mathbf{e}(t), \tag{5}$$

and then according to the stability analysis of the linear approximation in dynamical systems theory, synchronization between coupled chaotic systems (2) and (3) occurs if all eigenvalues of matrix $\mathbf{L}$ have negative real parts. Conversely, if matrix $\mathbf{L}$ does not have full rank: i.e., $\mathbf{L}$ has at least one zero eigenvalue, then we may yet have phase synchronization behavior.

3 Main Results

In this section, we present a stability criterion for synchronization. First, we introduce the concept of matrix measure. The matrix measure of a real square matrix $\mathbf{A} = (a_{ij})_{n\times n}$ is defined by

$$\mu_*(\mathbf{A}) = \lim_{\epsilon\to 0} \frac{\|\mathbf{I} + \epsilon\mathbf{A}\|_* - 1}{\epsilon},$$

where $\mathbf{I}$ is an $n \times n$ identity matrix and $\|\cdot\|_*$ is a matrix norm defined as follows:

$$\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^n |a_{ij}|, \qquad \|\mathbf{A}\|_2 = [\lambda \max(\mathbf{A}^T\mathbf{A})]^{1/2},$$

$$\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|, \qquad \|\mathbf{A}\|_\omega = \max_j \sum_{i=1}^n \frac{\omega_i}{\omega_j}|a_{ij}|,$$

where $\omega_i > 0$, we have the matrix measures

$$\mu_1(\mathbf{A}) = \max_j \Big\{a_{jj} + \sum_{i=1,\ i\neq j}^n |a_{ij}|\Big\}, \qquad \mu_2(\mathbf{A}) = \frac{1}{2}\lambda_{\max}(\mathbf{A}^T + \mathbf{A}),$$

$$\mu_\infty(\mathbf{A}) = \max_i \Big\{a_{ii} + \sum_{j=1,\ j\neq i}^n |a_{ij}|\Big\}, \qquad \mu_\omega(\mathbf{A}) = \max_j \Big\{a_{jj} + \sum_{i=1,\ i\neq j}^n \frac{\omega_i}{\omega_j}|a_{ij}|\Big\},$$

respectively.

Now suppose in error system (5), matrix $\mathbf{L}$ doesn't have a full rank and $\alpha = 0.5$. Then, as a consequence of the following theorem, we will show that under some conditions system (5) is globally asymptotically stable around a constant vector , on which $\mathbf{e}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$.

Theorem 3.1 *System (5) is globally asymptotically stable if there exists a non-singular time-varying matrix $\boldsymbol{B}(t)$ such that*

$$\lim_{t\to\infty} \exp\left(\int_{t_0}^t \mu_*(\dot{\boldsymbol{B}}\boldsymbol{B}^{-1} + \boldsymbol{B}\boldsymbol{L}\boldsymbol{B}^{-1})(s)ds\right) = 0,$$

for any $t_0 \geq 0$. Consequently, phase synchronization between systems (2) and (3) occurs which is globally asymptotically stable around a constant vector $\boldsymbol{c}$.

Proof. Let $\mathbf{e}(t)$ be a solution of error system (5) and $\mathbf{Y}(t) = \mathbf{B}(t)(\mathbf{e}(t) - \mathbf{c})$. Then for all $t \geq t_0$, we have

$$\begin{aligned}
D^+\|\mathbf{Y}(t)\|_* &= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left[\|\mathbf{Y}(t) + \epsilon\dot{\mathbf{Y}}(t)\|_* - \|\mathbf{Y}(t)\|_*\right] \\
&= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left[\left\|\mathbf{B}(t)(\mathbf{e}(t) - \mathbf{c}) + \epsilon\left(\dot{\mathbf{B}}(t)(\mathbf{e}(t) - \mathbf{c}) + \mathbf{B}\mathbf{L}(t)(\mathbf{e}(t) - \mathbf{c})\right)\right\|_*\right. \\
&- \left.\|\mathbf{B}(t)(\mathbf{e}(t) - \mathbf{c})\|_*\right] \\
&= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left[\left\|\mathbf{B}(t)(\mathbf{e}(t) - \mathbf{c}) + \epsilon(\dot{\mathbf{B}}\mathbf{B}^{-1} + \mathbf{B}\mathbf{L}\mathbf{B}^{-1})\mathbf{B}(t)(\mathbf{e}(t) - \mathbf{c})\right\|_*\right. \\
&- \left.\|\mathbf{B}(t)(\mathbf{e}(t) - \mathbf{c})\|_*\right] \\
&\leq \|\mathbf{B}(t)(\mathbf{e}(t) - \mathbf{c})\|_* \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left[\|\mathbf{I} + \epsilon\left(\dot{\mathbf{B}}\mathbf{B}^{-1} + \mathbf{B}\mathbf{L}\mathbf{B}^{-1}\right)\|_* - 1\right] \\
&= \|\mathbf{Y}(t)\|_*\mu_*\left(\dot{\mathbf{B}}\mathbf{B}^{-1} + \mathbf{B}\mathbf{L}\mathbf{B}^{-1}\right).
\end{aligned}$$

