Nonlinear Dynamics and Systems Theory, **15** (3) (2015) 221–230

# The Problem of Stability by Nonlinear Approximation

*to the 85th Birthday of Professor V.I. Zubov*

A.Yu. Aleksandrov [1*], A.A. Martynyuk [2] and A.P. Zhabko [1]

[1] *Saint Petersburg State University, 35 Universitetskij Pr., Petrodvorets, St. Petersburg 198504, Russia*
[2] *Institute of Mechanics National Academy of Science of Ukraine, Nesterov Str. 3, Kiev, 03057, Ukraine*

Received: December 18, 2014;  Revised: June 24, 2015

**Abstract:** In the present paper, Vladimir Zubov's results on the problem of stability by nonlinear approximation are surveyed together with their recent developments and extensions.

**Keywords:** *stability; homogeneous system; nonlinear approximation; Lyapunov function; perturbations; estimates of solutions.*

**Mathematics Subject Classification (2010):** 34A34, 34D20.

## 1 Introduction

The outstanding Russian mathematician and mechanical engineer Vladimir Ivanovich Zubov (1930-2000) made an invaluable contribution to the development of Stability Theory and Control Theory.

V. I. Zubov was born on April 14, 1930 in Kashira town, Moscow region, Russia. In 1945 he finished a middle school. At the age of 14, Vladimir was wounded by a hand grenade explosed accidently and soon failed eyesight. In 1949 he finished the Leningrad special school for blind and visually impaired children and entered the Mathematical and Mechanical Faculty of the Leningrad State University. In 1953, after graduating with honors, he joined the University faculty and since then his career was inseparably associated with the Leningrad (now, Saint Petersburg) State University.

In 1955, V. I. Zubov defended his PhD thesis "Boundaries of the Asymptotic Stability Domain" in which he proved the theorem on the asymptotic stability domain. This result is now known as *Zubov's theorem.*

* Corresponding author: mailto:alex43102006@yandex.ru

Further Zubov's activities involved both pure fundamental investigations and solution of applied real-life problems in several fields — from spacecraft to ship control.

In 1969, the Faculty of Applied Mathematics and Control Processes was founded at the Leningrad State University with Vladimir Zubov's appointment as its first dean. Two years later, a Research Institute of Computational Mathematics and Control Processes was set up by the USSR Government. Zubov became its brains-and-heart. In particular, he headed the projects on the design, development and operation of systems of self-guided winged missiles, and tactical schemes construction for the USSR Navy to oppose aircraft carriers of the potential enemy.

Zubov's scientific activities was surveyed in the paper [8] dedicated to his 80th Birthday. In the present review, we would like to focus on Zubov's works on the problem of stability by nonlinear approximation together with their ramifications in the last decade publications.

## 2 Stability Analysis by Nonlinear Approximation

The basic tool for the stability analysis of motions of differential equation systems is the Lyapunov direct method (or the Lyapunov functions method). However, it should be recalled that until now, a general algorithm has not been yet constructed for the Lyapunov function generation for an arbitrary nonlinear system. The most common approach to the problem consists in, firstly, reduction of an original system to a simpler one, secondly, stability investigation of the reduced system via the Lyapunov function construction, and, thirdly, subsequent testing of this function as a potential candidate for the Lyapunov function of the original system.

A. M. Lyapunov has determined conditions under which the conclusion on the stability of the zero solution for a nonlinear system can be obtained via the analysis of the corresponding system of linear approximation [22]. However, it is worth mentioning that in numerous applications it is required to study differential equation systems for which the expansions of the right-hand sides in powers of the phase variables do not contain linear terms at all. Thus, there arises a problem of stability by nonlinear approximation.

The first theorems on stability by nonlinear approximation were proved by I. G. Malkin, N. N. Krasovskii and V. I. Zubov [21, 23, 35, 36]. In these papers, systems with homogeneous right-hand sides were considered as the first approximation.

