Nonlinear Dynamics and Systems Theory, **15** (3) (2015) 272–289

Mild Solution for Impulsive Neutral Integro-Differential Equation of Sobolev Type with Infinite Delay

Alka Chadha [*]

Department of Mathematics, Indian Institute of Technology Roorkee,
Roorkee-247667, India

Received: September 18, 2014; Revised: June 26, 2015

Abstract: In this work, we consider an impulsive neutral integro-differential equation of Sobolev type with infinite delay in an arbitrary Banach space X. The existence of mild solution is obtained by using resolvent operator and Hausdorff measure of noncompactness. We give an example based on the theory and provide the conclusion at the end of the paper.

Keywords: *resolvent operator; impulsive differential equation; neutral integro-differential equation; measure of noncompactness.*

Mathematics Subject Classification (2010): 34K37, 34K30, 35R11, 47N20.

1 Introduction

In our recent work [19], we have studied the impulsive neutral integro-differential equation with infinite delay in a Banach space $(X, \|\cdot\|)$,

$$\begin{aligned}
\frac{d}{dt}[u(t) - F(t, u_t)] &= A[u(t) + \int_0^t f(t-s)u(s)ds] + G(t, u_t, \int_0^t \mathcal{E}(t, s, u_s)ds), \\
&\quad t \in J = [0, T_0],\ t \neq t_k,\ k = 1, 2, \cdots, m, && (1) \\
u_0 &= \phi \in \mathfrak{B}, && (2) \\
\Delta u(t_i) &= I_i(u_{t_i}),\ i = 1, 2, \cdots, m, && (3)
\end{aligned}$$

where $0 < T_0 < \infty$, A is a closed linear operator defined on a Banach space $(X; \|\cdot\|)$ with dense domain $D(A) \subset X$; $f(t), t \in [0, T_0]$ is a bounded linear operator. The functions $F : [0, T_0] \times \mathfrak{B} \to X$, $G : [0, T_0] \times \mathfrak{B} \times X \to X$,

---

[*] Corresponding author: `mailto:alkachadda23@gmail.com`

$\mathcal{E} : [0, T_0] \times [0, T_0] \times \mathfrak{B} \to X$, $I_i : X \to X$, $i = 1, \cdots, m$ are appropriate functions and $0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T_0$ are pre-fixed numbers. The symbol $\Delta u(t) = u(t^+) - u(t^-)$ denotes the jump of the function u at t i.e., $u(t^-)$ and $u(t^+)$ denotes the end limits of the $u(t)$ at t. The history $u_t : (-\infty, 0] \to X$ is a continuous function defined as $u_t(s) = u(t+s), s \leq 0$ belongs to the abstract phase space $\mathfrak{B}$ and $\mathfrak{B}$ is the phase space defined axiomatically later in Section 2. We have established the existence results by using Hausdorff measure of noncompactness and Darbo fixed point theorem with the assumption that A generates an analytic resolvent operator and G satisfies the Carathèodary condition.

In [20], the authors have discussed the regularity of solutions of the semilinear integro-differential equations of Sobolev type in Banach space which is illustrated as

$$\frac{d}{dt}[Ey(t)] = A[y(t) + \int_0^t f(t-s)y(s)ds] + F(t, y(t)), \tag{4}$$

$$y(0) = y_0, \qquad t \in [0, T_0],\ 0 < T_0 < \infty, \tag{5}$$

where E and A are considered as closed linear operators such that the domains contained in Banach space X and ranges contained in Banach space Y, $f(t), t \in [0, T_0]$ is a bounded linear operator such that Y is continuously and densely embedded in X. The nonlinear function $F : [0, T_0] \times X \to Y$ is a continuous function. The authors have obtained the results by using Banach fixed point theorem and resolvent operator.

As in the above mentioned work, our aim in this paper is to investigate the existence of mild solution of the following impulsive Sobolev type neutral integro-differential equation with infinite delay in a Banach space $(X, \|\cdot\|)$,

$$\frac{d}{dt}[Ey(t) + F(t, y_t, \int_0^t a(t, s, y_s)ds)] = A[y(t) + \int_0^t f(t-s)y(s)ds]$$
$$+G(t, u_t, \int_0^t \mathcal{E}(t, s, u_s)ds),\ t \in J = [0, T_0],\ t \neq t_i, \tag{6}$$

$$u_0 = \phi \in \mathfrak{B}, \tag{7}$$

$$\Delta u(t_i) = I_i(u_{t_i}),\ i = 1, 2, \cdots, m, \tag{8}$$

where E and A are the same operators as defined in equation (4). The functions $F : [0, T_0] \times \mathfrak{B} \times X \to Y$, $G : [0, T_0] \times \mathfrak{B} \times X \to Y$, $\mathcal{E} : [0, T_0] \times [0, T_0] \times \mathfrak{B} \to X$, $I_i : X \to X$, $i = 1, \cdots, m$ are appropriate functions satisfying some suitable conditions to be mentioned in Section 3.

Recently, impulsive differential equations have been rising as an important area of study due to their wide applicability in sciences and engineering such as physics, control theory, biology, population dynamics, medical domain and many others, and hence they have earned considerable attention of researchers. The process or phenomena subject to short-term external influences can be modeled by the impulsive differential equations which allow for discontinuities in the evolution of the state. For more study of such differential equations and their applications, we refer to the monographs [12], [24] and papers. Moreover, Sobolev type semilinear integrodifferential equation can be used to describe the flow of fluid through fissured rocks [2], thermodynamics and shear in second order fluids and many others. For wide study of Sobolev type differential equation, we refer to papers [20] – [23]. A lot of natural phenomena emerging from numerous areas, for example, fluid dynamics, electronics and kinetics, can be modeled in the form of the integro-differential equation. Integro-differential equation of neutral type with delay describe the system of rigid heat conduction with finite wave spaces.

The organization of the paper is as follows: Section 2 provides some basic facts, lemmas and theorems which will be used for establishing the result. Section 3 focuses on the existence of a mild solution by means of Hausdorff measure of noncompactness and analytic semigroup. Section 4 provides an example based on the obtained abstract theory. The last section of the paper is devoted to providing conclusion.

2 Preliminaries and Assumptions

In this section, we provide some fundamental definition, lemmas and theorems which will be utilized all around this paper.

Let X be a Banach space. The symbol $C([a,b];X)$, $(a,b\in\mathbb{R})$ stands for the Banach space of all the continuous functions from $[a,b]$ into X equipped with the norm $\| z(t)\|_C = \sup_{t\in[a,b]}\| z(t)\|_X$ and $L^p((a,b);X)$ stands for Banach space of all Bochner-measurable functions from (a,b) to X with the norm

$$\| z\|_{L^p} = (\int_{(a,b)}\| z(s)\|_X^p ds)^{1/p}.$$

For the differential equation with infinite delay, Kato and Hale [9] have proposed the phase space $\mathfrak{B}$ satisfying certain fundamental axioms.

Definition 2.1 The linear space of all functions from $(-\infty,0]$ into Banach space X with a seminorm $\|\cdot\|_{\mathfrak{B}}$ is known as phase space $\mathfrak{B}$. The fundamental axioms on $\mathfrak{B}$ are the following:

(A) If $y:(-\infty,d+T_0]\to X$, $T_0>0$ is a continuous function on $[d,d+T_0]$ such that $y_d\in\mathfrak{B}$ and $y|_{[d,d+T_0]}\in\mathfrak{B}\in\mathcal{PC}([d,d+T_0];X)$, then for every $t\in[d,d+T_0)$, the following conditions hold:

(i) $y_t\in\mathfrak{B}$,

(ii) $H\| y_t\|_{\mathfrak{B}}\geq\| y(t)\|$,

(iii) $\| y_t\|_{\mathfrak{B}}\leq N(t+d)\| y_d\|_{\mathfrak{B}}+K(t-d)\sup\{\| y(s)\| : d\leq s\leq t\}$,

where H is a positive constant; $N,\ K:[0,\infty)\to[1,\infty)$, N is locally bounded, K is continuous and $K,\ H,\ N$ are independent of $y(\cdot)$.

(A1) For the function y in $(A1)$, y_t is a $\mathfrak{B}$-valued continuous function for $t\in[d,d+T_0]$.

(B) The space $\mathfrak{B}$ is complete.

