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Passive Kinematic Synchronization of Dissimilar and Uncoupled Rotating Systems

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Abstract: Passive kinematic synchronization enables two dynamic physical systems to generate the same motion without any physical interaction or mediating control laws. In this paper, we demonstrate a generalized kinematic matching technique to model and passively synchronize two physically dissimilar and uncoupled rotating systems. The method is demonstrated by matching the motion of systems with different masses and mass distributions. Specifically, we matched the nonlinear motion between three single-link pendulums and the motions of two double-link pendulums. Despite the highly nonlinear dynamics of a double-link pendulum, temporal and spectral analysis results show that the two different and kinematically-matched systems generate nearly identical motion. The method is generalizable and can be used to describe and match the kinematics of any dissimilar open-ended rotating system chain such as rotors, discs, cams, or pendulum type systems. This method has implications for the modeling of physical rotating system dynamics, the study of swinging limbs in humans, animals, and robots, and in limb prosthesis design. We also present a step-by-step method to create a dissimilar but synchronized, rotating system, that is, passively matched to an original system. Further, we present a step-by-step method to kinematically synchronize two already available dissimilar rotating systems.

Keywords: *passive synchronization; nonlinear systems; uncoupled systems; pendulum; kinematic matching; physical rotating systems.*

Mathematics Subject Classification (2010): 93C10, 70K99.

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1 Introduction

Kinematic synchronization of systems is the matching of motion between two moving systems. The synchronization of any two rotating systems can be as simple as physically placing a joining spring or damper between the systems or may require sophisticatedly controlled actuators that augment natural system dynamics. Here, we focus on *dissimilar* rotating systems *without* any physical coupling. Our passive kinematic matching technique allows two independent systems to generate the same motion without any physical system coupling or actuator control law. To validate this method, this passive synchronization technique is applied to two open-ended rotating kinematic chains: single- and double- link pendulums with different masses at different mass locations along links. Even though double-link pendulums are highly nonlinear systems that are sensitive to changes in initial conditions and system parameters, our passive matching technique enables the same generated motion on dissimilar double-link pendulums.

The practical application of such a passive matching technique is the flexibility in mechanical design as one is able to describe the same kinematics with a variety of parameters (i.e., masses and mass distributions). In essence, one is able to decouple the mass and the first moment and second moment of inertia so systems with dissimilar masses and mass distributions will have the same motion. For example, the motion of a double-link pendulum modeled as two links with one mass per link can only be described by one unique combination of masses and mass locations along the links. However, having two masses per link allows the kinematics to be described with an infinite number of distinct systems with distinct masses and mass distribution that all have the same resulting motion. In fact, the minimum number of masses per rotating link to describe any arbitrary rotational kinematics is two masses, yet many models only include one mass. Using only one mass per link inherently couples the moments of inertia so that any change in the location of the mass necessarily affects both the first and second moments of inertia.

The modeling method to derive our synchronization technique can be used to simplify complicated rotational kinematics problems by simplifying the dynamics model of the system by assuming a finite distribution of point masses along swinging members. For example, the rotation of a fan blade can be represented with two masses distributed as specified using this method instead of finding detailed masses, mass distributions, or moments of inertias of the continuous system. This type of modeling can also be applied to human or robotic limbs and in prosthesis design. It is stressed that this point-mass modeling technique is not novel, however is used to develop our novel passive synchronization method that matches the rotational kinematics of two dissimilar and uncoupled rotational systems.

The only requirements for our passive kinematic synchronization of dissimilar systems are: identical degrees of freedom, initial conditions, and torques applied to the systems. These same requirements are also needed to cause two identical systems to have the same motion.

In the proceeding sections we will derive the essential and general model for an open-ended multi-degree of freedom rotational system, define the kinematic matching coefficients needed for system synchronization, and outline the step-by-step instructions on how to passively match the kinematics of newly created or already available systems.

We further form two examples providing proof and application of this system representation and unique passive matching method of dissimilar systems by mathematically and experimentally analyzing three dissimilar one-degree-of-freedom systems and also two dissimilar two-degree-of-freedom systems.

2 Background

2.1 Coupled synchronization

In 1657, in the quest to improve nautical navigation, Dutch mathematician Christiaan Huygenes invented the first pendulum clock [2]. Pendulum clocks were astounding mechanisms of their day. An interesting aspect is that they tend to synchronize and operate in phase or anti-phase when hung on the same wall with another pendulum clock. He deduced that the clocks were coupled by their common supporting structures which transferred small movements between clocks. This clock can be considered the first observation of a *synchronized coupled* oscillator.

The kinematic synchronization of two or more coupled mechanical systems such as Huygen's clock has been extensively studied since the time of Huygen himself. More recent such studies include the synchronization of coupled nonlinear oscillators [3], analysis of coupled multi-pendulum systems [5], and synchronization of double pendulums under the effects of external forces [18]. Osipov et al. [24] published a thorough review on synchronization in oscillatory networks, which mainly discusses different aspects of synchronization in chains and lattices of interconnected oscillatory elements.

As part of the rise of faster computing power came the ability to actively synchronize coupled mechanical systems with linear, nonlinear, passivity-based, or active control laws. There are hundreds of publications which demonstrate such control laws, some of these publications are on controlled motion synchronization for gyroscopes [23], inverted pendulum systems [22], and chaotic systems [19].

2.2 Uncoupled synchronization

Passive kinematic synchronization of physically uncoupled systems has been studied significantly less and the authors were only able to find two examples of uncoupled passive synchronization, both of which are rooted in sports science.

