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Weak Singular Solution of Six Coupled Nonlinear ODEs

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Abstract: In this paper we have investigated the complete integrability of the system of six coupled nonlinear ODEs (ordinary differential equations), which arose in the ODE reduction of uniformly stratified fluid contained in rotating rectangular box of dimension $L \times L \times H$. The reduced system is completely integrable if the Rayleigh number $Ra = 0$. Whereas, $Ra \neq 0$ is the case of non integrability and we have obtained the solutions in the form of logarithmic psi-series. We conclude that weak singular solutions exist with movable pole type singularity, which are cluster in a self-similar fashion.

Keywords: *completely integrable systems; non-integrable systems; Painlevé test; singular solutions.*

Mathematics Subject Classification (2010): 37K10, 37K15, 70H06, 70H07.

1 Introduction

In the fluid dynamics, the flow of fluid in the atmosphere and in the ocean is governed by Boussinesq equations. Majda and Shefter [3] analyzed certain ODE reduction of Boussinesq equations. Srinivasan et al. [15] extended this work and they gave the detail mathematical analysis of reduced system of six coupled ODEs. Whereas, Desale and Dasre [5] wrote the C-Programme to determine the numerical solutions on stable and unstable manifolds. Furthermore, Desale [4] had given the complete analysis of the system and also tested the system for complete integrability by determining four first integrals and used the Jacobi's theorem. Also, he has demonstrated the stability of non degenerate critical point. For the similar text of bifurcation analysis near the degenerate

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critical point one may refer to [14]. The rigorous mathematical analysis and special solutions of rotating stratified Boussinesq equations have been discussed by Desale and Sharma in their paper [7].

In his study of onset of instabilities in the stratified fluids at large Richardson number Paul Painlevé [12, 13] classified algebraic differential equations of first and second order whose solutions in the complex domain are devoid of movable essential singularities or movable branch point. The ODE possessing this property is said to be of Painlevé type. Painlevé test in view of partial differential equations is generally known as WTC (Weiss-Tabor-Carnevale [16]) test which is further modified by Kichenassamy and Srinivasan [9]. In their paper [8], Desale and Srinivasan tested the reduced system of stratified Boussinesq equations in the light of the ARS (Ablowitz, Ramani and Segur [1]) conjuncture. In continuation of this work Desale & Patil [6] tested the system of six coupled ODEs for complete integrability using the Painlevé test.

In this paper we have tested the system of six coupled nonlinear ODEs for its complete integrability via Painlevé test. We have the non integrable case for the Rayleigh number $Ra \neq 0$ causing the singular solution in the form of logarithmic psi-series, which is the weak solution. The presence of logarithm term in the series implies that the solution in question have singularity which is cluster in self similar fashion. This is sometime viewed as possible symptom for non-integrable behavior.

This paper consists of five sections. Section 1 is introduction, Section 2 gives ODE reduction of uniformly stratified fluid contained in rotating rectangular box. In Section 3, we provided the preliminary work which is the base for investigation of weak solutions in the non integrable case. Whereas, in Section 4, we determined the weak solutions. Finally, we conclude the result in Section 5.

2 Dynamics of an Uniformly Stratified Fluid Contained in Rotating Box

We now begin by describing the rotating stratified Boussinesq equations (see Majda [2])

$$\begin{aligned} \frac{D\vec{\mathbf{v}}}{Dt} + f(\hat{\mathbf{e_3}} \times \vec{\mathbf{v}}) &= -\nabla p + \nu(\Delta \vec{\mathbf{v}}) - \frac{g\tilde{\rho}}{\rho_b}\hat{\mathbf{e_3}}, \\ \operatorname{div} \vec{\mathbf{v}} &= 0, \\ \frac{D\tilde{\rho}}{Dt} &= \kappa \Delta \tilde{\rho}, \end{aligned} \tag{1}$$

where $\vec{\mathbf{v}}$ denotes the velocity field, ρ is the density which is the sum of constant reference density ρ_b and perturbation density $\tilde{\rho}$, p is the pressure, g is the acceleration due to gravity that points in $-\hat{\mathbf{e_3}}$ direction, f is the rotation frequency of earth, ν is the coefficient of viscosity, κ is the coefficient of heat conduction and $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{\mathbf{v}} \cdot \nabla)$ is a convective derivative. For more about rotating stratified Boussinesq equations one may see Majda [2].