Consider the following integro-differential equation

$$\frac{d}{dt}[Ey(t)] = A[y(t)+\int_0^t f(t-s)y(s)ds]. \tag{9}$$

To prove the result, we impose the following data on operators A and E. The following conditions are fulfilled by operators $A:D(A)\subset X\to Y$ and $E:D(E)\subset X\to Y$:
$(E1)$ A and E are closed linear operators,

$(E2)$ $D(E) \subset D(A)$ and E is bijective,
$(E3)$ $E^{-1}: Y \to D(E)$ is continuous operator and $E^{-1}B = BE^{-1}$,
$(E4)$ $AE^{-1}: Y \to Y$ is the infinitesimal generator of uniformly continuous semigroup of bounded linear operators in X.

To set the structure for our primary existence results, we have to introduce the following definitions.

Definition 2.2 A family $\{R(t)\}_{t\in[0,T_0]}$ of bounded linear operators is said to be a resolvent operator for equation (9) if the following conditions are satisfied

(i) $R(0) = I$, where I is the identity operator on X.

(ii) $R(t)$ is strongly continuous for $t \in [0, T_0]$.

(iii) $R(t) \in B(Z),\ t \in [0, T_0]$. For $z \in Z$ and $R(\cdot)z \in C([0, T_0]; Z) \cap C^1([0, T_0]; Z)$, we have

$$\begin{aligned} \frac{d}{dt}R(t)z &= AE^{-1}[R(t)z + \int_0^t f(t-s)R(s)zds], && (10) \\ &= R(t)AE^{-1}z + \int_0^t R(t-s)AE^{-1}f(s)zds, \quad t \in [0, T_0]. && (11) \end{aligned}$$

Here $B(Z)$ denotes the space of bounded linear operators defined on Z and Z is a Banach space formed from $D(A)$ with the graph norm.

Throughout the work, the resolvent operator $\{R(t)\}_{t\geq 0}$ is assumed to be analytic in Banach space X and there exist positive constants N_1 and N_2 such that $\| R(t)\| \leq N_1$ and $\|f(t)\| \leq N_2$ for each $t \in [0, T_0]$.

To consider the mild solution for the impulsive problem, we propose the set $\mathcal{PC}([0, T_0]; X) = \{y : [0, T_0] \to X : y$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$ and $y(t_i^+)$ exists, for all $i = 1, \cdots, m\}$. Clearly, $\mathcal{PC}([0, T_0]; X)$ is a Banach space endowed with the norm $\| u\|_{\mathcal{PC}} = \sup_{t\in[0,T_0]} \| u(s)\|$. For a function $y \in \mathcal{PC}([0, T_0]; X)$ and $i \in \{0, 1, \cdots, m\}$, we define the function $\widetilde{y_i} \in C([t_i, t_{i+1}], X)$ such that

$$\widetilde{y_i}(t) = \begin{cases} y(t), & \text{for } t \in (t_i, t_{i+1}], \\ y(t_i^+), & \text{for } t = t_i. \end{cases} \tag{12}$$

For $W \subset \mathcal{PC}([0, T_0]; X)$ and $i \in \{0, 1, \cdots, m\}$, we have $\widetilde{W_i} = \{\widetilde{y_i} : y \in W\}$ and the following Accoli-Arzelà type criteria.

Lemma 2.1 *[7]. A set $W \subset \mathcal{PC}([0, T_0]; X)$ is relatively compact if and only if each set $\widetilde{W_i} \subset C([t_i, t_{i+1}], X)$ $(i = 0, 1 \cdots, m)$ is relatively compact.*

Now, we discuss some basic definition of measure of noncompactness (MNC).

Definition 2.3 [10] The Hausdorff's measure of noncompactness (H'MNC) χ_Y is defined as

$$\chi_Y(U) = \inf\{\varepsilon > 0 : U \text{ can be covered by a finite number of balls with radius } \varepsilon\}, \tag{13}$$

for the bounded set $U \subset Y$, where Y is a Banach space.

Lemma 2.2 *[10] For any bounded set U, $V \subset Y$, where Y is a Banach space. Then the following conditions are fulfilled:*

- *(i)* $\chi_Y(U) = 0$ *if and only if U is pre-compact;*
- *(ii)* $\chi_Y(U) = \chi_Y(conv\ U) = \chi_Y(\overline{U})$, *where conv U and $\overline{U}$ denote the convex hull and closure of U respectively;*
- *(iii)* $\chi_Y(U) \subset \chi_Y(V)$, *when* $U \subset V$;
- *(iv)* $\chi_Y(U + V) \leq \chi_Y(U) + \chi_Y(V)$, *where* $U + V = \{u + v : u \in U,\ v \in V\}$;
- *(v)* $\chi_Y(U \cup V) \leq \max\{\chi_Y(U), \chi_Y(V)\}$;
- *(vi)* $\chi_Y(\lambda U) = \lambda \cdot \chi_Y(U)$, *for any* $\lambda \in \mathbb{R}$;
- *(vii) If the map $P : D(P) \subset Y \to \mathcal{Z}$ is continuous and satisfy the Lipschitsz condition with constant κ, then we have that $\chi_{\mathcal{Z}}(PU) \leq \kappa \chi_Y(U)$ for any bounded subset $U \subset D(P)$, where Y and $\mathcal{Z}$ are Banach spaces.*

Definition 2.4 [10] A bounded and continuous map $P : \mathcal{D} \subset Z \to Z$ is a χ_Z-contraction if there exists a constant $0 < \kappa < 1$ such that $\chi_Z(P(U)) \leq \kappa \chi_Z(U)$, for any bounded closed subset $U \subset \mathcal{D}$, where Z is a Banach space.

Lemma 2.3 *[16] Let $\mathcal{D} \subset Z$ be closed, convex with $0 \in \mathcal{D}$ and the continuous map $P : \mathcal{D} \to \mathcal{D}$ be a χ_Z-contraction. If the set $\{u \in \mathcal{D} : u = \lambda P u,\ for\ \ 0 < \lambda < 1\}$ is bounded, then the map P has a fixed point in $\mathcal{D}$.*

Lemma 2.4 *(Darbo-Sadovskii) [10]. Let $\mathcal{D} \subset Z$ be bounded, closed and convex. If the continuous map $P : \mathcal{D} \to \mathcal{D}$ is a χ_Z-contraction, then the map P has a fixed point in $\mathcal{D}$.*

In this paper, we consider that χ denotes the Hausdorff's measure of noncompactness (H'MNC)in X, χ_C denotes the Hausdorff's measure of noncompactness in $C([0, T_0]; X)$ and $\chi_{\mathcal{PC}}$ denotes the Hausdorff's measure of noncompactness in $\mathcal{PC}([0, T_0]; X)$.

Lemma 2.5 *([10]. If U is bounded subset of $C([0, T_0]; X)$, then we have that $\chi(U(t)) \leq \chi_C(U)$, $\forall\ \ t \in [0, T_0]$, where $U(t) = \{u(t); u \in U\} \subseteq X$. Furthermore, if U is equicontinuous on $[0, T_0]$, then $\chi(U(t))$ is continuous on the interval $[0, T_0]$ and*

$$\chi_C(U) = \sup_{t \in [0, T_0]} \{\chi(U(t))\}. \tag{14}$$

Lemma 2.6 *[10] If $U \subset C([0, T_0]; X)$ is bounded and equicontinuous, then $\chi(U(t))$ is continuous and*

$$\chi(\int_0^t U(s)ds) \leq \int_0^t \chi(U(s))ds,\ \forall\ \ t \in [0, T_0], \tag{15}$$

where $\int_0^t U(s)ds = \{\int_0^t u(s)ds, u \in U\}$.

Lemma 2.7 *[14]*

(1) If $U \subset \mathcal{PC}([0,T_0];X)$ *is bounded, then* $\chi(U(t)) \le \chi_{\mathcal{PC}}(U)$, $\forall\ t \in [0,T_0]$, *where* $U(t) = \{u(t) : u \in U\} \subset X$;

(2) If U *is piecewise equicontinuous on* $[0,T_0]$, *then* $\chi(U(t))$ *is piecewise continuous for* $t \in [0,T_0]$ *and*

$$\chi_{\mathcal{PC}}(U) = \sup\{\chi(U(t)) : t \in [0,T_0]\}; \tag{16}$$

(3) If $U \subset \mathcal{PC}([0,T_0];X)$ *is bounded and equicontinuous, then* $\chi(U(t))$ *is piecewise continuous for* $t \in [0,T_0]$ *and*

$$\chi(\int_0^t U(s)ds) \le \int_0^t \chi(U(s))ds, \forall\ t \in [0,T_0], \tag{17}$$

where $\int_0^t U(s)ds = \{\int_0^t u(s)ds : u \in U\}$.