**Definition 2.1** Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space. A function $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of the order $\mu$, where $\mu$ is a positive rational with the odd denominator, if

$$f(\lambda \mathbf{x}) = \lambda^{\mu} f(\mathbf{x}) \tag{1}$$

for $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. In the case when $\mu$ is a positive real number, and equality (1) holds for $\lambda \geq 0$ and $\mathbf{x} \in \mathbb{R}^n$, the function $f(\mathbf{x})$ is called positive homogeneous of the order $\mu$.

Consider the system of differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) \tag{2}$$

and the corresponding perturbed system

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) + \mathbf{G}(t, \mathbf{x}(t)). \tag{3}$$

Here $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector; components of the vector $\mathbf{F}(\mathbf{x})$ are homogeneous functions of the order $\mu > 0$ which are continuous for all $\mathbf{x} \in \mathbb{R}^n$; vector function $\mathbf{G}(t, \mathbf{x})$ is continuous for $t \geq 0$, $\|\mathbf{x}\| < H$ and satisfies the inequality $\|\mathbf{G}(t, \mathbf{x})\| \leq c\|\mathbf{x}\|^{\sigma}$, where $c$ and $\sigma$ are positive constants, $0 < H \leq +\infty$, and $\| \cdot \|$ denotes the Euclidean norm of a vector. Thus, systems (2) and (3) admit the zero solution.

It is required to determine conditions under which the asymptotic stability of the zero solution of (2) implies the same type of stability for the zero solution of the perturbed system (3).

In [21, 23], the case has been studied when components of the vector $\mathbf{F}(\mathbf{x})$ are homogeneous forms of an integer order $\mu > 1$. It was proved that if the inequality $\sigma > \mu$ holds, then the perturbations do not disturb the asymptotic stability of the zero solution.

It is worth mentioning that Malkin's proof was based on a geometric approach [23]. A family of closed surfaces surrounding the origin were constructed, and angles between these surfaces and trajectories of system (3) were estimated. To prove the theorem on the stability by nonlinear approximation, Krasovskii has used the Lyapunov direct method, see [21]. He has determined conditions under which for system (3) there exists a Lyapunov function solving the stability problem and satisfying estimates of a special form.

V. I. Zubov has extended the results of [21, 23] to wider classes of systems, see [34–36]. Unlike [21, 23], in [34–36] it was assumed that the components of the vector $\mathbf{F}(\mathbf{x})$ are, in general, not forms, but homogeneous functions of the order $\mu > 0$. Zubov has established the following properties of solutions of homogeneous systems:

(i) if $\mathbf{x}(t, \mathbf{x}_0)$ is a solution of (2) starting from the point $\mathbf{x}_0$ at $t = 0$, then, for any $c \in \mathbb{R}$, the function $c\mathbf{x}(c^{\mu-1}t, \mathbf{x}_0)$ is the solution of (2) as well;

(ii) the zero solution of (2) can be asymptotically stable only in the case when $\mu$ is a rational with the odd numerator and denominator;

(iii) if the zero solution of (2) is asymptotically stable, then it is globally asymptotically stable.

Zubov has investigated conditions under which for a homogeneous system there exists a homogeneous Lyapunov function satisfying the assumptions of the Lyapunov asymptotic stability theorem. He has obtained the following result, see [35, 36].

**Theorem 2.1** *Let for solutions of* (2) *the inequality* $\|\mathbf{x}(t, \mathbf{x}_0)\| \leq bt^{-\alpha}$ *be valid for* $t \geq T$, $\|\mathbf{x}_0\| = 1$, *where* $T, b, \alpha$ *are positive constants. Then there exist functions* $V(\mathbf{x})$ *and* $W(\mathbf{x})$ *possessing the properties:*

*(a)* $V(\mathbf{x})$ *and* $W(\mathbf{x})$ *are continuous for* $\mathbf{x} \in \mathbb{R}^n$ *positive homogeneous functions of the orders* $\gamma$ *and* $\gamma + \mu - 1$ *respectively, where* $\gamma$ *is sufficiently large positive number;*

*(b) functions* $V(\mathbf{x})$ *and* $W(\mathbf{x})$ *are positive definite;*

*(c) function* $V(\mathbf{x})$ *is differentiable with respect to solutions of system* (2), *and the equality* $\dot{V}\big|_{(2)} = -W(\mathbf{x})$ *holds.*