A golfer's technique as well as familiar equipment play an essential role in a golfer's performance. It is for this reason that all golf clubs in a set are matched (synchronized) statically and dynamically, so when swung, each club behaves and feels the same to the golfer [1]. Statically a golf club is matched by simply balancing it on a fulcrum, however dynamically matching the golf club can be achieved by matching the moment of inertia for each club in the set about the swinging axis [4]. Jorgensen presents a golf club dynamic synchronization technique by modeling the swing arm and golf club and matching overall moments of inertia about the wrist axis [17]. In these examples the kinematics of each uncoupled system (golf club) is synchronized given the same input torque (the golfer's swing). While this technique of golf club matching is practical in its specific application, it lacks generalization and flexibility to apply to other rotating systems to be synchronized.

Although very little can be found in the field of passive synchronization of uncoupled systems, a generalized passive synchronization method for physically uncoupled rotating systems has practical implications for locomotion robotics, lower limb gait analysis, and prosthetics. For instance, an individual's walk can largely be modeled as two inverted

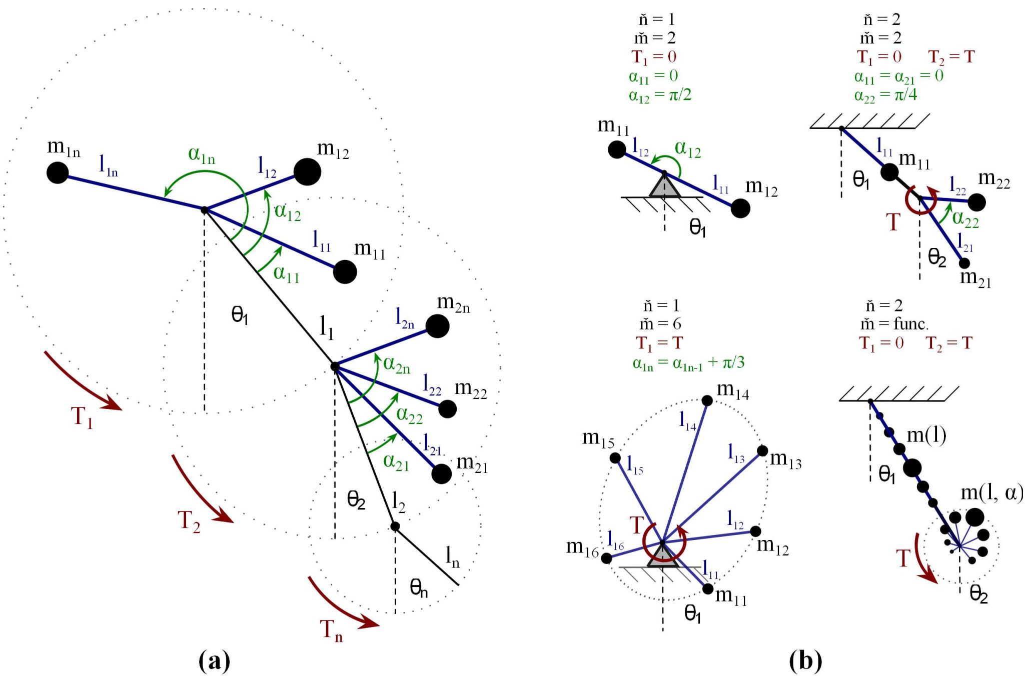

Figure 1: (a) General Rotating System Model. (b) The general rotating systems model can be adjusted to represent various configurations for rotating systems. These configurations can represent a sea-saw/rotor, double pendulum, cam, or a continuous mass distribution along rotating members.

pendulums (left and right step) rotating about the stance foot and progressing down a decline with gravity as the only source of energy [20]. Such models are called passive dynamic walkers (PDW) and have been shown to predict certain aspects of human gait dynamics [7, 13, 14]. Honeycutt et. al [16] used a brute force search through a numerical PDW model to show that asymmetric limbs can have symmetric kinematics, and moving a prosthetic knee joint lower while lowering the prosthetic mass can result in a spatially symmetric gait. Gregg [10, 11] examined symmetry from the other point of view by finding symmetric PDW parameters that yielded asymmetric kinematics. A leg synchronization technique for PDWs, general walking robots, and individuals can be helpful to design and implement devices and methods which either even out gait asymmetries [8], or intentionally exaggerate gait asymmetries for rehabilitation [12, 25]. These gait asymmetries can also arise from the asymmetric size and weight of a prosthetic limb [15].

3 Passive Kinematic Synchronization Technique Derivation

This section outlines the equations used to derive the kinematics of a two-dimensional general rotating system essential for our passive synchronization method. Subsequently we will use this generalized model to draw out a method to synchronize two or more dissimilar rotating systems with the same degrees of freedom, initial conditions, and torque input.