Now, we present the definition of mild solution for the system (6)-(8).

Definition 2.5 A piecewise continuous function $y : [-\infty, T_0]$ is said to be a mild solution for the system (6)-(8) if $y_0 = \phi$, $y(\cdot)|_{[0,T_0]} \in \mathcal{PC}$ and the following integral equation

$$\begin{aligned} y(t) &= E^{-1}R(t)E\phi(0) + E^{-1}R(t)F(0,\phi,0) - E^{-1}F(t,y_t,\int_0^t a(t,s,y_s)ds) \\ &\quad -E^{-1}\int_0^t R(t-s)AE^{-1}F(s,y_s,\int_0^s a(s,\tau,y_\tau)d\tau)ds \\ &\quad -E^{-1}\int_0^t R(t-s)AE^{-1}\int_0^s f(s-\tau)F(\tau,y_\tau,\int_0^\tau a(\tau,\xi,y_\xi)d\xi)d\tau ds \\ &\quad +E^{-1}\int_0^t R(t-s)G(s,y_s,\int_0^s \mathcal{E}(s,\tau,y_\tau)d\tau)ds \\ &\quad + \sum_{0<t_i<t} E^{-1}R(t-t_i)I_i(y_{t_i}), \quad t \in [0,T_0], \end{aligned} \tag{18}$$

is verified.

3 Main Results

We assume the following conditions which will be required to establish the result.

(E5) The function $F : [0,T_0] \times \mathfrak{B} \times X \to X$ is a continuous function and there exist positive constants L_{F_1} and L_{F_2} such that

$$\begin{aligned} \|F(t_1,w_1,z_1) - F(t_2,w_2,z_2)\| &\le L_{F_1}[|t_1-t_2| + \|w_1-w_2\|_{\mathfrak{B}} + \|z_1-z_2\|_X], \\ \|AF(t,w_1,z_1) - AF(t,w_2,z_2)\| &\le L_{F_2}[\|w_1-w_2\|_{\mathfrak{B}} + \|z_1-z_2\|_X], \end{aligned} \tag{19}$$

for all $t_1,t_2,t \in [0,T_0]$, $w_1,w_2 \in \mathfrak{B}$ and $z_1,z_2 \in X$ with $L_1 = \sup_{t\in[0,T_0]}\|F(t,0,0)\|$, $L_2 = \sup_{t\in[0,T_0]}\|AF(t,0,0)\|$.

(E6) (1). The function $a(t,s,\cdot) : \mathfrak{B} \to X$ is continuous for each $(t,s) \in [0,T_0] \times [0,T_0]$ and $a(\cdot,\cdot,w)$, $\mathcal{E}(\cdot,\cdot,w) : [0,T_0] \times [0,T_0] \to X$ are strongly measurable for all $w \in \mathfrak{B}$.

The function $a : J \times J \times \mathfrak{B} \to X$ is a continuous function and there exists constant $a_1 > 0$ such that

$$\| \int_0^t [a(t,s,w) - a(t,s,z)]ds\| \quad \leq \quad a_1 \|w - z\|_{\mathfrak{B}}, \tag{20}$$

for each $(t,s) \in J \times J$ and $z, w \in \mathfrak{B}$.
(2). There exist functions $m_a, m_{\mathcal{E}} : [0,T_0] \times [0,T_0] \to [0,+\infty)$ such that $m_a, m_{\mathcal{E}}$ are differentiable, a.e., with respect to the first variable and $\int_0^t m_a(t,s)ds$, $\int_0^t m_{\mathcal{E}}(t,s)ds$, $\int_0^t \frac{\partial, m_a(t,s)[\text{or } m_{\mathcal{E}}(t,s)]}{\partial t} ds$ are bounded on $[0,T_0]$ and $\frac{\partial m_{\mathcal{E}}}{\partial t} \geq 0$, for a.e., $0 \leq s < t \leq T_0$ such that

$$\begin{aligned} \|a(t,s,w)\| &\leq m_a(t,s) W_a(\|w\|_{\mathfrak{B}}), \\ \|\mathcal{E}(t,s,w)\| &\leq m_{\mathcal{E}}(t,s) W_{\mathcal{E}}(\|w\|_{\mathfrak{B}}), \end{aligned} \tag{21}$$

for each $0 \leq s < t \leq T_0,\ w \in \mathfrak{B}$ and $W_a, W_{\mathcal{E}} : [0,\infty) \to (0,\infty)$ are continuous nondecreasing functions.

(E7) $G : [0,T_0] \times \mathfrak{B} \times X \to X$ is a nonlinear function such that
(1) For each $y : (-\infty, T_0] \to X,\ y_0 = \phi \in \mathfrak{B},\ G(t,\cdot,\cdot)$ is continuous a.e. for $t \in [0,T_0]$ and function $t \mapsto G(t, y_t, \int_0^t \mathcal{E}(t,s,y_s)ds)$ is strongly measurable for $y \in \mathcal{PC}([0,T_0];X)$.
(2) There are integrable functions $\alpha, \beta : J \to [0,\infty)$ and continuously differentiable increasing functions $\Omega,\ \mathfrak{W} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\| G(\tau, w, z)\| \leq \alpha(\tau)\Omega(\| w\|_{\mathfrak{B}}) + \beta(\tau)\mathfrak{W}(\| z\|),\ \tau \in [0,T_0],\ (w,z) \in \mathfrak{B} \times X. \tag{22}$$

(3) There is an integrable function $\xi : J \to [0,\infty)$ such that for any bounded subsets $H_1 \subset \mathcal{PC}((-\infty,0];X)$, $H_2 \subset X$, we have that

$$\chi(R(\tau)G(\tau, H_1, H_2)) \leq \xi(\tau)\{\sup_{-\infty \leq \theta \leq 0} \chi(H_1(\theta)) + \chi(H_2)\}, \tag{23}$$

a.e. for $t \in [0,T_0]$. Where $H_1(\theta) = \{u(\theta) : u \in H_1\}$.

(E8) (1) The functions $I_i : \mathfrak{B} \to X,\ i = 1,2,\cdots,m$ are continuous and there are constants $L_i > 0$ $(i = 1,2,\cdots,m)$ such that

$$\| I_i(x) - I_i(y)\| \leq L_i \| x - y\|_{\mathfrak{B}},\ \forall\, x,y \in \mathfrak{B}. \tag{24}$$

(2) There exist positive constants K_i^1 and K_i^2,$(i = 1,\cdots,m)$ such that

$$\| I_i(x)\| = K_i^1 \| x\|_{\mathfrak{B}} + K_i^2,\ x \in \mathfrak{B}. \tag{25}$$

(E9)

$$\int_0^{T_0} b(s)ds \leq \int_e^{+\infty} [W_a(\vartheta) + \Omega(\vartheta) + \frac{W_{\mathcal{E}}(\vartheta)}{\Omega'(\vartheta)} \mathfrak{W}'(LW_{\mathcal{E}}(\vartheta))]^{-1} ds, \tag{26}$$

where

$$\begin{aligned}
b_1(t) &= \frac{1}{1-\mathcal{C}_2}[(N_{T_0}\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_2 N_1 T_0^2 L_{F_2})(m_a(t,t) \\
&\quad + \int_0^t \frac{\partial m_a(t,s)}{\partial t} ds)], \\
b_2(t) &= \frac{N_{T_0}\Lambda N_1 p(t)}{1-\mathcal{C}_2}, \;\; b_3(t) = m_{\mathcal{E}}(t,t) + \int_0^t \|\frac{\partial m_{\mathcal{E}}(t,s)}{\partial t}\| ds, \\
p(t) &= \max\{\alpha(t), \beta(t)\} \;\; b(t) = \max\{b_1(t), b_2(t), b_3(t)\} \;\; d = \frac{\mathcal{C}_1}{1-\mathcal{C}_2}, \\
\mathcal{C}_1 &= N_{T_0}[\Lambda N_1(L_{F_1} T_0 + L_1) + \Lambda L_1 + \Lambda^2 N_1 T_0 L_2(1 + N_2 T_0) + N_1 + \sum_{0<t_i<t} K_i^2] \\
&\quad + [N_1 L_{F_1} N_{T_0} + (N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})]\|\phi\|_{\mathfrak{B}}, \\
\mathcal{C}_2 &= N_{T_0}[\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_2 N_1 T_0^2 L_{F_2} + \Lambda N_1 \sum_{0<t_i<t} K_i^1] < 1, \\
e &= \Omega^{-1}(\Omega(d) + \mathfrak{W}(d)), \quad \int_0^t m_{\mathcal{E}}(t,s) ds < L_0, \\
&\quad \Omega_1 \text{ is arbitrary positive constant.}
\end{aligned}$$

We consider the function $z : (-\infty, T_0] \to X$ defined by $z_0 = \phi$ and $z(t) = E^{-1}R(t)E\phi(0)$ on $[0, T_0]$. It is easy to see that $\|z_t\| \leq [N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0}]\|\phi\|_{\mathfrak{B}}$, where $N_{T_0} = \sup_{t\in[0,T_0]} N(t)$, $K_{T_0} = \sup_{t\in[0,T_0]} K(t)$ and $\Lambda = \|E^{-1}\|$, $\Lambda' = \|E\|$.