In the frame of reference of an uniformly stratified fluid contained in rotating rectangular box of dimension $L \times L \times H$, which is temperature stratified with fixed zeroth order moments of mass and heat (so that there is neither net evaporation or precipitation, nor any net river input or output, and neither heating nor cooling). The container is assumed to be in steady uniform rotation on an f-plane. Maas [11] reduces the system of equations (1) into the following system of six coupled ODEs:

$$\begin{aligned} Pr^{-1}\frac{d\vec{\mathbf{w}}}{dt} + f'\hat{\mathbf{e_3}} \times \vec{\mathbf{w}} &= \hat{\mathbf{e_3}} \times \vec{\mathbf{b}} - (w_1,\, w_2,\, rw_3) + \hat{T}\vec{\mathbf{T}}, \\ \frac{d\vec{\mathbf{b}}}{dt} + \vec{\mathbf{b}} \times \vec{\mathbf{w}} &= -(b_1,\, b_2,\, \mu b_3) + Ra\vec{\mathbf{F}}. \end{aligned} \tag{2}$$

In these equations, $\vec{\mathbf{b}} = (b_1,\, b_2,\, b_3)$ is the center of mass, $\vec{\mathbf{w}} = (w_1,\, w_2,\, w_3)$ is the basin's averaged angular momentum vector, $\vec{\mathbf{T}}$ is the differential momentum, $\vec{\mathbf{F}}$ are the buoyancy fluxes, $f' = f/2r_h$ is the earth rotation, $r = r_v/r_h$ is the friction ($r_{v,h}$ are the Rayleigh damping coefficients), Ra is the Rayleigh number, Pr is the Prandtl number, μ is the diffusion coefficient and $\hat{T}$ is the magnitude of the wind stress torque.

Neglecting diffusive and viscous terms, Maas [11] considers the dynamics of an ideal rotating, uniformly stratified fluid in response to forcing. He assumes this to be due solely to differential heating in the meridional (y) direction. $\vec{\mathbf{F}} = (0,\, 1,\, 0)$, the wind effect is neglected i.e. $\vec{\mathbf{T}} = 0$. For Prandtl number Pr, equal to one, the system of equations (2) reduces to the following ideal rotating, uniformly stratified system of six coupled ODEs

$$\begin{aligned} \frac{d\vec{\mathbf{w}}}{dt} &= -f'\hat{\mathbf{e_3}} \times \vec{\mathbf{w}} + \hat{\mathbf{e_3}} \times \vec{\mathbf{b}}, \\ \frac{d\vec{\mathbf{b}}}{dt} &= -\vec{\mathbf{b}} \times \vec{\mathbf{w}} + Ra\vec{\mathbf{F}}. \end{aligned} \tag{3}$$

In his paper, Desale [4] has demonstrated the complete integrability of the system (3) for $Ra = 0$ using the first integrals and Jacobi's theory. Desale and Patil [6] continued this work and tested the system for complete integrability via Painlevé test. In this paper we investigate the case of non integrability $Ra \neq 0$. In the following section we consider the case of non integrability and obtain the weak singular solution in the form of logarithmic-psi series.

3 Preliminaries

We have a system of ODEs (3), which can be written component-wise as:

$$\begin{aligned} &\dot{w_1} = f'w_2 - b_2, \quad \dot{w_2} = -f'w_1 + b_1, \quad \dot{w_3} = 0, \\ &\dot{b_1} = w_2b_3 - w_3b_2, \quad \dot{b_2} = w_3b_1 - w_1b_3 + Ra, \quad \dot{b_3} = w_1b_2 - w_2b_1. \end{aligned} \tag{4}$$

Since $\dot{w_3} = 0$, which gives us $w_3 = \text{constant} = k_1$. Consequently, we have the following system of ODEs:

$$\begin{aligned} &\dot{w_1} = f'w_2 - b_2, \quad \dot{w_2} = -f'w_1 + b_1, \\ &\dot{b_1} = w_2b_3 - k_1b_2, \quad \dot{b_2} = k_1b_1 - w_1b_3 + Ra, \quad \dot{b_3} = w_1b_2 - w_2b_1. \end{aligned} \tag{5}$$