3.1 General rotating system model description

We begin by deriving the equation of motion for a general rotating system with $\check{n}$ *degrees of freedom* and $\check{m}$ *masses per degree of freedom*. Variable notation m symbolizes each individual mass whereas $\check{m}$ symbolizes the *total number* of masses per rotating member (or link). This generalized model is shown in Figure 1a, and can be described using Lagrangian mechanics where the Lagrangian is defined as the difference of kinetic and potential energy. Note that this following formulation of the generalized equation of motion is not novel, however it is used in the subsequently described kinematic matching technique

$$L(\theta,\dot{\theta},t) = K(\theta,\dot{\theta},t) - U(\theta,t). \tag{1}$$

To find the equation of motion, the Euler-Lagrange expression is applied:

$$\frac{d}{dt}\left(\frac{\partial L(\theta,\dot{\theta},t)}{\partial\dot{\theta}_{1,2\ldots\check{n}}}\right) = \frac{\partial L(\theta,\dot{\theta},t)}{\partial\theta_{1,2\ldots\check{n}}}. \tag{2}$$

Equation (2) produces $\check{n}$ equations for $\check{n}$ degrees of freedom of the system. After differentiating and collecting coefficients, the equations of motion of this general dynamic system are a set of $\check{n}$ number of first order nonlinear ordinary differential equations shown in matrix coefficient form in equation (3)

$$[M]\,\ddot{\Theta} \;+\; [N]\,\dot{\Theta}^2 \;+\; [G] \;=\; [T], \tag{3}$$

where the coefficient matrices [M], [N], and [G] are given in equations (4), (7), and (8), respectively. [M] is the inertia matrix coefficient, [N] is the velocity matrix coefficient, and [G] is the position/gravity coefficient matrix. [T] can represent any applied or non-conservative torque functions applied to the system such as actuator torque, joint friction torque, or air resistance experienced by a swinging member,

$$[M]^{\check{n},\check{n}}_{\text{sym}} = \begin{bmatrix} \mathbf{M}_{1,1} & \mathbf{M}_{1,2}\,cos(\theta_1-\theta_2) & \cdots & \mathbf{M}_{1,j}\,cos(\theta_1-\theta_j) \\ \mathbf{M}_{1,2}\,cos(\theta_1-\theta_2) & \mathbf{M}_{2,2} & & \vdots \\ \vdots & & \ddots & \mathbf{M}_{i-1,j}\,cos(\theta_{i-1}-\theta_j) \\ \mathbf{M}_{1,j}\,cos(\theta_1-\theta_j) & \cdots & & \mathbf{M}_{i,i} \end{bmatrix}. \tag{4}$$

Here, each of the coefficients on the diagonal are given by

$$\mathbf{M}_{i,i} = \sum_{p=1}^{\check{m}} l_{i,p}^2 m_{i,p} + l_i^2 \sum_{q=i+1}^{\check{n}} \sum_{p=1}^{\check{m}} m_{q,p} \tag{5}$$

and the remaining non-diagonal coefficients are given by

$$\mathbf{M}_{i,j} = l_i\left[\sum_{p=1}^{\check{m}} l_{j,p}m_{j,p} + \begin{cases} l_j \sum_{q=j+1}^{\check{n}} \sum_{p=1}^{\check{m}} m_{q,p} & j<\check{n} \\ 0 & j\ge\check{n}\end{cases}\right]. \tag{6}$$

The subscripts i and j represent the matrix entry indexes for matrix row and matrix column, respectively,

$$[N]^{\check{n},\check{n}} = \begin{bmatrix} 0 & M_{1,2}\,\sin(\theta_1-\theta_2) & \cdots & M_{1,j}\,\sin(\theta_1-\theta_j) \\ -M_{1,2}\,\sin(\theta_1-\theta_2) & 0 & & \vdots \\ \vdots & & \ddots & M_{i-1,j}\,\sin(\theta_{i-1}-\theta_j) \\ -M_{1,j}\,\sin(\theta_1-\theta_j) & \cdots & & 0 \end{bmatrix}, \tag{7}$$

$$[G]^{\check{n}} = \begin{bmatrix} \sum_{p=1}^{\check{m}} \mathbf{l_{1,p}m_{1,p}sin}(\alpha_{1,p}+\theta_1) + (\mathbf{l_1}\sum_{q=2}^{\check{n}}\sum_{p=1}^{\check{m}} \mathbf{m_{q,p}})\ sin(\theta_1) \\ \vdots \\ \sum_{p=1}^{\check{m}} \mathbf{l_{i,p}m_{i,p}}\,\mathbf{sin}(\alpha_{i,p}+\theta_i) + (\mathbf{l_i}\sum_{q=i+1}^{\check{n}}\sum_{p=1}^{\check{m}} \mathbf{m_{q,p}})\ sin(\theta_i) \\ \vdots \\ \sum_{p=1}^{\check{m}} \mathbf{l_{\check{n},p}m_{\check{n},p}sin}(\theta_{\check{n},p}+\theta_{\check{n}}) \end{bmatrix} g. \tag{8}$$

These are the coefficient matrices for the equations of motion of a general rotating system model with $\check{n}$ degrees of freedom and $\check{m}$ masses per degree of freedom. The [M] matrix is a symmetric matrix, while the [N] matrix is a negatively mirrored matrix with a zero diagonal. Note that the coefficients [equations (5) and (6)] are all unique matrix components in the $[N]$ matrix that all appear in the $[M]$ matrix. Also note that the last row of $[G]$ $(i = \check{n})$ is different since there are no masses from links further down the kinematic chain sequence. Masses (m) and mass distributions (l) are shown in Figure 1a.

Equation (3) can model any degree of rotating system or rotating system links. Degrees of freedom (links), mass, and mass distribution within each link can be easily modified to create models for such systems as shown in Figure 1b. These modified models can represent rotors, pendulums, cams, or rotating kinematic systems and open kinematic chains.