Theorem 3.1 *If the assumptions $(E1)-(E9)$ are fulfilled and*

$$N_{T_0}[\Lambda(1+a_1)(L_{F_1} + \Lambda N_1 T_0 L_{F_2} + \Lambda N_1 N_2 T_0^2 L_{F_2}) + \Lambda N_1 \sum_{0<t_i<t} L_i] + \Lambda(1 + L_0\Omega_1)\int_0^t \xi(s) ds < 1. \tag{27}$$

Then, there exists at least one solution for the system (6)-(8).

Proof. Let $\mathcal{S}(T_0) = \{y : (-\infty, T_0] \to X : y_0 = \phi,\ y|_{[0,T_0]} \in \mathcal{PC}\}$ with the supremum norm $(\|\cdot\|_{T_0})$ be the space. Now, we consider the operator $\Pi : \mathcal{S}(T_0) \to \mathcal{S}(T_0)$ defined by

$$\Pi y(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ E^{-1}R(t)F(0,\phi,0) - E^{-1}F(t, y_t + z_t, \int_0^t a(t,s,y_s+z_s)ds) & \\ -E^{-1}\int_0^t R(t-s)AE^{-1}F(s, y_s+z_s, \int_0^s a(s,\tau,y_\tau+z_\tau)d\tau)ds & \\ -E^{-1}\int_0^t R(t-s)AE^{-1}\int_0^s f(s-\tau)F(\tau, y_\tau+z_\tau, \int_0^\tau a(\tau,\xi,y_\xi+z_\xi)d\xi)d\tau ds & \\ +E^{-1}\int_0^t R(t-s)G(s, y_s+z_s, \int_0^s \mathcal{E}(s,\tau,y_\tau+z_\tau)d\tau)ds & \\ +\sum_{0<t_i<t} E^{-1}R(t-t_i)I_i(y_{t_i}+z_{t_i}), \quad t \in [0, T_0]. & \end{cases} \tag{28}$$

Clearly, we have $\|y_t + z_t\|_{\mathfrak{B}} \leq [N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0}]\|\phi\|_{\mathfrak{B}} + N_{T_0}\|y\|_t$, where $\|y\|_t = \sup_{s\in[0,t]}\|y(s)\|$. From the axioms A, our assumptions and the strong continuity of $R(t)$, we can see that $\Pi y \in \mathcal{PC}$. For $y \in S(T_0)$, we get

$$\|R(t-s)AE^{-1}F(s, y_s + z_s, \int_0^s a(s,\tau,y_\tau + z_\tau)d\tau)\| \leq \Lambda N_1[L_{F_2}(\|y_s + z_s\|_{\mathfrak{B}} + \int_0^t m_a(t,s)W_a(\|y_s + z_s\|_{\mathfrak{B}})) + L_2],$$

and

$$\|f(s-\tau)AE^{-1}F(\tau, y_\tau + z_\tau, \int_0^\tau a(\tau,\xi,y_\xi + z_\xi)d\xi)d\tau\| \leq N_2\Lambda[L_{F_2}(\|y_s + z_s\|_{\mathfrak{B}} + \int_0^t m_a(t,s)W_a(\|y_s + z_s\|_{\mathfrak{B}})) + L_2].$$

Thus, from the Bocher theorem it takes after that $AR(t-s)F(s, y_s + z_s, \int_0^s a(s,\tau,y_\tau + z_\tau)d\tau)$ is integrable. So, we deduce that Π is well defined on $\mathcal{S}(T_0)$. Next, we give the demonstration of Theorem 3.1 in numerous steps.

Step 1. The set $\{y \in \mathcal{PC}([0,T_0],X) : \ y(t) = \lambda \Pi y(t), \text{ for } 0 < \lambda < 1\}$ is bounded. For $\lambda \in (0,1)$, let y_λ be a solution for $y = \lambda \Pi y$. We obtain

$$\| y_{\lambda_t} + z_t\| \leq [N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0}]\|\phi\|_{\mathfrak{B}} + N_{T_0}\|y_\lambda\|_t. \tag{29}$$

Let $u_\lambda(t) = [N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0}]\|\phi\|_{\mathfrak{B}} + N_{T_0}\|y_\lambda\|_t$ for each $t \in [0,T_0]$ and $\lambda \in (0,1)$. $\|y_\lambda(t)\| = \|\lambda \Pi y_\lambda(t)\| \leq \|\Pi y_\lambda(t)\|$

$$\begin{aligned}
\leq \ & \|E^{-1}R(t)F(0,\phi,0)\| + \|E^{-1}F(t, y_{\lambda_t} + z_t, \int_0^t a(t,s,y_{\lambda_s} + z_s)ds)\| \\
& + \|E^{-1}\int_0^t R(t-s)AE^{-1}F(s, y_{\lambda_s} + z_s, \int_0^s a(s,\tau,y_{\lambda_\tau} + z_\tau)d\tau)ds\| \\
& + \|E^{-1}\int_0^t R(t-s)AE^{-1}\int_0^s f(s-\tau)F(\tau, y_{\lambda_\tau} + z_\tau, \int_0^\tau a(\tau,\xi,y_{\lambda_\xi} + z_\xi)d\xi)d\tau ds\| \\
& + \|\int_0^t R(t-s)E^{-1}G(s, y_{\lambda_s} + z_s, \int_0^s \mathcal{E}(s,\tau,y_{\lambda_\tau} + z_\tau)d\tau)ds\| \\
& + \sum_{0<t_i<t} \|E^{-1}R(t-t_i)I_i(y_{\lambda_{t_i}} + z_{t_i})\|, \\
\leq \ & \Lambda N_1(L_{F_1}(T_0 + \|\phi\|_{\mathfrak{B}}) + L_1) + \Lambda[L_{F_1}(u_\lambda(t) + \int_0^t m_a(t,s)W_a(u_\lambda(s))ds) + L_1] \\
& + \Lambda^2 N_1 T_0[L_{F_2}(u_\lambda(t) + \int_0^t m_a(t,s)W_a(u_\lambda(s))ds) + L_2] \\
& + \Lambda^2 N_2 N_1 T_0^2[L_{F_2}(u_\lambda(s) + \int_0^t m_a(t,s)W_a(u_\lambda(s))ds) + L_2] \\
& + \Lambda N_1 \int_0^t \alpha(s)\Omega(u_\lambda(s)) + \beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s,\tau)W_{\mathcal{E}}(u_\lambda(\tau))d\tau)ds \\
& + \Lambda N_1 \sum_{0<t_i<t}(K_i^1 u_\lambda(t) + K_i^2),
\end{aligned}$$

which gives that
$\|y_\lambda(t)\|$

$$\begin{aligned}
&\leq \Lambda N_1(L_{F_1}T_0+L_1)+\Lambda L_1+\Lambda^2N_1T_0L_2(1+N_2T_0)+N_1\sum_{0<t_i<t}K_i^2+N_1L_{F_1}\|\phi\|_{\mathfrak{B}}\\
&\quad+[\Lambda L_{F_1}+\Lambda^2N_1T_0L_{F_2}+\Lambda^2N_2N_1T_0^2L_{F_2}+\Lambda N_1\sum_{0<t_i<t}K_i^1]u_\lambda(t)\\
&\quad+[\Lambda L_{F_1}+\Lambda^2N_1T_0L_{F_2}+\Lambda^2N_2N_1T_0^2L_{F_2}]\int_0^t m_a(t,s)W_a(u_\lambda(s))ds\\
&\quad+\Lambda N_1\int_0^t \alpha(s)\Omega(u_\lambda(s))+\beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s,\tau)W_{\mathcal{E}}(u_\lambda(\tau))d\tau)ds.
\end{aligned}$$

