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NONLINEAR DYNAMICS & SYSTEMS THEORY

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Integral Estimates of Solutions to Nonlinear Systems and Their Applications

On the occasion of centenary of the birth of Professor A.N.Golubentsev

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March 29, 2016 marks the 100th birthday of Professor A.N. Golubentsev, the famous scientist in the field of machine mechanics and applied mathematics. For the detailed analysis of his scientific investigations and his contribution to the development of the Institute of Mechanics of NAS of Ukraine see the paper [13] and the book [14].

Abstract: The paper deals with the nonlinear systems of ordinary differential equations. New estimates of the norms of solutions for systems under consideration are established via nonlinear integral inequalities. The results are illustrated by the problems on boundedness of solutions, finite-time stability and exponential approximation of solution to a class of nonlinear systems.

Keywords: *nonlinear system; bounded solutions; finite-time stability.*

Mathematics Subject Classification (2010): 34A34, 34C11, 34C60, 93D05.

1 Introduction

For solution of problems of nonlinear dynamics different analytical and qualitative methods of general theory of equations are applied being adapted to a particular problem or a class of similar problems. For instance, in monograph [1] a method of dynamics analysis is considered for the systems described by the equations containing integrals with variable upper limit. The authors discuss physical meaning of the resolvent of integral equation and present basic analytical correlations relating the character of transient process in the system with its parameters. As to the dynamics of machines, a resolvent

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analytic expression is given for the systems of high order equations. In the investigation of nonlinear dynamics of machines the systems are treated which contain elastoplastic links, nonlinear couplings with hysteresis characteristic, etc.

Stability investigations of nonlinear system motions on finite and unbounded time interval for given estimates of the initial and subsequent perturbations were summarized in monograph [2].

In monographs [3 – 5] two classical theories of mathematics and mechanics have been developed. One was the theory of integral inequalities, and the other was a general theory of motion stability in terms of integral inequalities.

The present paper proposes estimates of norm of solutions to nonlinear systems based on the theory of nonlinear integral inequalities. Problems on boundedness of solutions, motion stability on finite time interval and exponential convergence of solutions for one class of nonlinear systems are considered as applications.

2 Statement of the Problem

Consider a model of some physical system described by a system of perturbed motion equations of the form

$$\frac{dx}{dt} = F(t,x), \tag{1}$$

$$x(t_0) = x_0, \tag{2}$$

where $x \in \mathbb{R}^n$, $F(t,x)$ is a vector-function definite and continuous with respect to $(t,x) \in \mathbb{R}^n_+$.

Further we shall assume that for the initial values $(t_0, x_0) \in J \times D$ the solution to the initial problem (1)–(2) is definite for all $t \in J$. Here $J \subset \mathbb{R}_+$ and $D \subseteq \mathbb{R}^n$ is an open domain in $\mathbb{R}^n$. It is known that the solution $x(t)$ of the initial problem (1)–(2) through the point (t_0, x_0) satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))ds \tag{3}$$

on the interval where the solution $x(t) = x(t, t_0, x_0)$ is definite.

Assume that for the right-hand part of nonlinear system (1) there exist nonnegative continuous functions $a(t)$ and $b(t)$ on any finite interval J such that

$$\|F(t,x)\| \le a(t)\|x\| + b(t)\|x\|^k, \tag{4}$$

where $k > 1$ and $\|\cdot\|$ is an Euclidean norm of the vector.

It is of interest to estimate the norm of solutions $x(t)$ to system (1) and to study behavior of the solutions on unbounded or finite time interval when inequality (4) is satisfied.

3 New Estimate of Solutions

We shall obtain uniform estimate of the norm of solutions to nonlinear system (1) with the initial conditions (2) when the condition (4) is satisfied.

The following result holds.

Lemma 1 (see [7, 8]) *For the right-hand part of system* (1) *assume that estimate* (4) *of the domain of values* $(t,x) \in J \times D$ *is satisfied and, besides,*

$$L(t) = 1 - (k-1)\|x_0\|^{k-1} \int\limits_{t_0}^{t} b(s) \exp\left[(k-1)\int\limits_{t_0}^{s} a(\tau)d\tau\right] ds > 0 \tag{5}$$

for all $t \in J$. *Then for the norm of solutions* $x(t)$ *of system* (1), *when* $(t_0, x_0) \in J \times D$ *the estimate*

$$\|x(t)\| \leq \|x_0\| \exp\left(\int\limits_{t_0}^{t} a(s)ds\right)(L(t))^{-\frac{1}{k-1}} \tag{6}$$

is valid for all $t \in J$.

Proof. From the integral equation (3) under condition (4) we have the estimate

$$\|x(t)\| \leq \|x_0\| + \int\limits_{t_0}^{t} (a(s)\|x(s)\| + b(s)\|x(s)\|^k)ds, \tag{7}$$

that is equivalent to the following one

$$\|x(t)\| \leq \|x_0\| + \int\limits_{t_0}^{t} (a(s) + b(s)\|x(s)\|^{k-1})\|x(s)\|ds \tag{8}$$

for all $t \in J$. Applying the Gronwall–Bellman lemma [6] to inequality (8) we get

$$\|x(t)\| \leq \|x_0\| \exp\left[\int\limits_{t_0}^{t} (a(s) + b(s)\|x(s)\|^{k-1})ds\right]. \tag{9}$$

Then, we represent inequality (9) as

$$\|x(t)\|^{k-1} \leq \|x_0\|^{k-1} \exp\left[(k-1)\int\limits_{t_0}^{t} (a(s) + b(s)\|x(s)\|^{k-1})ds\right] \tag{10}$$

and estimate from above the term

$$\exp\left[(k-1)\int\limits_{t_0}^{t} b(s)\|x(s)\|^{k-1}ds\right]. \tag{11}$$

Multiplying both parts of inequality (10) by the expression

$$-(k-1)b(t) \exp\left[-(k-1)\int\limits_{t_0}^{t} b(s)\|x(s)\|^{k-1}ds\right],$$

we arrive at

$$-(k-1)b(t)\|x(t)\|^{k-1}\exp\left[-(k-1)\int\limits_{t_0}^{t} b(s)\|x(s)\|^{k-1}ds\right]$$
$$\geq -(k-1)\|x_0\|^{k-1}b(t)\exp\left[(k-1)\int\limits_{t_0}^{t} a(s)ds\right].$$

Hence, it follows that

$$\begin{aligned}&\frac{d}{dt}\left(\exp\left[-(k-1)\int\limits_{t_0}^{t} b(s)\|x(s)\|^{k-1}ds\right]\right)\\ &\geq -(k-1)\|x_0\|^{k-1}b(s)\exp\left[(k-1)\int\limits_{t_0}^{t} a(s)ds\right].\end{aligned}\tag{12}$$

Integrating inequality (12) between t_0 and $t \in J$ we obtain

$$\exp\left[-(k-1)\int\limits_{t_0}^{t} b(s)\|x(s)\|^{k-1}ds\right] \geq L(t).$$

Under condition (5) the above inequality yields the estimate of the term (11) as follows

$$\exp\left[(k-1)\int\limits_{t_0}^{t} b(s)\|x(s)\|^{k-1}ds\right] \leq (L(t))^{-1} \quad \text{for all} \quad t \in J. \tag{13}$$

In view of estimate (13) we rewrite inequality (10) as

$$\|x(t)\|^{k-1} \leq \|x_0\|^{k-1}\exp\left[(k-1)\int\limits_{t_0}^{t} a(s)ds\right](L(t))^{-1}. \tag{14}$$

Since $k > 1$, we get from (14) the estimate (6), i.e.

$$\|x(t)\| \leq \|x_0\|\exp\left(\int\limits_{t_0}^{t} a(s)ds\right)(L(t))^{-\frac{1}{k-1}}$$

for all $t \in J$. This proves Lemma 1.

Corollary 1 In inequality (4) let the function $b(t) \equiv 0$ for all $t \in J$. Then estimate (6) becomes

$$\|x(t)\| \leq \|x_0\|\exp\left(\int\limits_{t_0}^{t} a(s)ds\right) \quad \text{for all} \quad t \in J. \tag{15}$$

This is the known Gronwall-Bellman estimate [6, p. 96].

Corollary 2 In inequality (4) let the function $a(t) \equiv 0$ for all $t \in J$. Then estimate (6) becomes

$$\|x(t)\| \le \|x_0\| \left\{ 1 - (k-1)\|x_0\|^{k-1} \int_{t_0}^{t} b(s)ds \right\}^{-\frac{1}{k-1}} \tag{16}$$

for all $t \in J$ whenever

$$1 - (k-1)\|x_0\|^{k-1} \int_{t_0}^{t} b(s)ds > 0.$$

Estimate (16) is obtained as well by the direct application of the Bihari lemma (see [9] to the inequality

$$\|x(t)\| \le \|x_0\| + \int_{t_0}^{t} b(s)\|x(s)\|^k ds.$$

Remark 1 In paper [8] new estimates of the norm of solutions are presented for some characteristic types of nonlinear mechanics equations.

4 Applications

We shall make use of the estimate (6) to solve some problems of system dynamics.

4.1 Boundedness of Motion

In system (1) let the vector-function $F(t, x)$ be definite and continuous on $J \times \mathbb{R}^n$. We shall cite some definitions according to [10].

Definition 1 The solution $x(t) = x(t, t_0, x_0)$ of system (1) is bounded, if there exists $\beta > 0$ such that $\|x(t, t_0, x_0)\| < \beta$ for all $t \ge t_0$, where β can depend on every solution.

Definition 2 The solution $x(t)$ of system (1) is equi-bounded, if for any $\alpha > 0$ and $t_0 \in J$ there exists $\beta(t_0, \alpha) > 0$ such that if $\|x_0\| < \alpha$, then $\|x(t, t_0, x_0)\| < \beta(t_0, \alpha)$ for all $t \ge t_0$.

Estimate (6) provides the following results.

Theorem 1 *If for any $x_0 \in \mathbb{R}^n$, $\|x_0\| < \infty$, all conditions of Lemma 1 are satisfied and, in addition, there exists $\beta > 0$ such that*

$$\exp\left(\int_{t_0}^{t} a(s)ds\right) (L(t))^{-\frac{1}{k-1}} < \frac{\beta}{\|x_0\|} \quad \textit{for all} \quad t \ge t_0,$$

then the motion described by the equation (1) *is bounded.*

Theorem 2 *If for $\|x_0\| < \alpha$ and*

$$L^*(t) = 1 - (k-1)\alpha^{k-1} \int_{t_0}^{t} b(s) \exp\left[(k-1)\int_{t_0}^{s} a(\tau)\,d\tau\right] ds > 0$$

all conditions of Lemma 1 are satisfied and there exists $\beta(t_0, \alpha) > 0$ *such that*

$$\exp\left(\int_{t_0}^{t} a(s)ds\right)(L^*(t))^{-\frac{1}{k-1}} < \frac{\beta(t_0,\alpha)}{\alpha} \quad \textit{for all} \quad t \ge t_0,$$

then the motion described by the equation (1) *is equi-bounded.*

Similar results can be established in terms of estimates (15) and (16) and Corollaries 1 and 2.

The proof of Theorems 1 and 2 follows immediately from the estimate (6) and Definitions 1 and 2.

4.2 Finite-Time Stability of Motion

For solution $x(t) = x(t, t_0, x_0)$ of the problem (1)–(2) we shall give the following definitions (see [2] and bibliography therein).

Definition 3 The motion of system (1) is:

(a) stable with respect to the values (λ, A, t_0, T), $0 < \lambda \le A$, if for any solution $x(t)$ with the initial conditions $x_0 \colon \|x_0\| < \lambda$ it follows that $\|x(t)\| < A$ for all $t \in [t_0, t_0 + T]$;

(b) uniformly stable with respect to the values (λ, A, t_0, T), $0 < \lambda \le A$, if for any solution $x(t)$ the condition $\|x(t_1)\| < \lambda$ implies $\|x(t)\| < A$ for any $t \ge t_1$, $(t, t_1) \in [t_0, t_0 + T]$.

Based on Lemma 1 we shall formulate the following result.

Theorem 3 *The motion of system* (1) *is:*

(a) *stable with respect to the values* (λ, A, t_0, T)*, if all conditions of Lemma 1 are satisfied as well as the inequality*

$$\exp\left(\int_{t_0}^{t} a(s)ds\right)(L(t))^{-\frac{1}{k-1}} < \frac{A}{\lambda} \quad \textit{for all} \quad t \in [t_0, t_0 + T]; \tag{17}$$

(b) *uniformly stable with respect to the values* (λ, A, t_0, T)*, if the inequality* (17) *is satisfied for any* $t_1 \in [t_0, t_0 + T]$ *such that* $\|x(t_1)\| < \lambda$.

In terms of estimates (15) and (16) we obtain the following results.

Theorem 4 *Let all conditions of Corollary 1 be satisfied as well as the inequality*

$$\int_{t_0}^{t} a(s)ds \le \ln\left(\frac{A}{\lambda}\right) \quad \textit{for all} \quad t \in [t_0, t_0 + T]. \tag{18}$$

Then the motion of system (1) *is stable with respect to the values* (λ, A, t_0, T).

Theorem 5 *Let all conditions of Corollary 2 be satisfied as well as the inequality*

$$\left\{1 - (k-1)\alpha^{k-1}\int_{t_0}^{t} b(s)ds\right\}^{-\frac{1}{k-1}} < \frac{A}{\lambda} \quad \textit{for all} \quad t \in [t_0, t_0 + T]. \tag{19}$$

Then the motion of system (1) *is stable with respect to the values* (λ, A, t_0, T).

The proof of Theorems 3–5 is based on the estimates (6), (15), (16) and Definition 3(a). The assumptions on motion uniform stability of system (1) with respect to the values (λ, A, t_0, T) are made in terms of estimates (18) and (19), provided that $\|x(t_1)\| < \lambda$ for any $t_1 \in [t_0, t_0 + T]$.

4.3 Exponential Convergence of Solutions to Systems with Quadratic Nonlinearity

Consider systems (1) with a particular type nonlinearity, namely, the systems with quadratic nonlinearity (see [11, 12] and bibliography therein)

$$\dot{x}(t) = Ax(t) + X^T(t)Bx(t), \quad x(0) = x_0. \tag{20}$$

Here $x \in \mathbb{R}^n$, A is a rectangular $n^2 \times n$-matrix consisting of symmetric square matrices B_i, $i = 1, 2, \ldots, n$,

$$B_i = \begin{bmatrix} b^i_{11} & b^i_{12} & \ldots & b^i_{1n} \\ b^i_{12} & b^i_{22} & \ldots & b^i_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b^i_{1n} & b^i_{2n} & \ldots & b^i_{nn} \end{bmatrix},$$

$X^T(t) = \{X_1(t), X_2(t), \ldots, X_n(t)\}$ is a rectangular $n \times n^2$-matrix consisting of square $n \times n$-matrices $X_i(t)$ with vectors $x(t)$ on their i-th lines, and the other elements are zero, i.e.

$$X_1(t) = \begin{bmatrix} x_1(t) & x_2(t) & \ldots & x_n(t) \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \quad X_2(t) = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ x_1(t) & x_2(t) & \ldots & x_n(t) \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \quad \ldots,$$

$$X_n(t) = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ x_1(t) & x_2(t) & \ldots & x_n(t) \end{bmatrix}.$$

Here and elsewhere the vector and matrix norms are specified by the formulas

$$\|x(t)\| = \left\{ \sum_{i=1}^{n} x_i^2(t) \right\}^{1/2}, \quad \|B\| = \{\lambda_{\max}(B^T B)\}^{1/2},$$

where $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ are extreme eigenvalues of the corresponding symmetric matrices.

Let the matrix A of the linear part of system (20) be asymptotically stable. Then, according to the theory of stability by first approximation (see [6]) the zero solution of nonlinear system (20) will also be asymptotically stable. We shall take the quadratic form $V(x) = x^T Hx$ as the Lyapunov function and compute its total derivative by virtue of system (20)

$$\begin{aligned} \frac{d}{dt}V(x(t)) &= [Ax(t) + X^T(t)Bx(t)]^T Hx(t) + x^t(t)H[Ax(t) + X^T(t)BNx(t)] \\ &= x^T(t)[(A^T H + HA) + (B^T X(t)H + HX^T(t)B)]x(t). \end{aligned} \tag{21}$$

Since the matrix A is asymptotically stable by assumption, for an arbitrary positive definite matrix C the matrix Lyapunov equation

$$A^T H + HA = -C \tag{22}$$

possesses a unique solution in the form of positive definite matrix H. In view of the fact that H is a solution of the Lyapunov equation (22) we get from (21) that

$$\frac{d}{dt}V(x(t)) = -x^T(t)[C - (B^T X(t)H + HX^T(t)B)]x(t). \tag{23}$$

The stability domain of the zero solution of system (20) is the interior of the surface of the level of the Lyapunov function $V(x) = r > 0$ located inside the domain

$$G_0 = \{x \in \mathbb{R}^n :\ C - B^T XH - HX^T B > \Theta\},$$

where the symbol

$$C - B^T XH - HX^T B > \Theta \tag{24}$$

is understood as positive definiteness of the matrix. We shall replace the condition (24) by a more "rough" one. Since for the chosen vector and matrix norms the correlation

$$\|X(t)\| = \|x(t)\|,$$

holds true, for the total derivative of the Lyapunov function (21) the estimate

$$\frac{d}{dt}V(x(t)) \leq -\left[\lambda_{\min}(C) - 2\|H\|\|B\|\|x(t)\|\right]\|x(t)\|^2. \tag{25}$$

is satisfied.

We designate

$$G_0 = \left\{x \in \mathbb{R}^n :\ \|x\| < \frac{\lambda_{\min}(C)}{2\|H\|\|B\|}\right\}. \tag{26}$$

Then the domain of "guaranteed" stability is specified by the expression

$$G_{r_0} = \max_{r>0}\{C_r :\ G_r \subset G_0\}, \quad G_r = \{x \in \mathbb{R}^n :\ x^T Hx < r^2\}. \tag{27}$$

From this dependence it follows that for the "maximal" stability domain be defined, it is necessary to "imbed" the ellipsoid $x^T HX = r^2$ into a sphere of radius $R = \dfrac{\lambda_{\min}(C)}{2\|H\|\|B\|}$ and to extend it for $r \to \infty$ until the ellipsoid surface touches the sphere.

Theorem 6 *Let the matrix of the linear part of system* (20) *be asymptotically stable. Then the zero solution of system* (20) *is asymptotically stable and for the solutions of the system satisfying the initial conditions*

$$\|x_0\| < \frac{\gamma(H)}{2\|B\|\varphi(H)}, \tag{28}$$

where

$$\varphi(H) = \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}, \quad \gamma(H) = \frac{\lambda_{\min}(C)}{\lambda_{\max}(H)},$$

the convergence of solutions obeys the estimate

$$\|x(t)\| \leq \frac{\gamma(H)\sqrt{\lambda_{min}(Y)}\|x_0\|}{\Big[\gamma(H) - 2\|B\|\varphi(H)\|x_0\|\Big]\, e^{\frac{1}{2}\gamma(H)t} + 2\|B\|\varphi(H)\|x_0\|.} \tag{29}$$

Proof. In order to obtain estimate (29) we use the Lyapunov function $V(x) = x^T Hx$ with total derivative (25). Since for the quadratic function $V(x) = x^T Hx$ the two-sided inequality

$$\lambda_{\min}(H)\|x\|^2 \leq V(x) \leq \lambda_{\max}(H)\|x\|^2 \tag{30}$$

is valid, the inequality (25) can be rewritten as

$$\frac{d}{dt}V(x(t)) \leq -\frac{\lambda_{\min}(C)}{\lambda_{\max}(H)}\, V(x(t)) + 2\lambda_{\max}(H)\|B\|\, \frac{V^{3/2}(x(t))}{\lambda_{\min}^{3/2}(H)}\,. \tag{31}$$

Using the designation (28) we rewrite the obtained expression as

$$\frac{d}{dt}V(x(t)) \leq -\gamma(H)V(x(t)) + 2\frac{\|B\|\varphi(H)}{\sqrt{\lambda_{\min}(H)}}\, V^{3/2}(x(t)).$$

Dividing this inequality by the expression $V^{3/2}(x)$ we get the estimate

$$V^{-3/2}(x(t))\, \frac{dV(x(t))}{dt} \leq -\gamma(H)V^{-1/2}(x(t)) + 2\frac{\|B\|\varphi(H)}{\sqrt{\lambda_{\min}(H)}}\,.$$

Hence, having designated $V^{-1/2}(x(t)) = z(t)$, we arrive at

$$-2\,\frac{dz(t)}{dt} \leq -\gamma(H)z(t) + 2\,\frac{\|B\|\varphi(H)}{\sqrt{\lambda_{\min}(H)}}\,,$$

and then

$$\frac{dz(t)}{dt} \geq -\frac{1}{2}\,\gamma(H)z(t) - \frac{\|B\|\varphi(H)}{\sqrt{\lambda_{\min}(H)}}\,.$$

Solving this inequality (in the same way as the linear inhomogeneous Bernoulli equation) we get

$$z(t) \geq \left[z_0 - 2\,\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right] e^{\frac{1}{2}\gamma(H)t} + 2\,\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\,.$$

Substitution $V^{-1/2}(x(t)) = z(t)$ yields the estimate

$$V^{-1/2}(x(t)) \geq \left[V^{-1/2}(x_0) - 2\,\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right] e^{\frac{1}{2}\gamma(H)t} + 2\,\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\,.$$

Hence

$$V^{1/2}(x(t)) \leq \left\{\left[V^{-1/2}(x_0) - 2\,\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right] e^{\frac{1}{2}\gamma(H)t} + 2\,\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right\}^{-1}.$$

Application of the two-sided inequality for the quadratic form (30) gives

$$\sqrt{\lambda_{\min}(H)}(\|x(t)\| \leq \left\{ \left[\frac{1}{\sqrt{V(\|x_0\|)}} - 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}} \right] e^{\frac{1}{2}\gamma(H)t} + 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}} \right\}^{-1}$$
$$\leq \left\{ \left[\frac{1}{\sqrt{\lambda_{\min}(H)}\|x_0\|} - 2\,\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}} \right] e^{\frac{1}{2}\gamma(H)t} + 2\,\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}} \right\}^{-1}$$
$$= \frac{\gamma(H)\sqrt{\lambda_{\min}(H)}\|x_0\|}{\left[\gamma(H) - 2\|B\|\varphi(H)\|x_0\|\right] e^{\frac{1}{2}\gamma(H)t} + 2\|B\|\varphi(H)\|x_0\|}\,.$$

Thus, for solutions $x(t)$ of system (20) with the initial conditions from the domain (27), i. e. $x_0 \in G_0$, we obtain the estimate of solutions convergence of (29) type. This completes the proof.

Remark 2 Consider the first order scalar equation

$$\dot{x}(t) = -ax(t) + bx^2(t), \quad a > 0, \quad x(0) = x_0. \tag{32}$$

This equation is an equation with separating variables and its exact solution is the function

$$x(t) = \frac{ax_0 e^{-at}}{a - bx_0[1 - e^{-at}]}\,. \tag{33}$$

Consider the application of the method of Lyapunov functions with the function $V(x) = x^2$ for the equation (32). For this function $\lambda_{\max}(H) = \lambda_{\min} = 1$. The total derivative by virtue of the linear part of system (32) is

$$\frac{d}{dt}V(x(t)) = -2ax^2(t).$$

Therefore, $\varphi(H) = 1$, $\gamma(H) = 2a$. The convergence estimate (29) for solutions of the equation with the initial conditions $\|x_0\| < a/|b|$ is of a similar form

$$\|x(t)\| \leq \frac{a\|x_0\|}{[a - |b|\|x_0\|]e^{at} + |b|\|x_0\|} = \frac{a\|x_0\|e^{-at}}{a - |b|[1 - e^{-at}]} \to 0.$$

Thus, for the scalar equation (32) with the exact solution (33) the convergence estimate coincides with the estimate obtained by the application of the quadratic Lyapunov function.

5 Concluding Remarks

The proposed method for estimating the norm of solutions to nonlinear systems possesses a considerable potential for application in the investigation of particular mechanical and other nature systems. Efficiency of the proposed estimation is illustrated by the example of a first order scalar equation, for which the convergence estimate is obtained by means of the direct Lyapunov method.

References

[1] Golubentsev, A.N. *Integral Methods in Dynamics.* Kiev: Tekhnika, 1967.

[2] Martynyuk, A.A. *Tecnical Stability in Dynamics.* Kiev: Tekhnika, 1973.

[3] Martynyuk, A.A. and Gutowski, R. *Integral Inequalities and Stability of Motion.* Kiev: Naukova Dumka, 1979.

[4] Martynyuk, A.A., Lakshmikantham, V., and Leela, S. *Stability of Motion. Method of Integral Inequalities.* Kiev: Naukova Dumka, 1989.

[5] Lakshmikantham, V., Leela, S., and Martynyuk, A.A. *Stability Analysis of Nonlinear Systems.* New York: Marcel Dekker, Inc., 1989.

[6] Bellman, R. *Stability Theory of Differential Equations.* New Yourk, Toronto, London, 1953.

[7] Louartassi, Y., Mazoudi, E.H.E. and Elalami, N. A new generalization of lemma of Gronwall-Bellman. *Applied Mathematical Sciences* (6) (2012) 621–628.

[8] Martynyuk, A.A. Novel bounds for solutions of nonlinear differential equations. *Applied Mathematics* (6) (2015) 182–194. http://dx.doi.org/10.4236/am.2015.61018

[9] Bihari, I. A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations. *Acta Math. Acad. Sci. Hung.* **7** (1) (1965) 81–94.

[10] Yoshizawa, T. *Stability Theory by Liapunov's Second Method.* Tokyo: The Math. Soc. of Japan, 1966.

[11] Khusainov, D.Ya., Dzhalladova I.A., and Shatyrko, O.A. Estimation of stability domain of differential systems with quadratic right-hand part. *Visnyk Kyiv Univ./ Seria Phiziko-Mathematical sciences* (3) (2011) 227–230.

[12] Dzhalladova, I.A., and Khusainov, D.Ya. Quadratic systems with delay. *Problems of control and computer science* (1) (2013) 5–13.

[13] Grigorenko, Ya.M., Larin, V.B., and Martynyuk, A.A. Professor A.N. Golubentsev. *Nonlinear Dynamics and Systems Theory* **11** (3) (2011) 223–226.

[14] Martynyuk, A.A. *Revisiting the Past with Neither Indifference nor Resentment.* Preface by Jewgeni H. Dshalalow. Kiev: Phoenix, 2014.

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On Exponential Domination of Some Graphs

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Abstract: Let G be a graph and $S \subseteq V(G)$. We denote by $\langle S \rangle$ the subgraph of G induced by S. For each vertex $u \in S$ and for each $v \in V(G) - S$, we define $\overline{d}(u,v) = \overline{d}(v,u)$ to be the length of the shortest path in $\langle V(G) - (S - \{u\}) \rangle$ if such a path exists, and ∞ otherwise. Let $v \in V(G)$. We define $w_s(v) = \sum_{u \in S} \frac{1}{2^{\overline{d}(u,v)-1}}$ if $v \notin S$, and $w_s(v) = 2$ if $v \in S$. If, for each $v \in V(G)$, we have $w_s(v) \geq 1$, then S is an *exponential dominating set*. The smallest cardinality of an exponential dominating set is the *exponential domination number* $\gamma_e(G)$. In this paper, we consider the *exponential domination number* in total graphs. We determine the *exponential domination number* of $T(G)$ for some specific graphs G.

Keywords: *graph vulnerability; network design and communication; domination; exponential domination number; total graph.*

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1 Introduction

In a communication network, the vulnerability measures the resistance of network to disruption of operation after the failure of certain stations or communication links. The stability of communication networks is of prime importance to network designers (see [9,10]). If we think of the graph as modeling a communication network, many graph theoretical parameters have been used to describe the stability of communication networks including connectivity, toughness, integrity, domination and its variations (see [1, 2, 4, 5]). The domination number is one of the measures of the graph vulnerability.

