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Synchronization of Dumbbell Satellites: Generalized Hamiltonian Systems Approach

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Abstract: In this paper, the attitude synchronization problem of two dumbbell satellite models is addressed. To achieve this purpose, a synchronization approach based on generalized Hamiltonian systems and state observer design reported in literature, is applied. Potential applications of attitude synchronization are multi-satellites arrays for self assembly structures, and resolution enhancement. Numerical results of the synchronization behavior achievement are presented.

Keywords: *dumbbell satellites; attitude synchronization; generalized Hamiltonian systems; nonlinear observers.*

Mathematics Subject Classification (2010): 34D06, 93B07, 93C10.

1 Introduction

Modern space missions involve the use of multiple small satellites, this scheme introduces several advantages compared to single satellite missions. An interesting topic regarding these missions, is the attitude synchronization of the satellites. This allows to handle larger structures than what can be launched. Some interesting applications include: resolution enhancement, interferometry or, super-sized focal length [1], this behavior is also useful for in-orbit-self-assembly operations [2].

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The mathematical model considered in this paper is reported in [3] and corresponds to a dumbbell satellite. This model represents a simple structure consisting of two point masses connected by a mass-less rod. This dumbbell satellite model is suitable for a straightforward investigation of the general properties of the rigid body motion in a gravity field and has attracted the attention of scientists since the middle of the past century [4].

For attitude synchronization of two dumbbell satellites, the approach used in this paper is the generalized Hamiltonian systems and design of nonlinear observer presented in [5] which has been successfully applied in synchronization of chaotic systems, see e.g. [6–11].

The paper is arranged as follows: Section 2 describes briefly the mathematical preliminaries on synchronization of nonlinear oscillators from the perspective of generalized Hamiltonian systems and design of nonlinear observer. Section 3 describes the dumbbell satellite mathematical model used for attitude synchronization purposes. Then, Section 4 presents the attitude synchronization of two dumbbell satellites in master-slave coupling via generalized Hamiltonian forms and state observer design approach. In Section 5, numerical results are discussed and finally some conclusions are given in Section 6.

2 Synchronization Via Generalized Hamiltonian Forms and Observer Design

In this section, briefly we describe the synchronization for two nonlinear dynamical systems via generalized Hamiltonian forms and nonlinear observer design approach, for details see [5].

2.1 Generalized Hamiltonian Systems

Consider the following nonlinear dynamical system described by the state equation

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n. \tag{1}$$

Following the approach provided in [5], many physical nonliner systems described by equation (1) can be written in the following generalized Hamiltonian canonical form,

$$\dot{\boldsymbol{x}} = \mathcal{J}(\boldsymbol{x})\frac{\partial H}{\partial x} + \mathcal{S}(\boldsymbol{x})\frac{\partial H}{\partial x}, \quad x \in \mathbb{R}^n, \tag{2}$$

where $H(x)$ denotes a smooth energy function which is globally positive definite in $\mathbb{R}^n$. The column *gradient vector* of H, denoted by $\partial H/\partial x$, is assumed to exist everywhere. One of the most frequently used functions $H(x)$ is the quadratic energy function of the form

$$H(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\mathcal{M}\boldsymbol{x} \tag{3}$$

with $\mathcal{M}$ being a symmetric, positive definite, constant matrix. In such case, $\partial H/\partial x = \mathcal{M}\boldsymbol{x}$. The square matrices $\mathcal{J}(\boldsymbol{x})$ and $\mathcal{S}(\boldsymbol{x})$, present in (2), satisfy, for all $x \in \mathbb{R}^n$, the following properties, which represent the energy managing structure of the system:

$$\mathcal{J}(\boldsymbol{x}) + \mathcal{J}^T(\boldsymbol{x}) = 0, \qquad \mathcal{S}(\boldsymbol{x}) = \mathcal{S}^T(\boldsymbol{x}). \tag{4}$$

The vector field $\mathcal{J}(\boldsymbol{x})\frac{\partial H}{\partial x}$ exhibits the *conservative* part of the system and it is also referred to as the *work-less part*, or *work-less forces* of the system. The matrix $\mathcal{S}(\boldsymbol{x})$

is, in general, a symmetric matrix describing the *working* or *nonconservative* part of the system. For certain systems, the symmetric matrix $\mathcal{S}(\boldsymbol{x})$ is *negative definite* or *negative semidefinite*, in such cases the vector field is known as the *dissipative* part of the system.