Thus, we estimate

$$\begin{aligned}
u_\lambda(t) \quad \leq \quad & \frac{\mathcal{C}_1}{1-\mathcal{C}_2}+\frac{N_{T_0}}{1-\mathcal{C}_2}[\Lambda L_{F_1}+\Lambda^2N_1T_0L_{F_2}\\
&+\Lambda^2N_2N_1T_0^2L_{F_2}]\int_0^t m_a(t,s)W_a(u_\lambda(s))ds\\
&+\frac{N_{T_0}\Lambda N_1}{1-\mathcal{C}_2}\int_0^t \alpha(s)\Omega(u_\lambda(s))+\beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s,\tau)W_{\mathcal{E}}(u_\lambda(\tau))d\tau)ds.
\end{aligned}$$

Take $d=\frac{\mathcal{C}_1}{1-\mathcal{C}_2}$ and get

$$\begin{aligned}
u_\lambda(t) \quad \leq \quad & d+\frac{N_{T_0}}{1-\mathcal{C}_2}[\Lambda L_{F_1}+\Lambda^2N_1T_0L_{F_2}+\Lambda^2N_2N_1T_0^2L_{F_2}]\int_0^t m_a(t,s)W_a(u_\lambda(s))ds\\
&+\frac{N_{T_0}\Lambda N_1}{1-\mathcal{C}_2}\int_0^t \alpha(s)\Omega(u_\lambda(s))+\beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s,\tau)W_{\mathcal{E}}(u_\lambda(\tau))d\tau)ds. \qquad (30)
\end{aligned}$$

Let

$$\begin{aligned}
\mu_\lambda(t) \quad = \quad & d+\frac{N_{T_0}}{1-\mathcal{C}_2}[\Lambda L_{F_1}+\Lambda^2N_1T_0L_{F_2}+\Lambda^2N_2N_1T_0^2L_{F_2}]\int_0^t m_a(t,s)W_a(u_\lambda(s))ds\\
&+\frac{N_{T_0}\Lambda N_1}{1-\mathcal{C}_2}\int_0^t \alpha(s)\Omega(u_\lambda(s))+\beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s,\tau)W_{\mathcal{E}}(u_\lambda(\tau))d\tau)ds, \qquad (31)
\end{aligned}$$

then, we get $\mu_\lambda(0)=d$ and $u_\lambda(t)\leq\mu_\lambda$ for each $t\in[0,T_0]$. Thus, we get

$$\begin{aligned}
\mu_\lambda^{'}(t) \quad \leq \quad & \frac{N_{T_0}}{1-\mathcal{C}_2}[\Lambda L_{F_1}+\Lambda^2N_1T_0L_{F_2}+\Lambda^2N_2N_1T_0^2L_{F_2}]\\
&\times(a_0(t,t)W_a(u_\lambda(t))+\int_0^t\frac{\partial m_a(t,s)}{\partial t}W_a(u_\lambda(t))ds)\\
&+\frac{N_{T_0}\Lambda N_1}{1-\mathcal{C}_2}[\alpha(t)\Omega(u_\lambda(t))+\beta(t)\mathfrak{W}(\int_0^t m_{\mathcal{E}}(t,s)W_{\mathcal{E}}(u_\lambda(s))ds)].
\end{aligned}$$

Let $\vartheta(t)$ be such that

$$\Omega(\vartheta)=\Omega(\mu_\lambda)+\mathfrak{W}(\int_0^t m_{\mathcal{E}}(t,s)W_{\mathcal{E}}(\mu_\lambda)ds). \qquad (32)$$

We also have $\vartheta \geq \mu_\lambda$. We differentiate the above equation and get

$$\begin{aligned}
\Omega^{'}(\vartheta)\vartheta^{'} &= \Omega^{'}(\mu_\lambda)\mu_\lambda^{'} + \mathfrak{W}^{'}(\int_0^t m_{\mathcal{E}}(t,s)W_{\mathcal{E}}(\mu_\lambda)ds) \\
&\quad \times[\int_0^t \frac{\partial m_{\mathcal{E}}}{\partial t}(t,s)W_{\mathcal{E}}(\mu_\lambda)ds + m_{\mathcal{E}}(t,t)W_{\mathcal{E}}(\mu_\lambda)], \\
\Omega^{'}(\vartheta)\vartheta^{'} &\leq \Omega^{'}(\vartheta)[\frac{N_{T_0}}{1-\mathcal{C}_2}(\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_2 N_1 T_0^2 L_{F_2}) \\
&\quad \times W_a(\vartheta)(a_0(t,t) + \int_0^t \frac{\partial m_a(t,s)}{\partial t}ds) \\
&\quad + \frac{N_{T_0}\Lambda N_1}{1-\mathcal{C}_2}p(t)\Omega(\vartheta)] + \mathfrak{W}^{'}(W_{\mathcal{E}}(\vartheta)\int_0^t m_{\mathcal{E}}(t,s)ds) \\
&\quad \times W_{\mathcal{E}}(\vartheta)[\int_0^t \|\frac{\partial m_{\mathcal{E}}}{\partial t}(t,s)\|ds + m_{\mathcal{E}}(t,t)] \qquad (33)
\end{aligned}$$

Furthermore, from the hypotheses on Ω, we get

$$\Omega^{'}(\vartheta) \geq \Omega^{'}(\mu_\lambda) \geq \Omega(\mu_\lambda(0)) \geq \Omega^{'}(\Lambda\Lambda N_1\|\phi\|_{\mathfrak{B}}) > 0.$$

Thus, we get

$$\begin{aligned}
\vartheta^{'} &\leq \frac{1}{1-\mathcal{C}_2}[(N_{T_0}\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_2 N_1 T_0^2 L_{F_2}) \times W_a(\vartheta)(a_0(t,t) \\
&\quad + \int_0^t \frac{\partial m_a(t,s)}{\partial t}ds) + N_{T_0}\Lambda N_1 p(t)\Omega(\vartheta)] + \frac{W_{\mathcal{E}}(\vartheta)}{\Omega'(\vartheta)}\mathfrak{W}^{'}(W_{\mathcal{E}}(\vartheta)\int_0^t m_{\mathcal{E}}(t,s)ds) \\
&\quad \times[\int_0^t \|\frac{\partial m_{\mathcal{E}}}{\partial t}(t,s)\|ds + m_{\mathcal{E}}(t,t)]. \qquad (34)
\end{aligned}$$

By the assumption $(E9)$, we estimate

$$\begin{aligned}
\vartheta^{'} &\leq [b_1 W_a(\vartheta) + b_2\Omega(\vartheta) + \frac{b_3 W_E(\vartheta)}{\Omega'(\vartheta)}\mathfrak{W}^{'}(LW_{\mathcal{E}}(\vartheta))], \\
&\leq b(t)(W_a(\vartheta) + \Omega(\vartheta) + \frac{W_{\mathcal{E}}(\vartheta)}{\Omega'(\vartheta)}\mathfrak{W}^{'}(LW_{\mathcal{E}}(\vartheta))). \qquad (35)
\end{aligned}$$

Thus, for $t \in [0, T_0]$

$$\begin{aligned}
&\int_{\vartheta(0)}^{\vartheta(t)}[W_a(\vartheta) + \Omega(\vartheta) + \frac{W_{\mathcal{E}}(\vartheta)}{\Omega'(\vartheta)}\mathfrak{W}^{'}(LW_{\mathcal{E}}(\vartheta))]^{-1}ds \\
&\qquad \leq \int_0^{T_0} b(s)ds, \\
&\qquad \leq \int_e^{+\infty}[W_a(\vartheta) + \Omega(\vartheta) + \frac{W_{\mathcal{E}}(\vartheta)}{\Omega'(\vartheta)}\mathfrak{W}^{'}(LW_{\mathcal{E}}(\vartheta))]^{-1}ds, \qquad (36)
\end{aligned}$$

it implies that the function $\vartheta(t)$ is bounded function on $[0, T_0]$. Thus, we obtain that the function $u_\lambda(t)$ is bounded on $[0, T_0]$. Hence, $y_\lambda(\cdot)$ is bounded on $[0, T_0]$.