Desale and Patil [6] obtained the solution of the system (5) in the form of the following power series:

$$\begin{aligned} &w_1(t) = \sum_{j=0}^{\infty} w_{1j}\tau^{j+m_1}, \quad w_2(t) = \sum_{j=0}^{\infty} w_{2j}\tau^{j+m_2}, \\ &b_1(t) = \sum_{j=0}^{\infty} b_{1j}\tau^{j+n_1}, \quad b_2(t) = \sum_{j=0}^{\infty} b_{2j}\tau^{j+n_2}, \quad b_3(t) = \sum_{j=0}^{\infty} b_{3j}\tau^{j+n_3}, \end{aligned} \tag{6}$$

where $\tau = t - t_0$ and t_0 is an arbitrary position of singularity. Also, the authors found that there were several possible cases of dominant balance of the system (5) and among those possible cases they obtained the singular solution only in the following case of principle dominant balance

$$\dot{w_1} = -b_2, \quad \dot{w_2} = b_1, \quad \dot{b_1} = w_2 b_3, \quad \dot{b_2} = -w_1 b_3, \quad \dot{b_3} = w_1 b_2 - w_2 b_1. \tag{7}$$

Consequently, they have determined the exponents as

$$m_1 = m_2 = -1, \quad n_1 = n_2 = n_3 = -2 \tag{8}$$

and possible branches of leading order coefficients as listed below

$$w_{10} = \pm\sqrt{-4 - k_2^2},\, w_{20} = k_2,\, b_{10} = -k_2,\, b_{20} = \pm\sqrt{-4 - k_2^2},\, b_{30} = 2. \tag{9}$$

Furthermore, the authors have given the following recursive relations to determine the coefficients w_{1j}, w_{2j}, b_{1j}, b_{2j} and b_{3j} for $j = 1, 2, 3 \ldots$.

$$\begin{pmatrix} j-1 & 0 & 0 & 1 & 0 \\ 0 & j-1 & -1 & 0 & 0 \\ 0 & -b_{30} & j-2 & 0 & -w_{20} \\ b_{30} & 0 & 0 & j-2 & w_{10} \\ -b_{20} & b_{10} & w_{20} & -w_{10} & j-2 \end{pmatrix} \begin{pmatrix} w_{1j} \\ w_{2j} \\ b_{1j} \\ b_{2j} \\ b_{3j} \end{pmatrix} = \begin{pmatrix} A_j \\ B_j \\ C_j \\ D_j \\ E_j \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned} A_j &= f' w_{2(j-1)}, \quad B_j = -f' w_{1(j-1)}, \\ C_j &= -k_1 b_{2(j-1)} + \sum_{k=1}^{j-1} w_{2k} b_{3(j-k)}, \\ D_j &= k_1 b_{1(j-1)} - \sum_{k=1}^{j-1} w_{1k} b_{3(j-k)}, \\ E_j &= \sum_{k=1}^{j-1} w_{1k} b_{2(j-k)} - \sum_{k=1}^{j-1} w_{2k} b_{1(j-k)}\ . \end{aligned} \tag{11}$$

The above recursive relations (10) determine the unknown expansion coefficients uniquely unless the determinant of coefficient matrix is zero. Those values of j at which the determinant of coefficient matrix vanishes are called the *resonances* and these are

$$j = -1,\, 0,\, 2,\, 3,\, 4. \tag{12}$$

We see that all resonances are simple. Here $j = -1$, is a usual resonance and $j = 0$ is corresponding to the arbitrariness of w_{20} in leading order coefficients.

Desale and Patil [6] have considered the following case of leading order coefficients

$$\begin{aligned} &w_{10} = \sqrt{-4 - k_2^2}, \quad w_{20} = k_2 \,(\text{arbitrary constant}), \\ &b_{10} = -k_2, \quad b_{20} = \sqrt{-4 - k_2^2}, \quad b_{30} = 2 \end{aligned} \tag{13}$$

and they have determined the singular solution passing through it. Ultimately they have checked the compatibility conditions at $j = 1$ and $j = 2$. They have obtained the following expansion coefficients:

$$\begin{aligned} &w_{11} = \tfrac{1}{2}(f' k_2 - k_1 k_2), \quad w_{21} = \tfrac{1}{2}(-f' + k_1)\sqrt{-4 - k_2^2}, \\ &b_{11} = f'\sqrt{-4 - k_2^2}, \quad b_{21} = f' k_2, \quad b_{31} = 0. \end{aligned} \tag{14}$$

$$\begin{array}{l} w_{12} = \frac{1}{2}(f'k_1 - k_3)\sqrt{-4-k_2^2}, \quad w_{22} = \frac{k_2}{2}(f'k_1 - k_3), \\ b_{12} = \frac{k_2}{2}\left[(f')^2 - k_3\right], \quad b_{22} = \frac{1}{2}\left[k_3 - (f')^2\right]\sqrt{-4-k_2^2}, \quad b_{32} = k_3. \end{array} \tag{15}$$

An arbitrary constant $b_{32} = k_3$ involved in (15) because of $j = 2$ is a resonance. While checking the compatibility conditions at resonance $j = 3$, they have concluded that the compatibility condition holds only if $Ra = 0$. Implying that $Ra \neq 0$ is a non integrable case. Thus, it motivates us to study this non integrable case and in the following section we are going to obtain weak singular solutions.