3.2 Passive kinematic synchronization using kinematically matched coefficients

Now that we have defined the general point-mass model for a rotational open-ended swinging system, we are able to utilize to create synchronized motion between two dissimilar systems.

Given the same torque input and initial conditions, two or more systems with the same degrees of freedom will exactly match in dynamics if all four coefficient matrices, $[M]$, $[N]$, $[G]$, and $[T]$ in equation (3) are matched between the systems. Since only the computed end values of these coefficients determine the dynamic behavior of the rotating systems, the masses and mass distribution do not have to match between them. This allows for two or more systems with dissimilar mass and mass distribution parameters to kinematically behave identically, that is, have identical dynamic coefficients $[M]$, $[N]$, $[G]$ and $[T]$. For instance, assuming identical torque input and initial conditions, a swinging single link pendulum with two masses can be designed to swing identically to another single link pendulum with two or more masses, where the masses are distributed differently along the pendulum link. This concept allows for the first and second moments of inertia to be decoupled and greater design flexibility is obtained. Given that each link has two or more masses distributed along the link ($\check{m} \geq 2$), there are infinite combinations of kinetmatically matched systems, that is, there is an infinite number of ways the masses can be distributed such that the four coefficient matrices in equation (3) match another system.

When the coefficient matrices are generalized for systems with $\check{n}$ degrees of freedom with $\check{m}$ masses per link (equations 4, 7 and 8), a pattern of repeating matrix entries emerges. It is seen that for the coefficient matrices to match between two rotating systems and cause synchronized dynamics, only unique parts of the coefficient matrices need to be matched between systems. We will call each unique term that appears in the coefficient matrices a *kinematically matched coefficient* (KMC). The KMCs are represented in equations (5), (6) and (8) and are written in **bold and highlighted** font. The total number of KMCs that have to be matched between kinematically synchronized systems is given in Table 1. For example, to synchronize the dynamics of a pair of one degree of freedom rotating systems, two KMCs need to be matched, while for a pair of three degree of freedom systems to be synchronized, nine KMCs need to be matched.

In the following section, we will review step-by-step instructions on how to apply the passive kinematic synchronization technique for dissimilar and rotating systems, while in Sections 4 and 5 we present two examples of this matching technique for one and two degree-of-freedom systems with experimental validation.

4 Example 1: Passive Single Link Pendulum

In this section, we utilize the method derived in Section 3 and experimentally demonstrate its validity. We start with creating two matched variations of a traditional passive ($[T] = 0$) single mass (m̌=1) single link (ň=1) pendulum that is shown in Figure 2a. Our created variations of the single link pendulum have two masses per link (m̌=2) (Figure 2b).

Although more masses could be utilized to match the motion of this single link pendulum, two masses are sufficient to describe any number of masses and mass

Table 1: Number of Kinematically Matched Coefficients For Synchronized Uncoupled Motion between Two or More Systems.

DOF (ň)	Number of KMCs
1	2
2	5
3	9
⋮	⋮
$\check{n}$	$KMC_{\check{n}-1} + (\check{n}+1)$

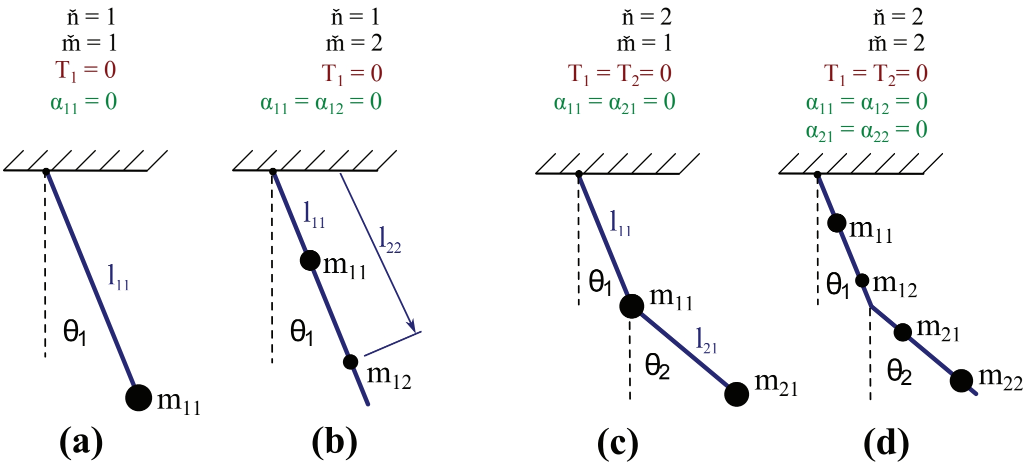

Figure 2: Single link and double (2-link) pendulum representation model. (a), (b), and (d) were used experimentally.

distributions. The parameters of all three dissimilar single link pendulums are shown in Table 2. Since a single link pendulum is one degree of freedom, only two KMCs had to be matched between systems ($M_{1,1}$=33,600 g-cm^2 and G_1=1,260 g-cm).

4.1 Experiment description

The three dissimilar single link pendulum systems were constructed from rigid foam board that was light (1.125g per link) relative to the entire pendulum. Mass and mass distributions were calculated using KMCs in equation 4, 7, and 8. Lead weights were used as pendulum masses and attached to the link at appropriate positions. The mass values listed in Table 2 were rounded to whole grams for the experimental pendulums. To ensure precise link dimensions, each pendulum was cut with a 60W laser cutter (Universal Systems VLS4.60).