*Moreover, in the case when the right-hand sides of* (2) *are* $k$ *times continuously differentiable functions for* $\mathbf{x} \in \mathbb{R}^n$, *where* $k \geq 1$, *while constructing functions* $V(\mathbf{x})$ *and* $W(\mathbf{x})$, *one can choose* $V(\mathbf{x})$ *in the class of* $k$ *times continuously differentiable functions.*

It was shown, see [36], that if the function $\mathbf{F}(\mathbf{x})$ is continuously differentiable for $\mathbf{x} \in \mathbb{R}^n$, then the functions $V(\mathbf{x})$ and $W(\mathbf{x})$ satisfy the system of partial differential equations

$$\left(\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\right)^T \mathbf{F}(\mathbf{x}) = -W(\mathbf{x}), \qquad \left(\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\right)^T \mathbf{x} = \gamma V(\mathbf{x}). \tag{4}$$

Zubov has studied the problem of solvability of this system [36]. In particular, in the case when $n = 2$, he has proposed a constructive approach for finding solutions of (4).

On the basis of Theorem 2.1, Zubov has determined the following stability and ultimate boundedness criteria for the perturbed system (3), see [35, 36].

**Theorem 2.2** *Let the vector function* $\mathbf{F}(\mathbf{x})$ *be continuously differentiable for* $\mathbf{x} \in \mathbb{R}^n$, *and inequality* $\sigma > \mu$ *hold. Then from the asymptotic stability of the zero solution of* (2) *it follows that the zero solution of* (3) *is asymptotically stable as well.*

**Theorem 2.3** *Let the vector function* $\mathbf{F}(\mathbf{x})$ *be continuously differentiable for* $\mathbf{x} \in \mathbb{R}^n$, *and the inequality* $\sigma < \mu$ *hold. Then from the asymptotic stability of the zero solution of* (2) *it follows that solutions of* (3) *are uniformly ultimately bounded.*

Moreover, new stability conditions were established in the critical case of several zero roots and in the critical case of several pairs of purely imaginary roots of characteristic equation, see [36, 37, 39, 40].

Zubov has also derived estimates for the convergence rate of solutions for asymptotically stable homogeneous system (2) and for the perturbed system (3), see [36]. He has proved that if $\mu > 1$, function $\mathbf{F}(\mathbf{x})$ is continuously differentiable for $\mathbf{x} \in \mathbb{R}^n$, and the zero solution of (2) is asymptotically stable, then, there exist positive constants $c_1, c_2, c_3, c_4$ such that for solutions of (2) the inequalities

$$\|\mathbf{x}_0\| \left(c_1 + c_2\|\mathbf{x}_0\|^{\mu-1}\, t\right)^{-\frac{1}{\mu-1}} \le \|\mathbf{x}(t,\mathbf{x}_0)\| \le \|\mathbf{x}_0\| \left(c_3 + c_4\|\mathbf{x}_0\|^{\mu-1}\, t\right)^{-\frac{1}{\mu-1}} \tag{5}$$

hold for any $\mathbf{x}_0 \in \mathbb{R}^n$ and for $t \ge 0$. For the case when $0 < \mu < 1$, Zubov has obtained conditions under which every solution of system (2) gets to the origin in a finite time, and remains at this point thereafter. In his later works [41, 43], this property of homogeneous systems with homogeneity orders less than one was used for the design of feedback controls providing finite-time synchronization of dynamical systems motions.

Furthermore, Zubov has extended the above results to systems with generally homogeneous right-hand sides [38, 39].

**Definition 2.2** A function $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is called generally homogeneous of the order $\nu$ with respect to the dilation $(m_1, \dots, m_n)$, where $\nu, m_1, \dots, m_n$ are positive rationals with the odd denominators, if

$$f(\lambda^{m_1}x_1, \dots, \lambda^{m_n}x_n) = \lambda^{\nu} f(\mathbf{x}) \tag{6}$$

for all $\lambda \in \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. In the case when $\nu, m_1, \dots, m_n$ are positive real numbers, and equality (6) holds for $\lambda \ge 0$ and $\mathbf{x} \in \mathbb{R}^n$, the function $f(\mathbf{x})$ is called positive generally homogeneous of the order $\nu$ with respect to the dilation $(m_1, \dots, m_n)$.