Sometimes, specially in the contex of state observer design, the system under observation will be written in the special form

$$\dot{\boldsymbol{x}} = \mathcal{J}(\boldsymbol{x})\frac{\partial H}{\partial x} + \mathcal{S}(\boldsymbol{x})\frac{\partial H}{\partial x} + \mathcal{F}(x), \tag{5}$$

where $\mathcal{F}(x)$ represents a locally destabilizing vector field and $\mathcal{S}(\boldsymbol{x})$ is a symmetric matrix, not necesarily of definite sign. However, many physical systems are already in the generalized Hamitlonian canonical form (2).

2.2 Nonlinear Observer Design for a Class of Systems in Generalized Hamiltonian Form

For a complete description of the synchronization method, the reader is encouraged to see [5]. A special class of generalized Hamiltonian systems with destabilizing vector field and linear output map y is given by

$$\begin{aligned} &\dot{x} = \mathcal{J}(y)\tfrac{\partial H}{\partial x} + (\mathcal{I}+\mathcal{S})\tfrac{\partial H}{\partial x} + \mathcal{F}(y), \qquad x \in \mathbb{R},^n \\ &y = \mathcal{C}\tfrac{\partial H}{\partial x}, \qquad y \in \mathbb{R}^m, \end{aligned} \tag{6}$$

where $\mathcal{S}$ is a *constant symmetric matrix*, not necessarily of definite sign. The matrix $\mathcal{I}$ is a *constant skew symmetric matrix.* The vector variable y is referred to as the *system output.* The matrix $\mathcal{C}$ is a constant matrix.

The estimate of the state vector x is denoted by ξ, and consider the Hamiltonian energy function $H(\xi)$ to be the particularization of H in terms of ξ, similarly, η is the estimated output computed in terms of the estimated state ξ. The gradient vector $\partial H(\xi)/\partial\xi$ is, naturally, of the form $\mathcal{M}\xi$ with $\mathcal{M}$ being a constant symmetric positive definite matrix.

A dynamic nonlinear state observer for the system (6) is obtained as

$$\begin{aligned} &\dot{\xi} = \mathcal{J}(y)\tfrac{\partial H}{\partial \xi} + (\mathcal{I}+\mathcal{S})\tfrac{\partial H}{\partial \xi} + \mathcal{F}(y) + K(y-\eta), \\ &\eta = C\tfrac{\partial H}{\partial \xi}, \end{aligned} \tag{7}$$

where K is a constant vector, known as the *observer gain.* The *state estimation error*, defined as $e = x - \xi$ and the *output estimation error*, defined as $e_y = y - \eta$, are governed by

$$\begin{aligned} &\dot{e} = \mathcal{J}(y)\tfrac{\partial H}{\partial e} + (\mathcal{I}+\mathcal{S}-\mathcal{K}\mathcal{C})\tfrac{\partial H}{\partial e}, \qquad e \in \mathbb{R}^n, \\ &e_y = C\tfrac{\partial H}{\partial e}, \qquad e_y \in \mathbb{R}^m, \end{aligned} \tag{8}$$

where the vector $\partial H(e)/\partial e$ with some abuse of notation, stands for the gradient vector of the modified energy function, $\partial H(e)/\partial e = \partial H(x)/\partial x - \partial H(\xi)/\partial\xi = \mathcal{M}(x-\xi) = \mathcal{M}e$. In the rest of this work, when needed, it is set that $\mathcal{I}+\mathcal{S} = \mathcal{W}$.