The links were attached to a short and rigid 0.375in (0.9525cm) aluminum rod using

Table 2: Single Pendulum (ň=1) System Synchronization Coefficient Equations and System Experimental Parameters.

	Coefficient Index	Coefficient Value	System 1 (m̌=1)	System 2 (m̌=2)	System 3 (m̌=2)
KMCs	$\mathbf{M}_{1,1}$	33,600 g-cm^2	$m_{11}l_{11}^2$	$m_{11}l_{11}^2 + m_{12}l_{12}^2$	
	$\mathbf{G}_1$	1,260 g-cm	$m_{11}l_{11}$	$m_{11}l_{11} + m_{12}l_{12}$	
Masses (g)			m_{11}=47.3	m_{11}=35.0	m_{11}=49.0
				m_{12}=21.0	m_{12}=31.8
Lengths (cm)			l_{11}=26.7	l_{11}=15.0	l_{11}=5.0
				l_{12}=35.0	l_{12}=31.9

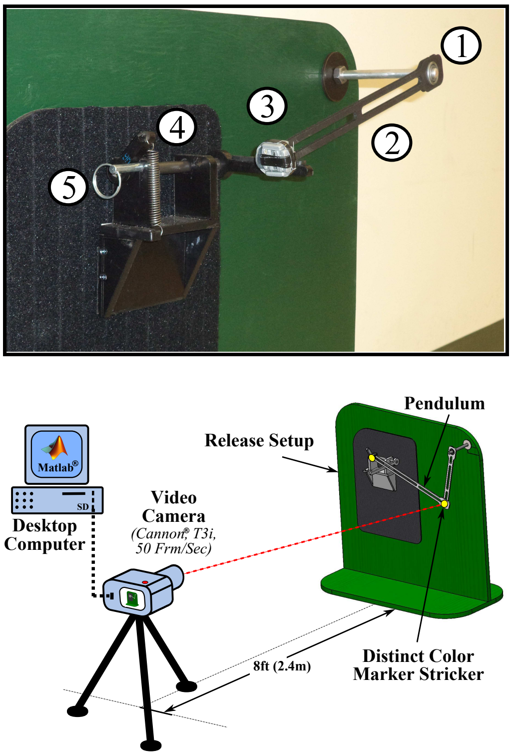

Figure 3: Release mechanism used for all pendulum measurements. (1) Ball Bearing (2) Rigid Foam Link (3) Lead Weights (4) Extension Spring (5) Release Pin.

a precision steel ball bearing to reduce friction. To minimize variability due to friction (negative torque), the exact same bearing was used for each system. Each pendulum system was dropped from the same initial position with an adjustable spring loaded release mechanism. This complete setup can be seen in Figure 3.

The pendulums were video recorded at 50 frames/second (50 Hertz) using a Cannon® T3i digital camera with a Cannon® EF 50mm f/1.8 II lens. Link angular position was interpreted with Matlab®, which was used to load video frames and identify each link's distinct color while in motion.

4.2 Results

Five videos of each pendulum were recorded (15 total). The recorded angular position was averaged and filtered using a low pass 2^{nd} order Butterworth filter at 6 Hz. This angular position data is presented in Figure 4 and compared with ideal predicted model behavior. Modeled systems have the same masses and mass distribution as measured physical systems. As predicted, all three ideal modeled systems have the same temporal kinematics and exactly overlap in Figure 4. Spectral analysis shows the same frequency peak between all measured physical systems, while all three modeled systems peaked 0.06 Hz below the measured system peaks.

While the recorded physical systems were affected by non-conservative forces, such as air resistance and friction, all three dissimilar pendulums matched kinematically. Their

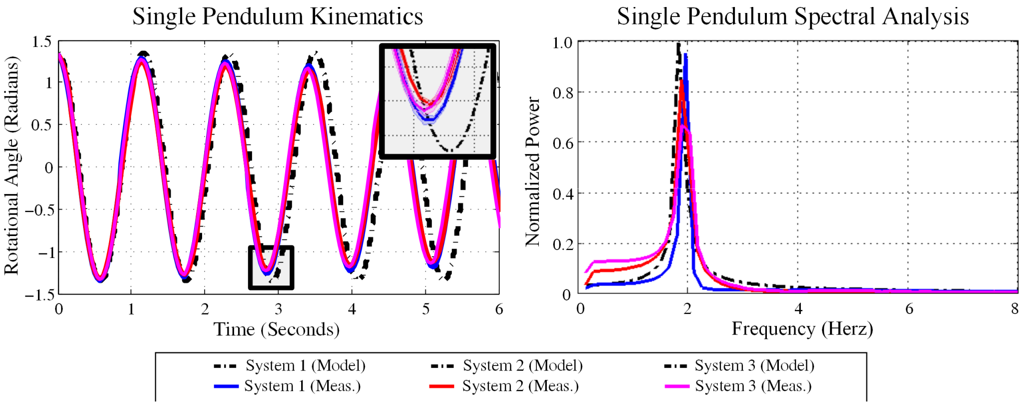

Figure 4: Temporal and spectral motion of three kinematically synchronized single link pendulums (ň=1) with dissimilar masses and mass distributions. The motion of the dissimilar modeled systems (dashed line) is matched exactly and overlaps while the measured motion of the three physical system is matched as well. The discrepancy of the modeled and physical system is due to non-conservative forces.

slight difference in amplitude can be explained by the variable mass and mass distribution in the pendulums that leads to variable weight and centripetal forces on the bearing, which in turn increases rotational friction. Similarly, the effect of the friction torque is affected by the inertia of the system. Although the kinematics are matched, the kinetics in these dissimilar systems does not match; the different masses will generate different forces. Despite these small effects, all three physically dissimilar pendulums had a frequency of $0.88 \pm 0.04\,\text{Hz}$.