**Definition 2.3** A vector field $\mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_n(\mathbf{x}))^T : \mathbb{R}^n \to \mathbb{R}^n$ is called generally homogeneous of the order $\mu$ with respect to the dilation $(m_1, \dots, m_n)$, where $\mu, m_1, \dots, m_n$ are rationals with the odd denominators, such that $m_i > 0$ and $\mu + m_i > 0$, $i = 1, \dots, n$, if $f_i(\lambda^{m_1}x_1, \dots, \lambda^{m_n}x_n) = \lambda^{\mu+m_i} f_i(x_1, \dots, x_n)$, $i = 1, \dots, n$, for all $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. System (2) is called generally homogeneous if its vector field $\mathbf{F}(\mathbf{x})$ is generally homogeneous.

In [38, 39], for generally homogeneous systems, conditions of the existence of generally homogeneous Lyapunov functions were obtained, and criteria of stability and ultimate boundedness by generally homogeneous approximation were found.

V. I. Zubov has also set up a problem of the stability by the first, in a broad sense, approximation. He has investigated the conditions for stability of the zero solution for arbitrary admissible functions included in the first approximation [39, 42].

In particular, systems of the form

$$\dot{x}_i(t) = \sum_{j=1}^{n} p_{ij}(t) f_j(x_j(t)) + g_i(t, \mathbf{x}(t)), \qquad i = 1, \dots, n, \tag{7}$$

have been considered [42]. Here coefficients $p_{ij}(t)$ are continuous for $t \geq 0$; functions $f_j(x_j)$ are continuous for $|x_j| < H$ $(0 < H \leq +\infty)$ and belong to a sector-like constrained set defined as follows: $x_j f_j(x_j) > 0$ for $x_j \neq 0$, $j = 1, \dots, n$; the perturbations $g_i(t, \mathbf{x})$ are given and continuous for $t \geq 0$, $\|\mathbf{x}\| < H$.

The following issues were investigated:

(i) under what conditions the zero solution of the unperturbed system ($g_i(t, \mathbf{x}) \equiv 0$, $i = 1, \dots, n$) is asymptotically stable for any admissible functions $f_j(x_j)$?

(ii) under what conditions perturbations do not destroy the asymptotic stability of the zero solution?

On the basis of the obtained results, Zubov has developed new and effective approaches to the problem of stability analysis of nonlinear systems in the cases being critical in the Lyapunov sense.

## 3 Some Extensions of Zubov's Rezults

### 3.1 Existence of homogeneous Lyapunov functions

From Zubov's results it follows that for system (2) with homogeneous polynomial right-hand sides possessing the asymptotic stability property for its zero solution, it is always possible to choose a Lyapunov function in the class of homogeneous functions. In [31], the problem has been discussed whether it is possible to choose this function in the class of homogeneous polynomials (forms) or not. The answer proves to be negative. For any given positive integer $\gamma$, there exists a system from the family

$$\dot{x}_1(t) = (\alpha - \varepsilon)x_1^3(t) - x_2^3(t), \quad \dot{x}_2(t) = x_1^3(t) - \alpha x_2^3(t), \quad 0 < \varepsilon < \alpha < 1,$$

such that the zero solution of this system is asymptotically stable but the derivative of any form of the order $\gamma$ with respect to this system is not sign-definite.

L. Rosier has proved that it is possible to guarantee the existence of continuously differentiable homogeneous functions for homogeneous systems under less conservative conditions than those imposed in Zubov's theorems, see [28].