2.3 Synchronization of dynamical systems

Definition 2.1 Synchronization problem ([12]): We say that the slave satellite (7) synchronizes with the master satellite (6), if

$$\lim_{t\to\infty} \|x(t) - \xi(t)\| = 0, \tag{9}$$

no matter which initial conditions $x(0)$ and $\xi(0)$ hold. Here the state estimation error $e(t) = x(t) - \xi(t)$ represents the synchronization error.

Theorem 2.1 ([5]) *The state $x(t)$ of the nonlinear system (6) can be globally, exponentially, asymptotically estimated by the state $\xi(t)$ of an observer of the form (7), if the pair of matrices $(\mathcal{C}, \mathcal{W})$, or the pair $(\mathcal{C}, \mathcal{S})$, is either observable or, at least, detectable.*

An observability condition on either of the pairs $(\mathcal{C}, \mathcal{W})$ or $(\mathcal{C}, \mathcal{S})$ is clearly a sufficient but not necessary condition for asymptotic state reconstruction. A necessary and sufficient condition for global asymtotic stability to zero of the state estimation error $e(t)$ is given by the following theorem.

Theorem 2.2 ([5]) *The state $x(t)$ of the nonlinear system (6) can be globally, exponentially, asymptotically estimated by the state $\xi(t)$ of an observer of the form (7) if and only if there exists a constant matrix $\mathcal{K}$ such that the symmetric matrix*

$$\begin{aligned}[\mathcal{W} - \mathcal{K}\mathcal{C}] + [\mathcal{W} - \mathcal{K}\mathcal{C}]^T &= [\mathcal{S} - \mathcal{K}\mathcal{C}] + [\mathcal{S} - \mathcal{K}\mathcal{C}]^T \\ &= 2\left[\mathcal{S} - \tfrac{1}{2}\left(\mathcal{K}\mathcal{C} + \mathcal{C}^T\mathcal{K}^T\right)\right]\end{aligned}$$

is negative definite.

The application of this method on the field of synchronization of chaotic circuits implies the design of a state observer of the form (7) to act as the receiver of the chaotic system in the form (6) considered as the emitter.

Several advantages of generalized Hamiltonian systems approach over other synchronization techiniques are reported in the literature, the following advantages are enumerated in [5] and [12] and reproduced below:

- It enables synchronization be achieved in a systematic way and clarifies the issue of deciding on the nature of the output signal to be transmitted.
- It can be successfully applied to several well-known chaotic systems.
- It does not require the computation of any Lyapunov exponent.
- It does not require initial conditions belonging to the same basin of attraction.

3 Dumbbell Satellite Model

Typical models of a dumbbell satellite are given in [3] and [4]. In Figure 1 a graphical interpretation can be observed. This model consists of two point masses coupled by a mass-less rod. In this case, θ represents the attitude of the satellite and the (r, ϕ)-tuple represents the position of the satellite with respect to a reference point.

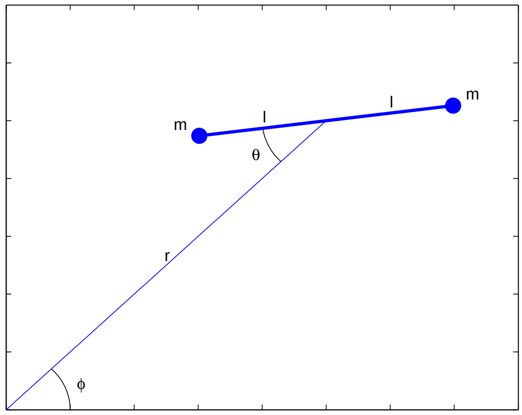

Figure 1: Dumbbell satellite representation.