Domination in graphs is a well-studied concept in graph theory. Domination based parameters reveal an underlying efficient communication network in which a vertex can

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affect all its neighborhood vertices in some sense. In real life applications, we can encounter that a vertex can affect both its neighborhood vertices and all vertices within a given distance. Distance domination is a kind of this situation. There has been no framework yet in which the effect of a vertex broadens beyond its neighborhood while decreasing by distance. It has been suggested (see [7]) that exponential domination is a model for the reliability of the spread of information or gossip. In this model, the dominating strategy of a vertex decreases exponentially with a distance, by the factor 1/2. Therefore, it is possible that a vertex v is dominated by one of its neighbors or by some vertices that are closer to v. The assumption is that gossip heard directly from a source is totally reliable, while gossip passed from person to person loses half its credibility with each individual in the chain. Finding the exponential domination number in this application amounts to determining the minimum number of sources needed so that each person gets fully reliable information.

In this paper, we consider simple finite undirected graphs without loops and multiple edges. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and an edge set $E = E(G)$. For vertices u of a graph G, the *open neighborhood* of u is $N(u) = \{v \in V(G)|(u, v) \in E(G)\}$. We define analogously for any $S \subseteq V(G)$ the open neighborhood $N(S) = \bigcup_{u \in S} N(u)$. The closed neighborhood of u is $N[u] = u \cup N(u)$. For a set $S \subseteq V$, its closed neighborhood $N[S] = N(S) \cup S$. A set S is dominating set of G if $N[S] = V$, or equivalently, every vertex in $V - S$ is adjacent to at least one vertex of S. The *dominating number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

The distance $d(u, v)$ between two vertices u and v in G is the length of the shortest path between them. If u and v are not connected, then $d(u, v) = \infty$, and for $u = v$, $d(u, v) = 0$. The diameter of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$ (see [3, 4]).

Throughout this paper, the largest integer not larger than x is denoted by $\lfloor x \rfloor$ and the smallest integer not smaller than x is denoted by $\lceil x \rceil$.

The paper proceeds as follows. In Sections 2 and 3, the definition of exponential domination number and known results are given, respectively. In Section 4, we give some results on the exponential domination number of total graphs. Formulas for the exponential domination number of the graphs obtained by binary graph operations are given in Section 5.

2 Exponential Domination Number

The exponential domination number of a graph is a new characteristic for graph vulnerability introduced in [7]. This definition is in the following:

This parameter is a variation of distance domination in which, as described in the motivation already given, the 'dominating power' radiating from a vertex declines exponentially with distance. Let G be a graph and $S \subseteq V(G)$. We denote by $\langle S \rangle$ the subgraph of G induced by S. For each vertex $u \in S$ and for each $v \in V(G) - S$, we define $\overline{d}(u, v) = \overline{d}(v, u)$ to be the length of the shortest path in $\langle V(G) - (S - \{u\}) \rangle$ if such a path exists, and ∞ otherwise. Let $v \in V(G)$. The definition is

$$w_s(v) = \begin{cases} \sum_{u \in S} \frac{1}{2^{\overline{d}(u,v)-1}}, & if\ v \notin S, \\ 2, & if\ v \in S. \end{cases}$$

We refer to $w_s(v)$ as the weight of S at v (note that we define $w_s(v) = 2$ if $v \in S$ since then v contributes $w_s(v)/2^d$ to every vertex it exponentially dominates at distance d).

If, for each $v \in V(G)$, we have $w_s(v) \geq 1$, then S is an *exponential dominating set*. The smallest cardinality of an exponential dominating set is the *exponential domination number*, $\gamma_e(G)$, and such a set is a *minimum exponential dominating set*, or $\gamma_e(G)$ -set for short. If $u \in S$ and $v \in V(G) - S$ and $\frac{1}{2^{\overline{d}(u,v)-1}} > 1$, then we say that u *exponentially dominates* v. Note that if S is an exponential dominating set, then every vertex of $V(G) - S$ is exponentially dominated, but the converse is not true (see [7, 8]).

3 Basic Results

Theorem 3.1 *[7] For every positive integer n, $\gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil$.*

Theorem 3.2 *[7] For every positive integer n,*

$$\gamma_e(C_n) = \begin{cases} 2, & if\ n = 4, \\ \lceil \frac{n}{4} \rceil, & if\ n \neq 4 \in S. \end{cases}$$

Theorem 3.3 *[7] If G is a connected graph of diameter d, then $\gamma_e(G) \geq \frac{\lceil d+2 \rceil}{4}$.*

Theorem 3.4 *[7] If G is a connected graph of order n, then $\gamma_e(G) \leq \frac{2}{5}(n+2)$.*

Theorem 3.5 *[7] Let G be a connected graph of order n and T be a spanning tree of G. Then $\gamma_e(G) \leq \gamma_e(T)$.*

4 Exponential Domination Number of Total Graphs

In this section, the exponential domination number of total graph of a graph is calculated and formula for the exponential domination number of $\gamma_e(\overline{T(G)})$ is given.

Definition 4.1 [3, 4] The vertices and edges of a graph are called its elements. Two elements of a graph are neighbors if they are either incident or adjacent. The *total graph* $T(G)$ of the graph $G = (V(G), E(G))$, has vertex set $V(G) \cup E(G)$, and two vertices of $T(G)$ are adjacent whenever they are neighbors in G.

Example 4.1

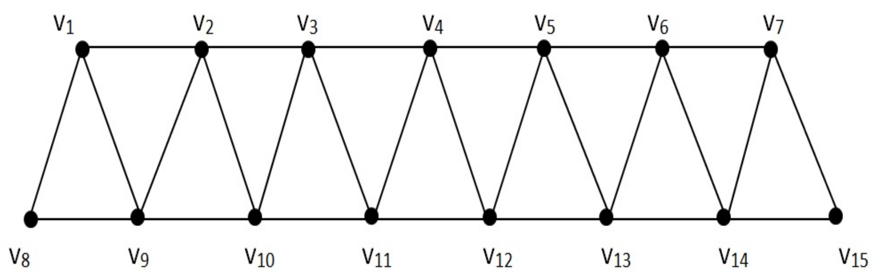

Figure 1: Total graph $T(P_8)$.

The following table shows us the weight of S_1 at all vertices of the graph $T(P_8)$, where $S_1 = \{v_2, v_8, v_{12}, v_{14}\}$.

v	$\overline{d}(v, v_2)$	$\overline{d}(v, v_8)$	$\overline{d}(v, v_{12})$	$\overline{d}(v, v_{14})$	$w_{s_1(v)}$
v_1	1	1	4	7	2.135
v_2	-	-	-	-	2
v_3	1	3	2	4	1.875
v_4	2	4	1	3	1.875
v_5	3	5	1	2	1.81
v_6	4	6	2	1	1.655
v_7	5	7	3	1	1.325
v_8	-	-	-	-	2
v_9	1	1	3	6	2.281
v_{10}	1	2	2	5	2.015
v_{11}	2	3	1	4	1.875
v_{12}	-	-	-	-	2
v_{13}	4	6	1	1	2.156
v_{14}	-	-	-	-	2
v_{15}	6	8	4	1	1.147

For S_1*_set*, $\forall v \in V(T(P_8))$, $w_s(v) \geq 1$ is satisfied. So, S_1*_set* is an exponential dominating set.

The following table shows us the weight of S_2 at all vertices of the graph $T(P_8)$, where $S_2 = \{v_5, v_{10}, v_{14}\}$.

v	$\overline{d}(v, v_5)$	$\overline{d}(v, v_{10})$	$\overline{d}(v, v_{14})$	$w_{s_2(v)}$
v_1	4	2	6	0.656
v_2	3	1	5	1.56
v_3	2	1	4	1.625
v_4	1	2	3	1.75
v_5	-	-	-	2
v_6	1	4	1	2.125
v_7	2	5	1	1.56
v_8	5	2	7	0.575
v_9	4	1	6	1.156
v_{10}	-	-	-	2
v_{11}	2	1	3	1.75
v_{12}	1	2	2	2
v_{13}	1	3	1	2.25
v_{14}	-	-	-	2
v_{15}	3	6	1	1.281

For S_2*_set*, $w_{S_2}(v_1) \geq 1$ and condition $w_{S_2}(v_8) \geq 1$ is not satisfied. So, S_2 is not an exponential dominating set.

The following table shows us the weight of S_3 at all vertices of the graph $T(P_8)$, where $S_3 = \{v_8, v_{11}, v_{14}\}$.

v	$\overline{d}(v, v_8)$	$\overline{d}(v, v_{11})$	$\overline{d}(v, v_{14})$	$w_{s3(v)}$
v_1	1	3	6	1.281
v_2	2	2	5	1.06
v_3	3	1	4	1.375
v_4	4	1	3	1.375
v_5	5	2	2	1.06
v_6	6	3	1	1.281
v_7	7	4	1	1.14
v_8	-	-	-	2
v_9	1	2	6	1.531
v_{10}	2	1	5	1.56
v_{11}	-	-	-	2
v_{12}	5	1	2	1.56
v_{13}	6	2	1	1.531
v_{14}	-	-	-	2
v_{15}	8	5	1	1.067

For S_3*_set*, $\forall v \in V(T(P_8))$, $w_s(v) \geq 1$ is satisfied. So, S_3*_set* is an exponential dominating set.

Similarly, we can get a lot of exponential dominating sets of the graph $T(P_8)$ but, for exponential domination number we need the minimum cardinality of among all exponential dominating sets. Here, $\gamma_e(T(P_n)) = min\{|S_1|, |S_3|\} = min\{4, 3\} = 3$.

Theorem 4.1 *Let P_n be a path graph with n vertices and $T(P_n) \cong G$ be a total graph of P_n with $2n-1$ vertices. Then $\gamma_e(G) = \lceil \frac{n}{3} \rceil$.*

Proof. The domination number of P_n is $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. If we add the vertices of the domination set to $\gamma_e - set$, every vertex v in $\gamma_e - set$ is adjacent to four vertices in graph G. For $\forall u \in N_{\gamma_e - set}(v)$, $w_s(u) \geq 1$. The length of the shortest path, from $\forall u \in V(G) - N_{\gamma_e - set}[v]$ remaining vertices to exactly two vertices in $\gamma_e - set$ is 2. So, $w_s(u) \geq 1$. Consequently, exponential domination number of G is

$$\gamma_e(G) = \lceil \tfrac{n}{3} \rceil.$$

The proof is completed.

Theorem 4.2 *Let C_n be a cycle graph with n vertices and $T(C_n) \cong G$ be a total graph of C_n with $2n$ vertices. Then, for $n > 3$ $\gamma_e(G) = \lceil \frac{n}{3} \rceil$.*

Proof. The proof is similar to the proof of Theorem 7.

Theorem 4.3 *Let $S_{1,n}$ be a star graph with $n+1$ vertices and $T(S_{1,n}) \cong G$ be a total graph of $S_{1,n}$ with $2n+1$ vertices. Then $\gamma_e(G) = 1$.*

Proof. Every vertex in G is adjacent to centre vertex c in G. So, we can add only centre vertex c to $\gamma_e - set$. Hence, we have $w_s(v) = 1$ for $\forall v \in V(G) - \{c\}$ and $w_s(c) = 2$. Therefore, the result is obvious.

The proof is completed.

Theorem 4.4 *Let K_n be a complete graph with n vertices and $T(K_n) \cong G$ be a total graph of K_n with $(n^2+n)/2$ vertices. Then, $\gamma_e(G) = 2$.*

Proof. Since in a complete graph all vertices are mutually adjacent, distance between each pair of vertex is 1. Distance between remaining vertices in $V(G) - V(K_n)$ and any vertex in K_n is at most 2. Hence, condition $w_s(v) \geq 1$ is not satisfied for some $v \in V(G) - V(K_n)$. As in the proof of Theorem 7, one more vertex in K_n should be added to $\gamma_e - set$ for the length of the path from vertices in $V(G) - V(K_n)$ to exactly two vertices in $\gamma_e - set$ to be 2. Hence, we have $\gamma_e(G) = 2$.

The proof is completed.

Theorem 4.5 *Let $W_{1,n}$ be a wheel graph with $n+1$ vertices and $T(W_{1,n}) \cong G$ be a total graph of $W_{1,n}$ with $3n+1$ vertices. Then, $\gamma_e(G) = \lceil \frac{n}{4} \rceil + 1$.*

Proof. Let the vertex-set of graph G be $V(G) = V_1(G) \cup V_2(G) \cup V_3(G) \cup V_4(G)$ where,

$V_1(G) =$ The set contains the center vertex c of graph $W_{1,n}$.

$V_2(G) =$ The set contains all vertices of graph $W_{1,n}$, except center vertex.

$V_3(G) =$ The set contains the edges of graph $W_{1,n}$, which are adjacent to center vertex; are the vertices of graph $T(W_{1,n})$.

$V_4(G) =$ The set contains the edges of the cycle of graph $W_{1,n}$ are the vertices of graph $T(W_{1,n})$.

The center vertex c is adjacent to every vertex in $V_2(G)$ and $V_3(G)$. So, the centre vertex c should be added to $\gamma_e - set$. But, the length of the path from $\forall u \in V_4(G)$ to every vertex in $\gamma_e - set$ is 2. Therefore, condition $w_s(u) \geq 1$ is not satisfied. As in the proof of Theorem 7, the length of the path from every vertex in $V_4(G)$ to exactly two vertices in $\gamma_e - set$ should be 2. The length of the path from every vertex in $V_2(G)$ to two vertices in $V_4(G)$ is 1 and two vertices in $V_4(G)$ is 2. Hence, $\lceil \frac{|V_2(G)|}{4} \rceil = \lceil \frac{n}{4} \rceil$ vertices in $V_2(G)$ should be added to $\gamma_e - set$. There is already the center vertex c in $\gamma_e - set$. Hence, we have $\gamma_e(G) = \lceil \frac{n}{4} \rceil + 1$.

The proof is completed.

5 Corona and join graphs, the exponential domination number

Definition 5.1 [3,4] The corona $G_1 o G_2$ is obtained by taking one copy of G_1 and $|G_1|$ copies of G_2 , and by joining each vertex of the *ith* copy of G_2 to the *ith* vertex of G_1, i=1,2,...,$|G_1|$.

Definition 5.2 [3,4] Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph $G = G_1 + G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{(u,v) : u \in V(G_1), v \in V(G_2)\}$.

Theorem 5.1 *Let $G_1 \cong P_n$ be a path graph with n vertices and G be any connected graph. Then, $\gamma_e(G_1 o G) = \lfloor \frac{n-2}{2} \rfloor + 2$.*

Proof. If $|V(G)| = n_1$, then $|V(G_1oG)| = n(n_1 + 1)$. Every vertex in G_1 except the end vertices is adjacent to n_1 vertices and two vertices in G_1. The path between every vertex in G_1 except the end vertices and $n(n_1+1)-(n_1-2)$ vertices in G_1oG is at least 2. So, we obtain the minimum exponential domination set S by adding some vertices in G_1 to S and $S \subseteq V(G_1)$. Two end vertices of graph G_1 should be added to exponential domination set S of G_1oG. Otherwise, for $\forall v \in V(G_1oG) - V(G_1)$ that are adjacent to these end vertices, $w_s(v) \geq 1$ is not satisfied, since the length of the path between v and one vertex in G_1oG is 2; the length of the path between v and the other remaining vertices in G_1oG is at least 3. If we add $\lfloor \frac{n-2}{2} \rfloor$ vertices in G_1 except these end vertices, to S, for $\forall u \in V(G_1oG)$ $w_s(u) \geq 1$ is satisfied. There are already two end vertices in S. Hence, we have $\gamma_e(G_1oG) = \lfloor \frac{n-2}{2} \rfloor + 2$.

The proof is completed.

Theorem 5.2 *Let $G_1 \cong C_n$ be a cycle graph with n vertices and G be any connected graph. Then, $\gamma_e(G_1oG) = \lceil \frac{n}{2} \rceil$.*

Proof. If $|V(G)| = n_1$, then $|V(G_1oG)| = n(n_1 + 1)$. Every vertex in G_1 is adjacent to n_1 vertices and two vertices in G_1. The path between every vertex in G_1 and $n(n_1 + 1) - (n_1 - 2)$ vertices in G_1oG is at least 2. So, we obtain the minimum exponential domination set S by adding some vertices in G_1 to S and $S \subseteq V(G_1)$. We obtain S by adding $\forall v \in S$ satisfies $d(u,v) \leq 2$ or $d(u,v) = \infty$ for $\forall u \in (V(G_1oG) - S)$. So, there must be $\lceil \frac{n}{2} \rceil$ vertices from G_1 in S . Consequently, $w_s(x) \geq 1$ satisfying for $\forall x \in V(G_1oG)$ and we have

$$\gamma_e(G_1oG) = \lceil \tfrac{n}{2} \rceil.$$

The proof is completed.

Corollary 5.1 *Let $G_1 \cong C_n$. Then, $\gamma_e(G_1oG) = diam(G_1)$.*

Theorem 5.3 *Let $G_1 \cong S_{1,n}$ be a star graph with $n+1$ vertices and G be any connected graph. Then, $\gamma_e(G_1oG) = 4$.*

Proof. We denote the centre vertex of G_1 by c. In G_1oG, for $\forall u \in V(G)$ and $\forall v \in V(G_1 - \{c\})$ $d(u,v) \leq 3$. If we set S with vertices from $G_1 - \{c\}$ then vertex v contributes at least $\frac{1}{2^{\bar{d}(u,v)-1}} = \frac{1}{2^2}$ to $w_s(u)$. Hence, adding any 4 vertices from $G_1 - \{c\}$ to S is sufficient and we have

$$\gamma_e(G_1oG) = 4.$$

The proof is completed.

Theorem 5.4 *Let $G_1 \cong W_{1,n}$ be a wheel graph with $n+1$ vertices and G be any connected graph. Then, $\gamma_e(G_1oG) = 4$.*

Proof. The proof is similar to the proof of Theorem 17.

Theorem 5.5 *Let $G_1 \cong K_n$ be a complete graph with n vertices and G be any connected graph. Then $\gamma_e(G_1oG) = 2$.*

Proof. The length of the path between $\forall v \in G_1oG$ and every vertex in G_1 is at most 2. So, it is easy to see that $diam(G_1oG) = 3$. Hence, $S \subseteq V(G_1)$ satisfying $w_s(v) \geq 1$. It is sufficient to add any two vertices to S. Therefore, we have

$$\gamma_e(G_1 o G) = 2.$$

The proof is completed.

Corollary 5.2 *For any two graphs G_1 and G_2, $G_1 o G_2 \geq \lceil \frac{diam(G_1 o G_2)}{2} \rceil$.*

Theorem 5.6 *Let G_1 and G_2 be any two graphs having respectively diameters d_1 and d_2. If $diam(G_1) = d_1 < diam(G_2) = d_2$, then $\gamma_e(G_1 + G_2) = \gamma_e(G_1)$.*

Proof. We assume that $diam(G_1) = d_1 < diam(G_2) = d_2$. By the definition of $\gamma_e(G_1)$, we can not reduce any vertex from $\gamma_e(G_1)$ and every vertex in G_2 is adjacent to every vertex in $\gamma_e(G_1)$. If we add every vertex in $\gamma_e(G_1)$ to S minimum exponential number of $G_1 + G_2$, for $\forall u \in G_1$ $w_s(u) \geq 1$ and for $\forall v \in V(G_2)$ $w_s(v) = 1$ is satisfied.

The proof is completed.

6 Conclusion

In an administrative setup, decisions are taken by a small group who have effective communication links with other members of the organization. Domination in graphs provides a model for such a concept. The domination in graphs is one of the concepts in graph theory which has attracted many researchers to work on it because of its many and varied applications in such fields as linear algebra and optimization, design and analysis of communication networks, and social sciences and military surveillance. Many variants of dominating models are available in the existing literature. Dankelmann et al. (see [7]) recently defined exponential domination. Hence, in this paper, we investigate the exponential domination number of some total graphs. Moreover some results about exponential domination number of graphs obtained by graph operations are established.

References

[1] Aytaç, A. and Turacı, T. The Average Lower Independence Number Of Total Graphs. *Bulletin of International Mathematical Virtual Institute* **2** (2012) 17–27.

[2] Barefoot, C.A., Entringer, R. and Swart, H. Vulnerability in Graphs – A Comparative Survey. *J. Comb. Math. Comb. Comput.* **1** (1987) 13–22.

[3] Chartrand, G. and Leisnak, L. *Graphs and Digraphs.* Second Edition. Wadsworth, Monterey, 1986.

[4] Harary, F. *Graph Theory.* Addision-Wesley Publishing Company, 1972.

[5] Haynes, T.W., Hedeniemi, S.T. and Slater, P.J. *Domination in graphs, Advanced Topic.* Marcel Dekker, Inc, New York, 1998.

[6] Slater P.J., R-Domination in graphs. *J. Assoc. Comput. Mach.* **23** (1976) 446–450.

[7] Dankelmann, P., Day, D., Erwin, D., Mukwembi, S. and Swart, H. Domination with exponential decay. *Discrete Mathematics* **309** (2009) 5877–5883.

[8] Anderson, M., Brigham, R.C., Carrington, J.R., Vitray, R.P. and Yellen, J. On Exponential Domination of $C_m x C_n$. *AKCE J. Graphs. Combin.* **6** (3) (2009) 341–351.

[9] Feng, Z. and Michel, A.N. Robustness Analysis of a Class of Discrete-Time Systems with Applications to Neural Networks. *Nonlinear Dynamics and Systems Theory* **3** (1) (2003) 75–86.

[10] Martunyuk, A.A. Stability in the Models of Real World Phenomena. *Nonlinear Dynamics and Systems Theory* **11** (1) (2011) 7–52.

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Investigation in the Technique of Adaptive Predictive Control Fed by a Hybrid Inverter Applied to a Permanent Magnetic Synchronous Machine

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Abstract: The purpose of this paper is to present an approach to control the nonlinear system represented here by permanent magnet synchronous machines with two forms of control. This approach results from a combination of the adaptive and predictive properties, and the interaction of continuous-time and discrete event systems. Such a hybrid system consists of a discrete program with an analog environment. Many of the control approaches are limited to discrete-time hybrid systems because many complex mathematical issues are removed. In many applications the command variables are intrinsically discrete, either because such a system design is simpler or for other technological reasons. Our system consists of a five level inverter which controls a synchronous permanent magnet machine by predictive adaptive control, also, multilevel inverter is an effective solution for increasing power and reducing harmonics of AC waveforms.

Keywords: *PMSM-GPC; adaptive predictive control; structure cascade hybrid inverter.*

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1 Introduction

The hybrid dynamic systems are systems that consist of coupled discrete and continuous components. Any electromechanical system with computerized controller is a hybrid system in general. In the past, the modeling and analysis of hybrid system have been done separately for its discrete and continuous components. The overall system is designed in a rather empirical fashion. Since computer-aided control is becoming more and more significant in modern system design practice, we face a major challenge: the development of intelligent, reliable, robust and safe computer-controlled systems [1–4]. The foundation for modeling and analysis systems must be established formatting [5].

Whatever be the electro-mechanical system it has be ruled by the following equation

$$[S] = [P] * [A] * [C], \tag{1}$$

where: S is an electro-mechanical system, P is a power supply, A is an actuator, C is a control. Hence, to make the system working at its optimum and running under the most efficient ability the parameters of the equations have to meet the following criteria:

$$[S]_r = [P]_p * [A]_s * [C]_o. \tag{2}$$

So, to construct a system that works in optimal status and in very favorable conditions, i.e. that tends towards to ideal, we must construct a highly reliable actuator with a good yield, good stability, and with a perfect power supply and robust control.

We need to add the third term so that the system operates in a closed loop. We explain the three terms of equation (2).

2 Electro-Mechanical System

Our actuator [A] is synchronous permanent magnet motor (PMSM), which has good characteristics such as high power density, high torque to inertia ratio and efficiency, The use of permanent magnet synchronous machine (PMSM) is in constant progress, in particular in the areas where significant performance is needed. The specific contributions of the synchronous machine are in relation to the gain in weight and volume, but also in the dynamic, thanks to more efficient control laws. For these reasons, this type of actuator is strongly preferred in the field of aeronautics [6, 7].

2.1 Machine model PMSM

The equations of electrical machines are described in reference d, q by the following equations [8]:

$$\begin{aligned}
\frac{di_d}{dt} &= -\frac{R}{Ld} i_d + \frac{L_q}{L_d} p\ \Omega\ i_q + \frac{1}{L_d} v_d, \\
\frac{di_q}{dt} &= -\frac{R}{L_q} i_q + \frac{L_d}{L_q} p\ \Omega\ i_d - \frac{\phi_f}{L_q} p\ \Omega + \frac{1}{L_q} v_q, \\
\frac{d\Omega}{dt} &= \frac{3p}{2J}(\phi_f i_q + (L_d - L_q) i_d i_q) - \frac{1}{J} T_r - \frac{F_c}{J}\Omega,
\end{aligned} \tag{3}$$

where v_d, v_q, i_d, i_q represent the stator voltages and currents returned to the axis d and q.

3 The Power Supply Study

The power supply is represented here by Multivel voltage-source inverters, that have been receiving more and more attention in the past few years for high- and medium-power induction-motor (IM) drive applications. Many multilevel inverter configurations and pulse width modulation (PWM) techniques are presented to improve the output voltage harmonic spectrum [9, 10]. Some of the popular multilevel configurations are the neutral point clamped (NPC), series-connected H-bridge, flying capacitor, etc. Although they can be configured for more than two levels, as the number of levels increases, the power circuit and control complexity due to a large number of devices, increase. An optimum topology for multilevel inverters with more than three levels has not been achieved until now [9, 11, 12].

A multilevel inverter has four main advantages over the conventional bipolar inverter. First, the voltage stress on each switch is decreased due to series connection of the switches. Therefore, the rated voltage and consequently the total power of the inverter could be safely increased. Second, the rate of change of voltage (dv/dt) is decreased due to the lower voltage swing of each switching cycle. Third, harmonic distortion is reduced due to more output levels. Forth, lower acoustic noise and electromagnetic interference (EMI) is obtained [13, 14].

Furthermore, the proposed hybrid PDPWM offers better harmonic performance compared to its conventional PWM counterpart [9], applying this technique for supplying the PMSM.

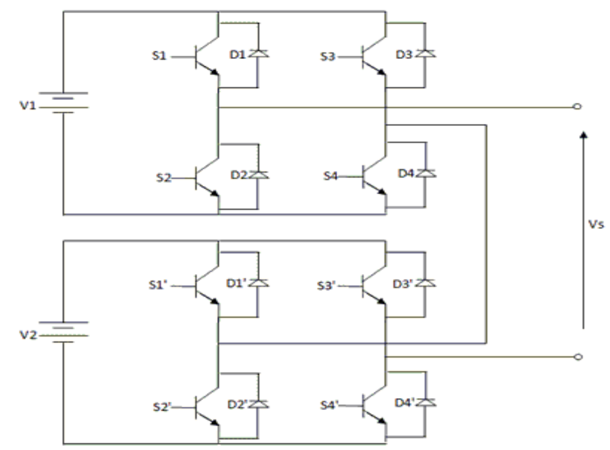

Figure 1: Schematic diagram of the inverter topology used to verify the proposed hybrid modulations.

Multilevel pulse width modulation is based on comparison of sinusoidal reference signal with each carrier to determine the voltage level that the inverter should switch to. Carrier based N level PWM operation consists of N-1 different carriers [13, 15]. The carriers have the same frequency f_c, the same peak to peak amplitude V, and are disposed so that the bands they occupy are contiguous. They are defined as [13]

$$C_i = V\left((-1)^{f(i)} y_c(\omega_c, \varphi) + i - \frac{N}{2}\right), \quad i = 1, \ldots, N-1, \tag{4}$$

where y_c is a normalized symmetrical triangular carrier defined as

$$y_c(\omega_c, \varphi) \quad = \quad (-1)^{[\alpha]} \left((\alpha \ mod \ 2) - 1\right) + \frac{1}{2}, \tag{5}$$

$$\alpha = \frac{\omega_c t + \varphi}{\pi}, \qquad \omega_c = 2\pi f_c, \tag{6}$$

where φ represents the phase angle of y_c, y_c is a periodic function with the period $T_c = 2\pi/\omega c$. It is shown that using symmetrical triangular carrier generates less harmonic distortion at the inverter's output [16].