**Theorem 3.1** *Let vector function $\mathbf{F}(\mathbf{x})$ be continuous for $\mathbf{x} \in \mathbb{R}^n$ positive generally homogeneous of the order $\mu \in \mathbb{R}$ with respect to the dilation $(m_1, \dots, m_n)$, where $m_i > 0$ and $\mu + m_i > 0$, $i = 1, \dots, n$. If the zero solution of* (2) *is asymptotically stable, then, for any positive integer $k$, there exists a Lyapunov function $V(\mathbf{x})$ possessing the following properties:*

*(a) the function $V(\mathbf{x})$ is $k$ times continuously differentiable at the point $\mathbf{x} = \mathbf{0}$, and it is infinitely differentiable for $\mathbf{x} \neq \mathbf{0}$;*

*(b) function $V(\mathbf{x})$ is positive definite;*

*(c) function $V(\mathbf{x})$ is positive generally homogeneous of the order $\gamma$ with respect to the dilation $(m_1, \ldots, m_n)$, where $\gamma$ is an arbitrary number greater than $k \max_{i=1,\ldots,n} m_i$;*

*(d) the derivative of $V(\mathbf{x})$ with respect to system (2) is negative definite.*

The application of Theorem 3.1 permits us to weaken the conditions of known criteria of the stability and the ultimate boundedness by nonlinear approximation.

For homogeneous systems, the problem of existence of homogeneous Lyapunov functions satisfying the assumptions of the first Lyapunov instability theorem was studied in [18].

### 3.2 Stability analysis of nonlinear systems via averaging

In [2, 3], nonlinear nonstationary systems whose right-hand sides are homogeneous with respect to phase variables have been studied. For such systems, an approach for Lyapunov functions constructing was proposed. Its application permits us to show that if the order of homogeneity of the right-hand sides of the time-varying system under consideration is greater than one, then the asymptotic stability of the zero solution of the corresponding averaged system implies the same property for the zero solution of the original system. These results have been further developed in [4, 5, 15, 26, 27, 30]. In particular, in [30], a modification of the approach for the Lyapunov functions construction was suggested. Other techniques for the determination of similar asymptotic stability conditions for time-varying homogeneous systems have been developed in [26, 27].

Compared with the known stability conditions obtained by the application of averaging technique, the principal novelty of the above results is that, to guarantee the asymptotic stability for a nonstationary homogeneous system, the right-hand sides of the system need not be fast time-varying. It is shown that in the averaging technique, instead of a small parameter providing the fast time-variation of a vector field, the orders of homogeneity can be used.

### 3.3 Stability of nonlinear complex and hybrid systems

In [24], a motion polystability problem for differential equation systems has been studied. In terms of matrix-valued Lyapunov functions, conditions of polystability for nonlinear systems with separable motions by nonlinear and psevdo-linear approximation were found.

Sufficient conditions of the asymptotic stability with respect to a part of variables for equilibrium positions of nolinear complex systems have been derived in [4, 29].

In [20], an approach for the stability analysis of multiconnected systems by nonlinear approximation was suggested. In [9, 10], the results of [20] were strengthened and extended to wider classes of systems. It is worth mentioning that the approach in [20] is based on the vector Lyapunov functions method, whereas in [9, 10] scalar Lyapunov functions were proposed.

The stability problem for hybrid homogeneous systems was studied in [7, 32]. Sufficient conditions were obtained under which a family of homogeneous subsystems admits a common Lyapunov function. The fulfilment of these conditions provides global asymptotic stability of the zero solution of the corresponding switched system for any admissible switching law. For the case when we can not guarantee the existence of such a function, in [7], the multiple Lyapunov function and the dwell-time approaches were used to determine the classes of switching signals for which the zero solution of the hybrid homogeneous system is locally or globally asymptotically stable. Stability conditions for some types of nonlinear multiconnected systems with a variable structure were found in [11, 33].

### 3.4 Preservation of stability under the digitization

In the papers [12, 14], the problem of preservation of stability under the digitization for certain classes of nonlinear differential equation systems was studied.

In [12], the homogeneous system (2) and the corresponding difference system

$$\mathbf{y}(k+1) = \mathbf{y}(k) + h\,\mathbf{F}(\mathbf{y}(k)) \tag{8}$$

have been considered. Here $\mathbf{y}(k) \in \mathbb{R}^n$; components of the vector $\mathbf{F}(\mathbf{x})$ are homogeneous functions of the order $\mu > 1$ which are are continuously differentiable for all $\mathbf{x} \in \mathbb{R}^n$; $h > 0$ is a digitisation step; $k = 0, 1, \ldots$.

The following theorem was proved.