In this model, the Lagrangian of the system with a normalized universal gravitational constant (G), is given by

$$L = m(\dot{r}^2 + r^2\dot{\phi}^2 + l^2(\dot{\theta} - \dot{\phi})^2) + \frac{m}{\sqrt{l^2 + r^2 - 2lrcos\theta}} + \frac{m}{\sqrt{l^2 + r^2 + 2lrcos\theta}}. \qquad (10)$$

Applying the Euler-Lagrange equation for θ, we can obtain the following differential equation

$$2l^2(\ddot{\theta} - \ddot{\phi}) + \frac{lrsin\theta}{(l^2 + r^2 + 2lrcos\theta)^{3/2}} - \frac{lrsin\theta}{(l^2 + r^2 - 2lrcos\theta)^{3/2}} = 0 \qquad (11)$$

by using a binomial approximation for both denominators, and taking into account that $r \gg l$, one can derive the differential equation of the attitude dynamics of a dumbbell satellite

$$\ddot{\theta} + \frac{3sin(2\theta)}{2r^3} = \ddot{\phi}. \qquad (12)$$

By using a similar procedure for r and ϕ, the differential equations are:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{1}{r^2}, \qquad (13)$$

$$\frac{d}{dt}(r^2\dot{\phi}) = 0. \qquad (14)$$

Equations (13) and (14) describe the Keplerian motion. By using the well-known solutions, $\ddot{\phi}$ can be computed. The equation (12) for the attitude dynamics of a dumbbell satellite is given by

$$\ddot{\theta} + \frac{3}{2}\frac{\sin(2\theta)}{r^3} = -\frac{2\varepsilon\sqrt{1-\varepsilon^2}\sin E}{a^3(1-\varepsilon\cos E)^4}. \qquad (15)$$

Here a and ε refer to the semi-major axis and the eccentricity of the dumbbell satellite's orbital motion, respectively. E denotes the so-called eccentric anomaly and is

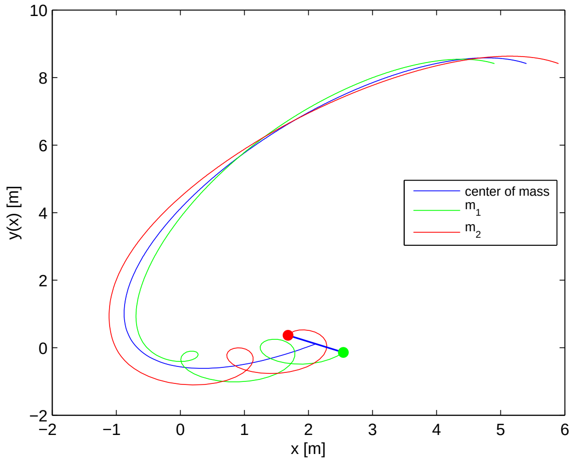

Figure 2: Motion trajectory of a single dumbbell satellite.

related to time t via Kepler's equation. If E is used eventually as an independent variable rather than t [3], the second order differential equation for $\ddot{\theta}$ can be obtained as follows

$$\frac{d^2\theta}{dE^2} - \frac{d\theta}{dE}\frac{\varepsilon \sin E}{1-\varepsilon \cos E} + \frac{3}{2}\frac{\sin(2\theta)}{1-\varepsilon\cos E} = -\frac{2\varepsilon\sqrt{1-\varepsilon^2}\sin E}{(1-\varepsilon\cos E)^2}. \tag{16}$$

Figure 2 shows the motion trajectory governed by the dynamics of the dumbbell satellite model (13)-(15). Recasting the second order equation as a first order system and writing x and t rather than θ and E, respectively, the attitude of the dumbbell satellite is described in the state space as

$$\dot{x_1} = x_2, \tag{17a}$$

$$\dot{x_2} = -\frac{3}{2}\frac{\sin(2x_1)}{1-\varepsilon\cos t} + \frac{\varepsilon\sin t}{1-\varepsilon\cos t}x_2 - \frac{2\varepsilon\sqrt{1-\varepsilon^2}\sin t}{(1-\varepsilon\cos t)^2}. \tag{17b}$$

In this case, x_1 represents the attitude (angular motion) while x_2 represents the angular velocity of the dumbbell satellite.