While the multilevel PWM techniques developed thus far have been extensions of two level PWM methods, the multiple levels in a cascaded inverter offer extra degrees of freedom and greater possibilities in terms of device utilization, state redundancies, and effective switching frequency.

In this paper, we proposed this method [13]. The hybrid multilevel PWM scheme is presented which takes advantage of the special properties available in conventional PWM methods and minimizes switching losses with better harmonic performance. Figure 2 shows the carriers and the reference signals for a five level PWM using PD technique with $mi = 0.8$ and carrier frequency $f_c = 1050hz$ [13].

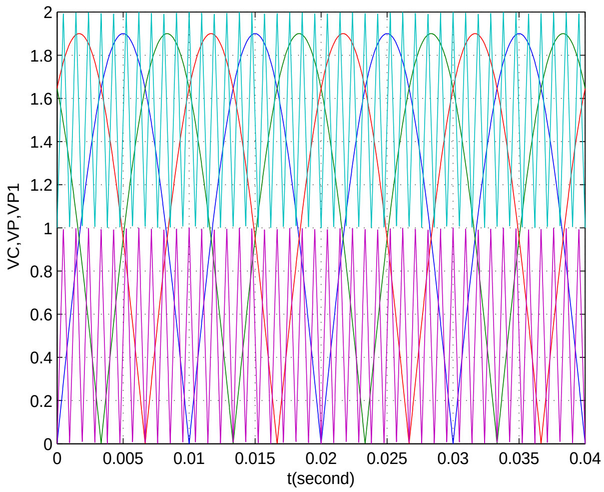

Figure 2: The references and carrier waves (triangular) for a five level inverter.

The proposed hybrid PWM is the combination of low frequency PWM and high frequency SPWM. In each cell of cascaded inverter, the four power devices are operated [13].

At two different frequencies, two being commutated at low frequency, i.e., the fundamental frequency of the output, while the other two power devices are pulse width modulated at high frequency. This arrangement causes the problem of differential switching losses among the switches [13].

An optimized sequential signal is added to the hybrid PWM pulses to overcome this problem. The low and high frequency PWM signals are shown in Figure 3. An optimized hybrid PDPWM method commutates the power switches at high frequency and

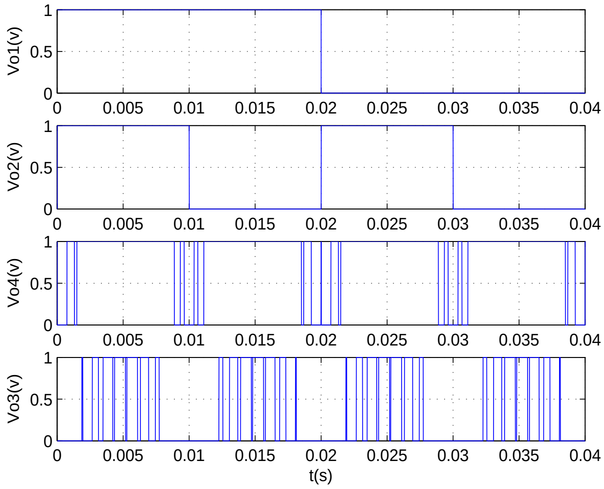

Figure 3: Low and high frequency hybrid PWM pulses at mi=0.8 and $f_c = 1050hz$.

low frequency sequentially. A common sequential signal and low frequency PWM signals are used for all cells in cascaded inverter. A high frequency SPWM for each cell is obtained by the comparison of the rectified modulation waveform with corresponding phase disposition carrier signal. The low frequency PWM signal should be synchronized with the modulation waveform. In Figure 4, the gate pulses are generated by a hybrid PWM controller. This controller is designed to mix the sequential signal, low frequency PWM and high frequency phase disposition sinusoidal PWM and to generate the appropriate gate pulses for cascaded inverter [17].

The previous section has presented the formulation of an optimized hybrid PDPWM switching pattern of a five level inverter. For completeness, the generalized formulation that suits N level inverter is presented [13].

4 Generalized Predictive Controller

The MPC provides various algorithms and the best algorithm is generalized predictive algorithm (GPC). MPC is one of the advanced control strategies, which can forecast the future response of the plant and optimize the control input with the help of a model of the plant. The prediction model will be augmented by the model of state space matrices [18].

In recent years, model predictive control (MPC) seems to be one of the most attractive advanced process control algorithms both in academia and in industry. The combination of new control design concepts in MPC, such as model prediction, receding horizon optimization and real-time correction, makes it possible to yield high performance for control systems. Among various MPC algorithms, general predictive control (GPC) has received particular attention. However, in contrast to the rapid development of MPC in application areas, the theoretical study of MPC properties seems still scarce. Only a little number of studies have been focused on the closed-loop properties of GPC and other MPC algorithms in relationship with the tuning parameters. Among these, excellent results

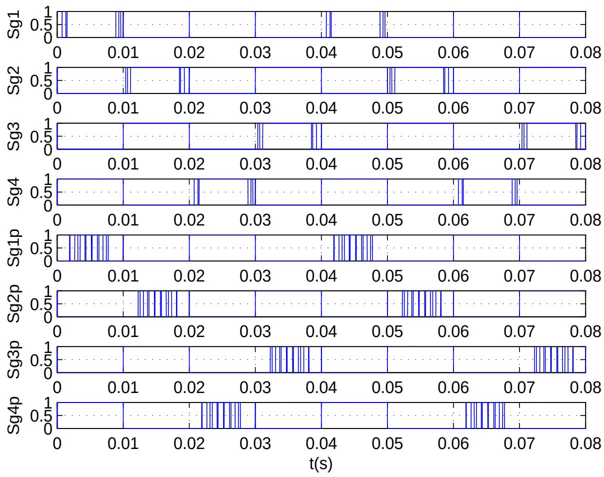

Figure 4: Optimized hybrid PWM switching pattern for five level cascaded multilevel inverter.

have been achieved by Clarke et al [19,20]. In the form of LQ problem, some new results on the GPC properties such as deadbeat control and stability were presented [19].

4.1 Formulation of Generalized Predictive Control

Most single-input single-output (SISO) plants, when considering operation around particular set points and after linearization, can be described by equation (7) [21].

$$A(q^{-1})y(t) = B(q^{-1})u(t) + C(q^{-1})\xi(t), \tag{7}$$

where $u(t)$ and $y(t)$ are the control and output sequence of the plant and $\xi(t)$ is a zero mean white noise. A, B and C are the following polynomials in the backward shift operator q^{-1}:

$$\begin{aligned} A(q^{-1}) &= 1 + a_1 q^{-1} + \cdots + a_{na} q^{-na}, \\ B(q^{-1}) &= q^{-d}\left(b_0 + b_1 q^{-1} + \cdots + b_{nb} q^{-nb}\right), \\ C(q^{-1}) &= 1 + c_1 q^{-1} + \cdots + c_{na} q^{-na}, \end{aligned} \tag{8}$$

where d is the dead time of the system. This model is known as a controller autoregressive moving-average (CARIMA) model. It has been argued that for many industrial applications in which disturbances are non-stationary an integrated CARMA (CARIMA) model is more appropriate. A CARIMA model is given by equation (9) [21]:

$$A(q^{-1})y(t) = B(q^{-1})u(t) + C(q^{-1})\frac{\xi(t)}{\Delta(q^{-1})} \tag{9}$$

with $\Delta(q^{-1}) = 1 - q^{-1}$. For simplicity, polynomial C in equation (9) is chosen to be 1. Notice that if C^{-1} can be truncated it can be absorbed into A and B.

From the previous equation (9), a polynomial optimal predictor is designed in the following form:

$$y(t+j) = [F_j(q^{-1})y(t) + H_j(q^{-1})\Delta u(t-1)] + [G_j(q^{-1})\Delta u(t+1) + J_j(q^{-1})\xi(t+j)], \quad (10)$$

where G_j, F_j, H_j, J_j are the terms representing respectively the future, present, past, and the term related to disturbance. The first bracketed expression in equation (10) represents the free response. The criterion is a weighted sum of square predicted future errors and square control signal increments.

Cost Function

GPC algorithm consists of applying a control sequence that minimizes a cost function of the form given in equation (11) [21]:

$$j = \sum_{N_1}^{N_2} (\hat{y}(t+j) - w(t+j))^2 + \lambda \sum_{N_1}^{N_u} \Delta u(t+j-1)^2. \quad (11)$$

Under the hypothesis

$$\Delta u(t+j) = 0 \quad \forall j \geqslant N_u \quad (12)$$

with: $w(t+j)$ reference applied at time $t+j$, $\hat{y}(t+j)$ predicted output at time $t+d$, $u(t+j-1)$ command increment at the instant $t+j-1$.

The relation (12) indicates that when the step of prediction j reaches the value fixed for the control horizon N_u, the change order will be canceled and therefore the future order will stabilize. This hypothesis will eventually simplify the control calculation.

The criterion requires the definition of four setting parameters, where N_u is the control horizon, N_1 is the minimum prediction horizon, N_2 is the maximum prediction horizon and λ are control weighting factors.

The control law is obtained by minimizing the previous criterion $\dfrac{\partial J}{\partial u} = 0$ such as

$$\widetilde{U} = M[w - \mathrm{if}(q^{-1})y(t) - \mathrm{ih}(q^{-1})\Delta u(t-1)]. \quad (13)$$

By reason of certain benefits introduced by the polynomial structure, we chose to formulate the control law in the canonical form of an RST controller.

Conventionally, in predictive control, only the first value of the sequence, equation (13) is finally applied to the system in agreement with the strategy of receding horizon, the whole process being effected again at the period of next sampling

$$\Delta u_{opt}(t) = -m_1' \left[\mathrm{if}(q^{-1})y(t) + \mathrm{ih}(q^{-1})\Delta u(t-1) - w\right] \quad (14)$$

with m_1': first row of the matrix M.

The GPC controller is implemented in a form of the RST by difference equation:

$$S(q^{-1})\Delta(q^{-1})u(t) = -R(q^{-1})y(t) + T(q)w(t). \quad (15)$$

This provides by identification the three polynomials R, S and T constituting the equivalent linear regulator [18]:

$$\begin{aligned} S(q^{-1}) &= 1 + m_1'\mathrm{ih}(q^{-1})q^{-1}, & d^\circ\left[S(q^{-1})\right] &= d^\circ\left[B(q^{-1})\right], \\ R(q^{-1}) &= m_1'\mathrm{if}(q^{-1})q^{-1}, & d^\circ\left[R(q^{-1})\right] &= d^\circ\left[A(q^{-1})\right], \\ T(q) &= m_1'\left[q^{N_1}\ldots q^{N_2}\right]', & d^\circ\left[T(q)\right] &= N_2, \end{aligned} \quad (16)$$

with:

$$\begin{aligned} \text{if}(q^{-1}) &= \left[F_{N_1}(q^{-1}) \dots F_{N_2}(q^{-1})\right]', \\ \text{ih}(q^{-1}) &= \left[H_{N_1}(q^{-1}) \dots H_{N_2}(q^{-1})\right]', \\ \widetilde{U} &= \left[\Delta u(t) \dots \Delta u(t+N_u-1)\right]', \\ \hat{y} &= \left[\hat{y}(t+N_1) \dots \hat{y}(t+N_2)\right]', \\ w &= \left[w(t+N_1) \dots w(t+N_2)\right]' \end{aligned} \tag{17}$$

$$G = \begin{bmatrix} g_{N_1}^{N_1} & g_{N_1-1}^{N_1} & \cdots & \cdots \\ g_{N_1+1}^{N_1+1} & g_{N_1}^{N_1+1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ g_{N_2}^{N_2} & g_{N_2-1}^{N_2} & \cdots & g_{N_2-N_u+1}^{N_2} \end{bmatrix}.$$

4.2 Reformulation of GPC control with adaptive control

We start with the definition of the performance error. Consider first the following regressor [22]. The starting point of this reformulation is constituted of setting equation presented in the previous paragraph, in particular, relationships to obtain the optimal control sequence.

4.3 Vectors parameters and regressor

The control law equation (13) may be transcribed in the form of the following matrix:

$$Mw = \theta' \Phi(t) \tag{18}$$

which involves the matrix of parameters θ of dimension $(n_a + n_b + N_u + 1) \times N_u$ with n_a and n_b being degrees of $A(q^{-1})$ and $B(q^{-1})$, respectively,

$$\theta' = [M\text{if} \ \ |_{Nu} \ \ M\text{ih}], \tag{19}$$

where if and ih matrices are formed of polynomial coefficients contained in $\text{if}(q^{-1})$ and $\text{ih}(q^{-1})$, and the following vector called regressor dimension $(n_a + n_b + N_u + 1)$:

$$\Phi(t) = \left[y(t) \dots y(t-n_a) \ \ \widetilde{u}' \ \ \Delta u(t-1) \dots \Delta u(t+n_b)\right]. \tag{20}$$

The matrix of parameters θ contains, on its first line, the coefficients of the polynomials R and S'. Indeed, from equation (14), the polynomial $m_1'\text{if}(q^{-1})$ corresponds to R and $m_1'\text{ih}(q^{-1})q^{-1}$ corresponds to S'. The regressor $\Phi(t)$ is the output vector and past orders including unknown commands $\widetilde{u}$ of dimension N_u.

We also note that when $N_u = 1$, the matrix θ is reduced to a vector including direct polynomial coefficients R and S'.

4.4 The method for updating

The matrix controller parameters can be updated as most of strategies. Here we can mention the gradient method and the recursive least squares method

$$\hat{\theta}(t+1) = \hat{\theta}(t) + F\phi(t)\varepsilon^0(t+1) \tag{21}$$

with the use of the algorithm of Trace constant for determining the adaptation gain at time t. To obtain a recursive algorithm, we consider the estimate $\hat{\theta}(t+1)$.

After development, it follows the A.A.P:

$$\hat{\theta}(t+1) = \hat{\theta}(t) + F(t+1)\phi(t)\varepsilon^0(t+1) \tag{22}$$

with

$$F(t+1) = F(t) - \frac{F(t)\phi(t)\phi(t)^T F(t)}{1 + \phi(t)^T F(t)\phi(t)}, \tag{23}$$

where $\hat{\theta}$ is the vector of the estimated parameters and $F(t+1)\phi(t)\varepsilon^0(t+1)$ represents the correction term, F is the adaptation gain, ϕ is the vector of observations (or measures) and ε is the prediction error (error adaptation), that is to say the difference between the measured process output and the predicted output [22].

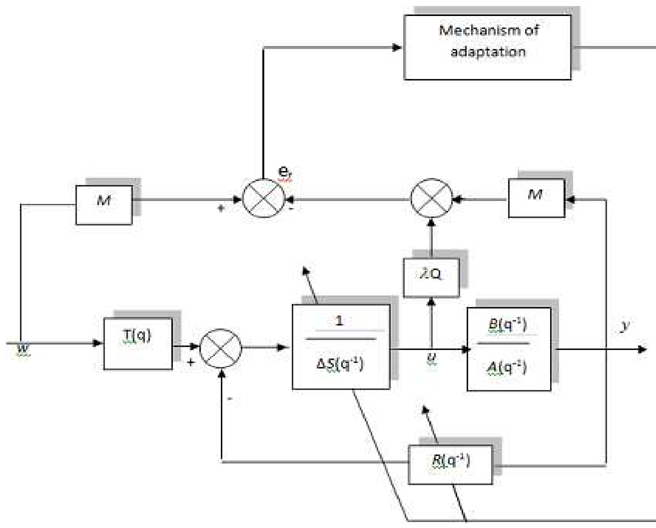

Figure 5: Structure equivalent of direct adaptive predictive control, control loop of RST and adaptation mechanism.

5 Simulation Results and Discussion

Figure 6 represents the overall structure of speed control of PMSM fed by a hybrid structure cascade five-level inverter, using the adaptive predictive control. To test the effectiveness of the proposed control strategy for adjusting the speed, we have used numerical simulation in the following cases:

- Step response of speed.
- Start unloading and then applying a torque resistant.
- Reverse speed.

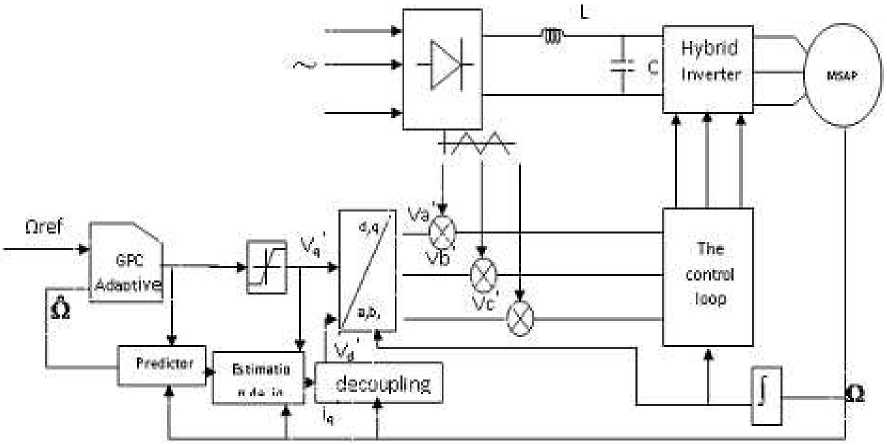

Figure 6: Global structure for regulating the PMSM.

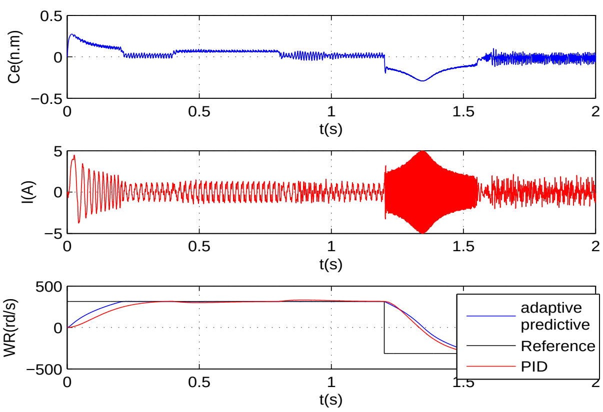

Figure 7: PMSM performance of the machine fed by the hybrid inverter.

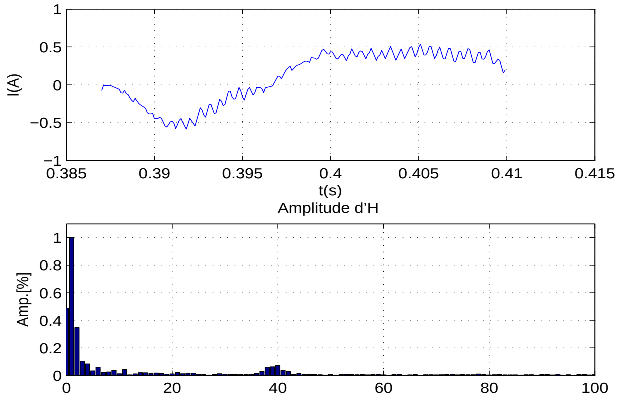

Figure 8: Top trace is phase current (ia). Second trace is normalized harmonic spectrum of phase current (technique of PI controller fed by five level inverter (NPC)).

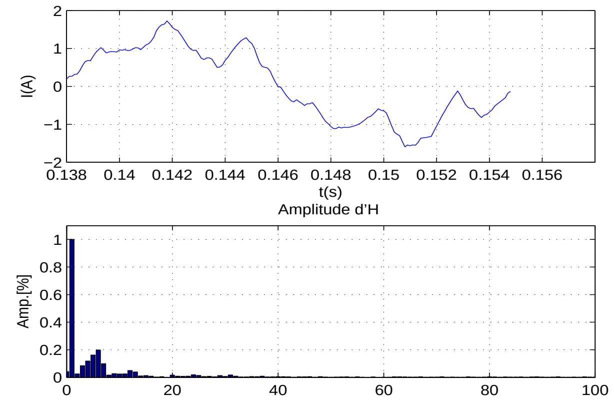

Figure 9: Top trace is phase current (ia). Second trace is normalized harmonic spectrum of phase current (technique of adaptive predictive control fed by five level inverter (NPC)).

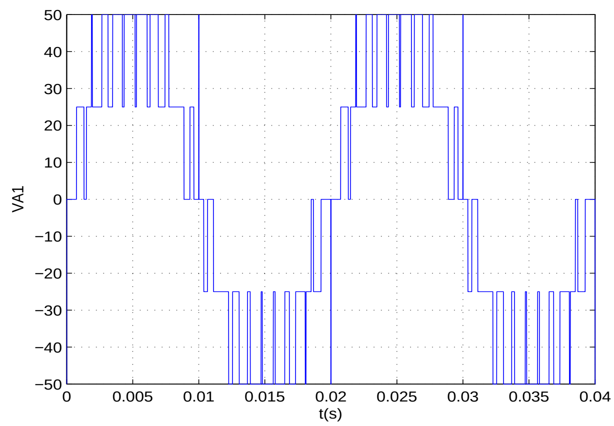

Figure 10: Output phase voltage waveform.

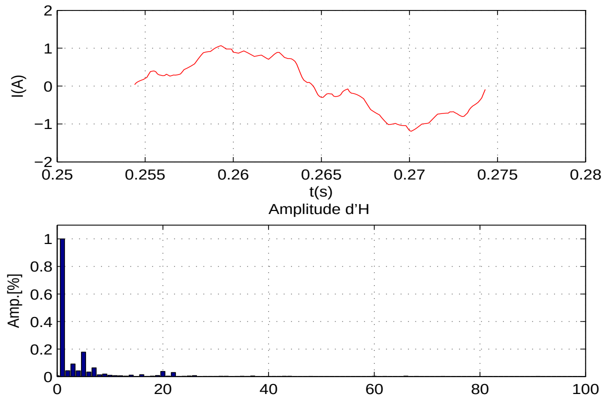

Figure 11: Top trace is phase current (ia). Second trace is normalized harmonic spectrum of phase current (technique of adaptive predictive control fed by hybrid inverter).

Table 1. Comparison of different strategies proposed.

Controller With power supply	PI with five livel (NPC)	adaptive Predictive with five livel(NPC)	adaptive Predictive with hybrid inverter.
Rotor speed	314(rd/s) at 0.4sec	314(rd/s) at 0.2 sec	314(rd/s) at 0.2 sec
THD	37.95	30.85	22.41

5.1 Discussion of the results of adaptive predictive control

As shown in Figure 7 it appears that for a reference of 314 rd/s during unloaded starting, the steady state is achieved at $t = 0.2s$, which is a very appreciable response time, compared with the conventional PID controller. The application of the load between $t = 0.4s$ and $0.8s$ causes a slight loss of speed that is quickly restored. Also note that this load has no influence on the direct current component, indicating that the vector control is effective. By analyzing the graph of the harmonic spectrum of the phase current, we notice that there is a very big improvement in the pace of the phase current compared to a five-level inverter. Finally, when reversing the speed reference we observe an excessive increase in the starting current, which is justified by the large variation subjected to the machine (from 314rd /s to -314rd /s). The time of the establishment of the speed increased slightly to reach $t = 0.34s$. However, upon reversal of the reference, we see an appearance of exceeding in terms of the response, so a runaway effect occurs, which led us to introduce an anti-windup device. The latter is not enough to limit the speed so it is recommended to act on the GPC parameters to remedy this problem.

5.2 Influence of the GPC parameters

As mentioned in references [23], for maintaining N_1, N_u, and λ_{opt} to the values 1, 1 and trace (G'G) respectively, and varying N_2 to reconcile between a rapid response and an acceptable startup current, it is necessary to find a set of parameters that can meet these requirements. To do this, the influence of parameters on the magnitudes of the PMSM is analyzed through the following figure: It appears that a strong increase for N_2 results

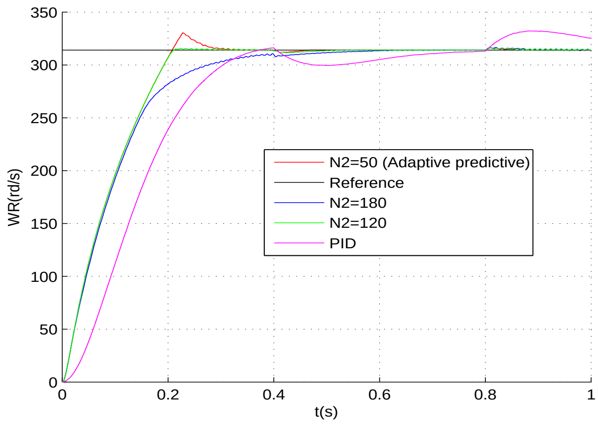

Figure 12: Parameter sets.

in a slow system response, while too large a decrease results in a large overshoot about the set-point (runaway). Note that when N_2 increases the response time increases. This leads to a supplementary computation time which, to be reduced, must be accompanied by an anti-windup device used primarily to limit both the speed around the set-point and the admissible starting current, in our case the best choice for N_2 or $N_{2_{optimum}} = 120$. It is clear that the time to response is very large in the case of conventional PID controller even if $N_2 = 180$, as well as the rejection of disturbance is very good in the adaptive predictive control (see Figure 12).

Also, the right choice of N_2 does not influence the response time only, but also the shape of the phase current. The following table clearly shows the THD of each value of N_2.

Table 2. Comparison of the THD for different values of N_2:

N_2:maximum prediction horizon	180	120	50
THD	31.78	22.41	48.75

6 Conclusion

The association between predictive control that has the ability to anticipate future events and can take control actions accordingly and the adaptive control whose main role is to eliminate the effect of disturbances in order to control better the system, relatively to the conventional controller. In addition, the proposed hybrid inverter gives better harmonic performance compared to its conventional homologue PWM. The simulation results show a vast improvement in the current waves and good agreement with the adaptive predictive control used to control the PMSM. Despite the introduction of the load and the inversion of the set-point, this system is characterized by a better control of the MASP transient regime, which conducts to good response times with an assured decoupling and a fast enough dynamic rejection of disturbances. With a good choice of the actuator (PMSM) and a robust control (adaptive predictive) and with a good fed (hybrid inverter) like ours, we could check the first formula of our paper. Therefore our system can provide superior performances in terms of increased efficiency and reduced noise.

References

[1] Tlemçani, A., Sebaa, K. and Henini, N. Indirect adaptive fuzzy control of multivariable nonlinear systems class with unknown parameters. *Nonlinear Dynamics and Systems Theory* **14** (2) (2014) 162–174.

[2] Hamidia, F., Larabi, A., Tlemçani, A. and Boucherit, M. S. AIDTC Techniques for Induction Motors. *Nonlinear Dynamics and Systems Theory* **13** (2) (2013) 147–156.

[3] Henini, N., Nezli, L., Tlemçani, A. and Mahmoudi, M. O. Improved Multimachine Multiphase Electric Vehicle Drive System Based on New SVPWM Strategy and Sliding Mode - Direct Torque Control. *Nonlinear Dynamics and Systems Theory* **11** (4) (2011) 425–438.

[4] Benmansour, K., Tlemçani, A., Djemai, M. and De Leon, J. A New Interconnected Observer Design in Power Converter: Theory and Experimentation. *Nonlinear Dynamics and Systems Theory* **10** (3) (2010) 211–224.

[5] Zhang, Y. and Mackworth, K. *Logical Foundations for Cognitive Agents* (1999) 370–396.

[6] Fadel, M. and Linh Nguyen, N. Different Solutions of Predictive Control for tow Synchronous Machines in Parallel. In: *International symposium on IEEE* (2013) 1–7.