4 Weak Singular Solution

In this section we have studied the non integrable case of system (5) that is, we have obtained the weak singular solutions in terms of logarithmic psi series.

We are going to find the singular solutions in the form of

$$t^{\nu} \sum_{m \geq l \geq 0} u_{m,l}(x) t^m (\ln t)^l, \tag{16}$$

which are suggested by Kichenassamy and Srinivasan [9]. They also made an interesting remark that $l = 1$ suffices if all the resonances are simple and 1 is not a resonance. In this case, we also have simple resonances $j = -1, 0, 2, 3, 4$. Therefore, our solution will be in the form of

$$t^{\nu} \sum_{m \geq 1} u_{m,1}(x) t^m (\ln t). \tag{17}$$

With above remarkable feature and compatibility conditions hold for $j = 0$, 1 and 2, we restructure the power series given by (6) as follows:

$$\begin{aligned} w_1(t) &= w_{10}\tau^{-1} + w_{11} + w_{12}\tau + \sum_{j=3}^{\infty} w_{1j}(\log\tau)\tau^{j-1}, \\ w_2(t) &= w_{20}\tau^{-1} + w_{21} + w_{22}\tau + \sum_{j=3}^{\infty} w_{2j}(\log\tau)\tau^{j-1}, \\ b_1(t) &= b_{10}\tau^{-2} + b_{11}\tau^{-1} + b_{12} + \sum_{j=3}^{\infty} b_{1j}(\log\tau)\tau^{j-2}, \\ b_2(t) &= b_{20}\tau^{-2} + b_{21}\tau^{-1} + b_{22} + \sum_{j=3}^{\infty} b_{2j}(\log\tau)\tau^{j-2}, \\ b_3(t) &= b_{30}\tau^{-2} + b_{31}\tau^{-1} + b_{32} + \sum_{j=3}^{\infty} b_{3j}(\log\tau)\tau^{j-2}. \end{aligned} \tag{18}$$

In the above equations (18) expansion coefficients w_{1j}, w_{2j}, b_{1j}, b_{2j} and b_{3j} for $j = 1, 2, 3$ are given by the equations (13), (14) and (15). The power series given by (18) provide us the weak singular solution in the form of logarithmic psi series.

• **Compatibility condition at the resonance** $j = 3$. Now we proceed to check the compatibility condition at the resonance $j = 3$. At the resonance level $j = 3$, we substitute equations (18) into the system of differential equations (5), then equating like powers of τ and $\tau(\log\tau)$ with $j = 3$, we get the following systems of non-homogeneous linear equations (19) and (20)

$$\begin{array}{l} w_{13} = f'w_{22}, \quad w_{23} = -f'w_{12}, \quad b_{13} = k_3 w_{21} - k_1 b_{22}, \\ b_{23} = k_1 b_{12} - k_3 w_{11} + Ra, \quad b_{33} = w_{11}b_{22} + w_{12}b_{21} - w_{21}b_{12} - w_{22}b_{11}. \end{array} \tag{19}$$

$$\begin{aligned}&2w_{13}=-b_{23},\quad 2w_{23}=b_{13},\quad b_{13}=b_{33}w_{20}+w_{23}b_{30},\\&b_{23}=-w_{10}b_{33}-w_{13}b_{30},\quad b_{33}=w_{13}b_{20}+w_{10}b_{23}-w_{23}b_{10}-w_{20}b_{13}.\end{aligned}\tag{20}$$

Solving (19) and (20) together, we obtain the system of linear equations, which is in matrix form as given below

$$\begin{pmatrix}0&0&0&1&0\\0&0&1&0&0\\0&2&0&0&k_2\\-2&0&0&0&-\sqrt{-4-k_2^2}\\\sqrt{-4-k_2^2}&-k_2&-k_2&\sqrt{-4-k_2^2}&0\end{pmatrix}\begin{pmatrix}w_{13}\\w_{23}\\b_{13}\\b_{23}\\b_{33}\end{pmatrix}=\begin{pmatrix}-2f'w_{22}\\-2f'w_{12}\\w_{21}b_{32}-k_1b_{22}\\k_1b_{12}-w_{11}b_{32}+Ra\\w_{11}b_{22}+w_{12}b_{21}-b_{12}w_{21}-w_{22}b_{11}\end{pmatrix}.\tag{21}$$