When comparing the collected and model data, the effects of damping become distinct. As a result, the amplitude and period decrease over time for the actual systems as shown in Figure 4. As previously explained, the model derivation did not include a damping coefficient, thus its effects on motion were not predicted. Despite this difference, the model and all three physically dissimilar pendulums have very similar motion.

5 Example 2: Passive Double (Two-Link) Pendulum

We further investigate our kinematic matching technique by passively synchronizing two passive ($[T] = 0$) dissimilar two degree-of-freedom (ň=2) systems with two masses per link (m̌=2). This double pendulum model is depicted in Figure 2c and 2d and KMCs are shown in Table 3. Either step-by-step kinematic synchronization matching technique could have been used to generate identical motion of these systems. That is, the second system may have been newly created or already available and subsequently matched by adding an additional mass.

Traditionally the double pendulum is modeled in Figure 2c, however this model is impractical from a design perspective considering that the pivot point between the upper and lower link is exactly where the mass is placed and the link is massless. Hence, for our comparison, we add design flexibility and utilize two masses per link.

Table 3: Double Pendulum (ň=2) Synchronization Coefficient Equations and Experimental System Parameters.

	Coefficient Index	Coefficient Value	System 1 (m̌=2)	System 2 (m̌=2)
KMCs	$\mathbf{M}_{1,1}$	28,175 g-cm^2	$l_{11}^2 m_{11} + l_{12}^2 m_{12} + l_1^2(m_{21} + m_{22})$	
	$\mathbf{M}_{1,2}$	23,800 g-cm^2	$l_1(l_{21}m_{21} + l_{22}m_{22})$	
	$\mathbf{M}_{2,2}$	32,900 g-cm^2	$l_{21}^2 m_{21} + l_{22}^2 m_{22}$	
	$\mathbf{G}_1$	1,715 g-cm	$l_{11}m_{11} + l_{12}m_{12} + l_1(m_{21} + m_{22})$	
	$\mathbf{G}_2$	1,190 g-cm	$l_{21}m_{21} + l_{22}m_{22}$	
Masses (g)			m_{11}= 5.0	m_{11}=52.6
			m_{12}=35.0	m_{12}=29.1
			m_{21}=14.0	m_{21}=23.0
			m_{22}=35.0	m_{22}=28.0
Lengths (cm)			l_1=20.0	l_1=20.0
			l_{11}=7.0	l_{11}=5.0
			l_{12}=14.0	l_{12}=15.0
			l_{21}=10.0	l_{21}=12.4
			l_{22}=30.0	l_{22}=32.4

5.1 Experiment description

Two double pendulums were created using the same fabrication technique and material as the single pendulum experiment in Section 4. An additional small ball bearing was placed at the pivot point between the upper and lower link with a 0.25in (6.25mm) wooden pin. Both small bearing and pin had a combined weight less than 2 grams.

The links were attached to the same aluminum rod, ball bearing, and were released with the same release mechanism shown in Figure 3. Specific colors were placed on each link to track their angular positions. Due to greater acceleration of links, the double pendulum nonlinear motion was again recorded at 50 frames/second with the same camera.

5.2 Results

As before, each pendulum's angular kinematics were recorded five times (10 total), averaged, and filtered with a 2$^{\text{nd}}$ order Butterworth filter at 6 Hz. The results of these angular positions are illustrated in Figure 5 and compared with the ideal predicted systems.

The motion for both link 1 (upper link) and link 2 (lower link) was in agreement with model conditions through around 4 seconds, but were in good agreement between experimental measurements throughout the whole trial, which was 12 seconds. This movement of the two dissimilar systems can be seen in Figure 6 and in the accompanying video. All collected data deviated less for link 1 than link 2, which can be explained by the more chaotic movement of the lower link and also because of more variability due to friction in the additional middle pivot. In summary, we have demonstrated two dissimilar chaotic systems that have the same motion by kinematically matching the two systems.