[7] Louis, J.P. *Control of Synchronous Motors.* London, Wiley-ISTE, 2011.

[8] Nguyen, D. Dutta, R. and Rahman, M. F. Investigating characteristics of a concentrated-winding interior permanent magnet synchronous machine for sensorless direct torque control. In: *International Symposium on IEEE*, 2013.

[9] Sivakumar, K. Ramchand, R. and Gopakumar, K. A Hybrid Multilevel Inverter Topology for an Open-End Winding Induction-Motor Drive Using Two-Level Inverters in Series With a Capacitor-Fed H-Bridge Cell. *IEEE transactions on industrial electronics* **57** (11) (2010) 3707–3714.

[10] Rodriguez, J., Bernet, S., Wu, B., Pontt, J.O. and Kouro, S. Multilevel voltage-source-converter topologies for industrial medium-voltage drives. *IEEE Trans. Ind. Electron.* **54** (6) (2007) 2930–2945.

[11] Busquets-Monge, S. Ortega, J. D. Bordonau, J. Beristain, J. A. and Rocabert, J. Closed-loop control of a three-phase neutral-point-clamped inverter using an optimized virtual-vector-based pulse width modulation. *IEEE Trans. Ind. Electron.* **55** (5) (2008) 2061–2071.

[12] Nabae, A. Takahashi, I. and Agaki, H. A new neutral point-clamped PWM inverter. *IEEE Trans. Ind. Appl.* **IA-17** (5) (1981) 518–523.

[13] Govindaraju, C. and Baskaran, K. Optimized Hybrid Phase Disposition PWM Control Method for Multilevel Inverter. *International Journal of Recent Trends in Engineering* **1** (3) (2010) p. 36–40.

[14] Rodriguez, J., Lai, J.S. and Peng, F.Z. multilevel inverter: Asurvey of topologies, controls and applications. *IEEE Trans. Ind. Electron.* **49** (4) (2002) 724–738.

[15] Mcgrath, B.P. and Holmes, D.G. Multicarrier PWM strategies for multilevel inverters. *IEEE Trans.Ind.Electron.* **49** (4) (2002) 858–867.

[16] Hamman, J. and Van Der Merwe, F.S. Voltage harmonics generated by voltage fed inverters using PWM natural sampling. *IEEE Trans. Power Electron* **PE-3** (3) (1988) 297–302.

[17] Hema Maheshwari, V. Soundradevi, G. Karthikeyan, B. and Sundari, G. A 11 level cascaded multilevel inverter using hybrid modulation technique. *International Journal of Emerging Trends in Engineering and Development* **2** (2013).

[18] Manimaran, M., Arumugam, A. and Ramkumar, K. Parameter Estimation based Optimal control for a Bubble Cap Distillation Column. *International Journal of Chem Tech Research IJCRGG* **6** (1) (2014) 790–799.

[19] Zhang, J. and Xi, Y. *Study on the Closed-loop Properties of GPC.* Chinese National Science Foundation (1991) 1–13.

[20] Clarke, D.W. and Mohtadi, C. Property of Generalized Predictive Control. *Automatica* **25** (6) (1989) 859–875.

[21] Chidrawar, S. and Patre, B. Generalized Predictive Control and Neural Generalized Predictive Control. *Leonardo Journal of Sciences* **13** (2008) 133–152.

[22] Dumur, D. Contribution L'analyse et au Développement D'algorithmes de Commande Prédictive, Rapport de Synthese (Habilitation à diriger des recherches), Universite Paris XI, 2002.

[23] Dumur, D. and Boucher, P. A review introduction to linear GPC and applications. *Journal A* **39** (4) (1998) 21–35.

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Approximate Controllability of Nonlinear Fractional Impulsive Stochastic Differential Equations with Nonlocal Conditions and Infinite Delay

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Abstract: This paper is concerned with the approximate controllability of nonlinear fractional impulsive stochastic differential equations with nonlocal conditions and infinite delay in Hilbert spaces. By using the Krasnoselskii-Schaefer-type fixed point theorem and stochastic analysis theory, some sufficient conditions are given for the approximate controllability of the system. At the end, an example is given to illustrate the application of our result.

Keywords: *approximate controllability; fixed point principle; fractional impulsive stochastic differential equations; mild solution; nonlocal conditions.*

Mathematics Subject Classification (2010): 65C30, 93B05, 34K40, 34K45.

1 Introduction

The controllability is one of the fundamental concepts in linear and nonlinear control theory, and plays a crucial role in both deterministic and stochastic control systems (see e.g. Zabczyk, [27]).The controllability of nonlinear systems represented by evolution equations or inclusions in abstract spaces and qualitative theory of fractional differential equations has been extensively considered in many publications and monographs, an

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extensive list of these publications can be found in Mahmudov [16] and the references contained therein.

On the other hand, the study of stochastic differential equations has attracted great interest due to their applications in characterizing many problems in physics, biology, chemistry, mechanics, and so on (see [6,7,9,12,17]) and the references contained therein). In practice, deterministic systems often fluctuate due to environmental noise. So it is important and necessary for us to discuss the stochastic control problems.

The problem with nonlocal condition, which is a generalization of the problem of classical condition, was motivated by physical problems. The pioneering work on nonlocal conditions is due to Byszewski (see [3–5]). Since it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems, stochastic differential equations with nonlocal conditions were studied by many authors and some basic results on nonlocal problems have been obtained. Slama and Boudaoui [26] obtained sufficient conditions for the existence of mild solutions for the fractional impulsive stochastic differential equation with nonlocal conditions and infinite delay. (For more details see [1, 24] and the references contained therein).

The approximate controllability of stochastic or deterministic systems has received extensive attention where a pioneering work has been reported by Bashirov and Mahmudov [2]. Mahmudov [15] investigated the controllability of infinite dimensional linear stochastic systems, and in [10] Dauer and Mahmudov extended the results to semilinear stochastic evolution equations with finite delay. Sakthivel et al. [23] studied the approximate controllability of nonlinear deterministic and stochastic evolution systems with unbounded delay in abstract spaces. Kumar and Sukavanam [13] established sufficient conditions of the approximate controllability for a class of fractional order semilinear control systems with bounded delay. Shukla et al. [25] studied the approximate controllability of semilinear stochastic control system with nonlocal conditions in a Hilbert space, the results are obtained by using Sadovskii's fixed point theorem.

Recently, the approximate controllability of fractional stochastic differential systems has been investigated. Sakthivel et al. [22] studied a class of control systems described by nonlinear fractional stochastic differential equations in Hilbert spaces. Sufficient conditions for approximate controllability of fractional stochastic differential equations are formulated by using fixed point technique, fractional calculus, and stochastic analysis technique. Rajiv Ganthi and Muthukumar [20] discussed the approximate controllability of fractional stochastic integral equation with finite delays in Hilbert spaces, and the results are obtained by using the assumption that the corresponding linear integral equation is an approximate controllable and a stochastic version of the Banach fixed point theorem. Muthukumar and Rajivganthi [18] studied the approximate controllability of fractional order neutral stochastic integro-differential system with nonlocal conditions and infinite delay in Hilbert spaces under the assumptions that the corresponding linear system is approximately controllable. Guendouzi [11] discussed the existence and approximate controllability for impulsive fractional-order stochastic infinite delay integro-differential equations in Hilbert space, sufficient conditions for the approximate controllability of impulsive fractional stochastic system are derived by using Krasnoselskii's fixed point theorem with stochastic analysis theory. Zang and Li [28] studied the approximate controllability of fractional impulsive neutral stochastic differential equations with nonlocal conditions and infinite delay. Sufficient conditions are given for the approximate controllability of the system by using the Krasnoselskii-Schaefer-type fixed point theorem and stochastic analysis theory.

For the best of our knowledge, there is no work reported on approximate controllability of nonlinear fractional impulsive stochastic differential equations with nonlocal conditions and infinite delay. Motivated by this consideration, in this paper we will study the approximate controllability of nonlinear fractional impulsive stochastic differential equations with nonlocal conditions and infinite delay in Hilbert space. Our approach is based on the fixed point theorem. The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries such as definitions of fractional calculus and some useful lemmas. In Section 3, we prove our main results. Finally in Section 4, an example is given to demonstrate the application of our results.

2 Preliminaries and Basic Properties

In this section, we introduce some notations and preliminary results, needed to establish our results. Throughout this paper, $\mathbb{H}, \mathbb{U}$ are two separable Hilbert spaces and $L(\mathbb{U}, \mathbb{H})$ is the space of bounded linear operators from $\mathbb{U}$ into $\mathbb{H}$. For convenience, we will use the same notation $\| \, . \, \|$ to denote the norms in $\mathbb{H}, \mathbb{U}$ and $L(\mathbb{U}, \mathbb{H})$, and use $\langle ., . \rangle$ to denote the inner product of $\mathbb{H}$ and $\mathbb{U}$ without any confusion. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions (i.e., it is increasing and right continuous, while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$). Let $W = (W_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the covariance operator Q such that $TrQ < \infty$. Let $W = W(t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the covariance operator Q, that is

$$E\langle W(t), x\rangle\langle W(s), y\rangle = (t \wedge s)\langle Qx, y\rangle \quad \forall x, y \in \mathbb{U} \quad \text{and} \quad t, s \in [0, T],$$

where Q is a positive, self-adjoint, trace class operator on $\mathbb{U}$.

Let $\mathscr{L}_2^0 = \mathscr{L}_2(\mathbb{U}, \mathbb{H})$ be the space of all Hilbert-Schmidt operators from $\mathbb{U}$ to $\mathbb{H}$ with the inner product $< \varphi, \psi >_{\mathscr{L}_2^0} = Tr[\varphi Q \psi^*]$. We consider the following fractional stochastic impulsive integro-differential systems with nonlocal conditions:

$$\begin{cases} D_t^\alpha x(t) & = Ax(t) + Bu(t) + f(t, x_t, B_1 x(t)) \\ & \quad +\sigma(t, x_t, B_2 x(t))dW(t), t \in J = [0, T], T > 0, t \neq t_k, \\ \Delta x(t_k) & = I_k(x(t_k^-)), \quad k = 1, \cdots, m, \\ x(0) + g(x) & = x_0 = \phi, \quad \phi \in B_h, \end{cases} \tag{1}$$

where D_t^α is the Caputo fractional derivative of order α, $0 < \alpha < 1$, the state variable $x(.)$ takes the value in the separable Hilbert space $\mathbb{H}$; $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous semigroup of a bounded linear operators $T(t), t \geq 0$ in the Hilbert space $\mathbb{H}$. The control function $u(.)$ is given in $L^2(J; \mathbb{U})$, $\mathbb{U}$ is a Hilbert space; B is a bounded linear operator from $\mathbb{U}$ into $\mathbb{H}$. The history $x_t : (-\infty, 0] \to \mathbb{H}, x_t(\theta) = x(t + \theta), \quad \theta \leq 0$ belongs to an abstract phase space $\mathcal{B}_h$; $f : J \times \mathcal{B}_h \times \mathbb{H} \to \mathbb{H}$, $\sigma : J \times \mathcal{B}_h \times \mathbb{H} \to \mathscr{L}_2^0$ and $g : B_h \to \mathbb{H}$ are appropriate functions to be specified later; $I_k : \mathbb{H} \to \mathbb{H}$, $(k = 1, 2, \cdots, m)$, are appropriate functions. The terms $B_1 x(t)$ and $B_2 x(t)$ are given by $B_1 x(t) = \int_0^t K(t, s)x(s)ds$ and $B_2 x(t) = \int_0^t P(t, s)x(s)ds$ respectively, where $K, P \in C(D, \mathbb{R}^+)$ are the set of all positive continuous functions on $D = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$. Here $0 = t_0 \leq t_1 \leq \cdots \leq t_m \leq t_{m+1} = T$, $\Delta x(t_k) = I_k(x(t_k^-)) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively. The initial data $\phi = \{\phi(t); t \in (-\infty, 0]\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{B}_h$-valued random variable independent of $W(t)$ with finite second moments.

Now, we present the abstract space phase $\mathcal{B}_h$. Assume that $h : (-\infty, 0] \to (0, +\infty)$ with $l = \int_{-\infty}^{0} h(t)dt < \infty$ is a continuous function. We define the abstract phase space $\mathcal{B}_h$ by

$$\mathcal{B}_h := \Big\{ \phi : (-\infty, 0] \to \mathbb{H}, \text{for any} \quad a > 0, (E \mid \phi(\theta \mid^2)^{\frac{1}{2}} \\ \text{is bounded and measurable function on} \\ [-a, 0] \quad \text{and} \quad \int_{-\infty}^{0} h(s) \sup_{s \leq \theta \leq 0} (E \mid \phi(\theta \mid^2)^{\frac{1}{2}} < +\infty \Big\}.$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} := \int_{-\infty}^{0} h(s) \sup_{s \leq \theta \leq 0} (E \mid \phi(\theta) \mid^2)^{\frac{1}{2}}, \phi \in \mathcal{B}_h,$$

then $(\mathcal{B}_h, \|.\|_{\mathcal{B}_h})$ is a Banach space [19,21].

Now we consider the space

$$\mathcal{B}_b := \{ x : (-\infty, T] \to \mathbb{H}, \quad \text{such that} \quad x|_{J_k} \in C(J_k, \mathbb{H}) \\ \text{and there exist} \quad x(t_k^+), \quad \text{and} \quad x(t_k^-) \\ \text{with} \quad x(t_k) = x(t_k^-), x_0 = \phi \in \mathcal{B}_h, k = 1, \cdots, m \big\},$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 0, 1, 2, \cdots, m$. We endow a seminorm $\|.\|_{\mathcal{B}_b}$on $\mathcal{B}_b$, it is defined by

$$\|x\|_{\mathcal{B}_b} = \|\phi\|_{\mathcal{B}_h} + \sup_{0 \leq s \leq T} (E\|x(s)\|^2)^{\frac{1}{2}}, x \in \mathcal{B}_b.$$

We recall the following lemma.

Lemma 2.1 *[21] Assume that $x \in \mathcal{B}_b$; then for $t \in J, x_t \in \mathcal{B}_h$. Moreover*

$$l(E\|x(t)\|^2)^{\frac{1}{2}} \leq l \sup_{s \in [0,t]} E\|x(s)\|^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{B}_h},$$

where $l = \int_{-\infty}^{0} h(s)ds < \infty$.

Definition 2.1 [8] The Caputo derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$, which is at least n-times differentiable can be defined as

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = I_a^{(n-\alpha)} \left(\frac{d^n f}{dt^n} \right)(t) \tag{2}$$

for $n - 1 \leq \alpha < n, n \in \mathbb{N}$. If $0 < \alpha \leq 1$, then

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} \left(\frac{df(s)}{ds} \right) ds. \tag{3}$$

Obviously, the Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$L\{D_t^{\alpha} f(t); \lambda\} = \lambda^{\alpha} \widehat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} f^{(k)}(0) \quad n - 1 \leq \alpha < n.$$

Definition 2.2 The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (s-t)^{\alpha-1}f(s)ds \tag{4}$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where Γ is the gamma function.

Definition 2.3 A stochastic process $x : J\times\Omega \to \mathbb{H}$ is called a mild solution of the system (1) if

(i) $x(t)$ is measurable and $\mathcal{F}_t$-adapted, for each $t \geq 0$;

(ii) $x(t) \in \mathbb{H}$ has càdlàg paths on $t \in [0,T]$ a.s., and satisfies the following integral equation

$$\begin{array}{rl} x(t) = & T_\alpha(t)(\phi(0)-g(x)) + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)Bu(s)ds \\ & +\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s,x_s,B_1x(s))ds \\ & +\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\sigma(s,x_s,B_2x(s))dW(t) \\ & +\sum_{0<t_k<t} T_\alpha(t-t_k)I_k(x(t_k^-)), \quad t\in J; \end{array} \tag{5}$$

(iii) $x_0 = \phi \in \mathcal{B}_h$ on $(-\infty,0]$ satisfying $\|\phi\|_{\mathcal{B}_h} < \infty$, where

$$T_\alpha(t) = \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)d\theta, \quad S_\alpha(t) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\varpi(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta\in(0,\infty),$$

ξ_α is a probability density function defined on $(0,\infty)$, that is,

$$\xi_\alpha \geq 0, \quad \theta\in(0,\infty), \quad \text{and} \quad \int_0^\infty \xi_\alpha(\theta)d\theta = 1.$$

Lemma 2.2 *[29] The operators T_α and S_α have the following properties:*

(i) For any fixed $t \geq 0$, $T_\alpha(t)$ and $S_\alpha(t)$ are linear and bounded operators, i.e., for any $x \in X$,

$$\|T_\alpha(t)x\| \leq M\|x\|, \quad \|S_\alpha(t)x\| \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|x\|.$$

(ii) $\{T_\alpha(t), t\geq 0\}$ and $\{S_\alpha(t), t \geq 0\}$ are strongly continuous, which means that for every $x\in\mathbb{H}$ and for $0 \leq t^{'} < t^{''} \leq T$, we have

$$\|T_\alpha(t^{''})x - T_\alpha(t^{'})x\| \to 0 \quad \textit{and} \quad \|S_\alpha(t^{''})x - S_\alpha(t^{'})x\| \to 0, \quad \textit{as} \quad t^{'} \to t^{''}.$$

(iii) For every $t \geq 0$, $T_\alpha(t)$ and $S_\alpha(t)$ are also compact operators if $T(t)$ is compact for every $t > 0$.

In order to study the approximate controllability for the fractional control system (1), we introduce the following linear fractional differential system

$$\begin{cases} D_t^\alpha x(t) = \quad Ax(t) + Bu(t), \quad t \in J, \\ x(0) = x_0. \end{cases} \tag{6}$$

The controllability operator associated with (6) is defined by

$$\Gamma_0^T = \int_O^T (T-s)^{\alpha-1} S_\alpha(t-s) BB^* S_\alpha^*(T-s) ds,$$

where B^* and S_α^* denote the adjoint of B and S_α, respectively.

Let $x(T;\phi,u)$ be the state value of (1) at terminal time T, corresponding to the control u and the initial value ϕ. Denote by $R(T,\phi) = \{x(T;\phi,u) : u \in L^2(J,\mathbb{U})\}$ the reachable set of system (1) at terminal time T, its closure in $\mathbb{H}$ is denoted by $\overline{R(T,\phi)}$.

Definition 2.4 The system (1) is said to be approximately controllable on J if $\overline{R(T,\phi)} = L^2(\Omega,\mathbb{H})$.

Lemma 2.3 *[14] The linear fractional control system* (6) *is approximately controllable on J if and only if $\lambda(\lambda I + \Gamma_0^T) \to 0$ as $\lambda \to 0^+$ in the strong operator topology.*

Lemma 2.4 *[29] (Krasnoselskii's fixed point theorem) Let E be a Banach space, let $\hat{E}$ be a bounded closed and convex subset of E, and let F_1, F_2 be maps of $\hat{E}$ into E such that $F_1x + F_2y \in \hat{E}$ for every pair $x, y \in \hat{E}$. If F_1 is a contraction and F_2 is completely continuous, then the equation $F_1x + F_2x = x$ has a solution on $\hat{E}$.*

3 Main Results

In this section, we formulate sufficient conditions for the approximate controllability of system (1). For this purpose, we first prove the existence of solutions for system (1). Second, in Theorem 3.2, we shall prove that system (1) is approximately controllable under certain assumptions.

In order to establish the results, we impose the following conditions

(H1) $f : J \times \mathcal{B}_h \times \mathbb{H} \to \mathbb{H}$ is continuous and there exist μ_1, $\mu_2 > 0$ such that

$$E\|f(t,\gamma,x) - f(t,\psi,y)\|_{\mathbb{H}}^2 \le \mu_1\|\gamma-\psi\|_{\mathcal{B}_h}^2 + \mu_2 E\|x-y\|_{\mathbb{H}}^2,$$

and there exist two continuous functions $\mu_1, \mu_2 : J \to (0,\infty)$ such that

$$E\|f(t,\psi,x)\|_{\mathbb{H}}^2 \le \mu_1(t)\|\psi\|_{\mathcal{B}_h}^2 + \mu_2(t) E\|x\|_{\mathbb{H}}^2, \quad (t,\psi,x) \in J \times \mathcal{B}_h \times \mathbb{H},$$

where $\mu_1^* = \sup_{s\in[0,t]} \mu_1(s)$ and $\mu_2^* = \sup_{s\in[0,t]} \mu_2(s)$.

(H2) There exist ν_1, $\nu_2 > 0$ such that

$$E\|\sigma(t,\gamma,x) - f(t,\psi,y)\|_{\mathscr{L}_2^0}^2 \le \nu_1\|\gamma-\psi\|_{\mathcal{B}_h}^2 + \nu_2 E\|x-y\|_{\mathbb{H}}^2,$$

and there exist two continuous functions $\nu_1, \nu_2 : J \to (0,\infty)$ such that

$$E\|\sigma(t,\psi,x)\|_{\mathscr{L}_2^0}^2 \le \nu_1(t)\|\psi\|_{\mathcal{B}_h}^2 + \nu_2(t) E\|x\|_{\mathbb{H}}^2, \quad (t,\psi,x) \in J \times \mathcal{B}_h \times \mathscr{L}_2^0,$$

where $\nu_1^* = \sup_{s\in[0,t]} \nu_1(s)$ and $\nu_2^* = \sup_{s\in[0,t]} \nu_2(s)$.

(H3) g is continuous, and there exist some positive constants δ_1 such that

$$E\|g(x)\|^2_{\mathbb{H}} \leq \delta_1 \|x\|^2_{\mathcal{B}_h}.$$

(H4) The function $I_k : \mathbb{H} \to \mathbb{H}$ is continuous and there exist continuous nondecreasing functions L_k such that, for each $x \in \mathbb{H}$,

$$E\|I_k(x)\|^2_{\mathbb{H}} \leq L_k E\|x\|^2_{\mathbb{H}} \quad \text{and} \quad \lim_{r \longrightarrow +\infty} \frac{L_k(r)}{r} = \beta_k < \infty, \quad k = \cdots, n.$$

(H5) The linear stochastic system (6) is approximately controllable on $[0, T]$.

The following lemma is required to define the control function.

Lemma 3.1 *[15] For any $\overline{x}_T \in L^2(\mathcal{F}_T, H)$, there exists $\eta(.) \in L^2_{\mathcal{F}}(\Omega; L^2(J; L^0_2))$ such that $\overline{x}_T = E\overline{x}_T + \int_0^T \eta(s)dW(s)$.*

Now, for any $\lambda > 0$ and $\overline{x}_T \in L^2(\mathcal{F}_T, H)$, we define the control function

$$\begin{aligned}
u^\lambda(t) \quad = \quad & B^* S^*_\alpha(T-t)(\lambda I + \Gamma^T_0)^{-1} \\
& \times \Big[E\overline{x}_T + \int_0^t \eta(s)dW(s) + T_\alpha(T)(\phi(0) - g(x)) \Big] \\
& - B^* S^*_\alpha(T-t) \int_0^t (\lambda I + \Gamma^T_s)^{-1} (T-s)^{\alpha-1} S_\alpha(T-t) f(s, x_s, B_1 x(s)) ds \\
& - B^* S^*_\alpha(T-t) \int_0^t (\lambda I + \Gamma^T_s)^{-1} (T-s)^{\alpha-1} S_\alpha(T-t) g(s, x_s, B_2 x(s)) dW(s) \\
& - B^* S^*_\alpha(T-t)(\lambda I + \Gamma^T_0)^{-1} \sum_{0<t_k<T} T_\alpha(T-t_k) I_k(x(t_k^-)).
\end{aligned}$$

Theorem 3.1 *Assume that the conditions $(H1) - (H4)$ hold. Then for each $\lambda > 0$, the system (1) has a mild solution on $[0, T]$, provided that*

$$\begin{aligned}
\Big[4l^2 M^2 \delta_1 + & \left(\tfrac{MT^\alpha}{\Gamma(1+\alpha)}\right)^2 (4l^2 \mu_1^* + \mu_2^* B_1^*) + \tfrac{T^{2\alpha-1}}{2\alpha-1} \left(\tfrac{\alpha M}{\Gamma(1+\alpha)}\right)^2 (4l^2 \nu_1^* + \nu_2^* B_2^*) \\
& + 4l^2 m M^2 \textstyle\sum_{k=1}^m \beta_k \Big] . \Big[5 + \tfrac{30 T^{2\alpha}}{\lambda^2 \alpha^2} \left(\tfrac{\alpha M M_B}{\Gamma(1+\alpha)}\right)^4 \Big] \leq 1
\end{aligned}$$

and

$$2 \left[\frac{T^{2\alpha}}{\alpha^2} \Big(\frac{\alpha M}{\Gamma(1+\alpha)} \Big)^2 (\mu_1 l + \mu_2 B_1^*) + \frac{T^{2\alpha-1}}{2\alpha-1} \Big(\frac{\alpha M}{\Gamma(1+\alpha)} \Big)^2 (\nu_1 l + \nu_2 B_2^*) \right] < 1,$$

where $B_1^* = \sup_{t\in[0,T]} \int_0^t K(t,s)ds < \infty$ and $B_2^* = \sup_{t\in[0,T]} \int_0^t P(t,s)ds < \infty$.

Proof. For any $\lambda > 0$, define the operator $\Psi : \mathcal{B}_b \to \mathcal{B}_b$ by

$\Psi x(t) = \phi(t), \quad t \in (-\infty, 0],$

$$\Psi x(t) = T_\alpha(t)(\phi(0) - g(x)) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B u^\lambda(s) ds$$
$$+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, x_s, B_1(x(s))) ds$$
$$+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \sigma(s, x_s, B_2(x(s))) dW(t)$$
$$+ \sum_{0<t_k<t} T_\alpha(t-t_k) I_k(x(t_k^-)), t \in J.$$

We shall show that the operator Ψ has a fixed point in the space $\mathcal{B}_b$, which is the mild solution of (1).

For $\phi \in \mathcal{B}_h$, we define $\widehat{\phi}$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ T_\alpha(t)\phi(0), & t \in J; \end{cases} \quad \text{then } \widehat{\phi} \in \mathcal{B}_b.$$

Let $x(t) = y(t) + \widehat{\phi}(t), -\infty < t < T$. It is evident that y satisfies $y_0 = 0, t \in (-\infty, 0]$ and

$$\begin{aligned} y(t) =& T_\alpha(t)(-g(y + \widehat{\phi})) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B u^\lambda(s) ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, y_s + \widehat{\phi}_s, B_1(y(s) + \widehat{\phi}(s))) ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \sigma(s, y_s + \widehat{\phi}_s, B_2(y(s) + \widehat{\phi}(s))) dW(s) \\ &+ \sum_{0<t_k<t} T_\alpha(t-t_k) I_k(y(t_k^-) + \widehat{\phi}(t_k^-)), \ t \in J. \end{aligned}$$

Set $\mathcal{B}_b^0 = \{y \in \mathcal{B}_b, \quad \text{such that} \quad y_0 = 0 \in \mathcal{B}_h\}$ and for any $y \in \mathcal{B}_b^0$ we have

$$\|y\|_{\mathcal{B}_b^0} = \|y_0\|_{\mathcal{B}_h} + \sup_{t \in J}(E\|y(t)\|^2)^{\frac{1}{2}} = \sup_{t \in J}(E\|y(t)\|^2)^{\frac{1}{2}},$$

thus $(\mathcal{B}_b^0, \|.\|_{\mathcal{B}_b^0})$ is a Banach space.