Further, we solve the system and expansion coefficient are uniquely determined, which are listed below

$$\begin{aligned}&w_{13}=\frac{(2f'^2k_1-Rak_2-2f'k_3)k_2}{4(2+k_2^2)},\quad w_{23}=\frac{(2(f'^2)k_1-Rak_2-2f'k_3)\sqrt{-4-k_2^2}}{4(2+k_2^2)},\\&b_{13}=-f'(k_1f'-k_3)\sqrt{-4-k_2^2},\quad b_{23}=f'k_2(-f'k_1+k_3),\\&b_{33}=\frac{(Ra+f'^2k_1k_2-f'k_2k_3)\sqrt{-4-k_2^2}}{2(2+k_2^2)}.\end{aligned}\tag{22}$$

• **Compatibility condition at the resonance** $j=4$. Again we substitute (18) into the system (5) and in these equations, we substitute the earlier determined expansion coefficients which are given by (13), (14), (15) and (22). We simplify the both sides of resultant equations and equating the powers of τ^2 and $\tau^2(\log\tau)$, we obtain the following non-homogeneous systems of linear equations, which are given by the following equations

$$w_{14}=0,\quad w_{24}=0,\quad b_{14}=w_{22}b_{32},\quad b_{24}=-w_{12}b_{32},\quad b_{34}=-w_{22}b_{12}+w_{12}b_{22}.\tag{23}$$

$$\begin{aligned}3w_{14}&=f'w_{23}-b_{24},\\3w_{24}&=-f'w_{13}+b_{14},\\2b_{14}&=w_{21}b_{33}-w_{30}b_{23}+w_{20}b_{34}+w_{24}b_{30},\\2b_{24}&=w_{30}b_{13}-w_{10}b_{34}-w_{11}b_{33}-w_{14}b_{30},\\2b_{34}&=w_{11}b_{23}+w_{13}b_{21}-w_{21}b_{13}+w_{23}b_{11}\\&+\ w_{14}b_{20}-w_{24}b_{10}-w_{20}b_{14}+w_{10}b_{24}.\end{aligned}\tag{24}$$

We solve the equations (23) and (24) together in the similar way as we adopted in the previous case and determine the expansion coefficients uniquely at this resonance level, which are listed below

$$\begin{aligned}w_{14}&=\tfrac{1}{16(2+k_2^2)}\big[\big(-2f'Rak_2-2f'^2k_1^2(8+3k_2^2)-16k_3^2+2k_2^2k_3f'^2-4k_2^2k_3^2\\&+\ 2k_2^4k_3^2+2Rak_1k_2+32f'k_1k_3-2f'^3k_2^2+10f'k_2^2k_3-2f'k_2^4k_3\big)\sqrt{-4-k_2^2}\end{aligned}$$

$$
\begin{aligned}
& + && \big(k_2^6 k_3 + 8k_2^2 k_3 + 6k_2^4 k_3 - k_2^6 f' - 8f' k_2^2 - 6f' k_2^4\big)(-f'^2 + k_3)\big],\\
w_{24} & = && \tfrac{1}{16(2+k_2^2)}\big[\big(8k_2 k_3 + 6k_2^3 k_3 + k_2^5 k_3 - 8f' k_1 k_2 - 6f' k_2^3 - f' k_2^5\big)\\
& && (f'^2 - k_3)\sqrt{-4-k_2^2} + 2(k_1 Ra - f' Ra - f' k_1 k_2 (f'^2 - k_3))(k_2^2 + 4)\\
& - && 2f'^2 k_1^2 k_2 (4 + 3k_2^2) + 8f'^2 k_2 k_3 + 2k_2^3 k_3 (f'^2 + 2k_3 + k_2^2 k_3^2) - 2f' k_1 k_2^5 k_3\big]\\
b_{14} & = && \frac{f' k_2 (2f'^2 k_1 - Rak_2 - 2f' k_3)}{4(2+k_2^2)}, \quad b_{24} = \frac{f'\sqrt{-4-k_2^2}(2f'^2 k_1 - Rak_2 - 2f' k_3)}{4(2+k_2^2)},\\
b_{34} & = && \tfrac{1}{8}\big[k_2^2((f'^3)k_1 - f'^2 k_3 - f' k_1 k_3 + k_3^2)\sqrt{-4-k_2^2} + 2k_2^2 k_3 (f' k_1 - k_3)\big].
\end{aligned} \tag{25}
$$