**Theorem 3.2** *If the zero solution of system* (2) *is asymptotically stable, then the zero solution of* (8) *is asymptotically stable for any value of* $h > 0$.

Thus, unlike the case of linear systems, for essentially nonlinear homogeneous systems, the preservation of stability while passing from differential systems to difference ones can be guaranteed for an arbitrary digitization step.

Furthermore, in [12, 14], theorems on the stability by nonlinear approximation were obtained for various classes of difference systems.

### 3.5 Stability analysis of nonlinear time-delay systems

In the papers [5, 6, 13, 15], certain classes of nonlinear time-delay systems have been studied. It was assumed that the trivial solution of a system is asymptotically stable when delay is equal to zero. The Lyapunov direct method and the Razumikhin theorem were used to show that if the system is essentially nonlinear, i.e., the right-hand sides of the system do not contain linear terms, then the asymptotic stability of the zero solution is preserved for an arbitrary positive value of the delay. On the basis of the proposed approach, new delay-independent stability conditions have been obtained for wide classes of nonlinear systems, see [5, 6, 13, 15].

In particular, in [6], homogeneous time-delay system of the form

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{x}(t-\tau)) \tag{9}$$

has been considered. Here $\mathbf{x}(t) \in \mathbb{R}^n$; the components of the vector $\mathbf{F}(\mathbf{x}, \mathbf{y})$ are homogeneous functions of the order $\mu > 1$, defined for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and continuous with respect to their variables, and continuously differentiable with respect to $\mathbf{y}$; $\tau$ is a constant positive delay. This means that system (9) admits the zero solution.

**Theorem 3.3** *Let the zero solution of the corresponding delay free system* $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{x}(t))$ *be asymptotically stable. Then the zero solution of* (9) *is asymptotically stable for any value of* $\tau > 0$.

### 3.6 Estimates of the convergence rate of solutions

Zubov's results on the finite-time stability and synchronization have been vigorously developed during the past decades, see [16, 17] and the references cited therein.

In [12], a discrete-time counterpart of estimates (5) was obtained for nonlinear homogeneous difference systems.

In [15], in terms of the Razumikhin approach, a procedure for the estimation of the convergence rate of solutions for essentially nonlinear time-delay systems was developed.

### 3.7 Stability by the first, in a broad sense, approximation

Some results on the stability by the first, in a broad sense, approximation have been obtained in [1, 5, 9, 10, 19, 20]. For instance, in [9, 10], a generalization of system (7) was studied. With the aid of the well-known Martynyuk–Obolenskij stability criteria for autonomous Wazewskij systems, see [25], an approach to the construction of Lyapunov functions for the system in question was proposed, and existence conditions for such functions were found. By the use of the Lyapunov functions constructed, new theorems on the stability and ultimate boundedness by nonlinear approximation have been proved [9, 10].

## 4 Conclusion

Vladimir Zubov was a prominent scholar, engineer and university lecturer. In the previous sections we have reviewed just only one area of scientific activity of his own and his successors.

Zubov is the author of about 200 publications including 31 monographs and text books. He was an advisor for 20 DSc and about 100 PhD dissertations. Under Zubov's supervision, a worldwide famous school in control theory was developed in St. Petersburg.

In 1968 V. I. Zubov became the USSR State Prize winner for his pioneer works in Control Theory. In 1981 he was elected a corresponding member of the Soviet Union Academy of Sciences, and in 1998 he was awarded the title of the Honor Scholar of the Russian Federation. In 1996, the Zubov scientific school "Processes of control and stability" was the winner of the competition for the State support of leading scientific schools of Russia. In 2001, the Research Institute of Computational Mathematics and Control Processes of St. Petersburg State University was named after him.

For outstanding merits to the world science, Zubov's name was perpetuated as a name of minor planet 'ZUBOV 10022'. This asteroid has a size of 6 km, a brightness of 13.8 magnitude, and the greatest orbit's semiaxis of 2.369 astronomical units.

## Acknowledgment

This work was partially supported by the Saint Petersburg State University, project nos. 9.38.674.2013 and 9.37.157.2014, and by the Russian Foundation for Basic Research, grant nos. 13-01-00376-a and 15-58-53017.

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