Let $\mathcal{B}_r = \left\{y \in \mathcal{B}_b^0, \quad \|y\|^2_{\mathcal{B}_b^0} \le r, \ r > 0\right\}$. The set $\mathcal{B}_r$ is clearly a bounded closed convex set in $\mathcal{B}_b^0$ for each $r > 0$ and for each $y \in \mathcal{B}_r$. By Lemma 2.1 we have

$$\begin{aligned} \|y_t + \widehat{\phi}_t\|^2_{\mathcal{B}_h} \le \ & 2(\|y_t\|^2_{\mathcal{B}_h} + \|\widehat{\phi}_t\|^2_{\mathcal{B}_h}) \\ & \le 4(l^2 \sup\nolimits_{s \in [0,t]} E\|y(s)\|^2_{\mathbb{H}} + \|y_0\|^2_{\mathcal{B}_h}) \\ & +4(l^2 \sup\nolimits_{s \in [0,t]} E\|\widehat{\phi}(s)\|^2_{\mathbb{H}} + \|\widehat{\phi}_0\|^2_{\mathcal{B}_h}) \\ & \le 4(\|\phi\|^2_{\mathcal{B}_h} + l^2(r + M^2 E\|\phi(0)\|^2_{\mathbb{H}})). \end{aligned}$$

For the sake of convenience, we divide the proof into several steps.

Step 1. We claim that there exists a positive number r such that $\Psi(\mathcal{B}_r) \subset \mathcal{B}_r$. If this is not true, then, for each positive integer r, there exists $y^r \in \mathcal{B}_r$ such that $E\|\Psi(y^r)(t)\|^2 > r$ for $t \in (-\infty, T]$, t may depending upon r. However, on the other hand, we have

$$\begin{array}{rcl} r & \leq & E\|\Psi(y^r)(t)\|^2 \\ & \leq & 5E\|T_\alpha(t)[-g(y^r+\widehat{\phi})\|^2 \\ & & +5E\|\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)Bu^\lambda(s)ds\|^2 \\ & & +5E\|\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s,y_s^r+\widehat{\phi}_s,B_1(y_s^r+\widehat{\phi}_s))ds\|^2 \\ & & +5E\|\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)\sigma(s,y_s^r+\widehat{\phi}_s,B_2(y_s^r+\widehat{\phi}_s))dW(t)\|^2 \\ & & +5E\|\sum_{0<t_k<t}\|T_\alpha(t-t_k)I_k(y(t_k^-)+\widehat{\phi}(t_k^-))\|^2, \quad t\in J. \end{array}$$

By using (H1)-(H4), Lemma 2.1 and Hölder's inequality, we obtain

$$\begin{array}{rcl} r & \leq & E\|(\Psi y^r)(t)\|^2 \\ & \leq & 5M^2\delta_1\|y^r+\widehat{\phi}\|^2_{\mathcal{B}_h}+5\frac{T^\alpha}{\alpha}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^2\int_0^t(t-s)^{\alpha-1}E\|u^\lambda(s)\|^2ds \\ & & +5\frac{T^\alpha}{\alpha}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2\int_0^t(t-s)^{\alpha-1}E\|f(s,y_s^r+\widehat{\phi}_s,B_1(y_s^r+\widehat{\phi}_s))\|^2ds \\ & & +5\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2\int_0^t(t-s)^{2(\alpha-1)}E\|\sigma(s,y_s^r+\widehat{\phi}_s,B_2(y_s^r+\widehat{\phi}_s))\|^2_{\mathscr{L}_2^0}ds \\ & & +5mM^2\sum_{0=1}^m E\|I_k(y(t_k^-)+\widehat{\phi}(t_k^-))\|^2 \\ & \leq & 5M^2\delta_1 r'+\frac{30T^{2\alpha}}{\lambda^2\alpha^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^4\delta_2 \\ & & +5\left(\frac{MT^\alpha}{\Gamma(1+\alpha)}\right)^2(\mu_1^*r'+\mu_2^*B_1^*(\sup_{s\in[0,T]}E\|x\|^2) \\ & & +5\frac{T^{2\alpha-1}}{2\alpha-1}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2(\nu_1^*r'+\nu_2^*B_2^*(\sup_{s\in[0,T]}E\|x\|^2) \\ & & +5mM^2\sum_{0=1}^m L_k(r'), \\ & \leq & 5M^2\delta_1 r'+\frac{30T^{2\alpha}}{\lambda^2\alpha^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^4\delta_2 \\ & & +5\left(\frac{MT^\alpha}{\Gamma(1+\alpha)}\right)^2(\mu_1^*r'+\mu_2^*B_1^*r)+5\frac{T^{2\alpha-1}}{2\alpha-1}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2(\nu_1^*r'+\nu_2^*B_2^*r) \\ & & +5mM^2\sum_{k=1}^m L_k r', \end{array}$$

where $r'=4(\|\phi\|^2_{\mathcal{B}_h}+l^2(r+M^2E\|\phi(0)\|^2_{\mathbb{H}}))$,$\|B\|\leq M_B$ and

$$\begin{array}{rcl} \delta_2 & = & 2E\|\bar{x}_T\|^2+2\int_0^t E\|\eta(s)\|^2_{\mathscr{L}_2^0}ds+M^2\|\phi\|^2_{\mathcal{B}_h}+M^2\delta_1 r' \\ & & +\left(\frac{MT^\alpha}{\Gamma(1+\alpha)}\right)^2(\mu_1^*r'+\mu_2^*B_1^*r)+\frac{T^{2\alpha-1}}{2\alpha-1}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2(\nu_1^*r'+\nu_2^*B_2^*r) \\ & & +mM^2\sum_{k=1}^m L_k r'. \end{array}$$

Dividing both sides by r and taking the limit as $r\longrightarrow\infty$, we obtain

$$\begin{array}{rl} 1\leq\Big[4l^2M^2\delta_1+ & \left(\frac{MT^\alpha}{\Gamma(1+\alpha)}\right)^2(4l^2\mu_1^*+\mu_2^*B_1^*)+\frac{T^{2\alpha-1}}{2\alpha-1}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2(4l^2\nu_1^*+\nu_2^*B_2^*) \\ & +4l^2mM^2\sum_{k=1}^m\beta_k\Big].\Big[5+\frac{30T^{2\alpha}}{\lambda^2\alpha^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^4\Big] \end{array}$$

which is a contradiction to our assumption. Thus, for each $\lambda>0$, there exists some positive number r such that $\Psi(\mathcal{B}_r)\subset\mathcal{B}_r$.

Next, we show that the operator Ψ is condensing, for convenience, we decompose Ψ as $\Psi=\Psi_1+\Psi_2$, where

$$(\Psi_1 y)(t)=\begin{cases} \int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s,y_s+\widehat{\phi}_s,B_1(y(s)+\widehat{\phi}(s)))ds \\ +\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)\sigma(s,y_s+\widehat{\phi}_s,B_2(y(s)+\widehat{\phi}(s)))dW(t), \end{cases} \tag{7}$$

and

$$(\Psi_2 y)(t) = \begin{cases} T_\alpha(t)(-g(y+\widehat{\phi})) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B u^\lambda(s) ds \\ \qquad + \sum_{0<t_k<t} T_\alpha(t-t_k) I_k(y(t_k^-) + \widehat{\phi}(t_k^-)), \qquad t \in [0,T]. \end{cases} \tag{8}$$

Step 2. We prove that Ψ_1 is a contraction on $\mathcal{B}_r$. Let $t \in J$ and $y,\, y^* \in \mathcal{B}_r$, we have

$$\begin{aligned}
&\|(\Psi_1 y)(t) - (\Psi_1 y^*)(t)\|_{\mathbb{H}}^2 \\
&\leq 2E\| \int_0^t (T-s)^{\alpha-1} S_\alpha(T-s) \Big[f(s, y_s + \widehat{\phi}_s, B_1(y(s) + \widehat{\phi}(s))) \\
&-f(s, y_s^* + \widehat{\phi}_s, B_1(y^*(s) + \widehat{\phi}(s))) \Big] ds \|_{\mathbb{H}}^2 \\
&+2E\| \int_0^t (T-s)^{\alpha-1} S_\alpha(T-s) \Big[\sigma(s, y_s + \widehat{\phi}_s, B_2(y(s) + \widehat{\phi}(s))) \\
&-\sigma(s, y_s^* + \widehat{\phi}_s, B_2(y^*(s) + \widehat{\phi}(s))) \Big] dW(t) \|_{\mathbb{H}}^2 \\
&\leq 2\tfrac{T^\alpha}{\alpha} \Big(\tfrac{\alpha M}{\Gamma(1+\alpha)} \Big)^2 \int_0^t (T-s)^{\alpha-1} \big[\mu_1 \|y(s) - y^*(s)\|_{\mathcal{B}_h}^2 \\
&+\mu_2 E\|B_1(y(s) + \widehat{\phi}(s)) - B_1(y^*(s) + \widehat{\phi}(s)\|_{\mathbb{H}}^2 \big] ds \\
&+2\Big(\tfrac{\alpha M}{\Gamma(1+\alpha)} \Big)^2 \int_0^t (T-s)^{2(\alpha-1)} \big[\nu_1 \|y_s - y_s^*\|_{\mathcal{B}_h}^2 \\
&+\nu_2 E\|B_2(y(s) + \widehat{\phi}(s)) - B_2(y^*(s) + \widehat{\phi}(s))\|_{\mathbb{H}}^2 \big] ds \\
&\leq 2\left[\tfrac{T^{2\alpha}}{\alpha^2} \Big(\tfrac{\alpha M}{\Gamma(1+\alpha)} \Big)^2 (\mu_1 l + \mu_2 B_1^*) \right. \\
&\left. + \tfrac{T^{2\alpha-1}}{2\alpha-1} \Big(\tfrac{\alpha M}{\Gamma(1+\alpha)} \Big)^2 (\nu_1 l + \nu_2 B_2^*) \right] \|y - y^*\|_{\mathcal{B}_b^0}^2,
\end{aligned}$$

where $2\left[\frac{T^{2\alpha}}{\alpha^2} \Big(\frac{\alpha M}{\Gamma(1+\alpha)} \Big)^2 (\mu_1 l + \mu_2 B_1^*) + \frac{T^{2\alpha-1}}{2\alpha-1} \Big(\frac{\alpha M}{\Gamma(1+\alpha)} \Big)^2 (\nu_1 l + \nu_2 B_2^*) \right] < 1$, hence Ψ_1 is a contraction.

Step 3. Ψ_2 maps bounded sets into bounded sets in $\mathcal{B}_r$, Let us prove that for $r > 0$ there exists a $\widehat{r} > 0$ such that for each $y \in \mathcal{B}_r$ we have $E\|(\Psi_2 y)(t)\|_{\mathbb{H}}^2 < \widehat{r}$ for $t \in J$. Now we have

$$\begin{aligned}
E\|\Psi_2 y(t)\|_{\mathbb{H}}^2 \quad &\leq \quad 3E\|T_\alpha(t)(-g(y+\widehat{\phi}))\|^2 \\
&\quad\ +3E\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B u^\lambda(s) ds \|^2 \\
&\quad\ +3E\| \sum_{0<t_k<t} T_\alpha(t-t_k) I_k(y(t_k^-) + \widehat{\phi}(t_k^-)) \|^2 \\
&\leq \quad 3M^2 \delta_1 r' + \tfrac{18}{\lambda^2} \tfrac{T^{2\alpha}}{\alpha^2} \Big(\tfrac{\alpha M M_B}{\Gamma(1+\alpha)} \Big)^4 \delta_2 + 3M^2 m^2 \sum_{k=1}^m L_k r' \\
&= \widehat{r}.
\end{aligned}$$

Step 4. The map Ψ_2 is equicontinuous. Let $u, v \in J$, $0 \leq u < v \leq T$, $y \in \mathcal{B}_r$, we obtain

$$\begin{aligned}
&E\|(\Psi_2 y)(v) - (\Psi_2 y)(u)\|_{\mathbb{H}}^2 \leq \\
&5E\|T_\alpha(v) - T_\alpha(u)(-g(y+\widehat{\phi}))\|^2 \\
&+ 5E\|\int_0^u (u-s)^{\alpha-1}[S_\alpha(v-s) - S_\alpha(u-s)]Bu^\lambda(s)ds\|^2 \\
&+ 5E\|\int_0^u [(v-s)^{\alpha-1} - (u-s)^{\alpha-1}]S_\alpha(v-s)Bu^\lambda(s)ds\|^2 \\
&+ 5E\|\int_u^v (v-s)^{\alpha-1}S_\alpha(v-s)Bu^\lambda(s)ds\|^2 \\
&+ 5E\|\sum_{0\leq t_k\leq T}[T_\alpha(v-t_k) - T_\alpha(u-t_k)]I_k(y(t_k^-) + \widehat{\phi}(t_k^-))\|^2.
\end{aligned}$$

Noting the fact that for every $\epsilon > 0$, there exists a $\delta > 0$ such that, whenever $|s_1 - s_2| < \delta$ for every $s_1, s_2 \in J$, $\|T_\alpha(s_1) - T_\alpha(s_2)\| < \epsilon$ and $\|S_\alpha(s_1) - S_\alpha(s_2)\| < \epsilon$. Therefore, when $|v-u| < \delta$, we have

$$\begin{aligned}
E\|(\Psi_2 y)(v) - (\Psi_2 y)(u)\|_{\mathbb{H}}^2 \leq & 5\epsilon^2\delta_1 r' + \frac{30\epsilon^2 M_B^2}{\lambda^2}\frac{T^{2\alpha}}{\alpha^2}\delta_2 \\
&+ \frac{30\delta_2}{\alpha^2\lambda^2}\Big(\frac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^4[v^\alpha - u^\alpha - (v-u)^\alpha]^2 \\
&+ \frac{30\delta_2}{\alpha^2\lambda^2}\Big(\frac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^4(v-u)^{2\alpha} + 5m\epsilon^2\sum_{k=1}^m L_k r'.
\end{aligned}$$

The right hand of the inequality above tends to 0 as $v \longrightarrow u$ and $\epsilon \longrightarrow 0$, hence the set $\{\Psi_1 y, y \in \mathcal{B}_r\}$ is equicontinuous.

Step 5. The set $V(t) = \{\Psi_2 y(t), y \in \mathcal{B}_r\}$ is relatively compact in $\mathcal{B}_r$. Let $0 < t \leq T$ be fixed and $0 < \epsilon < t$. For $\delta > 0$, $y \in \mathcal{B}_r$, we define

$$\begin{aligned}
\Psi_2^{\epsilon,\delta}y(t) \;\leq\;& \textstyle\int_\delta^\infty \xi_\alpha(\theta)T(t^\alpha\theta)(-g(y+\widehat{\phi}))d\theta \\
&+\alpha \textstyle\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)Bu^\lambda(s)d\theta ds \\
&+\textstyle\sum_{0<t_k<t}\int_\delta^\infty \xi_\alpha(\theta)T((t-t_k)^\alpha\theta)I_k(y(t_k^-) + \widehat{\phi}(t_k^-))d\theta \\
=\;& T(\epsilon^\alpha\delta)\textstyle\int_\delta^\infty \xi_\alpha(\theta)T(t^\alpha\theta - \epsilon^\alpha\delta)(-g(y+\widehat{\phi})d\theta \\
&+\alpha T(\epsilon^\alpha\delta)\textstyle\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-\theta)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta - \epsilon^\alpha\delta)Bu^\lambda(s)d\theta ds \\
&+\textstyle\sum_{0<t_k<t}T(\epsilon^\alpha\delta)\int_\delta^\infty \xi_\alpha(\theta)T((t-t_k)^\alpha\theta - \epsilon^\alpha\delta)I_k(y(t_k^-) + \widehat{\phi}(t_k^-))d\theta.
\end{aligned}$$

Then from the compactness of $T(\epsilon^\alpha\delta)$, we obtain that $V_{\epsilon,\delta}(t) = \{\Psi_2^{\epsilon,\delta}y(t) : y \in \mathcal{B}_r\}$ is relatively compact in H for every ϵ, $0 < \epsilon < t$. Moreover, for $y \in \mathcal{B}_r$, we can easily prove that $\Psi_2^{\epsilon,\delta}y(t)$ is convergent to $\Psi_2 y(t)$ in $\mathcal{B}_r$ as $\epsilon \longrightarrow 0$ and $\delta \longrightarrow 0$, hence the set $V(t) = \{\Psi_2 y(t) : y \in \mathcal{B}_r\}$ is also relatively compact in $\mathcal{B}_r$. Thus, by Arzela-Ascoli theorem Ψ_1 is completely continuous. Consequently, from Lemma 2.4 Ψ has a fixed point, which is a mild solution of (1).

Theorem 3.2 *Assume that (H1)-(H5) are satisfied, and the conditions of Theorem 3.1 hold. Further, if the functions f and σ are uniformly bounded, and $T(t)$ is compact, then the system (1) is approximately controllable on $[0, T]$.*

Proof. Let x^λ be a solution of (1), then we can easily get that

$$\begin{aligned} x^\lambda(t) &= \bar{x}_T - \lambda(\lambda I + \Gamma_0^T)^{-1}\Big[E\overline{x}_T + \int_0^t \eta(s)dW(s) - T_\alpha(T)(\phi(0) - g(x))\Big] \\ &\quad +\lambda\int_0^T (\lambda I + \Gamma_s^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-t)f(s, x_s^\lambda, B_1x^\lambda(s))ds \\ &\quad +\lambda\int_0^T (\lambda I + \Gamma_s^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-t)\sigma(s, x_s^\lambda, B_2x^\lambda(s))dW(s) \\ &\quad +\lambda(\lambda I + \Gamma_0^T)^{-1}\sum_{0<t_k<T} T_\alpha(T-t_k)I_k(x(t_s^\lambda)). \end{aligned}$$

In view of the assumptions that f and σ are uniformly bounded on J, there is a subsequence still denoted by $f(s, x_s^\lambda, B_1x^\lambda(s))$ and $\sigma(s, x_s^\lambda, B_2x^\lambda(s))$, which converges weakly to, say $f(s)$ in H, and $\sigma(s)$ in $L(U,H)$. On the other hand, by assumption $(H5)$, the operator $\lambda(\lambda I + \Gamma_s^T)^{-1} \longrightarrow 0$ strongly as $\lambda \longrightarrow 0^+$ for all $0 \le s \le T$, and, moreover, $\|\lambda(\lambda I + \Gamma_s^T)^{-1}\| \le 1$. Thus, the Lebesgue dominated convergence theorem and the compactness of S yield

$$\begin{aligned} E\|x^\lambda(t) - \bar{x}_T\|^2 &\le 4\|\lambda(\lambda I+\Gamma_0^T)^{-1}\|^2 E\|E\overline{x}_T + \int_0^T \eta(s)dW(s) - T_\alpha(T)(\phi(0) - g(x))\|^2 \\ &+ 4E\Big(\int_0^T \|\lambda(\lambda I+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-t)f(s, x_s^\lambda, B_1x^\lambda(s))\|ds\Big)^2 \\ &+ 4E\Big\|\int_0^T \|\lambda(\lambda I+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}S_\alpha(T-t)\sigma(s, x_s^\lambda, B_1x^\lambda(s))dW(s)\Big\|^2 \\ &+ 4\|\lambda(\lambda I+\Gamma_0^T)^{-1}\|^2 E\|\sum_{0<t_k<T} T_\alpha(T-t_k)I_k(x(t_s^\lambda))\|^2 \to 0, \text{as } \lambda \to 0^+. \end{aligned}$$

This gives the approximate controllability of (1), the proof is complete.

4 An Example

As an application, we consider an impulsive stochastic partial differential equation of the following form

$$\begin{cases} D_t^\alpha x(t,y) = \frac{\partial^2}{\partial y^2}x(t,y) + b(y)u(t) + \int_{-\infty}^0 H(t,y,s-t)Q(x(s,y))ds \\ +\int_0^t K(s,t)e^{-x(s,y)}ds + \Big[\int_{-\infty}^0 V(t,y,s-t)U(x(s,y))ds \\ +\int_0^t p(s,t)e^{-x(s,y)}ds\Big]dW(t) \; y \in [0,\pi], \quad t \in [0,T], T > 0, t \ne t_k, \\ I_k(x(t_k^-,y)) = x(t_k^+,y) - x(t_k^-,y), \quad k = 1,\cdots,m, \\ x(t,0) = x(t,\pi) = 0, \quad t \in [0,T], \\ x(0,y) + \int_0^\pi G(y,z)x(t,z)dz = \phi(t,y), \quad t \in (-\infty, 0]. \end{cases} \tag{9}$$

Let $\mathbb{U} = \mathbb{H} = L^2([0,\pi])$ and $h(t) = e^{2t}$, $t < 0$, Then $l = \int_{-\infty}^0 h(s)ds = \frac{1}{2}$ .To study the approximate controllability of (9), assume that H, Q, V and U are continuous; $\phi \in \mathcal{B}_h$.

We define the operator A by $Ax = \frac{\partial^2 x}{\partial y^2}$. with domain $D(A) = \{x \in \mathbb{H}, \frac{\partial x}{\partial y}, \frac{\partial^2 x}{\partial y^2} \in \mathbb{H}$ and $x(0) = x(\pi) = 0\}$. It is well known that A generates an analytic semigroup $T(t), t \ge 0$ given by $T(t)x = \sum_{n=1}^\infty e^{-n^2 t}\langle x, e_n\rangle e_n$, $x \in H$, and $e_n(y) = (2/\pi)^{1/2}\sin(ny), n = 1, 2, \cdots$, is the orthogonal set of eigenvectors of A.

Define the operators $f : J \times \mathcal{B}_h \times L^2([0,\pi]) \to \mathbb{H}$, $\sigma : J \times \mathcal{B}_h \times L^2([0,\pi]) \to \mathscr{L}_2^0(\mathbb{U},\mathbb{H})$, $g : \mathcal{B}_h \to L^2([0,\pi])$,

$$f(t,\phi,B_1x(t))(y) = \int_{-\infty}^{t} H(t,y,s-t)Q(x(s,y))ds + \int_0^t K(s,t)e^{-x(s,y)}ds,$$

$$\sigma(t,\phi,B_2x(t))(y) = \int_{-\infty}^{0} V(t,y,s-t)U(x(s,y))ds + \int_0^t p(s,t)e^{-x(s,y)}ds,$$

$$g(y) = \int_0^{\pi} G(y,z)x(t,z)dz.$$

With the choice of A, f, σ and g can be rewritten as the abstract form of system (1). Under the appropriate conditions on the functions f, σ, g and I_k as those in (H1)-(H5), system (9) is approximately controllable.

5 Conclusion

Approximate controllability of nonlinear fractional impulsive stochastic differential equations with nonlocal conditions and infinite delay in Hilbert spaces has been investigated. By employing fractional calculus, Krasnoselskii-Schaefer-type fixed point theorem and stochastic analysis theory, sufficient conditions for the approximate controllability of nonlinear fractional impulsive stochastic differential equations are formulated and proved under the assumption that the associated linear system is approximately controllable.

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References

[1] Balasubramaniam, P., Park, J.Y. and Vincent, A. Antony Kumar. Existence of solutions for semilinear neutral stochastic functional differential equations with nonlocal conditions. *Nonlinear Anal. TMA* **71** (2009) 1049–1058.

[2] Bashirov, A.E. and Mahmudov, N.I. On concepts of controllability for deterministic and stochastic systems. *SIAM J. Control Optim.* **37** (1999) 1808–1821.

[3] Byszewski, L. and Lakshmikantham, V. Theorem about the existence and uniqueness of solutions of a nonlocal Cauchy problem in a Banach space. *Appl. Anal.* **40** (1990) 11–19.

[4] Byszewski, L. Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. *J. Math. Anal. Appl.* **162** (1991) 494–506.

[5] Byszewski, L. Existence and uniqueness of solutions of semilinear evolution nonlocal Cauchy problem. *Zesz. Nauk. Politech. Rzesz., Mat. Fiz.* **18** (1993) 109–112.

[6] Cao, J., Yang, Q., Huang, Z., and Liu, Q. Asymptotically almost periodic solutions of stochastic functional differential equations. *Appl. Math. Comput.* **218** (2011) 1499–1511.

[7] Cao, J., Yang, Q. and Huang, Z. On almost periodic mild solutions for stochastic functional differential equations. *Nonlinear Anal. TMA* **13** (2012) 275–286.

[8] Caputo, M. *Elasticite Dissipazione.* Zanichelli, Bologna, 1969.

[9] Chang, Y.K., Zhao, Z.H., NGurkata, G.M. and Ma, R. Stepanov-like almost automorphy for stochastic processes and applications to stochastic differential equations. *Nonlinear Anal. RWA* **12** (2) (2011) 1130–1139.

[10] Dauer, J.P. Mahmudov, N.I. Controllability of stochastic semilinear functional differential equations in Hilbert spaces. *J. Math. Anal. Appl.* **290** (2004) 373–394.

[11] Guendouzi, T. Existence and controllability of fractional-order impulsive stochastic system with infinite delay. *Discussiones Mathematicae, Diffferential Inclusions, Control and Optimization* **33** (2013) 65–87.

[12] Karatzas, I. and Shreve, S. E. *Brownian Motion and Stochastic Calculus.* Springer-Verlag, Berlin, 1991.

[13] Kuma, S. and Sukavanam, N. On the Approximate Controllability of Fractional Order Control Systems with Delay. *Nonlinear Dynamics and Systems Theory* **13** (1) (2013) 69–78.

[14] Mahmudov, N.I. and Denker, A. On controllability of linear stochastic systems. *Int. J. Control* **73** (2000) 144–151.

[15] Mahmudov, N.I. Controllability of linear stochastic systems in Hilbert spaces. *J. Math. Anal. Appl.* **259** (2001) 64–82.

[16] Mahmudov, N.I. and Zorlu, S. Approximate controllability of fractional integro-differential equations involving nonlocal initial conditions. *Boundary Value Problems* (2013) **2013**:118.

[17] Mao, X. *Stochastic Differential Equations and Applications.* Horwood, Chichester, 1997.

[18] Muthukumar, P. and Rajivganthi, C. Approximate controllability of fractional order neutral stochastic integro-differential system with nonlocal conditions and infinite delay. *Taiwanese Journal of Mathematics* **17** (5) (2013) 1693–1713.

[19] Ren, Y. and Sun, D.D. Second-order neutral stochastic evolution equations with infinite delay under carathodory conditions. *J. Optim. Theory Appl.* **147** (2010) 569–582.

[20] Rajiv Ganthi, C. and Muthukumar, P. Approximate controllability of fractional stochastic integral equation with finite delays in Hilbert spaces. *ICMMSC* CCIS **283** (2012) 302–309.

[21] Ren, Y., Zhou, Q. and Chen, L. Existence, uniqueness and stability of mild solutions for time-dependent stochastic evolution equations with Poisson jumps and infinite delay. *J. Optim. Theory Appl.* **149** (2011) 315–331.

[22] Sakthivel, R., Suganyab, S. and Anthonib, S.M. Approximate controllability of fractional stochastic evolution equations. *Comput. Math. Appl.* **63** (2012) 660–668.

[23] Sakthivel, R., Nieto, J.J. and Mahmudov, N.I. Approximate controllability of nonlinear deterministic and stochastic systems with unbounded delay. *Taiwan. J. Math.* **14** (2010) 1777–1797.

[24] Sakthivel, R., Revathi, P. and Yong Renc, B. Existence of solutions for nonlinear fractional stochastic differential equations. *Nonlinear Analysis* **81** (2013) 70–86.

[25] Shukla, A., Sukavanam, N. and Pandey, D.N. Approximate controllability of semilinear stochastic sontrol system with nonlocal conditions. *Nonlinear Dynamics and Systems Theory* **15** (3) (2015) 321–333.

[26] Slama, A. and Boudaoui, A. Existence of solutions for nonlinear fractional impulsive stochastic differential equations with nonlocal conditions and infinite delay. *International Journal of Differential Equations and Applications* **13** (4) (2014) 185–201.

[27] Zabczyk, J. *Mathematical Control Theory.* Basel: Birkhauser, 1992.

[28] Zang, Y. and Li, J. Approximate controllability of fractional impulsive neutral stochastic differential equations with nonlocal conditions. *Boundary Value Problems* (2013) **2013**:193.

[29] Zhou, Y. and Jiao, F. Existence of mild solution for fractional neutral evolution equations. *Comput. Math. Appl.* **59** (2010) 1063–1077.