By integrating both sides of $D^+\|\mathbf{Y}(t)\|_* \le \|\mathbf{Y}(t)\|_*\mu_*\left(\dot{\mathbf{B}}\mathbf{B}^{-1} + \mathbf{B}\mathbf{L}\mathbf{B}^{-1}\right)$ from t_0 to t, we obtain

$$\|\mathbf{B}(t)(\mathbf{e}(t) - \mathbf{c})\|_* \le \|\mathbf{B}(0)(\mathbf{e}(0) - \mathbf{c})\|_* \exp\left(\int_{t_0}^{t} \mu_*(\dot{\mathbf{B}}\mathbf{B}^{-1} + \mathbf{B}\mathbf{L}\mathbf{B}^{-1})(s)ds\right).$$

Therefore,

$$\begin{aligned} \|\mathbf{e}(t) - \mathbf{c}\|_* &= \|\mathbf{B}^{-1}(t)\mathbf{B}(t)(\mathbf{e}(t) - \mathbf{c})\|_* \le \|\mathbf{B}^{-1}(t)\|_*\|\mathbf{B}(t)(\mathbf{e}(t) - \mathbf{c})\|_* \\ &\le \|\mathbf{B}^{-1}(t)\|_*\|\mathbf{B}(0)(\mathbf{e}(0) - \mathbf{c})\|_* \exp\left(\int_{t_0}^{t} \mu_*(\dot{\mathbf{B}}\mathbf{B}^{-1} + \mathbf{B}\mathbf{L}\mathbf{B}^{-1})(s)ds\right). \end{aligned}$$

Therefore, $\lim_{t\to\infty} \|\mathbf{e}(t) - \mathbf{c}\|_* = 0$ since $\lim_{t\to\infty} \exp\left(\int_{t_0}^{t} \mu_*(\dot{\mathbf{B}}\mathbf{B}^{-1} + \mathbf{B}\mathbf{L}\mathbf{B}^{-1})(s)ds\right) = 0$ and $\|\mathbf{B}^{-1}\| > 0$. Therefore, system (5) is globally asymptotically stable around a constant vector $\mathbf{c}$ and note that the constant vector $\mathbf{c}$ depends upon the initial conditions. This completes the proof.

In the case when $\mathbf{B}(t)$ is a constant matrix, by Theorem 3.1, we have the following result.

Corollary 3.1 *System (5) is globally asymptotically stable if there exists a non-singular matrix* $\boldsymbol{B}$ *such that*

$$\int_{t_0}^{\infty} \mu_*(\boldsymbol{B}\boldsymbol{L}(s)\boldsymbol{B}^{-1})ds = -\infty,$$

for any $t_0 \ge 0$. *Consequently, phase synchronization between systems (2) and (3) occurs which is globally asymptotically stable around a constant vector* $\boldsymbol{c}$.

In Corollary 1, when $\mathbf{B}$ is an identity matrix, then the main result in [13, 23] is obtained.

Corollary 3.2 *System (5) is globally asymptotically stable if*

$$\int_{t_0}^{\infty} \mu_*(\boldsymbol{L}(s))ds = -\infty,$$

for any $t_0 \ge 0$.

4 Numerical Results

In this section, we give some examples to show the efficiency of the above theory.

Example 1. Consider the following forced Duffing system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = ax - by - x^3 + c\cos(2\pi dt). \end{cases}$$

This system is chaotic for parameter values $a = c = 0.3, b = 0.35$ and $d = 0.2$. Using a nonlinear coupling function to couple two identical copies of this system yields

$$\begin{cases} \dot{x_1} = -x_1 + y_1 + x_1 + \alpha|x_2 - x_1|, \\ \dot{y_1} = ax_1 - by_1 - x_1^3 + c\cos(2\pi dt) + \alpha|x_1^3 - x_2^3|, \end{cases} \tag{6}$$

and

$$\begin{cases} \dot{x_2} = -x_2 + y_2 + x_2 + \alpha|x_1 - x_2|, \\ \dot{y_2} = ax_2 - by_2 - x_2^3 + c\cos(2\pi dt) + \alpha|x_2^3 - x_1^3|, \end{cases} \tag{7}$$

where the linear and nonlinear matrices are defined by

$$\mathbf{L} = \begin{bmatrix} -1 & 1 \\ 0.3 & -0.35 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} -x \\ x^3 - 0.3\cos(0.4\pi t) \end{bmatrix}.$$

By taking $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, we have $\mathbf{B}^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{BLB}^{-1} = \begin{bmatrix} -0.11 & -1.1 \\ 1.35 & -2.05 \end{bmatrix}$. Now, by using matrix measure $\mu_2(\cdot)$, we have

$$\frac{1}{2}\lambda_{\max}\left((\mathbf{BLB}^{-1})^T + \mathbf{BLB}^{-1}\right) = \frac{1}{2}\lambda_{\max}\begin{bmatrix} -0.2 & 0.25 \\ 0.25 & -4.1 \end{bmatrix} = -0.09202.$$

Therefore, according to Corollary 3.1, synchronization of systems (7) and (8) is globally asymptotically stable. See Figure 1.