• **Compatibility condition for** $j \geq 5$. Here, we provide the recursion relations by which we can determine expansion coefficients of logarithmic psi series (18) for $j \geq 5$. These relations will be obtained by substituting (18) into the system (5) and then equating the powers of τ^j and $\tau^j(\log \tau)$. This will result into two non homogeneous systems of linear equations. Further, we combine these two systems together, the resultant system is as given below that lead us to determine all the expansion coefficients

$$
\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & b_{30} & 0 & 0 & w_{20} \\ b_{30} & 0 & 0 & 0 & w_{10} \\ -b_{20} & b_{10} & w_{20} & -w_{10} & 0 \end{pmatrix}
\begin{pmatrix} w_{1j} \\ w_{2j} \\ b_{1j} \\ b_{2j} \\ b_{3j} \end{pmatrix}
=
\begin{pmatrix} A_j{}^* \\ B_j{}^* \\ C_j{}^* \\ D_j{}^* \\ E_j{}^* \end{pmatrix}, \tag{26}
$$

where

$$
\begin{aligned}
A_j{}^* & = f' w_{2(j-1)}, \quad B_j{}^* = f' w_{1(j-1)},\\
C_j{}^* & = k_1 b_{2(j-1)} - b_{31} w_{2(j-1)} - b_{32} w_{2(j-2)} - w_{21} b_{3(j-1)} - w_{22} b_{3(j-2)},\\
D_j{}^* & = -k_1 b_{1(j-1)} + w_{11} b_{3(j-1)} + w_{12} b_{3(j-2)} + b_{31} w_{1(j-1)} + b_{32} w_{1(j-2)},\\
E_j{}^* & = w_{11} b_{2(j-1)} + w_{12} b_{2(j-2)} + b_{21} w_{1(j-1)} + b_{22} w_{1(j-2)}\\
& + w_{21} b_{1(j-1)} + w_{22} b_{1(j-2)} - b_{11} w_{2(j-1)} - b_{12} w_{2(j-2)}.
\end{aligned} \tag{27}
$$

From the equation (26), we see that the determinant of coefficient matrix is nonzero for the given leading order coefficients this implies that all expansion coefficients for $j \geq 5$ are determined uniquely in terms of predetermined coefficients.

During the implementation of Painlevé algorithm with logarithmic terms, we observed that all compatibility conditions were satisfied. Hence, the system (5) passes the Painlevé test which indicate that the weak singular solution of the system (5) exists. The weak singular solution of (3) in the considered case of leading order coefficients is as follows

$$
\begin{aligned}
w_1(t) & = && \sqrt{-4-k_2^2}\tau^{-1} + \tfrac{1}{2}(f' k_2 - k_1 k_2) + \big[\tfrac{1}{2}(f' k_1 - k_3)\sqrt{-4-k_2^2}\big]\tau\\
& + && \left[\frac{(2f'^2 k_1 - Rak_2 - 2f' k_3)k_2}{4(2+k_2^2)}\right](\log \tau)\tau^2 + \frac{1}{16(2+k_2^2)}\big[\big(-2f' Rak_2\\
& - && 2f'^2 k_1^2 (8 + 3k_2^2) - 16k_3^2 + 2k_2^2 k_3 f'^2 - 4k_2^2 k_3^2 + 2k_2^4 k_3^2 + 2Rak_1 k_2\\
& + && 32f' k_1 k_3 - 2f'^3 k_2^2 + 10f' k_2^2 k_3 - 2f' k_2^4 k_3\big)\sqrt{-4-k_2^2}\\
& + && \big(k_2^6 k_3 + 8k_2^2 k_3 + 6k_2^4 k_3 - k_2^6 f' - 8f' k_2^2 - 6f' k_2^4\big)(-f'^2 + k_3)\big](\log \tau)\tau^3\\
& + && \sum_{j=5}^{\infty} w_{1j}(\log \tau)\tau^{j-1},
\end{aligned}
$$