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Weak Singular Solution of Six Coupled Nonlinear ODEs

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Abstract: In this paper we have investigated the complete integrability of the system of six coupled nonlinear ODEs (ordinary differential equations), which arose in the ODE reduction of uniformly stratified fluid contained in rotating rectangular box of dimension $L \times L \times H$. The reduced system is completely integrable if the Rayleigh number $Ra = 0$. Whereas, $Ra \neq 0$ is the case of non integrability and we have obtained the solutions in the form of logarithmic psi-series. We conclude that weak singular solutions exist with movable pole type singularity, which are cluster in a self-similar fashion.

Keywords: *completely integrable systems; non-integrable systems; Painlevé test; singular solutions.*

Mathematics Subject Classification (2010): 37K10, 37K15, 70H06, 70H07.

1 Introduction

In the fluid dynamics, the flow of fluid in the atmosphere and in the ocean is governed by Boussinesq equations. Majda and Shefter [3] analyzed certain ODE reduction of Boussinesq equations. Srinivasan et al. [15] extended this work and they gave the detail mathematical analysis of reduced system of six coupled ODEs. Whereas, Desale and Dasre [5] wrote the C-Programme to determine the numerical solutions on stable and unstable manifolds. Furthermore, Desale [4] had given the complete analysis of the system and also tested the system for complete integrability by determining four first integrals and used the Jacobi's theorem. Also, he has demonstrated the stability of non degenerate critical point. For the similar text of bifurcation analysis near the degenerate

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critical point one may refer to [14]. The rigorous mathematical analysis and special solutions of rotating stratified Boussinesq equations have been discussed by Desale and Sharma in their paper [7].

In his study of onset of instabilities in the stratified fluids at large Richardson number Paul Painlevé [12, 13] classified algebraic differential equations of first and second order whose solutions in the complex domain are devoid of movable essential singularities or movable branch point. The ODE possessing this property is said to be of Painlevé type. Painlevé test in view of partial differential equations is generally known as WTC (Weiss-Tabor-Carnevale [16]) test which is further modified by Kichenassamy and Srinivasan [9]. In their paper [8], Desale and Srinivasan tested the reduced system of stratified Boussinesq equations in the light of the ARS (Ablowitz, Ramani and Segur [1]) conjuncture. In continuation of this work Desale & Patil [6] tested the system of six coupled ODEs for complete integrability using the Painlevé test.

In this paper we have tested the system of six coupled nonlinear ODEs for its complete integrability via Painlevé test. We have the non integrable case for the Rayleigh number $Ra \neq 0$ causing the singular solution in the form of logarithmic psi-series, which is the weak solution. The presence of logarithm term in the series implies that the solution in question have singularity which is cluster in self similar fashion. This is sometime viewed as possible symptom for non-integrable behavior.

This paper consists of five sections. Section 1 is introduction, Section 2 gives ODE reduction of uniformly stratified fluid contained in rotating rectangular box. In Section 3, we provided the preliminary work which is the base for investigation of weak solutions in the non integrable case. Whereas, in Section 4, we determined the weak solutions. Finally, we conclude the result in Section 5.

2 Dynamics of an Uniformly Stratified Fluid Contained in Rotating Box

We now begin by describing the rotating stratified Boussinesq equations (see Majda [2])

$$\begin{aligned} \frac{D\vec{\mathbf{v}}}{Dt} + f(\hat{\mathbf{e_3}} \times \vec{\mathbf{v}}) &= -\nabla p + \nu(\Delta \vec{\mathbf{v}}) - \frac{g\tilde{\rho}}{\rho_b}\hat{\mathbf{e_3}}, \\ \operatorname{div} \vec{\mathbf{v}} &= 0, \\ \frac{D\tilde{\rho}}{Dt} &= \kappa \Delta \tilde{\rho}, \end{aligned} \tag{1}$$

where $\vec{\mathbf{v}}$ denotes the velocity field, ρ is the density which is the sum of constant reference density ρ_b and perturbation density $\tilde{\rho}$, p is the pressure, g is the acceleration due to gravity that points in $-\hat{\mathbf{e_3}}$ direction, f is the rotation frequency of earth, ν is the coefficient of viscosity, κ is the coefficient of heat conduction and $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{\mathbf{v}} \cdot \nabla)$ is a convective derivative. For more about rotating stratified Boussinesq equations one may see Majda [2].

In the frame of reference of an uniformly stratified fluid contained in rotating rectangular box of dimension $L \times L \times H$, which is temperature stratified with fixed zeroth order moments of mass and heat (so that there is neither net evaporation or precipitation, nor any net river input or output, and neither heating nor cooling). The container is assumed to be in steady uniform rotation on an f-plane. Maas [11] reduces the system of equations (1) into the following system of six coupled ODEs:

$$
\begin{aligned}
Pr^{-1}\frac{d\vec{\mathbf{w}}}{dt} + f'\hat{\mathbf{e_3}} \times \vec{\mathbf{w}} &= \hat{\mathbf{e_3}} \times \vec{\mathbf{b}} - (w_1,\, w_2,\, rw_3) + \hat{T}\vec{\mathbf{T}}, \\
\frac{d\vec{\mathbf{b}}}{dt} + \vec{\mathbf{b}} \times \vec{\mathbf{w}} &= -(b_1,\, b_2,\, \mu b_3) + Ra\vec{\mathbf{F}}.
\end{aligned} \tag{2}
$$

In these equations, $\vec{\mathbf{b}} = (b_1,\, b_2,\, b_3)$ is the center of mass, $\vec{\mathbf{w}} = (w_1,\, w_2,\, w_3)$ is the basin's averaged angular momentum vector, $\vec{\mathbf{T}}$ is the differential momentum, $\vec{\mathbf{F}}$ are the buoyancy fluxes, $f' = f/2r_h$ is the earth rotation, $r = r_v/r_h$ is the friction ($r_{v,h}$ are the Rayleigh damping coefficients), Ra is the Rayleigh number, Pr is the Prandtl number, μ is the diffusion coefficient and $\hat{T}$ is the magnitude of the wind stress torque.

Neglecting diffusive and viscous terms, Maas [11] considers the dynamics of an ideal rotating, uniformly stratified fluid in response to forcing. He assumes this to be due solely to differential heating in the meridional (y) direction. $\vec{\mathbf{F}} = (0,\, 1,\, 0)$, the wind effect is neglected i.e. $\vec{\mathbf{T}} = 0$. For Prandtl number Pr, equal to one, the system of equations (2) reduces to the following ideal rotating, uniformly stratified system of six coupled ODEs

$$
\begin{aligned}
\frac{d\vec{\mathbf{w}}}{dt} &= -f'\hat{\mathbf{e_3}} \times \vec{\mathbf{w}} + \hat{\mathbf{e_3}} \times \vec{\mathbf{b}}, \\
\frac{d\vec{\mathbf{b}}}{dt} &= -\vec{\mathbf{b}} \times \vec{\mathbf{w}} + Ra\vec{\mathbf{F}}.
\end{aligned} \tag{3}
$$

In his paper, Desale [4] has demonstrated the complete integrability of the system (3) for $Ra = 0$ using the first integrals and Jacobi's theory. Desale and Patil [6] continued this work and tested the system for complete integrability via Painlevé test. In this paper we investigate the case of non integrability $Ra \neq 0$. In the following section we consider the case of non integrability and obtain the weak singular solution in the form of logarithmic-psi series.

3 Preliminaries

We have a system of ODEs (3), which can be written component-wise as:

$$
\begin{aligned}
&\dot{w_1} = f'w_2 - b_2, \quad \dot{w_2} = -f'w_1 + b_1, \quad \dot{w_3} = 0, \\
&\dot{b_1} = w_2 b_3 - w_3 b_2, \quad \dot{b_2} = w_3 b_1 - w_1 b_3 + Ra, \quad \dot{b_3} = w_1 b_2 - w_2 b_1.
\end{aligned} \tag{4}
$$

Since $\dot{w_3} = 0$, which gives us $w_3 = \text{constant} = k_1$. Consequently, we have the following system of ODEs:

$$
\begin{aligned}
&\dot{w_1} = f'w_2 - b_2, \quad \dot{w_2} = -f'w_1 + b_1, \\
&\dot{b_1} = w_2 b_3 - k_1 b_2, \quad \dot{b_2} = k_1 b_1 - w_1 b_3 + Ra, \quad \dot{b_3} = w_1 b_2 - w_2 b_1.
\end{aligned} \tag{5}
$$

Desale and Patil [6] obtained the solution of the system (5) in the form of the following power series:

$$
\begin{aligned}
&w_1(t) = \sum_{j=0}^{\infty} w_{1j}\tau^{j+m_1}, \quad w_2(t) = \sum_{j=0}^{\infty} w_{2j}\tau^{j+m_2}, \\
&b_1(t) = \sum_{j=0}^{\infty} b_{1j}\tau^{j+n_1}, \quad b_2(t) = \sum_{j=0}^{\infty} b_{2j}\tau^{j+n_2}, \quad b_3(t) = \sum_{j=0}^{\infty} b_{3j}\tau^{j+n_3},
\end{aligned} \tag{6}
$$

where $\tau = t - t_0$ and t_0 is an arbitrary position of singularity. Also, the authors found that there were several possible cases of dominant balance of the system (5) and among those possible cases they obtained the singular solution only in the following case of principle dominant balance

$$\dot{w_1} = -b_2, \quad \dot{w_2} = b_1, \quad \dot{b_1} = w_2 b_3, \quad \dot{b_2} = -w_1 b_3, \quad \dot{b_3} = w_1 b_2 - w_2 b_1. \tag{7}$$

Consequently, they have determined the exponents as

$$m_1 = m_2 = -1, \quad n_1 = n_2 = n_3 = -2 \tag{8}$$

and possible branches of leading order coefficients as listed below

$$w_{10} = \pm\sqrt{-4 - k_2^2},\ w_{20} = k_2,\ b_{10} = -k_2,\ b_{20} = \pm\sqrt{-4 - k_2^2},\ b_{30} = 2. \tag{9}$$

Furthermore, the authors have given the following recursive relations to determine the coefficients w_{1j}, w_{2j}, b_{1j}, b_{2j} and b_{3j} for $j = 1, 2, 3 \ldots$.

$$\begin{pmatrix} j-1 & 0 & 0 & 1 & 0 \\ 0 & j-1 & -1 & 0 & 0 \\ 0 & -b_{30} & j-2 & 0 & -w_{20} \\ b_{30} & 0 & 0 & j-2 & w_{10} \\ -b_{20} & b_{10} & w_{20} & -w_{10} & j-2 \end{pmatrix} \begin{pmatrix} w_{1j} \\ w_{2j} \\ b_{1j} \\ b_{2j} \\ b_{3j} \end{pmatrix} = \begin{pmatrix} A_j \\ B_j \\ C_j \\ D_j \\ E_j \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned} A_j &= f' w_{2(j-1)}, \quad B_j = -f' w_{1(j-1)}, \\ C_j &= -k_1 b_{2(j-1)} + \sum_{k=1}^{j-1} w_{2k} b_{3(j-k)}, \\ D_j &= k_1 b_{1(j-1)} - \sum_{k=1}^{j-1} w_{1k} b_{3(j-k)}, \\ E_j &= \sum_{k=1}^{j-1} w_{1k} b_{2(j-k)} - \sum_{k=1}^{j-1} w_{2k} b_{1(j-k)} \ . \end{aligned} \tag{11}$$

The above recursive relations (10) determine the unknown expansion coefficients uniquely unless the determinant of coefficient matrix is zero. Those values of j at which the determinant of coefficient matrix vanishes are called the *resonances* and these are

$$j = -1,\ 0,\ 2,\ 3,\ 4. \tag{12}$$

We see that all resonances are simple. Here $j = -1$, is a usual resonance and $j = 0$ is corresponding to the arbitrariness of w_{20} in leading order coefficients.

Desale and Patil [6] have considered the following case of leading order coefficients

$$\begin{aligned} &w_{10} = \sqrt{-4 - k_2^2}, \quad w_{20} = k_2 \,\text{(arbitrary constant)}, \\ &b_{10} = -k_2, \quad b_{20} = \sqrt{-4 - k_2^2}, \quad b_{30} = 2 \end{aligned} \tag{13}$$

and they have determined the singular solution passing through it. Ultimately they have checked the compatibility conditions at $j = 1$ and $j = 2$. They have obtained the following expansion coefficients:

$$\begin{aligned} &w_{11} = \tfrac{1}{2}(f'k_2 - k_1 k_2), \quad w_{21} = \tfrac{1}{2}(-f' + k_1)\sqrt{-4 - k_2^2}, \\ &b_{11} = f'\sqrt{-4 - k_2^2}, \quad b_{21} = f' k_2, \quad b_{31} = 0. \end{aligned} \tag{14}$$

$$\begin{array}{l} w_{12} = \frac{1}{2}(f'k_1 - k_3)\sqrt{-4-k_2^2}, \quad w_{22} = \frac{k_2}{2}(f'k_1 - k_3), \\ b_{12} = \frac{k_2}{2}\left[(f')^2 - k_3\right], \quad b_{22} = \frac{1}{2}\left[k_3 - (f')^2\right]\sqrt{-4-k_2^2}, \quad b_{32} = k_3. \end{array} \tag{15}$$

An arbitrary constant $b_{32} = k_3$ involved in (15) because of $j = 2$ is a resonance. While checking the compatibility conditions at resonance $j = 3$, they have concluded that the compatibility condition holds only if $Ra = 0$. Implying that $Ra \neq 0$ is a non integrable case. Thus, it motivates us to study this non integrable case and in the following section we are going to obtain weak singular solutions.

4 Weak Singular Solution

In this section we have studied the non integrable case of system (5) that is, we have obtained the weak singular solutions in terms of logarithmic psi series.

We are going to find the singular solutions in the form of

$$t^{\nu} \sum_{m \geq l \geq 0} u_{m,l}(x) t^m (\ln t)^l, \tag{16}$$

which are suggested by Kichenassamy and Srinivasan [9]. They also made an interesting remark that $l = 1$ suffices if all the resonances are simple and 1 is not a resonance. In this case, we also have simple resonances $j = -1, 0, 2, 3, 4$. Therefore, our solution will be in the form of

$$t^{\nu} \sum_{m \geq 1} u_{m,1}(x) t^m (\ln t). \tag{17}$$

With above remarkable feature and compatibility conditions hold for $j = 0$, 1 and 2, we restructure the power series given by (6) as follows:

$$\begin{aligned} w_1(t) &= w_{10}\tau^{-1} + w_{11} + w_{12}\tau + \sum_{j=3}^{\infty} w_{1j}(\log\tau)\tau^{j-1}, \\ w_2(t) &= w_{20}\tau^{-1} + w_{21} + w_{22}\tau + \sum_{j=3}^{\infty} w_{2j}(\log\tau)\tau^{j-1}, \\ b_1(t) &= b_{10}\tau^{-2} + b_{11}\tau^{-1} + b_{12} + \sum_{j=3}^{\infty} b_{1j}(\log\tau)\tau^{j-2}, \\ b_2(t) &= b_{20}\tau^{-2} + b_{21}\tau^{-1} + b_{22} + \sum_{j=3}^{\infty} b_{2j}(\log\tau)\tau^{j-2}, \\ b_3(t) &= b_{30}\tau^{-2} + b_{31}\tau^{-1} + b_{32} + \sum_{j=3}^{\infty} b_{3j}(\log\tau)\tau^{j-2}. \end{aligned} \tag{18}$$

In the above equations (18) expansion coefficients w_{1j}, w_{2j}, b_{1j}, b_{2j} and b_{3j} for $j = 1, 2, 3$ are given by the equations (13), (14) and (15). The power series given by (18) provide us the weak singular solution in the form of logarithmic psi series.

• **Compatibility condition at the resonance** $j = 3$. Now we proceed to check the compatibility condition at the resonance $j = 3$. At the resonance level $j = 3$, we substitute equations (18) into the system of differential equations (5), then equating like powers of τ and $\tau(\log\tau)$ with $j = 3$, we get the following systems of non-homogeneous linear equations (19) and (20)

$$\begin{array}{l} w_{13} = f'w_{22}, \quad w_{23} = -f'w_{12}, \quad b_{13} = k_3 w_{21} - k_1 b_{22}, \\ b_{23} = k_1 b_{12} - k_3 w_{11} + Ra, \quad b_{33} = w_{11}b_{22} + w_{12}b_{21} - w_{21}b_{12} - w_{22}b_{11}. \end{array} \tag{19}$$

$$\begin{aligned}&2w_{13} = -b_{23}, \quad 2w_{23} = b_{13}, \quad b_{13} = b_{33}w_{20} + w_{23}b_{30},\\&b_{23} = -w_{10}b_{33} - w_{13}b_{30}, \quad b_{33} = w_{13}b_{20} + w_{10}b_{23} - w_{23}b_{10} - w_{20}b_{13}.\end{aligned} \tag{20}$$

Solving (19) and (20) together, we obtain the system of linear equations, which is in matrix form as given below

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 2 & 0 & 0 & k_2\\ -2 & 0 & 0 & 0 & -\sqrt{-4-k_2^2}\\ \sqrt{-4-k_2^2} & -k_2 & -k_2 & \sqrt{-4-k_2^2} & 0 \end{pmatrix}\begin{pmatrix} w_{13}\\ w_{23}\\ b_{13}\\ b_{23}\\ b_{33}\end{pmatrix}$$
$$= \begin{pmatrix} -2f'w_{22}\\ -2f'w_{12}\\ w_{21}b_{32} - k_1b_{22}\\ k_1b_{12} - w_{11}b_{32} + Ra\\ w_{11}b_{22} + w_{12}b_{21} - b_{12}w_{21} - w_{22}b_{11}\end{pmatrix}. \tag{21}$$

Further, we solve the system and expansion coefficient are uniquely determined, which are listed below

$$\begin{aligned}&w_{13} = \frac{(2f'^2k_1 - Rak_2 - 2f'k_3)k_2}{4(2+k_2^2)}, \quad w_{23} = \frac{(2(f'^2)k_1 - Rak_2 - 2f'k_3)\sqrt{-4-k_2^2}}{4(2+k_2^2)},\\&b_{13} = -f'(k_1f' - k_3)\sqrt{-4-k_2^2}, \quad b_{23} = f'k_2(-f'k_1 + k_3),\\&b_{33} = \frac{(Ra + f'^2k_1k_2 - f'k_2k_3)\sqrt{-4-k_2^2}}{2(2+k_2^2)}.\end{aligned} \tag{22}$$

• **Compatibility condition at the resonance** $j = 4$. Again we substitute (18) into the system (5) and in these equations, we substitute the earlier determined expansion coefficients which are given by (13), (14), (15) and (22). We simplify the both sides of resultant equations and equating the powers of τ^2 and $\tau^2(\log\tau)$, we obtain the following non-homogeneous systems of linear equations, which are given by the following equations

$$w_{14} = 0, \quad w_{24} = 0, \quad b_{14} = w_{22}b_{32}, \quad b_{24} = -w_{12}b_{32}, \quad b_{34} = -w_{22}b_{12} + w_{12}b_{22}. \tag{23}$$

$$\begin{aligned}3w_{14} &= f'w_{23} - b_{24},\\ 3w_{24} &= -f'w_{13} + b_{14},\\ 2b_{14} &= w_{21}b_{33} - w_{30}b_{23} + w_{20}b_{34} + w_{24}b_{30},\\ 2b_{24} &= w_{30}b_{13} - w_{10}b_{34} - w_{11}b_{33} - w_{14}b_{30},\\ 2b_{34} &= w_{11}b_{23} + w_{13}b_{21} - w_{21}b_{13} + w_{23}b_{11}\\ &+ w_{14}b_{20} - w_{24}b_{10} - w_{20}b_{14} + w_{10}b_{24}.\end{aligned} \tag{24}$$

We solve the equations (23) and (24) together in the similar way as we adopted in the previous case and determine the expansion coefficients uniquely at this resonance level, which are listed below

$$\begin{aligned}w_{14} &= \tfrac{1}{16(2+k_2^2)}\big[\big(-2f'Rak_2 - 2f'^2k_1^2(8+3k_2^2) - 16k_3^2 + 2k_2^2k_3f'^2 - 4k_2^2k_3^2\\ &+ 2k_2^4k_3^2 + 2Rak_1k_2 + 32f'k_1k_3 - 2f'^3k_2^2 + 10f'k_2^2k_3 - 2f'k_2^4k_3\big)\sqrt{-4-k_2^2}\end{aligned}$$

$$
\begin{aligned}
& + \quad \big(k_2^6 k_3 + 8k_2^2 k_3 + 6k_2^4 k_3 - k_2^6 f' - 8f' k_2^2 - 6f' k_2^4\big)(-f'^2 + k_3)\big], \\
w_{24} &= \tfrac{1}{16(2+k_2^2)}\big[\big(8k_2 k_3 + 6k_2^3 k_3 + k_2^5 k_3 - 8f' k_1 k_2 - 6f' k_2^3 - f' k_2^5\big) \\
& \quad (f'^2 - k_3)\sqrt{-4-k_2^2} + 2(k_1 Ra - f' Ra - f' k_1 k_2 (f'^2 - k_3))(k_2^2 + 4) \\
& - \quad 2f'^2 k_1^2 k_2 (4 + 3k_2^2) + 8f'^2 k_2 k_3 + 2k_2^3 k_3 (f'^2 + 2k_3 + k_2^2 k_3^2) - 2f' k_1 k_2^5 k_3\big] \\
b_{14} &= \frac{f' k_2 (2f'^2 k_1 - Rak_2 - 2f' k_3)}{4(2+k_2^2)}, \quad b_{24} = \frac{f'\sqrt{-4-k_2^2}(2f'^2 k_1 - Rak_2 - 2f' k_3)}{4(2+k_2^2)}, \\
b_{34} &= \tfrac{1}{8}\big[k_2^2((f'^3)k_1 - f'^2 k_3 - f' k_1 k_3 + k_3^2)\sqrt{-4-k_2^2} + 2k_2^2 k_3 (f' k_1 - k_3)\big].
\end{aligned} \tag{25}
$$

• **Compatibility condition for** $j \geq 5$. Here, we provide the recursion relations by which we can determine expansion coefficients of logarithmic psi series (18) for $j \geq 5$. These relations will be obtained by substituting (18) into the system (5) and then equating the powers of τ^j and $\tau^j(\log \tau)$. This will result into two non homogeneous systems of linear equations. Further, we combine these two systems together, the resultant system is as given below that lead us to determine all the expansion coefficients

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & b_{30} & 0 & 0 & w_{20} \\
b_{30} & 0 & 0 & 0 & w_{10} \\
-b_{20} & b_{10} & w_{20} & -w_{10} & 0
\end{pmatrix}
\begin{pmatrix} w_{1j} \\ w_{2j} \\ b_{1j} \\ b_{2j} \\ b_{3j} \end{pmatrix}
=
\begin{pmatrix} A_j{}^* \\ B_j{}^* \\ C_j{}^* \\ D_j{}^* \\ E_j{}^* \end{pmatrix}, \tag{26}
$$

where

$$
\begin{aligned}
A_j{}^* &= f' w_{2(j-1)}, \quad B_j{}^* = f' w_{1(j-1)}, \\
C_j{}^* &= k_1 b_{2(j-1)} - b_{31} w_{2(j-1)} - b_{32} w_{2(j-2)} - w_{21} b_{3(j-1)} - w_{22} b_{3(j-2)}, \\
D_j{}^* &= -k_1 b_{1(j-1)} + w_{11} b_{3(j-1)} + w_{12} b_{3(j-2)} + b_{31} w_{1(j-1)} + b_{32} w_{1(j-2)}, \\
E_j{}^* &= w_{11} b_{2(j-1)} + w_{12} b_{2(j-2)} + b_{21} w_{1(j-1)} + b_{22} w_{1(j-2)} \\
& + \quad w_{21} b_{1(j-1)} + w_{22} b_{1(j-2)} - b_{11} w_{2(j-1)} - b_{12} w_{2(j-2)}.
\end{aligned} \tag{27}
$$

From the equation (26), we see that the determinant of coefficient matrix is nonzero for the given leading order coefficients this implies that all expansion coefficients for $j \geq 5$ are determined uniquely in terms of predetermined coefficients.

During the implementation of Painlevé algorithm with logarithmic terms, we observed that all compatibility conditions were satisfied. Hence, the system (5) passes the Painlevé test which indicate that the weak singular solution of the system (5) exists. The weak singular solution of (3) in the considered case of leading order coefficients is as follows

$$
\begin{aligned}
w_1(t) &= \sqrt{-4-k_2^2}\tau^{-1} + \tfrac{1}{2}(f' k_2 - k_1 k_2) + \big[\tfrac{1}{2}(f' k_1 - k_3)\sqrt{-4-k_2^2}\big]\tau \\
& + \quad \left[\frac{(2f'^2 k_1 - Rak_2 - 2f' k_3)k_2}{4(2+k_2^2)}\right](\log \tau)\tau^2 + \frac{1}{16(2+k_2^2)}\big[\big(-2f' Rak_2 \\
& - \quad 2f'^2 k_1^2 (8 + 3k_2^2) - 16k_3^2 + 2k_2^2 k_3 f'^2 - 4k_2^2 k_3^2 + 2k_2^4 k_3^2 + 2Rak_1 k_2 \\
& + \quad 32f' k_1 k_3 - 2f'^3 k_2^2 + 10f' k_2^2 k_3 - 2f' k_2^4 k_3\big)\sqrt{-4-k_2^2} \\
& + \quad \big(k_2^6 k_3 + 8k_2^2 k_3 + 6k_2^4 k_3 - k_2^6 f' - 8f' k_2^2 - 6f' k_2^4\big)(-f'^2 + k_3)\big](\log \tau)\tau^3 \\
& + \quad \sum_{j=5}^{\infty} w_{1j}(\log \tau)\tau^{j-1},
\end{aligned}
$$

$$
\begin{aligned}
w_2(t) &= k_2\tau^{-1} + \big[\tfrac{1}{2}(-f' + k_1)\sqrt{-4-k_2^2}\big] + \big[\tfrac{k_2}{2}(f'k_1 - k_3)\big]\tau \\
&+ \big[\frac{(2(f'^2)k_1 - Rak_2 - 2f'k_3)\sqrt{-4-k_2^2}}{4(2+k_2^2)}\big](\log\tau)\tau^2 \\
&+ \tfrac{1}{16(2+k_2^2)}\big[\big(8k_2k_3 + 6k_2^3k_3 + k_2^5k_3 - 8f'k_1k_2 - 6f'k_2^3 - f'k_2^5\big) \\
&\quad (f'^2 - k_3)\sqrt{-4-k_2^2} + 2(k_1Ra - f'Ra - f'k_1k_2(f'^2 - k_3))(k_2^2+4) \\
&- 2f'^2k_1^2k_2(4+3k_2^2) + 8f'^2k_2k_3 + 2k_2^3k_3(f'^2 + 2k_3 + k_2^2k_3^2) \\
&- 2f'k_1k_2^5k_3\big](\log\tau)\tau^3 + \sum_{j=5}^{\infty} w_{2j}(\log\tau)\tau^{j-1}, \\
w_3(t) &= k_1 \text{ (arbitrary constant)}, \\
b_1(t) &= -k_2\tau^{-2} + \big[f'\sqrt{-4-k_2^2}\big]\tau^{-1} + \tfrac{k_2}{2}\big[(f')^2 - k_3\big] + (-f'(k_1f' - k_3)) \\
&\quad \sqrt{-4-k_2^2}(\log\tau)\tau + \big[\frac{f'k_2(2f'^2k_1 - Rak_2 - 2f'k_3)}{4(2+k_2^2)}\big](\log\tau)\tau^2 \\
&+ \sum_{j=5}^{\infty} b_{1j}(\log\tau)\tau^{j-2}, \\
b_2(t) &= \sqrt{-4-k_2^2}\tau^{-2} + f'k_2\tau^{-1} + \big[\tfrac{1}{2}\big((k_3 - (f')^2\big)\sqrt{-4-k_2^2}\big] \\
&+ (f'k_2(-f'k_1 + k_3))(\log\tau)\tau \\
&+ \big[\frac{f'(2f'^2k_1 - Rak_2 - 2f'k_3)\sqrt{-4-k_2^2}}{4(2+k_2^2)}\big](\log\tau)\tau^2 + \sum_{j=5}^{\infty} b_{2j}(\log\tau)\tau^{j-2}, \\
b_3(t) &= 2\tau^{-2} + k_3 + \frac{(Ra + f'^2k_1k_2 - f'k_2k_3)\sqrt{-4-k_2^2}}{2(2+k_2^2)}(\log\tau)\tau \\
&+ \tfrac{1}{8}\big[k_2^2((f'^3)k_1 - f'^2k_3 - f'k_1k_3 + k_3^2)\sqrt{-4-k_2^2} \\
&+ 2k_2^2k_3(f'k_1 - k_3)\big](\log\tau)\tau^2 + \sum_{j=5}^{\infty} b_{3j}(\log\tau)\tau^{j-2}.
\end{aligned} \tag{28}
$$

Equations (28) contain four arbitrary constants k_1, k_2, k_3, and arbitrary position of singularity t_0 satisfying the system of ODEs (3). The convergence of such logarithmic psi series solutions is guaranteed by Kichenassamy and Littman [10].