Step 2. Π is a χ-contraction.
Now, we introduce the decomposition of $\Pi = \Pi_1 + \Pi_2$ defined by

$$\begin{aligned}
\Pi_1 y(t) &= E^{-1}R(t)F(0,\phi,0) - E^{-1}F(t, y_t + z_t, \int_0^t a(t,s,y_s+z_s)ds) \\
&\quad -E^{-1}\int_0^t R(t-s)AE^{-1}F(s, y_s+z_s, \int_0^s a(s,\tau,y_\tau+z_\tau)d\tau)ds \\
&\quad -E^{-1}\int_0^t R(t-s)AE^{-1}\int_0^s f(s-\tau)F(\tau, y_\tau+z_\tau, \int_0^\tau a(\tau,\xi,y_\xi+z_\xi)d\xi)d\tau ds \\
&\quad + \sum_{0<t_i<t} E^{-1}R(t-t_i)I_i(y_{t_i}+z_{t_i}), \qquad (37) \\
\Pi_2 y(t) &= E^{-1}\int_0^t R(t-s)G(s, y_s+z_s, \int_0^s E(s,\tau,y_\tau+z_\tau)d\tau)ds. \qquad (38)
\end{aligned}$$

Now, we firstly show that Π is Lipschitz continuous with Lipschitz constant $\mathcal{K}_1$. Let $y_1,\ y_2 \in \mathcal{S}(T_0)$. Then, we obtain

$$\begin{aligned}
&\|\Pi_1 y_1(t) - \Pi_1 y_2(t)\| \leq \\
&\quad \|E^{-1}F(t, {y_1}_t + z_t, \int_0^t a(t,s,{y_1}_s+z_s)ds) - E^{-1}F(t, {y_2}_t + z_t, \int_0^t a(t,s,{y_2}_s+z_s)ds)\| \\
&\quad +\|E^{-1}\| \int_0^t \|R(t-s)AE^{-1}[F(s, {y_1}_s+z_s, \int_0^s a(s,\tau,{y_1}_\tau+z_\tau)d\tau) \\
&\quad -F(s, {y_2}_s+z_s, \int_0^s a(s,\tau,{y_2}_\tau+z_\tau)d\tau)]\|ds \\
&\quad +\|E^{-1}\| \int_0^t \|R(t-s)AE^{-1}\int_0^s f(s-\tau)F(\tau, {y_1}_\tau+z_\tau, \int_0^\tau a(\tau,\xi,{y_1}_\xi+z_\xi)d\xi) \\
&\quad -F(\tau, {y_2}_\tau+z_\tau, \int_0^\tau a(\tau,\xi,{y_2}_\xi+z_\xi)d\xi)]d\tau\|ds \\
&\quad + \sum_{0<t_i<t} \|E^{-1}R(t-t_i)\| \cdot \|I_i({y_1}_{t_i}+z_{t_i}) - I_i({y_2}_{t_i}+z_{t_i})\|, \\
&\leq \Lambda L_{F_1}(1+a_1)\|{y_1}_t - {y_2}_t\|_{\mathfrak{B}} + \Lambda^2 N_1 T_0 L_{F_2}(1+a_1)\|{y_1}_t - {y_2}_t\|_{\mathfrak{B}} \\
&\quad +\Lambda^2 N_1 N_2 T_0^2 L_{F_2}(1+a_1)\|{y_1}_t - {y_2}_t\|_{\mathfrak{B}} + \Lambda N_1 \sum_{0<t_i<t} L_i \|{y_1}_t - {y_2}_t\|_{\mathfrak{B}}, \\
&\leq N_{T_0}[\Lambda(1+a_1)(L_{F_1} + \Lambda N_1 T_0 L_{F_2} + \Lambda N_1 N_2 T_0^2 L_{F_2}) + \Lambda N_1 \sum_{0<t_i<t} L_i] \\
&\quad \times \|y_1 - y_2\|_{T_0}, \qquad (39)
\end{aligned}$$

which implies that Π_1 is Lipschitz continuous with Lipschitz constant $\mathcal{K}_1 = N_{T_0}[\Lambda(1+a_1)(L_{F_1} + \Lambda N_1 T_0 L_{F_2} + \Lambda N_1 N_2 T_0^2 L_{F_2}) + \Lambda N_1 \sum_{0<t_i<t} L_i] < 1$.

Let B be an arbitrary subset of $\mathcal{S}(T_0)$. Besides, $R(t)$ is equicontinuous resolvent operator. Therefore, from the assumption (HG) and the strong continuity of $R(t)$, we have that $R(t-s)G(s, x_s + y_s, \int_0^s \mathcal{E}(s,\tau,x_\tau+y_\tau)d\tau)$ is piecewise equicontinuous. Then, by Lemma 2.6 we have

$$
\begin{aligned}
&\chi(\Pi_2(B(t)))\\
&\quad\leq \chi(E^{-1}\int_0^t R(t-s)G(s,B_s+z_s,\int_0^s \mathcal{E}(s,\tau,B_\tau+z_\tau)d\tau)ds),\\
&\quad\leq \Lambda\int_0^t \xi(s)\cdot(\sup_{-\infty<\theta\leq 0}\chi(B(s+\theta)+z(s+\theta))+\chi(\int_0^s E(s,\tau,B_\tau+z_\tau)d\tau))ds,\\
&\quad\leq \Lambda\int_0^t \xi(s)\sup_{-\infty<\theta\leq 0}[\chi(B(s+\theta)+z(s+\theta))+L_0\chi(W_{\mathcal{E}}(B(s+\theta)+z(s+\theta)))]ds,\\
&\quad\leq \Lambda\int_0^t \xi(s)\sup_{0\leq\tau\leq s}(\chi(B(\tau))+L_0\chi(W_{\mathcal{E}}(B(\tau))))ds,\\
&\quad\leq \Lambda\ \chi_{\mathcal{PC}}(B)[1+\Omega_1L_0]\int_0^t\xi(s)ds,\ [\because\ \chi(W_{\mathcal{E}}(B(\tau)))\leq\Omega_1\chi(B(\tau))], \qquad (40)
\end{aligned}
$$

for every bounded set $B\subset\mathcal{PC}$. Here Ω_1 is constant and $\int_0^t m_{\mathcal{E}}(t,s)ds\leq L_0$.

Now we can see that for any bounded subset $B\in\mathcal{PC}$

$$
\begin{aligned}
\chi_{\mathcal{PC}}(\Pi(B)) &= \chi_{\mathcal{PC}}(\Pi_1B+\Pi_2B),\\
&\leq \chi_{\mathcal{PC}}(\Pi_1B)+\chi_{\mathcal{PC}}(\Pi_2B),\\
&\leq (\mathcal{K}_1+\Lambda(1+L_0\Omega_1)\int_0^t\xi(s)ds)\chi_{\mathcal{PC}}(B), \qquad (41)
\end{aligned}
$$

from the above inequality we obtain that Π is χ-contraction. Hence Π has at least one fixed point in B by Darbo fixed point theorem. Let y be the fixed point of the map Π on $S(T_0)$. Thus $u=y+z$ is a mild solution for the problem (6)-(8). Therefore, this completes the proof of the theorem.

Theorem 3.2 *Let us assume that the hypotheses* $(E1)$*-*$(E4)$ *and* $(E5)$*-*$(E9)$ *are satisfied and*

$$
\begin{aligned}
&N_{T_0}[\Lambda L_{F_1}+\Lambda^2N_1T_0L_{F_2}+\Lambda^2N_1N_2T_0^2L_{F_2}+N_1\Lambda\sum_{0<t_i<t}K_i^1]\\
&+(\Lambda L_{F_1}+\Lambda^2N_1T_0L_{F_2}+\Lambda^2N_1N_2T_0^2L_{F_2})\times\int_0^{T_0}m_a(T_0,s)\lim_{\tau\to\infty}\sup\frac{W_a(\tau)}{\tau}ds\\
&+\Lambda N_1\int_0^{T_0}[\alpha(s)\lim_{\tau\to\infty}\sup\frac{\Omega(\tau)}{\tau}+\beta(s)\lim_{\tau\to\infty}\sup\frac{\mathfrak{W}(\tau)}{\tau}]ds<1. \qquad (42)
\end{aligned}
$$

Then, there exists at least one mild solution for Sobolev type equation (6)-(8).