$$
\begin{aligned}
w_2(t) &= k_2\tau^{-1} + \left[\tfrac{1}{2}(-f' + k_1)\sqrt{-4-k_2^2}\right] + \left[\tfrac{k_2}{2}(f'k_1 - k_3)\right]\tau \\
&+ \left[\frac{(2(f'^2)k_1 - Rak_2 - 2f'k_3)\sqrt{-4-k_2^2}}{4(2+k_2^2)}\right](\log\tau)\tau^2 \\
&+ \tfrac{1}{16(2+k_2^2)}\left[\left(8k_2k_3 + 6k_2^3k_3 + k_2^5k_3 - 8f'k_1k_2 - 6f'k_2^3 - f'k_2^5\right)\right. \\
&\quad (f'^2 - k_3)\sqrt{-4-k_2^2} + 2(k_1Ra - f'Ra - f'k_1k_2(f'^2 - k_3))(k_2^2 + 4) \\
&- 2f'^2k_1^2k_2(4 + 3k_2^2) + 8f'^2k_2k_3 + 2k_2^3k_3(f'^2 + 2k_3 + k_2^2k_3^2) \\
&- \left. 2f'k_1k_2^5k_3\right](\log\tau)\tau^3 + \sum_{j=5}^{\infty} w_{2j}(\log\tau)\tau^{j-1}, \\
w_3(t) &= k_1 \text{ (arbitrary constant)}, \\
b_1(t) &= -k_2\tau^{-2} + \left[f'\sqrt{-4-k_2^2}\right]\tau^{-1} + \tfrac{k_2}{2}\left[(f')^2 - k_3\right] + (-f'(k_1f' - k_3)) \\
&\quad \sqrt{-4-k_2^2}(\log\tau)\tau + \left[\frac{f'k_2(2f'^2k_1 - Rak_2 - 2f'k_3)}{4(2+k_2^2)}\right](\log\tau)\tau^2 \\
&+ \sum_{j=5}^{\infty} b_{1j}(\log\tau)\tau^{j-2}, \\
b_2(t) &= \sqrt{-4-k_2^2}\tau^{-2} + f'k_2\tau^{-1} + \left[\tfrac{1}{2}\left((k_3 - (f')^2\right)\sqrt{-4-k_2^2}\right] \\
&+ (f'k_2(-f'k_1 + k_3))(\log\tau)\tau \\
&+ \left[\frac{f'(2f'^2k_1 - Rak_2 - 2f'k_3)\sqrt{-4-k_2^2}}{4(2+k_2^2)}\right](\log\tau)\tau^2 + \sum_{j=5}^{\infty} b_{2j}(\log\tau)\tau^{j-2}, \\
b_3(t) &= 2\tau^{-2} + k_3 + \frac{(Ra + f'^2k_1k_2 - f'k_2k_3)\sqrt{-4-k_2^2}}{2(2+k_2^2)}(\log\tau)\tau \\
&+ \tfrac{1}{8}\left[k_2^2((f'^3)k_1 - f'^2k_3 - f'k_1k_3 + k_3^2)\sqrt{-4-k_2^2}\right. \\
&+ \left. 2k_2^2k_3(f'k_1 - k_3)\right](\log\tau)\tau^2 + \sum_{j=5}^{\infty} b_{3j}(\log\tau)\tau^{j-2}.
\end{aligned} \tag{28}
$$

Equations (28) contain four arbitrary constants k_1, k_2, k_3, and arbitrary position of singularity t_0 satisfying the system of ODEs (3). The convergence of such logarithmic psi series solutions is guaranteed by Kichenassamy and Littman [10].

In the similar way of calculations, we can find the singular solution to the system (3) corresponding to the following branch of leading order coefficients:

$$
\begin{aligned}
& w_{10} = -\sqrt{-4-k_2^2}, \quad w_{20} = k_2 \text{ (arbitrary constant)}, \\
& b_{10} = -k_2, \quad b_{20} = -\sqrt{-4-k_2^2}, \quad b_{30} = 2.
\end{aligned} \tag{29}
$$

The weak singular solution to the system (3) for this branch of leading order coefficients (29) is given by the following equations (30) and (31)

$$
\begin{aligned}
w_1(t) &= -\sqrt{-4-k_2^2}\tau^{-1} + \tfrac{1}{2}(f'k_2 - k_1k_2) + \left[\tfrac{1}{2}(-f'k_1 + k_3)\sqrt{-4-k_2^2}\,\right]\tau \\
&+ \left[\frac{(2f'^2k_1 - Rak_2 - 2f'k_3)k_2}{4(2+k_2^2)}\right](\log\tau)\tau^2 + \frac{1}{16(2+k_2^2)}\left[\left(-2f'Rak_2\right.\right. \\
&- 2f'^2k_1^2(8 + 3k_2^2) - 16k_3^2 + 2k_2^2k_3f'^2 - 4k_2^2k_3^2 + 2k_2^4k_3^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 2Rak_1k_2 + 32f'k_1k_3 - 2f'^3k_2^2 + 10f'k_2^2k_3 - 2f'k_2^4k_3\big)(-\sqrt{-4-k_2^2}) \\
&+ \quad \big(k_2^6k_3 + 8k_2^2k_3 + 6k_2^4k_3 - k_2^6f' - 8f'k_2^2 - 6f'k_2^4\big)(-f'^2 + k_3)\big]\,(\log\tau)\tau^3 \\
&+ \quad \sum_{j=5}^{\infty} w_{1j}(\log\tau)\tau^{j-1},
\end{aligned}
$$