In the similar way of calculations, we can find the singular solution to the system (3) corresponding to the following branch of leading order coefficients:

$$
\begin{aligned}
&w_{10} = -\sqrt{-4-k_2^2}, \quad w_{20} = k_2 \text{ (arbitrary constant)}, \\
&b_{10} = -k_2, \quad b_{20} = -\sqrt{-4-k_2^2}, \quad b_{30} = 2.
\end{aligned} \tag{29}
$$

The weak singular solution to the system (3) for this branch of leading order coefficients (29) is given by the following equations (30) and (31)

$$
\begin{aligned}
w_1(t) &= -\sqrt{-4-k_2^2}\tau^{-1} + \tfrac{1}{2}(f'k_2 - k_1k_2) + \big[\tfrac{1}{2}(-f'k_1 + k_3)\sqrt{-4-k_2^2}\,\big]\tau \\
&+ \left[\frac{(2f'^2k_1 - Rak_2 - 2f'k_3)k_2}{4(2+k_2^2)}\right](\log\tau)\tau^2 + \frac{1}{16(2+k_2^2)}\big[\big(-2f'Rak_2 \\
&- 2f'^2k_1^2(8+3k_2^2) - 16k_3^2 + 2k_2^2k_3f'^2 - 4k_2^2k_3^2 + 2k_2^4k_3^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 2Rak_1k_2 + 32f'k_1k_3 - 2f'^3k_2^2 + 10f'k_2^2k_3 - 2f'k_2^4k_3\big)(-\sqrt{-4-k_2^2})\\
&+ \quad \big(k_2^6k_3 + 8k_2^2k_3 + 6k_2^4k_3 - k_2^6f' - 8f'k_2^2 - 6f'k_2^4\big)(-f'^2 + k_3)\big]\,(\log\tau)\tau^3\\
&+ \quad \sum_{j=5}^{\infty} w_{1j}(\log\tau)\tau^{j-1},\\
w_2(t) \;&= \quad k_2\tau^{-1} + \big[\tfrac{1}{2}(f'-k_1)\sqrt{-4-k_2^2}\big] + \big[\tfrac{k_2}{2}(f'k_1 - k_3)\big]\tau\\
&+ \quad \big[\frac{(-2(f'^2)k_1 + Rak_2 + 2f'k_3)\sqrt{-4-k_2^2}}{4(2+k_2^2)}\big](\log\tau)\tau^2\\
&+ \quad \tfrac{1}{16(2+k_2^2)}\big[\big(8k_2k_3 + 6k_2^3k_3 + k_2^5k_3 - 8f'k_1k_2 - 6f'k_2^3 - f'k_2^5\big)\\
&\quad\quad (f'^2 - k_3)(-\sqrt{-4-k_2^2}) + 2(k_1Ra - f'Ra - f'k_1k_2(f'^2 - k_3))(k_2^2+4)\\
&- \quad 2f'^2k_1^2k_2(4+3k_2^2) + 8f'^2k_2k_3 + 2k_2^3k_3(f'^2 + 2k_3 + k_2^2k_3^2)\\
&- \quad 2f'k_1k_2^5k_3\big](\log\tau)\tau^3 + \sum_{j=5}^{\infty} w_{2j}(\log\tau)\tau^{j-1},\\
w_3(t) \;&= \quad k_1 \text{ (arbitrary constant)},\\
b_1(t) \;&= \quad -k_2\tau^{-2} - \big[f'\sqrt{-4-k_2^2}\big]\tau^{-1} + \tfrac{k_2}{2}\big[(f')^2 - k_3\big] + (f'(k_1f' - k_3))\\
&\quad\quad \sqrt{-4-k_2^2}(\log\tau)\tau + \big[\frac{f'k_2(2f'^2k_1 - Rak_2 - 2f'k_3)}{4(2+k_2^2)}\big](\log\tau)\tau^2\\
&+ \quad \sum_{j=5}^{\infty} b_{1j}(\log\tau)\tau^{j-2},\\
b_2(t) \;&= \quad -\sqrt{-4-k_2^2}\tau^{-2} + f'k_2\tau^{-1} + \big[\tfrac{1}{2}\left((-k_3 + (f')^2\right)\sqrt{-4-k_2^2}\big]\\
&+ \quad (f'k_2(-f'k_1 + k_3))(\log\tau)\tau\\
&+ \quad \big[\frac{f'(-2f'^2k_1 + Rak_2 + 2f'k_3)\sqrt{-4-k_2^2}}{4(2+k_2^2)}\big](\log\tau)\tau^2 + \sum_{j=5}^{\infty} b_{2j}(\log\tau)\tau^{j-2},
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
b_3(t) \;&= \quad 2\tau^{-2} + k_3 + \frac{(-Ra - f'^2k_1k_2 + f'k_2k_3)\sqrt{-4-k_2^2}}{2(2+k_2^2)}(\log\tau)\tau\\
&+ \quad \tfrac{1}{8}\big[k_2^2(-(f'^3)k_1 + f'^2k_3 + f'k_1k_3 - k_3^2)\sqrt{-4-k_2^2}\\
&+ \quad 2k_2^2k_3(f'k_1 - k_3)\big](\log\tau)\tau^2 + \sum_{j=5}^{\infty} b_{3j}(\log\tau)\tau^{j-2}.
\end{aligned} \tag{31}
$$

The result of this section can be summarized in the form of the following theorem.

Theorem 4.1 *An ideal rotating, uniformly stratified system of six coupled ODEs (3) is completely integrable for Rayleigh number $Ra = 0$. Whereas, $Ra \neq 0$ is the case of non integrability and system (3) admits weak singular solutions in the form of logarithmic psi series given by equations (28) and (30), (31) for two different branches of leading order coefficients given by equations (9).*

5 Conclusion

The reduced system of ODEs (3) which arose in the reduction of uniformly stratified fluid contained in the rotating box of dimension $L \times L \times H$ is completely integrable if the

Rayleigh number $Ra = 0$. If $Ra \neq 0$ then the system (3) is non integrable. In this case of non integrability we have determined the weak solutions (28) and (30), (31) in the different branches of leading order. The solutions are in the form of logarithmic psi series and the convergence of the series is guaranteed by Kichenassamy and Littman [10]. We see that the nature of movable singularities are pole type singularities which are cluster in a self similar fashion.

References

[1] Ablowitz, M. J., Ramani, A. and Segur, H. A. A Connection between Nonlinear Evolution Equations and Ordinary Differential Equations of P-type. I. *Journal of Math. Phys.* **21** (1980) 715–721.

[2] Majda, A.J. *Introduction to PDEs and Waves for the Atmosphere and Ocean.* Courant Lecture Notes in Mathematics 9, American Mathematical Society. Providence, Rhode Island, 2003.

[3] Majda, A.J. and Shefter, M.G. Elementary stratified flows with instability at large Richardson number. *J. Fluid Mechanics* **376** (1998) 319–350.

[4] Desale, B.S. Complete Analysis of an Ideal Rotating Uniformly Stratified System of ODEs. *Nonlinear Dynamics and System Theory* **9** (3) (2009) 263–275.

[5] Desale, B.S. and Dasre, N.R. Numerical Solutions of System of Nonlinear ODEs by Euler Modified Method. *Nonlinear Dyanamics and Systems Theory* **12** (3) (2012) 215–236.

[6] Desale, B.S. and Patil, K.D. Painlevé Test to a Reduced System of Six Coupled Nonlinear ODEs. *Nonlinear Dynamics and System Theory* **10** (4) (2010) 349–361.

[7] Desale, B.S. and Sharma Vivek. Special Solutions to Rotating Stratified Boussinesq Equations. *Nonlinear Dyanamics and Systems Theory* **10** (1) (2010) 29–38.

[8] Desale, B.S. and Srinivasan, G.K. Singular Analysis of the System of ODE Reductions of the Stratified Boussinesq Equations. *IAENG International Journal Of Applied Mathematics* **38** (4) (2008) 181–191.

[9] Kichenassamy, S. and Srinivasan, G. The Structure of WTC expansions and applications. *Journal of Phys. A: Math. Gen.* **28** (1995) 1977–2004.

[10] Kichenassamy, S. and Littman, W. Blow-up surfaces for nonlinear wave equations I. *Commun. PDE* **18** (1993) 431–452.

[11] Leo R. M. Maas. Theory of Basin Scale Dynamics of a Stratified Rotating Fluid. *Surveys in Geophysics* **25** (2004) 249–279.

[12] Painlev, Paul. *Leçons. sur la théorie analytique des équations différentielles professées á Stockholm (Autumn 1895) Sur línvitation de S. M. le roide Suede et de Norwege.* Librairie Scientifique, A Hermann, Paris, 1897.

[13] Painlev, Paul. Sur les équations differentielles du Second Ordre et dórdre supérieur dont l'integrale générale est uniform. *Acta Math.* **25** (1902) 1–86.

[14] Scarpello, G.M. and Ritelli, D. Nonlinear Dynamics of a Two-Degrees of Freedom Hamiltonian System: Bifurcations and Integration. *Nonlinear Dynamics and Systems Theory* **8** (1) (2008) 97–108.

[15] Srinivasan, G. K., Sharma, V. D. and Desale, B. S. An integrable system of ODE reductions of the stratified Boussinesq equations. *Computers and Mathematics with Applications* **53** (2007) 296–304.

[16] Weiss, J., Tabor, M. and Carnevale, G. The Painlevé property for partial Differential Equations. *Journal of Math. Phys.* **24** (1983) 522–526.

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On Tractable Functionals in Antagonistic Games with a Constant Initial Condition

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Abstract: This paper continues modeling of an antagonistic game with two players initiated in Dshalalow and Ke [4] which dealt with a stochastic game with player A losing to player B. Theorem 1 in [4] gave an explicit functional of several key components of the game, including the ruin time of A and the total casualties to both players at the exit, i.e. at A's ruin time. The claim of why the formula in Theorem 1 of [4] for the above mentioned functional was explicit is fully justified. Here we work on a particular case calculating Laplace-Carson inverse transforms and probability density functions followed by numerics.

Keywords: *noncooperative stochastic games; fluctuation theory; marked point processes; Poisson process; ruin time; exit time; first passage time; modified Bessel functions.*

Mathematics Subject Classification (2010): 82B41, 60G51, 60G55, 60G57, 91A10, 91A05, 91A60, 60K05.

1 Introduction

This paper models an antagonistic game with two players earlier initiated in Dshalalow and Ke [4]. The first part of [4] dealt with a basic game when player A lost the game to player B. Theorem 1 in [4] gave an explicit functional of several major components of the game, including the ruin (exit) time, the total casualties to both players at the exit. The claim of why the formula in Theorem 1 of [4] for the above mentioned functional was explicit is finally justified in this paper. Here we analyze a particular case evaluating Laplace-Carson inverse transforms and probability density functions followed by numerical calculations.

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In short, the game initiated in [4] was modeled by a complex marked point process. It included two marked Poisson processes representing incremental casualties to players A and B during the conflict as well as the hitting times. Both processes were supposed to be observed by a third party point process which preserved more or less crude information about the course of the game. So, the ruin time as well as other events were cumulative upon observation epochs. The literature on antagonistic games is very rich. We mention just a few articles and books: [1, 5, 7–8, 11, 12]. The contemporary work on antagonistic games finds its applications to economics [1, 7, 8, 11] and warfare [4, 5, 12]. The techniques used in this paper are based on fluctuation theory developed by the first author in [4] and his earlier papers. Related work on fluctuation theory is in [9,10].

The paper is organized as follows. In Section 2, we give a brief description of the model in [4]. Section 3 formalizes a special case making an assumption about the distributions of casualties and observation process. The double inverse of the Laplace-Carson transform is evaluated explicitly in terms of the modified Bessel functions of order zero and one. Section 4 deals with one marginal functional of the ruin time and casualties to player A, all in terms of the Laplace-Stieltjes transform. Other results, such as casualties to player B and inverse of the Laplace-Stieltjes transform (that yields associated probability density functions), are dealt in paper [6].

2 The Model

For consistency, we present some descriptional details of the model before we turn to the special case. Let $(\Omega, \mathcal{F}(\Omega), P)$ be a probability space and let $\mathcal{F}_A, \mathcal{F}_B, \mathcal{F}_\tau \subseteq \mathcal{F}(\Omega)$ be independent sub- σ-algebras. Suppose

$$\mathcal{A} := \sum_{j\geq 1} w_j \varepsilon_{s_j} \quad \text{and} \quad \mathcal{B} := \sum_{k\geq 1} z_k \varepsilon_{t_k}, \quad s_1,\, t_1 > 0, \tag{2.1}$$

are $\mathcal{F}_A$-measurable and $\mathcal{F}_B$-measurable marked Poisson random measures (ε_a is a point mass at a) with respective intensities λ_A and λ_B and position independent marking. They are specified by their transforms

$$Ee^{-\alpha\mathcal{A}(\cdot)} = e^{\lambda_A|\cdot|[g(\alpha)-1]}, \quad g(\alpha) = Ee^{-\alpha w_1}, \quad Re(\alpha) \geq 0, \tag{2.2}$$

$$Ee^{-\beta\mathcal{B}(\cdot)} = e^{\lambda_B|\cdot|[h(\beta)-1]}, \quad h(\beta) = Ee^{-\beta z_1}, \quad Re(\beta) \geq 0, \tag{2.3}$$

$|\cdot|$ is the Borel-Lebesgue measure, and w_j and z_k are nonnegative r.v.'s. Furthermore, let

$$\tau := \sum_{i\geq 0} \varepsilon_{\tau_i}, \quad \tau_0 > 0, \tag{2.4}$$

be an $\mathcal{F}_\tau$-measurable delayed renewal process.

If

$$(A(t), B(t)) := \mathcal{A} \otimes \mathcal{B}\left((-\infty, t]\right), \tag{2.5}$$

then

$$(A_j, B_j) := (A(\tau_j), B(\tau_j)) = \mathcal{A} \otimes \mathcal{B}\left((-\infty, \tau_j]\right), \quad j = 0, 1, \ldots, \tag{2.6}$$

forms the observation process upon $\mathcal{A} \otimes \mathcal{B}$ embedded over τ, with respective increments

$$(X_j, Y_j) = \mathcal{A} \otimes \mathcal{B}((\tau_{j-1}, \tau_j]), \quad j = 1, 2, \ldots, \tag{2.7}$$

and

$$X_0 = A_0, \quad Y_0 = B_0. \tag{2.8}$$

Obviously, the bivariate marked point process

$$\mathcal{A}_\tau \otimes \mathcal{B}_\tau := \sum_{j\geq 0}(X_j, Y_j)\varepsilon_{\tau_j}, \tag{2.9}$$

where

$$\mathcal{A}_\tau = \sum_{i\geq 0} X_i \varepsilon_{\tau_i} \quad \text{and} \quad \mathcal{B}_\tau = \sum_{i\geq 0} Y_i \varepsilon_{\tau_i}. \tag{2.10}$$

are with position dependent marking and with X_j and Y_j being interdependent. With the notation

$$\Delta_j := \tau_j - \tau_{j-1}, \quad j = 1, 2, \ldots, \tag{2.11}$$

we evaluate the functional

$$\gamma(\alpha, \beta, \theta) = Ee^{-\alpha X_j - \beta Y_j - \theta\Delta_j} = \delta\{\theta + \lambda_A(1 - g(\alpha)) + \lambda_B(1 - h(\beta))\}, \quad j = 1, 2, \ldots, \tag{2.12}$$

$$Re(\alpha) \geq 0, \quad Re(\beta) \geq 0, \quad Re(\theta) \geq 0, \tag{2.13}$$

where

$$\delta(\theta) = Ee^{-\theta\Delta_1}, \quad Re(\theta) \geq 0. \tag{2.14}$$

is the common marginal Laplace-Stieltjes transform of Δ_1, $\Delta_2, \ldots$.

Analogously,

$$\gamma_0(\alpha, \beta, \theta) = Ee^{-\alpha A_0 - \beta B_0 - \theta\tau_0} = \delta_0\{\theta + \lambda_A(1 - g(\alpha)) + \lambda_B(1 - h(\beta))\}, \tag{2.15}$$

where

$$\delta_0(\theta) = Ee^{-\theta\tau_0}. \tag{2.16}$$

The *game* in this case is stochastic process $\mathcal{A}_\tau \otimes \mathcal{B}_\tau$ describing the evolution of a conflict between players A and B known to an observer only upon process $\tau = \{\tau_0, \tau_1, \ldots\}$. The game is over when on the kth observation epoch τ_k (for some k), the cumulative damage to player A or B (A_k or B_k, respectively) exceeds its respective threshold M or N (some positive real numbers). But we are looking into the paths of the game where player A is losing first.

With the *exit indices*

$$\mu := \inf\{j \geq 0 : \; A_j = X_0 + X_1 + \ldots + X_j > M\} \tag{2.17}$$

and

$$\nu := \inf\{k \geq 0 : B_k = Y_0 + Y_1 + \ldots + Y_k > N\}, \tag{2.18}$$

A_μ and B_ν are the respective cumulative damages to players A and B at their ruin times. We will be concerned, however, with the ruin time of player A and thus restrict our game to the confined trace σ-algebra $\mathcal{F}(\Omega) \cap \{\mu < \nu\}$. In paper [4] (Dshalalow-Ke) the authors studied a game between two players, A and B, in particular, the functional

$$\Phi_{\mu\nu} = \Phi_{\mu\nu}(\alpha, \beta, \theta) = E\left[e^{-\alpha A_\mu - \beta B_\mu - \theta\tau_\mu}\mathbf{1}_{\{\mu<\nu\}}\right] \tag{2.19}$$

of the game. It represented the multivariate Laplace-Stieltjes transform of joint distribution of the exit time τ_μ of the game and the status of the casualties to both players at the exit. The evolution of the game is followed here when player A loses the game to player B.

Theorem 1 [4] below gives an explicit formula for $\varPhi_{\mu\nu}$. With (2.12) and (2.15) we abbreviate

$$\gamma_0(\alpha,\beta,\theta) := Ee^{-\alpha X_0-\beta Y_0-\theta\Delta_0}, \quad Re(\alpha)\geq 0, \quad Re(\beta)\geq 0, \quad Re(\theta)\geq 0, \tag{2.20}$$

$$\gamma(\alpha,\beta,\theta) := Ee^{-\alpha X_j-\beta Y_j-\theta\Delta_j}, \quad Re(\alpha)\geq 0, \quad Re(\beta)\geq 0, \quad Re(\theta)\geq 0, \quad j>0, \tag{2.21}$$

$$\varGamma_0 := \gamma_0(\alpha+x,\beta+y,\theta), \quad \varGamma_0^1 := \gamma_0(\alpha,\beta+y,\theta), \tag{2.22}$$

$$\varGamma := \gamma(\alpha+x,\beta+y,\theta), \quad \varGamma^1 := \gamma(\alpha,\beta+y,\theta). \tag{2.23}$$

The results are presented in terms of the inverse of the Laplace-Carson transform defined as

$$\mathcal{LC}_{pq}(\cdot)(x,y) := xy\int_{p=0}^{\infty}\int_{q=0}^{\infty} e^{-xp-yq}(\cdot)d(p,q), \quad Re(x)>0, \quad Re(y)>0. \tag{2.24}$$

Denote its inverse

$$\mathcal{LC}_{xy}^{-1}(\cdot)(p,q) = \mathcal{L}_{xy}^{-1}\left(\cdot\,\frac{1}{xy}\right), \tag{2.25}$$

where $\mathcal{L}^{-1}$ is the inverse of the bivariate Laplace transform.

Theorem 1 [4] *In light of abbreviations* (2.20)–(2.23)*, the functional* $\varPhi_{\mu\nu}$ *of the game on the trace* σ*-algebra* $\mathcal{F}(\Omega)\cap\{\mu<\nu\}$ *satisfies the following formula:*

$$\varPhi_{\mu\nu} = \mathcal{LC}_{xy}^{-1}\left(\varGamma_0^1 - \varGamma_0 + \frac{\varGamma_0}{1-\varGamma}(\varGamma^1-\varGamma)\right)(M,N), \tag{2.26}$$

which for the restricted functional (2.19) *of only three major components can be rewritten as*

$$\varPhi_{\mu\nu} = \mathcal{LC}_{xy}^{-1}\left(\varGamma_0^1 - \varGamma_0\frac{1-\varGamma^1}{1-\varGamma}\right)(M,N). \tag{2.27}$$

3 A Special Case

We assume that the intervals $\Delta_1,\Delta_2,\dots$ between the successive observation times $\tau_0,\tau_1,\dots$ are exponentially distributed with parameter δ, i.e.

$$\delta(\theta) := Ee^{-\theta\Delta} = \frac{\delta}{\delta+\theta}. \tag{3.1}$$

We assume that the game starts from zero, i.e., X_0 and Y_0 are some constants and that

$$\Delta_0 := 0. \tag{3.2}$$

Furthermore, we assume that the marks in the processes $\mathcal{A}$ and $\mathcal{B}$ specified by g and h in (2.2) and (2.3), respectively, are exponential with parameters g and h, i.e.

$$g(\alpha) = \frac{g}{g+\alpha} \quad \text{and} \quad h(\beta) = \frac{h}{h+\beta}. \tag{3.3}$$

Our goal is to simplify $\Phi_{\mu\nu}$ of (2.27) for this special case to a form for which we can find the Laplace-Carson inverse explicitly. We start with the first factor, Γ_0^1 of (2.27) by unfolding notation (2.12):

$$\Gamma_0^1 = \gamma_0(\alpha, \beta + y, \theta) = E[e^{-\alpha X_0 - (\beta+y)Y_0 - \theta\Delta_0}] = Ee^{-\alpha X_0 - (\beta+y)Y_0}. \tag{3.4}$$

Now we apply the Laplace-Carson inverse to (3.4):

$$\begin{aligned} \mathcal{LC}_{xy}^{-1}(\Gamma_0^1)(p,q) &= \mathfrak{L}_{xy}^{-1}\left(\frac{1}{xy}e^{-\alpha X_0 - (\beta+y)Y_0}\right)(p,q) = \mathfrak{L}_y^{-1}\left(\frac{1}{y}e^{-\alpha X_0 - (\beta+y)Y_0}\right)(q) \\ &= e^{-\alpha X_0 - \beta Y_0}\mathbf{1}_{(Y_0,\infty)}(q) = \psi\mathbf{1}_{(Y_0,\infty)}(q). \end{aligned} \tag{3.5}$$

Turn to the second term $\Gamma_0\frac{1-\Gamma^1}{1-\Gamma}$ of (2.27). Firstly,

$$\Gamma_0 = \gamma_0(\alpha + x, \beta + y, \theta) = e^{-(\alpha+x)X_0 - (\beta+y)Y_0}. \tag{3.6}$$

Recall from (2.13) that $\gamma(\alpha, \beta, \theta) = \delta\big[\theta + \lambda_A(1 - g(\alpha)) + \lambda_B(1 - h(\beta))\big]$. Using (3.1) we get

$$1 - \gamma(\alpha, \beta, \theta) = \frac{\theta(g+\alpha)(h+\beta) + \lambda_A\alpha(h+\beta) + \lambda_B\beta(g+\alpha)}{(\delta+\theta)(g+\alpha)(h+\beta) + \lambda_A\alpha(h+\beta) + \lambda_B\beta(g+\alpha)}. \tag{3.7}$$

Denote $X := X(x) = g + \alpha + x$ and $Y := Y(y) = h + \beta + y$. Then

$$\begin{aligned} \frac{1-\Gamma^1}{1-\Gamma} &= \frac{1-\gamma(\alpha, \beta+y, \theta)}{1-\gamma(\alpha+x, \beta+y, \theta)} \\ &= \frac{\theta(g+\alpha)Y + \lambda_A\alpha Y + \lambda_B(\beta+y)(g+\alpha)}{(\delta+\theta)(g+\alpha)Y + \lambda_A\alpha Y + \lambda_B(\beta+y)(g+\alpha)} \cdot \frac{1}{\frac{\theta XY + \lambda_A(\alpha+x)Y + \lambda_B(\beta+y)X}{(\delta+\theta)XY + \lambda_A(\alpha+x)Y + \lambda_B(\beta+y)X}}. \end{aligned}$$

Continuing with calculations we have

$$\frac{1-\Gamma^1}{1-\Gamma} = \frac{GY - \lambda_B h(g+\alpha)}{G_\delta Y - \lambda_B h(g+\alpha)} \cdot \frac{(\delta+\Lambda)XY - \lambda_A gY - \lambda_B hX}{\Lambda XY - \lambda_A gY - \lambda_B hX} = f_1(Y)f_2(X,Y), \tag{3.8}$$

where

$$\Lambda := \theta + \lambda_A + \lambda_B, \quad G := \Lambda(g+\alpha) - \lambda_A g, \quad G_\delta := (\delta+\Lambda)(g+\alpha) - \lambda_A g, \tag{3.9}$$

$$f_1(Y) := \frac{GY - \lambda_B h(g+\alpha)}{G_\delta Y - \lambda_B h(g+\alpha)}, \quad f_2(X,Y) := \frac{(\delta+\Lambda)XY - \lambda_A gY - \lambda_B hX}{\Lambda XY - \lambda_A gY - \lambda_B hX}. \tag{3.10}$$

Here is how $f_2(X,Y)$ can be evaluated to separate x and $Y = Y(y)$:

$$f_2(X,Y) = \frac{(\delta+\Lambda)XY - \lambda_A gY - \lambda_B hX}{\Lambda XY - \lambda_A gY - \lambda_B hX} = f_3(Y) + \frac{\xi}{x+a}, \tag{3.11}$$

where

$$f_3(Y) := 1 + \frac{\delta}{\Lambda} + \frac{\lambda_B h\delta}{\Lambda} \cdot \frac{1}{\Lambda Y - \lambda_B h}, \tag{3.12}$$

$$\xi = \xi(Y) := \frac{\lambda_A g\delta Y^2}{(\Lambda Y - \lambda_B h)^2}, \tag{3.13}$$

$$a = a(Y) := g + \alpha - \frac{\lambda_A gY}{\Lambda Y - \lambda_B h}. \tag{3.14}$$

For the upcoming calculations we rewrite the function $f_3(Y)$ as

$$f_3(Y) = b + c \cdot \frac{1}{Y - r}, \tag{3.15}$$

where

$$b := 1 + \frac{\delta}{\Lambda}, \quad c := \frac{\lambda_B h \delta}{\Lambda^2}, \quad r := \frac{\lambda_B h}{\Lambda}. \tag{3.16}$$

(3.11) substituted in (3.8) gives

$$\frac{1 - \mathit{\Gamma}^1}{1 - \mathit{\Gamma}} = f_1(Y)\left(f_3(Y) + \frac{\xi}{x + a}\right). \tag{3.17}$$

Due to (3.5) and (3.6),

$$\mathit{\Gamma}_0 = e^{-(\alpha+x)X_0-(\beta+y)Y_0} = e^{-\alpha X_0-\beta Y_0}e^{-xX_0-yY_0} = \psi \cdot e^{-xX_0-yY_0}. \tag{3.18}$$

With (3.17) and (3.18) substituted in $\mathit{\Gamma}_0\frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}$, we arrive at

$$\mathit{\Gamma}_0\frac{1 - \mathit{\Gamma}^1}{1 - \mathit{\Gamma}} = \psi \cdot e^{-xX_0-yY_0} f_1(Y)\left(f_3(Y) + \frac{\xi}{x + a}\right). \tag{3.19}$$

Now we apply the Laplace-Carson inverse to (3.19):

$$\begin{aligned}
&\mathcal{LC}_{xy}^{-1}\left(\mathit{\Gamma}_0\frac{1 - \mathit{\Gamma}^1}{1 - \mathit{\Gamma}}\right)(p, q) = \mathfrak{L}_{xy}^{-1}\left(\frac{1}{xy} \cdot \mathit{\Gamma}_0\frac{1 - \mathit{\Gamma}^1}{1 - \mathit{\Gamma}}\right)(p, q) \\
&= \mathfrak{L}_{xy}^{-1}\left\{\frac{1}{y} \cdot \psi \cdot e^{-yY_0} f_1(Y)\left(f_3(Y) \cdot \frac{1}{x}e^{-xX_0} + \frac{\xi}{a} \cdot e^{-xX_0}\left(\frac{1}{x} - \frac{1}{x + a}\right)\right)\right\}(p, q).
\end{aligned}$$

By Fubini's theorem, we can apply univariate Laplace inverses first in x and then in y. So

$$\begin{aligned}
&\mathcal{LC}_{xy}^{-1}\left(\mathit{\Gamma}_0\frac{1 - \mathit{\Gamma}^1}{1 - \mathit{\Gamma}}\right)(p, q) \\
&= \mathfrak{L}_{y}^{-1}\left\{\psi \cdot e^{-yY_0}\left[\frac{1}{y}f_1(Y)f_3(Y) + \frac{1}{y}f_1(Y)\frac{\xi}{a}\left(1 - e^{-a(p-X_0)}\right)\right]\mathbf{1}_{(X_0,\infty)}(p)\right\}(q).
\end{aligned} \tag{3.20}$$

To make (3.20) inversely transformable in a closed form we decompose the underlying expressions with respect to y. The partial fraction decomposition will be rendered throughout.