Proof. The proof of the theorem is similar to the proof of the previous Theorem 3.1. We consider the operator Π defined by the equation (28). Next, we show that there exist a positive constant k such that $\Pi(B_k)\subset B_k$, here B_k denotes the closed and convex ball with center at the origin and radius k i.e.,$B_k=\{y\in\mathcal{S}(T_0):\|y\|_{T_0}\leq k\}$. To show the claim, we assume that for any $k>0$, there exists $y_k\in B_k$ and $t_k\in[0,T_0]$ such that $k<\|\ \Pi y_k(t_k)\|$. For $y_k\in B_k$ and $t_k\in[0,T_0]$, we get

$$
\begin{aligned}
k &< \|\Pi y_k(t_k)\| \\
&\leq \Lambda N_1(L_{F_1}T_0 + L_1)\|\phi\|_{\mathfrak{B}} + \Lambda[L_{F_1}(\|y_{k_{t_k}} + z_{t_k}\|_{\mathfrak{B}} \\
&\quad + \int_0^{t_k} m_a(t_k, s)W_a(\|y_{k_{t_k}} + z_{t_k}\|_{\mathfrak{B}})ds) + L_1] \\
&\quad + \Lambda^2 N_1 T_0[L_{F_2}(\|y_{k_{t_k}} + z_{t_k}\|_{\mathfrak{B}} + \int_0^{t_k} m_a(t_k,\tau)W_a(\|y_{k_\tau} + z_\tau\|_{\mathfrak{B}})d\tau) + L_2] \\
&\quad + \Lambda^2 N_1 N_2 T_0^2[L_{F_2}(\|y_{k_s} + z_s\|_{\mathfrak{B}} + \int_0^{t_k} m_a(t_k,\tau)W_a(\|y_{k_\tau} + z_\tau\|_{\mathfrak{B}})d\tau) + L_2] \\
&\quad + \Lambda N_1 \int_0^{t_k} \alpha(s)\Omega(\|y_{k_s} + z_s\|_{\mathfrak{B}}) + \beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s,\tau)W_{\mathcal{E}}(\|y_{k_\tau} + z_\tau\|_{\mathfrak{B}})d\tau)ds \\
&\quad + N_1\Lambda \sum_{0<t_i<t}(K_i^1\|y_{k_{t_k}} + z_{t_k}\|_{\mathfrak{B}} + K_i^2), \\
&\leq N_1(L_{F_1}T_0 + L_1)\|\phi\|_{\mathfrak{B}} + \Lambda L_1 + \Lambda^2 N_1 T_0 L_2 + \Lambda^2 N_1 N_2 T_0^2 L_2 + N_1\Lambda \sum_{0<t_i<t} K_i^2 \\
&\quad + [\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2} + N_1\Lambda \sum_{0<t_i<t} K_i^1] \times \|y_{k_{t_k}} + z_{t_k}\|_{\mathfrak{B}} \\
&\quad + (\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2}) \int_0^{t_k} m_a(t_k,s)W_a(\|y_{k_{t_k}} + z_{t_k}\|_{\mathfrak{B}})ds \\
&\quad + \Lambda N_1 \int_0^{t_k} [\alpha(s)\Omega(\|y_{k_s} + z_s\|_{\mathfrak{B}}) + \beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s,\tau)W_{\mathcal{E}}(\|y_{k_\tau} + z_\tau\|_{\mathfrak{B}})d\tau)]ds, \\
&\leq N_1(L_{F_1}T_0 + L_1)\|\phi\|_{\mathfrak{B}} + \Lambda L_1 + \Lambda^2 N_1 T_0 L_2 + \Lambda^2 N_1 N_2 T_0^2 L_2 + N_1\Lambda \sum_{0<t_i<t} K_i^2 \\
&\quad + [\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2} + N_1\Lambda \sum_{0<t_i<t} K_i^1] \\
&\quad \times [(N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})\|\phi\|_{\mathfrak{B}} + N_{T_0}k] + (\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2}) \\
&\quad \times \int_0^{t_k} m_a(t_k,s)W_a((N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})\|\phi\|_{\mathfrak{B}} + N_{T_0}k)ds \\
&\quad + \Lambda N_1 \int_0^{t_k} [\alpha(s)\Omega((N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})\|\phi\|_{\mathfrak{B}} + N_{T_0}k) \\
&\quad + \beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s,\tau)W_{\mathcal{E}}(N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})\|\phi\|_{\mathfrak{B}} + N_{T_0}k)d\tau)]ds \qquad (43)
\end{aligned}
$$

Dividing the above inequality by k and taking $k \to \infty$, we conclude

$$
\begin{aligned}
1 \quad &< \quad N_{T_0}[\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2} + N_1\Lambda \sum_{0<t_i<t} K_i^1] \\
&\quad + (\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2}) \\
&\quad \times \int_0^{T_0} m_a(T_0,s) \lim_{k\to\infty} \sup \frac{W_a((N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})\|\phi\|_{\mathfrak{B}} + N_{T_0}k)}{k} ds \\
&\quad + \Lambda N_1 \int_0^{T_0} [\alpha(s) \lim_{k\to\infty} \sup \frac{\Omega((N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})\|\phi\|_{\mathfrak{B}} + N_{T_0}k)}{k}
\end{aligned}
$$

$$
\begin{aligned}
&+\beta(s) \lim_{k\to\infty}\sup \frac{\mathfrak{W}(\int_0^{T_0} m_{\mathcal{E}}(T_0,\tau)W_{\mathcal{E}}(N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})\|\phi\|_{\mathfrak{B}} + N_{T_0}k)d\tau)}{k}]ds \\
\leq\ & N_{T_0}[\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2} + N_1\Lambda \sum_{0<t_i<t} K_i^1] \\
&+(\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2}) \times \int_0^{T_0} m_a(T_0,s) \lim_{\tau\to\infty}\sup \frac{W_a(\tau)}{\tau} ds \\
&+\Lambda N_1 \int_0^{T_0} [\alpha(s) \lim_{\tau\to\infty}\sup \frac{\Omega(\tau)}{\tau} + \beta(s) \lim_{\tau\to\infty}\sup \frac{\mathfrak{W}(\tau)}{\tau}]ds \qquad (44)
\end{aligned}
$$

which gives a contradiction with the inequality (42). Hence, we obtain that $\Pi(B_k) \subset B_k$. As in the proof of Theorem 3.1, we conclude that there exists at least one mild solution for the system (6)-(8).

4 Application

Consider the following first order impulsive Sobolev type integro-differential equation with unbounded delay in a Banach space $(X, \|\cdot\|)$

$$
\begin{aligned}
&\frac{d}{dt}[x(t,u) + x_{uu}(t,u) - F(t, x(t-k,u), \int_0^t g_1(t,s,x(s-k,u))ds)] \\
&= \frac{\partial^2}{\partial u^2}[x(t,u) + \int_0^t f(t-s,u)x(s,u)ds] \\
&+ \int_0^t a(t,u,s-t)G(x(s,u), \int_0^s E(s,\tau,x_\tau)d\tau)ds, \quad t\in[0,T_0], \quad u\in[0,\pi], \qquad (45) \\
&x(t,0) = x(t,\pi) = 0, \quad t\in[0,T_0], \qquad (46) \\
&x(\tau,u) = \phi(\tau,u), \quad \tau\leq 0,\ 0\leq u\leq\pi, \qquad (47) \\
&\Delta x(t_i)(u) = \int_{-\infty}^t c_i(t_i-s)x(s,u)ds, \qquad (48)
\end{aligned}
$$

where $\phi \in C_0 \times L^2(h,X)$ ($\mathfrak{B}$-Phase space) and $0 < t_1 < t_2 < \cdots < t_m < b$ are fixed numbers.

The functions f, a, G, E, c_i, F satisfy the following conditions:

(A1) The operator $f(t), t \geq 0$ is bounded and $\| f(t,u)\| \leq N_2$;

(A2) $a(t,u,\tau)$ is continuous function on $[0,T_0]\times[0,\pi]\times(-\infty,0]$ with $\int_{-\infty}^0 a(t,u,\tau)d\tau = n(t,u) < \infty$;

(A3) G is a continuous function, satisfying $G(x_1,x_2) \leq \Omega_1(\| x_1\|) + \Omega_2(\| x_2\|)$, where $\Omega_1(\cdot)$ and $\Omega_2(\cdot)$ are continuous, increasing and positive functions on $[0,\infty)$;

(A4) The function $E(\cdot)$ is a continuous function, satisfying $0 \leq E(t,s,u) \leq m_E(t,s)\omega(\| u\|)$, where ω is a positive increasing continuous function on $[0,\infty)$ and m_E is differentiable a.e., with respect to the first variable with $\int_0^t m_E(t,s)ds$, $\int_0^t \frac{\partial m_E(t,s)}{\partial t}ds$ are bounded on $[0,T_0]$ and $\frac{\partial m_E(t,s)}{\partial t} \geq 0$;

(A5) The functions $c_i \in C([0,\infty);\mathbb{R})$ and $K_i^3 = (\int_{-\infty}^0 \frac{(c_i(s))^2}{h(s)}ds)^{1/2} < 0, \forall\ i = 1,\cdots,m$;

(A6) F is an appropriate Lipschitz continuous function satisfying assumption $(E5)$.