$$
\begin{aligned}
w_2(t) &= k_2\tau^{-1} + \big[\tfrac{1}{2}(f'-k_1)\sqrt{-4-k_2^2}\big] + \big[\tfrac{k_2}{2}(f'k_1 - k_3)\big]\tau \\
&+ \big[\frac{(-2(f'^2)k_1 + Rak_2 + 2f'k_3)\sqrt{-4-k_2^2}}{4(2+k_2^2)}\big](\log\tau)\tau^2 \\
&+ \tfrac{1}{16(2+k_2^2)}\big[\big(8k_2k_3 + 6k_2^3k_3 + k_2^5k_3 - 8f'k_1k_2 - 6f'k_2^3 - f'k_2^5\big) \\
&\quad (f'^2 - k_3)(-\sqrt{-4-k_2^2}) + 2(k_1Ra - f'Ra - f'k_1k_2(f'^2 - k_3))(k_2^2+4) \\
&- 2f'^2k_1^2k_2(4+3k_2^2) + 8f'^2k_2k_3 + 2k_2^3k_3(f'^2 + 2k_3 + k_2^2k_3^2) \\
&- 2f'k_1k_2^5k_3\big](\log\tau)\tau^3 + \sum_{j=5}^{\infty} w_{2j}(\log\tau)\tau^{j-1}, \\
w_3(t) &= k_1 \text{ (arbitrary constant)}, \\
b_1(t) &= -k_2\tau^{-2} - \big[f'\sqrt{-4-k_2^2}\big]\tau^{-1} + \tfrac{k_2}{2}\big[(f')^2 - k_3\big] + (f'(k_1f' - k_3)) \\
&\quad \sqrt{-4-k_2^2}(\log\tau)\tau + \big[\frac{f'k_2(2f'^2k_1 - Rak_2 - 2f'k_3)}{4(2+k_2^2)}\big](\log\tau)\tau^2 \\
&+ \sum_{j=5}^{\infty} b_{1j}(\log\tau)\tau^{j-2}, \\
b_2(t) &= -\sqrt{-4-k_2^2}\tau^{-2} + f'k_2\tau^{-1} + \big[\tfrac{1}{2}\big((-k_3 + (f')^2\big)\sqrt{-4-k_2^2}\big] \\
&+ (f'k_2(-f'k_1 + k_3))(\log\tau)\tau \\
&+ \big[\frac{f'(-2f'^2k_1 + Rak_2 + 2f'k_3)\sqrt{-4-k_2^2}}{4(2+k_2^2)}\big](\log\tau)\tau^2 + \sum_{j=5}^{\infty} b_{2j}(\log\tau)\tau^{j-2},
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
b_3(t) &= 2\tau^{-2} + k_3 + \frac{(-Ra - f'^2k_1k_2 + f'k_2k_3)\sqrt{-4-k_2^2}}{2(2+k_2^2)}(\log\tau)\tau \\
&+ \tfrac{1}{8}\big[k_2^2(-(f'^3)k_1 + f'^2k_3 + f'k_1k_3 - k_3^2)\sqrt{-4-k_2^2} \\
&+ 2k_2^2k_3(f'k_1 - k_3)\big](\log\tau)\tau^2 + \sum_{j=5}^{\infty} b_{3j}(\log\tau)\tau^{j-2}.
\end{aligned} \tag{31}
$$

The result of this section can be summarized in the form of the following theorem.

Theorem 4.1 *An ideal rotating, uniformly stratified system of six coupled ODEs (3) is completely integrable for Rayleigh number $Ra = 0$. Whereas, $Ra \neq 0$ is the case of non integrability and system (3) admits weak singular solutions in the form of logarithmic psi series given by equations (28) and (30), (31) for two different branches of leading order coefficients given by equations (9).*

5 Conclusion

The reduced system of ODEs (3) which arose in the reduction of uniformly stratified fluid contained in the rotating box of dimension $L \times L \times H$ is completely integrable if the

Rayleigh number $Ra = 0$. If $Ra \neq 0$ then the system (3) is non integrable. In this case of non integrability we have determined the weak solutions (28) and (30), (31) in the different branches of leading order. The solutions are in the form of logarithmic psi series and the convergence of the series is guaranteed by Kichenassamy and Littman [10]. We see that the nature of movable singularities are pole type singularities which are cluster in a self similar fashion.

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