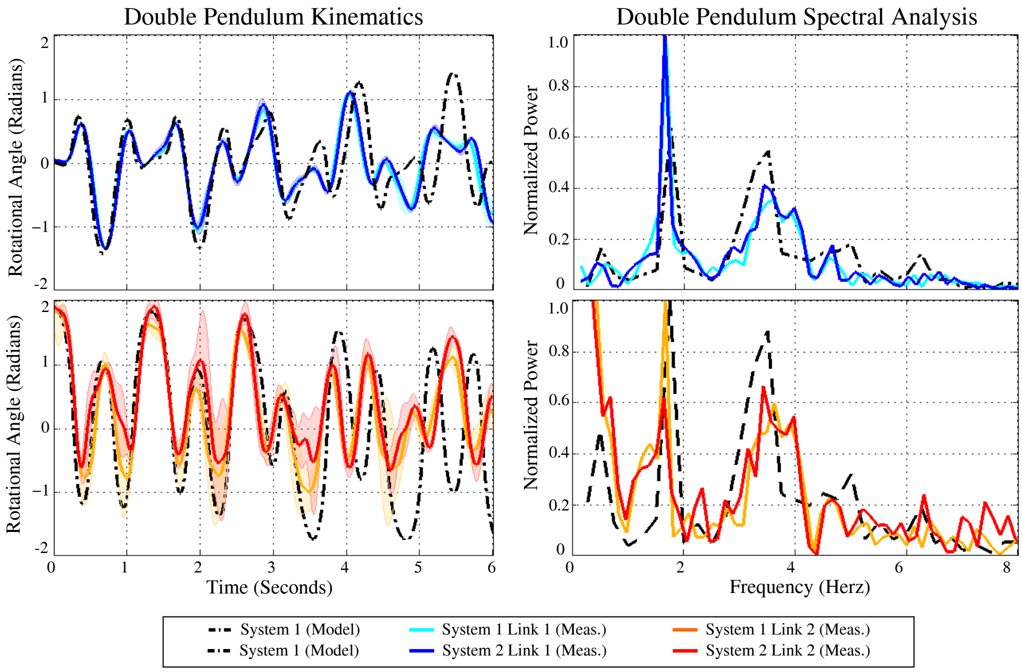

Figure 5: Double pendulum ($\check{n}$=2) model and experimental rotational link position and spectral analysis.

6 Practical Application

The preceding sections presented the derivation and validation of the kinematic synchronization technique. In this section we present a step-by-step tutorial for passive synchronization of two dissimilar and rotating systems and some possible applications of this method.

6.1 Creating a rotating system that is synchronized to an existing system

When one complete rotating system is available and another rotating system is to be created to precisely match the rotational kinematics of the available systems, the following steps can be applied to accomplish this.

Step 1: Determine the degrees of freedom for the original and available system (A) ($\check{n}^A$). For example, a swinging arm as a whole may be represented as a single degree of freedom rotating system, while a swinging leg may be represented as a double degree of freedom system as it bends at the knee. This is the number of degrees of freedom the newly created and synchronized rotation system will have ($\check{n}^A = \check{n}^B$).

Step 2: Measure and represent the mass distribution of this system as lumped point masses along each link. Make sure that each link in a system has the same number of masses ($\check{m}$) as any other link in that system, even though some may be set to zero. For example, for a three degree of freedom system, link one, two, and three each has five point mass representations along each link. However, link one and

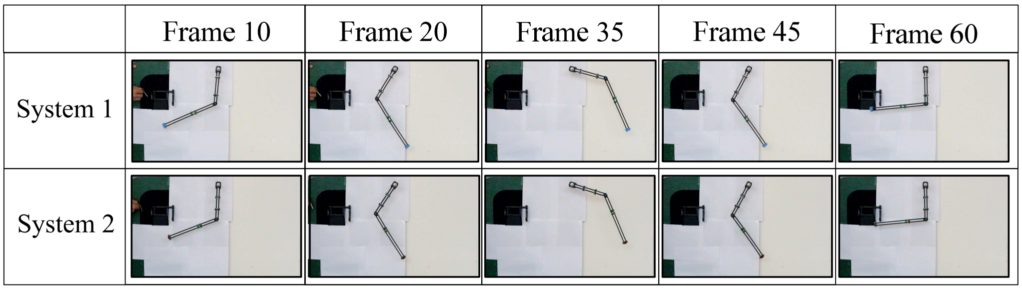

Figure 6: Temporal and spectral motion of two kinematically synchronized double link pendulums (ň=2) with dissimilar masses and mass distributions. The motion both dissimilar modeled systems is exactly the same, while of both dissimilar physically measured systems are also synchronized. The discrepancy of the modeled and physical system is due to non-conservative forces.

two could be represented as five masses along each link, but link three may be represented as four non-zero masses and one mass set to zero.

Step 3: Calculate the total numerical values of KMCs of the available system (A) using equations 4, 7, and 8. For example:
$M^A_{1,1}$ = 30,000 g-cm^2
G^A_1 = 1000 g-cm
(see Table 2 and Table 3 for other KMC examples)

Step 4: Model a newly created rotational system (B) with the same degrees of freedom ($\check{n}^A = \check{n}^B$) and represent the mass distribution of each link with the same number of masses per link. The number of masses per link must be equal to or greater than two masses ($\check{m}^B \geq 2$). As before in Step 2, some masses on links may be set to zero.

Step 5: Set the numerical KMCs of the available system (A) equal to the symbolic KMCs of the newly created system (B). For example:
$M_{1,1}$ = 30,000 g-cm^2 $= m^B_{11} l^{2B}_{11} + m^B_{12} l^{2B}_{12}$
G_1 = 1,000 g-cm $= m^B_{11} l^B_{11} + m^B_{12} l^B_{12}$
etc.

Step 6: Input approximate values for the masses, mass locations, and link lengths of the newly created system (B). Leave as many unknown parameter variables as variables as there are KMCs. That is, the number of variables to be found should equal the number of KMCs. For example:
$M_{1,1}$ = 30,000 g-cm^2 $= (35g) l^{2B}_{11} + m^{2B}_{12}(31.9cm)$
G_1 = 1,000 g-cm $= (35g) l^B_{11} + m^B_{12}(31.9cm)$
etc.

Step 7: Solve for the unknown system parameter for the newly created system (B).

6.2 Synchronizing two existing rotating systems

Two already available, dissimilar, and rotating systems with equal degrees of freedom can be passively synchronized in their independent rotational motion by augmenting one of the systems to match the other. The following succession of steps describes how to passively synchronize two such independent, dissimilar, and uncoupled systems.