We define the operators $A : D(A) \subset X \to X$ and $E : D(E) \subset X \to X$ such that

$$Ax = x'', \qquad Ex = x + x'',$$

where $D(A)$ and $D(B)$ are defined by

$$\{x \in X : x,\ x_u \text{ are absolutely continuous, } \ x_{uu} \in X,\ x(0) = x(\pi) = 0\}. \tag{49}$$

Then, we get

$$\begin{aligned} Ax &= \sum_{n=1}^{\infty} n^2 < x, x_n > x_n, \quad x \in D(A), \\ Ez &= \sum_{n=1}^{\infty} (1 + n^2) < x, x_n > x_n, \quad x \in D(E), \end{aligned} \tag{50}$$

with $x_n(u) = \sqrt{2/\pi}\sin(nu), \ \ n = 1, \cdots$, is the orthogonal set of vectors of A. Moreover, $x \in X$, we get

$$\begin{aligned} E^{-1}z &= \sum_{n=1}^{\infty} \frac{1}{1+n^2} < x_n, x > x_n, \\ AE^{-1} &= \sum_{n=1}^{\infty} \frac{n^2}{1+n^2} < x_n, x > x_n, \\ R(t)x &= \sum_{n=1}^{\infty} \exp(\frac{n^2 t}{1+n^2}) < x_n, x > x_n. \end{aligned} \tag{51}$$

Clearly, AE^{-1} is the infinitesimal generator of a strongly continuous resolvent operator $R(t)$ on Y. Applying Theorem 3.1, we conclude that there exists at least one mild solution for the system (45)-(48).

5 Conclusion

The existence of mild solution for an impulsive neutral integro-differential equation of Sobolev type was investigated. The sufficient condition for ensuring the existence of mild solution was provided by using Darbo-Sadovskii fixed point theorem, analytic semigroup and Hausdorff measure of noncompactness without assuming Lipschitz continuity of nonlinear part G and compactness of semigroup. An example was studied for explaining the feasibility of the discussed results.

Acknowledgment

The author would like to thank the referee for valuable comments and suggestions. The work of the author is supported by the University Grants Commission (UGC), Government of India, New Delhi and Indian Institute of Technology, Roorkee.

References

[1] Pazy, A. Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York, 1983.

[2] Barenblat, G., Zheltor, J. and Kochiva, I. Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks. *J. Appl. Math. Mech.* **24** (1960) 1286–1303.

[3] Hern*á*ndez, E. and Henr*í*quez, H. R. Existence results for partial neutral functional differential equations with bounded delay. *J. Mathematical Anal. Appl.* **221** (1998) 452–475.

[4] Hern*á*ndez, E., Pierri, M. and Goncalves, G. Existence results for an impulsive abstract partial differential equation with state-dependent delay. *Comput. Math. Appl.* **52** (2006) 411–420.

[5] Hern*á*ndez, E., Sakthivel, R. and Aki, S.T. Existence results for impulsive evolution differential equations with state-dependent delay. *Elect. J. Diff. Equ.* **28** (2008) 1–11.

[6] Henr*í*quez, H. R. and Dos Santos, J. P. C. Existence results for abstract partial neutral integro-differential equation with unbounded delay. *Elect. J. Qualit. Theo. Diff. Equ.* **29** (2009) 1–23.

[7] Hern*á*ndez, E. and O' Regan, D. On a new class of abstract impulsive differential equations. *Proc. Amer. Math. Soc.* **141** (2012) 1641–1649.

[8] Heinz, H.P. On the behavior of measure of noncompactness with respect to differentiation and integration of vector-valued functions. *Nonlinear Analysis: TMA* **7** (1983) 1351–1371.

[9] Hale, J. K. and Kato, J. Phase space for retarded equations with infinite delay. *Funkcial. Ekvac.* **21** (1978) 11–41.

[10] J*ó*zef Banas and Kazimierz Goebel. Measure of Noncompactness in Banach Spaces. *Lecture Notes in Pure and Applied Mathematics.* Marcel Dekker, New York, USA, 1980.

[11] Pr*ü*ss, J. Evolutionary Integral Equations and Applications. In: *Monographs Math.*, Vol. 87. Birkhauser-Verlag, 1993.

[12] Benchohra, M., Henderson, J. and Ntouyas, S. K. Impulsive differential equations and inclusions. *Contemporary Mathematics and Its Applications, Vol.2.* Hindawi Publishing Corporation, New York, 2006.

[13] Agarwal, R. P., Benchohra, M. and Seba, D. On the application of measure of noncompactness to the existence of solutions for fractional differential equations. *Results Math.* **55** (2009) 221–230.

[14] Ye, R. Existence of solutions for impulsive partial neutral functional differential equations with infinite delay. *Nonlinear Analysis: TMA* **73** (2010) 155–162.

[15] Ye, R.S, Dong, Q. and Li G. Existence of solutions of nonlinear abstract neutral integrodifferential equations with infinite delay. *Nonlinear Funct. Anal. Appl.* **17** (2012) 405–420.

[16] Agarwal, R., Meehan, M. and O' Regan, D. Fixed point theory and applications. In: *Cambridge Tracts in Mathematics.* Cambridge University Press, New York, 2001 178–179.

[17] Akhmerov, R. R., Kamenski*ǐ*, M. I., Potapov, A. S., Rodkina A. E. and Sadovski*ǐ*, B. N. Measures of noncompactness and Condensing operators. *Birkhäuser, Boston-Basel*, Berlin, Germany, 1992.

[18] Gunasekar, T., Samuel, F. P. and Arjunan, M. M. Existence results for impulsive neutral functional integro-differential equation with infinite delay. *J. Nonlinear Sci. Appl.* **6** (2013) 234–243.

[19] Chadha, A. and Pandey, D. N. Existence results for an impulsive neutral integro-differential equation with infinite delay via fractional operators. *Malya Journal of Mathematik* **2** (2014) 203–214.

[20] Balachandran, K. and Karunanithi, S. Regularity of solutions of Sobolev type semilinear integrodifferential equations in Banach spaces. *Elect. J. Diff. Equ.* **2003** (114) (2003) 1–8.

[21] Radhakrishnan, B., Mohanraj, A. and Vinoba, V. Existence of nonlinear neutral impulsive integrodifferential evolution equations of Sobolev type with time varying delays. *J. Nonlinear Anal. Optimiz.* **4** (2013) 205–218.

[22] Radhakrishnan, B., Mohanraj, A. and Vinoba, V. Existence of solutions for nonlinear impulsive neutral integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces. *Elect. J. Diff. Equ.* **2013** (18) (2013) 1–13.

[23] Agarwal, S. and Bahuguna, D. Exsitence of solutions to Sobolev-type partial neutral differential equations. *J. Applied Math. Stoch. Anal.* **2006** (2006) 1–10.

[24] Lakshmikantham, V., Bǎinov, D. and Simeonov, P. S. Theory of Impulsive Differential Equations, Series in Modern Applied Mathematics. *World Scientific Publishing Co., Inc.*, Teaneck, NJ, 1989.

[25] Zhang, X., Huang X. and Liu, Z. The Existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay. *Noninear Analysis: HS* **4** (2010) 775–781.

[26] Hino, Y., Murakami, S. and Naito, T. Functional Differential Equations with Infinite Delay. In: *Lecture Notes in Math.*, vol. 1473. Springer-Verlag, Berlin, 1991.

[27] Chang, Y. K. and Li, W. S. Solvability for impulsive neutral integro-differential equations with State-dependent delay via Fractional Operator. *J. Optimi. The. Appl.* **144** (2010) 445–459.

[28] Chang, Y. K. and Nieto, J. J. Existence of solutions for impulsive neutral integro-differential inclusions with nonlocal initial conditions via fractional operators. *Num. Funct. Anal. Optimi.* **30** (2009) 227–244.