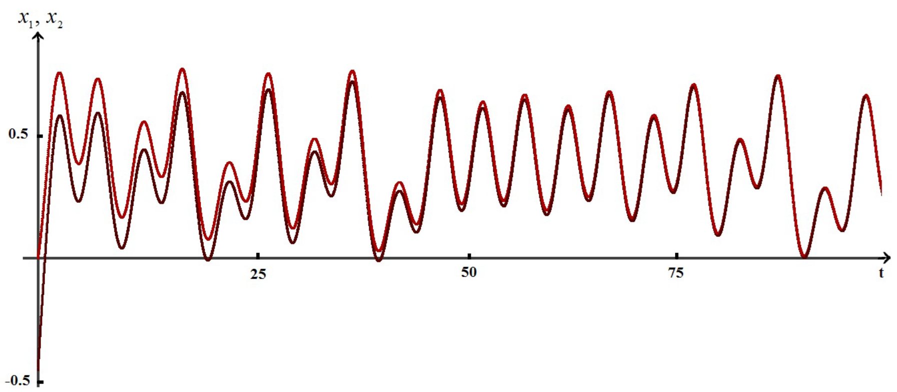

Figure 1: Global asymptotic stability of synchronization between two chaotic systems (7) and (8) in Example 1.

Remark. The above results in Corollary 3.1 and 3.2 are useful to proof the global asymptotic stability of phase synchronization in coupled chaotic systems. As discussed, this synchronization occurs whenever the maximum real part of the eigenvalues of $\mathbf{L}$ is zero. In this case, even the linear stability analysis is not useful for (local) stability analysis of phase synchronization. Nevertheless, using the results of these two corollaries, if in the hypothesis we replace $\int_{t_0}^{\infty} \mu_*(\mathbf{BL}(s)\mathbf{B}^{-1})ds = -\infty$ or $\int_{t_0}^{\infty} \mu_*(\mathbf{L}(s))ds = -\infty$ by $\int_{t_0}^{\infty} \mu_*(\mathbf{BL}(s)\mathbf{B}^{-1})ds = 0$ or $\int_{t_0}^{\infty} \mu_*(\mathbf{L}(s))ds = 0$, respectively, then the error vector in the coupled chaotic systems remains constant. That is, if there is phase synchronization between two coupled chaotic systems, then this synchronization is globally asymptotically stable.

Example 2. Consider the same Duffing system in Example 1 with parameter values $a = b = 0.35, c = 0.3$ and $d = 0.2$. Then this system is again chaotic. Now, with the same nonlinear coupling method as above, we have

$$\mathbf{L} = \begin{bmatrix} -1 & 1 \\ 0.35 & -0.35 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} -x \\ x^3 - 0.3\cos(0.4\pi t) \end{bmatrix}.$$

By taking identity matrix for $\mathbf{B}$ and choosing $\omega_1 = 7$ and $\omega_2 = 20$, we get $\mu_\omega(\mathbf{BLB}^{-1}) = \mu_\omega(\mathbf{L}) = 0$. Therefore, phase synchronization occurring between systems (7) and (8) is globally asymptotically stable. See Figure 2.

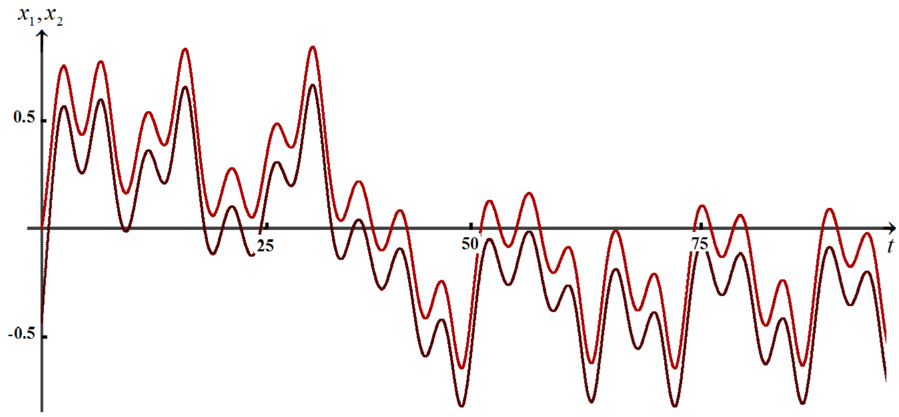

Figure 2: Global asymptotical stability of phase synchronization between two chaotic systems (7) and (8) in Example 2.

5 Conclusion

We have discussed a sufficient analytical condition for the stability of synchronization in coupled chaotic systems. As we have seen using a method of nonlinear feedback function and the scheme of matrix measure together with numerical results have justified this analytical stability. In particular, we have shown that our stability analysis is useful to proof the global asymptotic stability of phase synchronization in coupled chaotic systems.

Acknowledgment

This publication was made possible by NPRP grant NPRP 6-137-1-026 from the Qatar National Research Fund (a member of Qatar Foundation). The statements made herein are solely the responsibility of the authors.

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