Step 1: Verify that the first system (A), the reference system, and second system (B) are of equal degrees of freedom ($\check{n}^A = \check{n}^B$). For example, the kinematics of a one degree of freedom rotational system such as a rotating blade may only be matched to the motion of another one degree of freedom rotational system.

Step 2: Measure and represent the mass distribution of this system as lumped point masses along each link. Make sure that each link in a system has the same number of masses ($\check{m}$) as any other link in that system, even though some may be set to zero. For example, for a three degree of freedom system, link one, two, and three each has five point mass representation along each link. However, link one and two could be represented as five masses along each link, but link three may be represented as four masses with one mass set to zero.

Step 3: Calculate the total numerical values of KMCs of the available system (A) using equations 4, 7, and 8. For example:
$M^A_{1,1}$ = 30,000 g-cm^2
G^A_1 = 1,000 g-cm
etc.

Step 4: For the second system (B), add one additional mass for each link. This additional mass per link and its location on the link are to be determined subsequently.

Step 5: Using equations 4, 7, and 8, find the KMC equations for the second system (B) ($M^B_{1,1}$, G^B_1, etc.), and input the known (measured) lumped point masses and their locations.

Step 6: Set the numerical KMC values for the first system (A) equal to the KMC equations found for the second system (B) For example:
$M_{1,1}$ = 30,000 g-cm^2 $= m^B_{11} l^{2B}_{11} + m^B_{12} l^{2B}_{12}$
G_1 = 1,000 g-cm $= m^B_{11} l^B_{11} + m^B_{12} l^B_{12}$
etc.

Step 7: Solve for the added and unknown point masses and their locations (from Step 4) for each link of the second system (B). There should be as many unknown parameters (added masses, mass locations, and link lengths) as there are KMCs.

6.3 Kinematic system simplification technique

We have shown that given the same degrees of freedom and torque input, two dissimilar rotating systems can be motion matched. A minimum of two masses per degree of freedom are required to mimic the motion of a matching system. In essence, this kinematic matching technique can be used to simplify a complicated rotating system. For example, a rotating fan blade, gear, or cam of arbitrary shape can be modeled as one link with two masses, while an open ended chain with any number of links can be modeled as two masses per link. This can greatly simplify computation resulting in the same kinematics.

6.4 Gait pattern manipulation

In humans [7, 13, 14], animals [6], and some insects [21], the limbs can be modeled as swinging pendulums that swing in accordance to their masses and mass distribution. It is possible to manipulate limb movements by simply changing mass and mass distributions such as adding mass to a specific location of the limb. For example, a gait asymmetry (walking limp) can be created in an individual by attaching an extra weight to one leg [14], while in contrast a symmetric gait can be restored from an asymmetric walking pattern by adding weight to a specific location [8]. With the presented kinematic matching technique, we can match two swinging limbs, such as human legs, so they move symmetrically, but out 180° out of phase. While walking kinematics are the most obvious application, other parts of the body can be synchronized such as swinging arms during walking or moving fingers while playing an instrument or typing on a keyboard. This technique can also be used for the kinematic behavior prediction of swinging robotic limbs [6, 9].

6.5 Prosthetics

Wearing a prosthesis that does not have the exact size and weight of the missing limb can create gait asymmetries [15]. Prosthetics research commonly tries to mimic the lost limb in regards to size, weight, and length; however this design constraint can often times seem unrealistic and overconstraining. Using a numerical passive dynamic walker model, Sushko et. al [26] showed that this design constraint can be alleviated by changing left and right limb mass and mass distribution parameters to obtain symmetric gait with asymmetric limb parameters. As previously stated the presented kinematic matching technique can analytically match two limbs with symmetric limb mass and mass distribution parameters. That is, we can apply this technique to match the healthy limb with the other limb with a prosthetic by adding masses to one or both limbs, yielding a symmetric gait.

7 Conclusions and Future Work

We derived a general equation of motion for two-dimensional $\check{n}$ degree-of-freedom $\check{m}$ masses per degree of freedom open ended rotating systems. Further we developed a passive kinematic matching technique that is applicable to such systems. In order to match the same rotating kinematics, only two masses per degree of freedom are necessary. The motion analysis of three matched one-degree-of-freedom unactuated single link pendulums with dissimilar masses and mass distribution showed that these dissimilar systems were kinematically identical, although unmodeled nonconservative forces created slight deviations between ideal model predictions and actual measurements. While chaotic in motion, the same results were shown in the motion analysis of two two-degree-of-freedom unactuated double link pendulums with synchronous motion lasting for about 4 seconds before nonconservative forces caused deviation. Measured kinematics of the two dissimilar experimental double pendulums matched for more than 12 seconds.

It is possible to alter the mass distribution of a rotating system by moving masses along system links in order to kinematically match it to another system. It is also possible to add or remove masses at key locations along a rotating link. These methods could be utilized to synchronize the kinematics of two swinging legs while walking. However, although dissimilar kinematically synchronized systems move identically, the kinetics can vary. This was seen in our first example between three dissimilar single link pendulums. While system kinematics matched, pendulum bearing reaction forces varied, yielding dissimilar damping forces. Unless mass and mass distribution parameters are exactly matched, the internal forces throughout the system will not match. Future work includes the analysis and possible synchronization of inter-system kinetics. The authors hypothesis is that either the kinematics or kinetics can be matched, but not both simultaneously in dissimilar systems.

It is also presumed that similar passive synchronization techniques can also be derived in all three dimensions; further derivations are needed.

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