4 Synchronization of Two Dumbbell Satellites

As seen in the previous section, the equations (17) govern the attitude dynamics of the dumbbell satellite. Therefore, take the state vector as $x^T = [x_1, x_2]$ and define an energy function as $H(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\mathcal{I}\boldsymbol{x}$ where $\mathcal{I}$ is the 2×2 identity matrix. The system (17) can be rewritten in its generalized Hamiltonian form, according to equation (6), so in this way the *master dummbell satellite* in generalized Hamiltonian form is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\frac{\partial H}{\partial x} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\partial H}{\partial x} + \begin{bmatrix} 0 \\ -\frac{3}{2}\frac{\sin(2x_1)}{(1-\varepsilon\cos(t))} + \frac{\varepsilon\sin(t)}{1-\varepsilon\cos(t)}x_2 - \frac{2\varepsilon\sqrt{1-\varepsilon^2}\sin(t)}{(1-\varepsilon\cos(t))^2} \end{bmatrix}. \quad (18)$$

If we select $y = x_1$ as the output, then the $\mathcal{J}$, $\mathcal{S}$, and $\mathcal{C}$ matrices are given by

$$\mathcal{J} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{S} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \quad (19)$$

From equation (19) it can be seen that the pair $(\mathcal{C}, \mathcal{S})$ is observable. Therefore the observer for the system (18) according to equation (7) (*slave dumbbell satellite*) has the following form

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\frac{\partial H}{\partial \xi} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\partial H}{\partial \xi} + \begin{bmatrix} 0 \\ -\frac{3}{2}\frac{\sin(2y)}{(1-\varepsilon\cos(t))} + \frac{\varepsilon\sin(t)}{1-\varepsilon\cos(t)}\xi_2 - \frac{2\varepsilon\sqrt{1-\varepsilon^2}\sin(t)}{(1-\varepsilon\cos(t))^2} \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}(x_1 - \xi_1), \quad (20)$$

where k_1 and k_2 are the observer gains. If the synchronization error is defined as $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{\xi}(t)$, then the dynamics of this error are described as

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -k_1 & 1+k_2 \\ -(1+k_2) & 0 \end{bmatrix}\frac{\partial H}{\partial e} + \frac{1}{2}\begin{bmatrix} -k_1 & 1-k_2 \\ 1-k_2 & 0 \end{bmatrix}\frac{\partial H}{\partial e} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{\varepsilon\sin(t)}{1-\varepsilon\cos(t)} \end{bmatrix}\frac{\partial H}{\partial e}. \quad (21)$$

Next, we examine the stability of the synchronization error (21) between the master dumbbel satellite (18) in Hamiltonian form and slave dumbbell satellite (20) state observer. Invoking to Theorem 2.2, we have that

$$2\left[\mathcal{S} - \frac{1}{2}\left(\mathcal{K}\mathcal{C} + \mathcal{C}^T\mathcal{K}^T\right)\right] < 0,$$

and

$$\begin{bmatrix} -2k_1 & 1-k_2 \\ 1-k_2 & 0 \end{bmatrix} < 0 \quad (22)$$

by applying the Sylvester's criterion – which provides a test for negative definiteness of a matrix – thus, we have the mentioned 2×2 matrix will be negative definite matrix, if we choose k_1 and k_2 such that the condition (22) holds. In the following numerical results, we have used k_1, $k_2 > 0$ to satisfy the stability condition (22).

5 Numerical Results

In this section, numerical results are reported for synchronization of the attitude and angular velocity of two dumbbell satellites, by using generalized Hamiltonian forms and

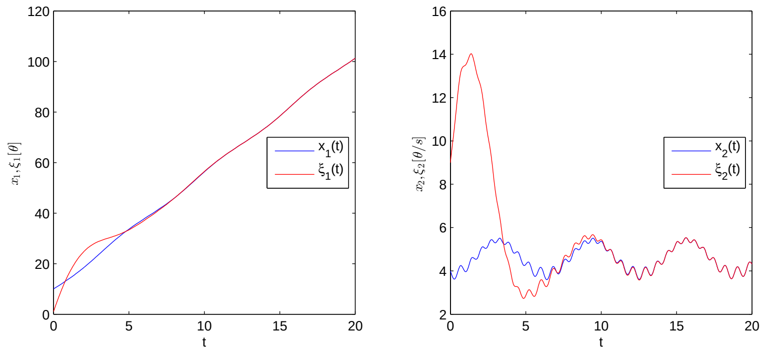

Figure 3: State attitudes $x_1(t)$, $\xi_1(t)$ (left) and state angular velocities $x_2(t)$, $\xi_2(t)$ (right) for master and slave dumbbell satellites.

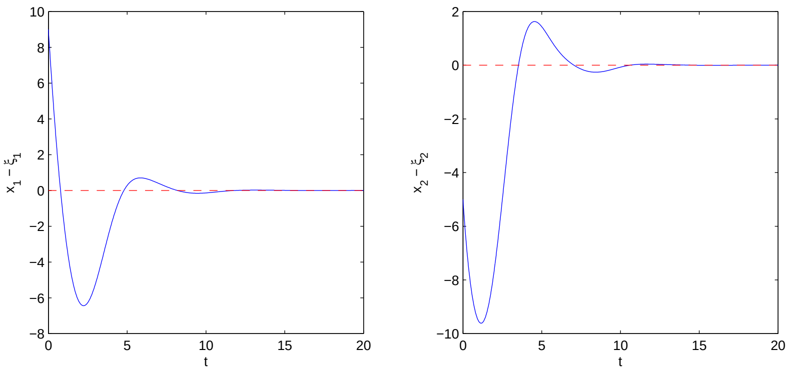

Figure 4: Error dynamics of the attitude (left) and its angular velocity (right) for the numerical simulation in Figure 3.

observer design (equations (18) and (20), respectively). Figure 3 shows the state trajectories of master and slave satellites for the following values: initial conditions $x_1(0) = 10$, $x_2(0) = 4$, $\xi_1(0) = 1$, and $\xi_2(0) = 9$, the eccentricity of the dumbell satellites $\varepsilon = 0.3$, and the gains for slave satellite dumbbell $k_1 = k_2 = 1$.

The synchronization error dynamics between the master dumbbell satellite (18) and its slave dumbbell satellite (20) are shown in Figure 4.

Figure 5 illustrates the synchronization between two dumbbell satellites x_i vs ξ_i, $i = 1, 2$.

6 Conclusion

In this paper, we have presented synchronization between two dumbbell satellites, in particular for the attitude and for the angular velocity, from the perspective of generalized Hamiltonian forms and state nonlinear observer design, an approach that has proven its

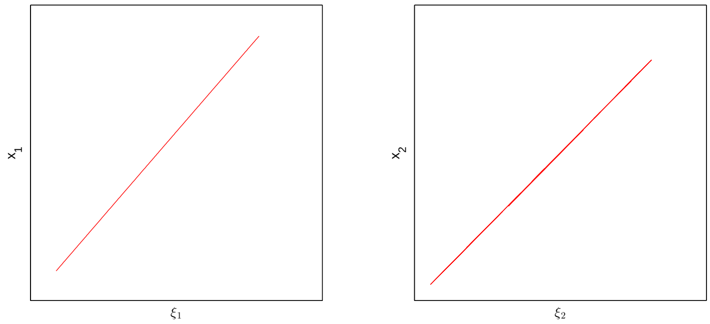

Figure 5: Synchronization of two dumbbell satellites for x_i vs ξ_i, $i = 1, 2$.

efficiency in the literature. The numerical results reported support the control laws designed for attitude synchronization of two dumbbell satellites.

Attitude synchronization for satellites is intended to serve as a first control loop for large array satellite missions; in which a large number of small satellites forms a bigger system functioning as a whole for capabilities enhancement. Thus, in future, a formation controller and the one presented above, can be used together for this type of synchronization space missions with small dumbbell satellites, via synchronization approach used in this paper.

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