Let

$$\sigma := \lambda_B h(g + \alpha) \quad \text{and} \quad f_1(Y) = \frac{GY - \lambda_B h(g + \alpha)}{G_\delta Y - \lambda_B h(g + \alpha)} = \frac{GY - \sigma}{G_\delta Y - \sigma}. \tag{3.21}$$

Then the partial fraction decomposition of $\frac{1}{y}f_1(Y)$ gives

$$\frac{1}{y}f_1(Y) = \frac{A}{y} + \frac{B}{G_\delta Y - \sigma}, \tag{3.22}$$

with

$$A = \frac{G(h + \beta) - \sigma}{G_\delta(h + \beta) - \sigma}, \quad B = \frac{\sigma(G_\delta - G)}{G_\delta(h + \beta) - \sigma}. \tag{3.23}$$

Continuing working on the first term $\frac{1}{y}f_1(Y)f_3(Y)$ of (3.20) and using (3.15) and (3.22) we get

$$\frac{1}{y}f_1(Y)f_3(Y) = \frac{Ab}{y} + \frac{Bb}{G_\delta Y - \sigma} + \frac{Ac}{y}\cdot\frac{1}{Y-r} + \frac{Bc}{G_\delta Y - \sigma}\cdot\frac{1}{Y-r},$$

in notation,

$$= \varphi_1(y) + \varphi_2(y) + \varphi_3(y) + \varphi_4(y). \tag{3.24}$$

Here is the partial fraction decomposition of $\varphi_3(y)$, and $\varphi_4(y)$. We distinguish three cases with various combinations of $\alpha \neq 0$, $\alpha = 0$, $\delta \neq \lambda_A$, and $\delta = \lambda_A$.

***(i)* Case $\boldsymbol{\alpha \neq 0}$.**

$$\varphi_3(y) = \frac{A_3}{y} + \frac{B_3}{Y-r} \quad \text{and} \quad \varphi_4(y) = \frac{A_4}{G_\delta Y - \sigma} + \frac{B_4}{Y-r}, \tag{3.25}$$

where

$$A_3 = -B_3 = \frac{Ac}{h+\beta-r}, \quad A_4 = (-G_\delta)B_4 = \frac{G_\delta Bc}{\sigma - rG_\delta}. \tag{3.26}$$

Substituting (3.25) into (3.24) we have

$$\begin{aligned}\frac{1}{y}f_1(Y)f_3(Y) &= \frac{Ab}{y} + \frac{Bb/G_\delta}{Y-\sigma/G_\delta} + \left(\frac{A_3}{y} - \frac{A_3}{Y-r}\right) + \left(\frac{B_4}{Y-r} - \frac{B_4}{Y-\sigma/G_\delta}\right)\\ &= \Big(Ab + A_3\Big)\frac{1}{y} + \Big(B_4 - A_3\Big)\frac{1}{Y-r} + \left(\frac{Bb}{G_\delta} - B_4\right)\frac{1}{Y-\sigma/G_\delta}\end{aligned} \tag{3.27}$$

***(ii)* Case $\boldsymbol{\alpha = 0}$ and $\boldsymbol{\delta \neq \lambda_A}$.** Here we have

$$\varphi_1(y) = \frac{\theta(h+\beta) + \lambda_B\beta}{(\delta+\theta)(h+\beta) + \lambda_B\beta}\left(1 + \frac{\delta}{\Lambda}\right)\cdot\frac{1}{y}, \tag{3.28}$$

$$\varphi_2(y) = \frac{\lambda_B h\delta}{(\delta+\theta)(h+\beta) + \lambda_B\beta}\cdot\frac{1}{\delta+\theta+\lambda_B}\left(1 + \frac{\delta}{\Lambda}\right)\cdot\frac{1}{Y - \frac{\lambda_B h}{\delta+\theta+\lambda_B}}, \tag{3.29}$$

$$\varphi_3(y) = \frac{\theta(h+\beta) + \lambda_B\beta}{(\delta+\theta)(h+\beta) + \lambda_B\beta}\cdot\frac{c}{h+\beta-r}\left(\frac{1}{y} + \frac{-1}{Y-r}\right), \tag{3.30}$$

$$\varphi_4(y) = \frac{\lambda_B h\delta^2}{(\delta+\theta)(h+\beta) + \lambda_B\beta}\cdot\frac{1}{\Lambda(\delta-\lambda_A)}\left(\frac{-1}{Y - \frac{\lambda_B h}{\delta+\theta+\lambda_B}} + \frac{1}{Y-r}\right). \tag{3.31}$$

Substituting (3.28)–(3.31) into (3.24) we obtain

$$\begin{aligned}\frac{1}{y}f_1(Y)f_3(Y) &= \frac{\theta(h+\beta) + \lambda_B\beta}{(\delta+\theta)(h+\beta) + \lambda_B\beta}\left(1 + \frac{\delta(h+\beta)}{\Lambda(h+\beta-r)}\right)\frac{1}{y}\\ &+ \frac{-\lambda_A\lambda_B h\delta}{(\delta-\lambda_A)(\delta+\theta+\lambda_B)}\cdot\frac{1}{(\delta+\theta)(h+\beta) + \lambda_B\beta}\cdot\frac{1}{Y - \frac{\lambda_B h}{\delta+\theta+\lambda_B}}\\ &+ \frac{\lambda_A\lambda_B h\delta}{\Lambda(\delta-\lambda_A)}\cdot\frac{1}{\Lambda(h+\beta) - \lambda_B h}\cdot\frac{1}{Y-r}.\end{aligned} \tag{3.32}$$

***(iii)* Case $\boldsymbol{\alpha = 0}$ and $\boldsymbol{\delta = \lambda_A}$.** Here we have

$$\varphi_1(y) = \frac{\theta(h+\beta)+\lambda_B\beta}{\Lambda(h+\beta)-\lambda_B h}\left(1+\frac{\lambda_A}{\Lambda}\right)\cdot\frac{1}{y}, \tag{3.33}$$

$$\varphi_2(y) = \frac{\lambda_A\lambda_B h}{\Lambda(h+\beta)-\lambda_B h}\cdot\frac{1}{\Lambda}\left(1+\frac{\lambda_A}{\Lambda}\right)\cdot\frac{1}{Y-r}, \tag{3.34}$$

$$\varphi_3(y) = \frac{\theta(h+\beta)+\lambda_B\beta}{(h+\beta-r)^2}\cdot\frac{\lambda_A\lambda_B h}{\Lambda^3}\left(\frac{1}{y}+\frac{-1}{Y-r}\right), \tag{3.35}$$

$$\varphi_4(y) = \frac{\lambda_A\lambda_B^2 h^2\delta}{\Lambda(h+\beta)-\lambda_B h}\cdot\frac{1}{\Lambda^3}\left(\frac{1}{Y-r}\right)^2. \tag{3.36}$$

Substituting (3.33)–(3.36) into (3.24) we finally have

$$\begin{aligned}\frac{1}{y}f_1(Y)f_3(Y) &= \frac{\theta(h+\beta)+\lambda_B\beta}{\Lambda(h+\beta)-\lambda_B h}\left(1+\frac{\lambda_A(h+\beta)}{\Lambda(h+\beta)-\lambda_B h}\right)\frac{1}{y}\\ &+\frac{\lambda_A^2\lambda_B h}{\Lambda^2}\cdot\frac{2\Lambda(h+\beta)-\lambda_B h}{[\Lambda(h+\beta)-\lambda_B h]^2}\cdot\frac{1}{Y-r}\\ &+\frac{1}{\Lambda^3}\cdot\frac{\lambda_A^2\lambda_B^2 h^2}{\Lambda(h+\beta)-\lambda_B h}\left(\frac{1}{Y-r}\right)^2.\end{aligned} \tag{3.37}$$

Now turn to the factor $\frac{1}{y}f_1(Y)\frac{\xi}{a}$ in the second term of (3.20). We begin with evaluation of $\frac{\xi}{a}$ substituting (3.13) and (3.14),

$$\frac{\xi}{a} = \eta\cdot\frac{Y^2}{Y-r}\cdot\frac{1}{Y-R}, \tag{3.38}$$

where

$$\eta := \frac{\lambda_A g\delta}{\Lambda\big[\Lambda(g+\alpha)-\lambda_A g\big]} = \frac{\lambda_A g\delta}{\Lambda G} \quad\text{and}\quad R := \frac{\lambda_B h(g+\alpha)}{\Lambda(g+\alpha)-\lambda_A g} = \frac{\sigma}{G}. \tag{3.39}$$

Represent the last two factors $\frac{Y^2}{Y-r}\cdot\frac{1}{Y-R}$ of (3.38)–(3.39) as

$$\frac{Y^2}{Y-r}\cdot\frac{1}{Y-R} = 1+\frac{r^2}{(Y-r)(r-R)}-\frac{R^2}{(Y-R)(r-R)}. \tag{3.40}$$

With (3.40) substituted in (3.38) we have

$$\frac{\xi}{a} = \eta\cdot\frac{Y^2}{Y-r}\cdot\frac{1}{Y-R} = \eta\cdot\left(1+\frac{r^2}{(Y-r)(r-R)}-\frac{R^2}{(Y-R)(r-R)}\right). \tag{3.41}$$

Further, substituting (3.22) and (3.41) into the second term $\frac{1}{y}f_1(Y)\frac{\xi}{a}$ of (3.20), we arrive at

$$\begin{aligned}\frac{1}{y}f_1(Y)\frac{\xi}{a} &= \eta\cdot\left(\frac{A}{y}+\frac{B/G_\delta}{Y-\sigma/G_\delta}\right)+\alpha_1\cdot\frac{1}{y}\cdot\frac{1}{Y-r}+\alpha_2\cdot\frac{1}{Y-\sigma/G_\delta}\cdot\frac{1}{Y-r}\\ &+\alpha_3\cdot\frac{1}{y}\cdot\frac{1}{Y-R}+\alpha_4\cdot\frac{1}{Y-\sigma/G_\delta}\cdot\frac{1}{Y-R},\end{aligned} \tag{3.42}$$

where

$$\alpha_1 := \frac{Ar^2\eta}{r-R}, \quad \alpha_2 := \frac{Br^2\eta}{G_\delta(r-R)}, \quad \alpha_3 := -\frac{AR^2\eta}{r-R}, \quad \alpha_4 := -\frac{BR^2\eta}{G_\delta(r-R)}. \tag{3.43}$$

Continuing with calculations, after a decomposition and some algebra, we arrive at:

(*i*) Case $\boldsymbol{\alpha \neq 0}$.

$$\frac{1}{y}f_1(Y)\frac{\xi}{a} = a_1 \cdot \frac{1}{y} + a_2 \cdot \frac{1}{Y-\sigma/G_\delta} + a_3 \cdot \frac{1}{Y-R} + a_4 \cdot \frac{1}{Y-r}, \tag{3.44}$$

where

$$a_1 = \eta A + \frac{\alpha_1}{h+\beta-r} + \frac{\alpha_3}{h+\beta-R}, \tag{3.45}$$

$$a_2 = \frac{\eta B}{G_\delta} + \frac{\alpha_2}{\sigma/G_\delta - r} + \frac{\alpha_4}{\sigma/G_\delta - R}, \tag{3.46}$$

$$a_3 = \frac{-\alpha_3}{h+\beta-R} + \frac{-\alpha_4}{\sigma/G_\delta - R}, \tag{3.47}$$

$$a_4 = \frac{-\alpha_1}{h+\beta-r} + \frac{-\alpha_2}{\sigma/G_\delta - r}. \tag{3.48}$$

(*ii*) Case $\boldsymbol{\alpha = 0}$ and $\boldsymbol{\delta \neq \lambda_A}$.

$$\begin{aligned}\frac{1}{y}f_1(Y)\frac{\xi}{a} &= \frac{\lambda_A\delta(h+\beta)^2}{\Lambda(h+\beta)-\lambda_B h} \cdot \frac{1}{(\delta+\theta)(h+\beta)+\lambda_B\beta} \cdot \frac{1}{y} \\ &\quad + \frac{-\lambda_A\lambda_B h\delta}{\Lambda(\delta-\lambda_A)} \cdot \frac{1}{\Lambda(h+\beta)-\lambda_B h} \cdot \frac{1}{Y-r} \\ &\quad + \frac{\lambda_A\lambda_B h\delta}{(\delta-\lambda_A)(\delta+\theta+\lambda_B)} \cdot \frac{1}{(\delta+\theta)(h+\beta)+\lambda_B\beta} \cdot \frac{1}{Y-\frac{\lambda_B h}{\delta+\theta+\lambda_B}}.\end{aligned} \tag{3.49}$$

(*iii*) Case $\boldsymbol{\alpha = 0}$ and $\boldsymbol{\delta = \lambda_A}$.

$$\begin{aligned}\frac{1}{y}f_1(Y)\frac{\xi}{a} &= \frac{\lambda_A^2(h+\beta)^2}{[\Lambda(h+\beta)-\lambda_B h]^2} \cdot \frac{1}{y} + \frac{-\lambda_A^2\lambda_B h}{\Lambda^2} \cdot \frac{2\Lambda(h+\beta)-\lambda_B h}{[\Lambda(h+\beta)-\lambda_B h]^2} \cdot \frac{1}{Y-r} \\ &\quad + \frac{-\lambda_A^2\lambda_B^2 h^2}{\Lambda^3} \cdot \frac{1}{\Lambda(h+\beta)-\lambda_B h}\left(\frac{1}{Y-r}\right)^2\end{aligned} \tag{3.50}$$

With (3.32) and (3.44) substituted into (3.20) we have

(*i*) Case $\boldsymbol{\alpha \neq 0}$.

$$\begin{aligned}\mathcal{L}\mathcal{C}_{xy}^{-1}\left(\mathit{\Gamma}_0\frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}\right)(p,q) &= \mathfrak{L}_y^{-1}\Bigg\{\psi \cdot e^{-yY_0}\Bigg[\Big(Ab + A_3 + a_1\Big)\frac{1}{y} \\ &\quad + \left(\frac{Bb}{G_\delta} - B_4 + a_2\right)\frac{1}{Y-\sigma/G_\delta} + a_3 \cdot \frac{1}{Y-R} + \Big(B_4 - A_3 + a_4\Big)\frac{1}{Y-r} \\ &\quad - e^{-a(p-X_0)}\left(a_1 \cdot \frac{1}{y} + a_2 \cdot \frac{1}{Y-\sigma/G_\delta} + a_3 \cdot \frac{1}{Y-R} + a_4 \cdot \frac{1}{Y-r}\right)\Bigg] \\ &\quad \times \mathbf{1}_{(X_0,\infty)}(p)\Bigg\}(q).\end{aligned} \tag{3.51}$$

Correspondingly, we modify the above components in (3.51). After some algebra in (3.26) and (3.45) and the use of notation (3.16), (3.23), (3.39), and (3.43) we arrive at

$$A_3 = \frac{Ac}{h+\beta-r} = \frac{A}{h+\beta-r}\cdot\frac{\lambda_B h\delta}{\Lambda^2} = \frac{A}{h+\beta-r}\cdot\frac{r\delta}{\Lambda}, \tag{3.52}$$

$$a_1 = \eta A + \frac{\alpha_1}{h+\beta-r} + \frac{\alpha_3}{h+\beta-R} = \frac{\lambda_A g\delta(h+\beta)^2}{\Lambda(h+\beta-r)[\,G_\delta(h+\beta)-\sigma]}. \tag{3.53}$$

With (3.16) and (3.52)–(3.53) substituted into $Ab + A_3 + a_1$ we finally have

$$Ab + A_3 + a_1 = A(1+\frac{\delta}{\Lambda}) + \frac{A}{h+\beta-r}\cdot\frac{r\delta}{\Lambda} + \frac{\lambda_A g\delta(h+\beta)^2}{\Lambda(h+\beta-r)[\,G_\delta(h+\beta)-\sigma]} = 1. \tag{3.54}$$

We continue calculating $\frac{Bb}{G_\delta} - B_4 + a_2$ in (3.51). After some algebra we arrive at

$$\begin{aligned}\mathcal{LC}_{xy}^{-1}\Big(\mathit{\Gamma}_0\frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}\Big)(p,q) &= \mathfrak{L}_y^{-1}\Big\{\psi\cdot e^{-yY_0}\Big[\frac{1}{y} - e^{-a(p-X_0)}\Big(a_1\cdot\frac{1}{y}\\ &+ a_2\cdot\frac{1}{Y-\sigma/G_\delta} + a_4\cdot\frac{1}{Y-r}\Big)\Big]\times\mathbf{1}_{(X_0,\infty)}(p)\Big\}(q),\end{aligned} \tag{3.55}$$

where

$$a_1 = \frac{\lambda_A g\delta(h+\beta)^2}{\Lambda(h+\beta-r)[\,G_\delta(h+\beta)-\sigma]},\quad a_2 = \frac{\lambda_A g B}{G_\delta G'_\delta}, \tag{3.56}$$

$$a_4 = \frac{-Ar^2\lambda_A g\delta}{\Lambda G(r-R)(h+\beta-r)} + \frac{Br\lambda_A g\delta}{\Lambda G G'_\delta(r-R)}. \tag{3.57}$$

***(ii)* Case $\boldsymbol{\alpha = 0}$ and $\boldsymbol{\delta \neq \lambda_A}$.** Substituting (3.32) and (3.49) into (3.20), we have

$$\begin{aligned}&\mathcal{LC}_{xy}^{-1}\big(\mathit{\Gamma}_0\frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}\big)(p,q)\\ &= \mathfrak{L}_y^{-1}\Big\{\psi\cdot e^{-yY_0}\Big[\frac{1}{y} + \Big(\frac{-\lambda_A\delta(h+\beta)^2}{\Lambda(h+\beta)-\lambda_B h}\cdot\frac{1}{(\delta+\theta)(h+\beta)+\lambda_B\beta}\cdot\frac{1}{y}\\ &+ \frac{\lambda_A\lambda_B h\delta}{\Lambda(\delta-\lambda_A)}\cdot\frac{1}{\Lambda(h+\beta)-\lambda_B h}\cdot\frac{1}{Y-r}\\ &+ \frac{-\lambda_A\lambda_B h\delta}{(\delta-\lambda_A)(\delta+\theta+\lambda_B)}\cdot\frac{1}{(\delta+\theta)(h+\beta)+\lambda_B\beta}\cdot\frac{1}{Y-\frac{\lambda_B h}{\delta+\theta+\lambda_B}}\Big)e^{-a(p-X_0)}\Big]\\ &\times\mathbf{1}_{(X_0,\infty)}(p)\Big\}(q).\end{aligned} \tag{3.58}$$

***(iii)* Case $\boldsymbol{\alpha = 0}$ and $\boldsymbol{\delta = \lambda_A}$.** Substituting (3.37) and (3.50) into (3.20), we get

$$\begin{aligned}&\mathcal{LC}_{xy}^{-1}\big(\mathit{\Gamma}_0\frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}\big)(p,q)\\ &= \mathfrak{L}_y^{-1}\Big\{\psi\cdot e^{-yY_0}\Big[\frac{1}{y} + \Big(\frac{-\lambda_A^2(h+\beta)^2}{[\Lambda(h+\beta)-\lambda_B h]^2}\cdot\frac{1}{y} + \frac{\lambda_A^2\lambda_B h}{\Lambda^2}\cdot\frac{2\Lambda(h+\beta)-\lambda_B h}{[\Lambda(h+\beta)-\lambda_B h]^2}\cdot\frac{1}{Y-r}\\ &+ \frac{\lambda_A^2\lambda_B^2 h^2}{\Lambda^3}\cdot\frac{1}{\Lambda(h+\beta)-\lambda_B h}\big(\frac{1}{Y-r}\big)^2\Big)e^{-a(p-X_0)}\Big]\mathbf{1}_{(X_0,\infty)}(p)\Big\}(q).\end{aligned} \tag{3.59}$$

Now we need to handle $e^{-a(p-X_0)}$ in (3.59) before inversely transforming the rest of the terms. Unfold (3.14) by using notation (3.16), we have

$$a = g + \alpha - \frac{\lambda_A g}{\Lambda} \cdot \frac{Y}{Y - \frac{\lambda_B h}{\Lambda}} = \xi_1 - \xi_2 \cdot \frac{1}{Y - r}, \tag{3.60}$$

where

$$\xi_1 = g + \alpha - \frac{\lambda_A g}{\Lambda}, \quad \xi_2 = \frac{\lambda_A \lambda_B hg}{\Lambda^2}. \tag{3.61}$$

We finally have

$$e^{-a(p-X_0)} = e^{-(\xi_1 - \xi_2 \cdot \frac{1}{Y-r})(p-X_0)} = e^{-\xi_1(p-X_0)} e^{\xi_2(p-X_0)\cdot\frac{1}{Y-r}}, \tag{3.62}$$

where $\xi_2(p - X_0)$ is positive (if $p > X_0$) since $\xi_2 > 0$.

Now, we will apply the univariate Laplace-Carson inverse in y to (3.55), (3.58), and (3.59). We will make use of the following formulas for the Laplace inverse (cf. [2,3]):

$$\mathcal{L}_y^{-1}\left(e^{-\alpha y} \cdot \frac{1}{y+b}\right)(q) = e^{-b(q-\alpha)} \mathbf{1}_{(\alpha,\infty)}(q), \tag{3.63}$$

$$\mathcal{L}_y^{-1}\left(e^{-\alpha y} \cdot \frac{e^{\frac{a}{y+b}}}{y+b}\right)(q) = e^{-b(q-\alpha)} I_0\big(2\sqrt{a(q-\alpha)}\big) \mathbf{1}_{(\alpha,\infty)}(q), \tag{3.64}$$

$$\begin{aligned}\mathcal{L}_y^{-1}\left(e^{-\alpha y} \cdot \frac{e^{\frac{a}{y+b_1}}}{y+b_2}\right)(q) &= e^{-b_1(q-\alpha)} I_0\big(2\sqrt{a(q-\alpha)}\big) \mathbf{1}_{(\alpha,\infty)}(q) \\ &+ (b_1 - b_2) \cdot e^{-b_2(q-\alpha)} \int_{z=0}^{q-\alpha} e^{(b_2-b_1)z} I_0\big(2\sqrt{az}\big) dz \mathbf{1}_{(\alpha,\infty)}(q),\end{aligned} \tag{3.65}$$

$$\mathcal{L}_y^{-1}\left(e^{-\alpha y} \cdot \frac{e^{\frac{a}{y+b}}}{(y+b)^2}\right)(q) = \sqrt{\frac{q-\alpha}{a}} \cdot e^{-b(q-\alpha)} I_1\big(2\sqrt{a(q-\alpha)}\big) \mathbf{1}_{(\alpha,\infty)}(q), \tag{3.66}$$

where I_0 and I_1 are the modified Bessel functions of order zero and one, respectively. Equation (3.65) can be readily proved, while the rest of the above formulas can be found in references [2,3].

***(i)* Case $\boldsymbol{\alpha \neq 0}$.** Using (3.63)–(3.66) in (3.55), then combining it with (3.5) we finally have

$$\begin{aligned}&\mathcal{LC}_{xy}^{-1}\left(\Gamma_0^1 - \Gamma_0 \frac{1-\Gamma^1}{1-\Gamma}\right)(p,q) \\ &= \psi\Big\{(a_1 + a_2 + a_4) e^{-\xi_1(p-X_0)} e^{-(h+\beta-r)(q-Y_0)} I_0\big(2\sqrt{\xi_2(p-X_0)(q-Y_0)}\big) \\ &\quad + a_1(h+\beta-r) e^{-\xi_1(p-X_0)} \int_{z=0}^{q-Y_0} e^{-(h+\beta-r)z} I_0\big(2\sqrt{\xi_2(p-X_0)z}\big) dz \\ &\quad + a_2\left(\frac{\sigma}{G_\delta} - r\right) e^{-\xi_1(p-X_0)} e^{-(h+\beta-\frac{\sigma}{G_\delta})(q-Y_0)} \\ &\quad \times \int_{z=0}^{q-Y_0} e^{(r-\frac{\sigma}{G_\delta})z} I_0\big(2\sqrt{\xi_2(p-X_0)z}\big) dz\Big\} \mathbf{1}_{(X_0,\infty)}(p) \mathbf{1}_{(Y_0,\infty)}(q).\end{aligned} \tag{3.67}$$

Calculating $a_1 + a_2 + a_4$ and other terms we arrive at

$$\begin{aligned}
&\mathcal{LC}_{xy}^{-1}\big(\mathit{\Gamma}_0^1 - \mathit{\Gamma}_0 \frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}\big)(p,q) \\
&\quad = \psi \Big\{ \frac{\lambda_A g \delta}{\Lambda G_\delta} \cdot e^{-\xi_1(p-X_0)} e^{-(h+\beta-r)(q-Y_0)} I_0\big(2\sqrt{\xi_2(p-X_0)(q-Y_0)}\big) \\
&\qquad + \frac{\lambda_A g \delta (h+\beta)^2}{\Lambda[\,G_\delta(h+\beta)-\sigma]} \cdot e^{-\xi_1(p-X_0)} \int_{z=0}^{q-Y_0} e^{-(h+\beta-r)z} I_0\big(2\sqrt{\xi_2(p-X_0)z}\big)dz \\
&\qquad + \frac{-\lambda_A g \delta \sigma^2}{\Lambda G_\delta^2 [G_\delta(h+\beta)-\sigma]} \cdot e^{-\xi_1(p-X_0)} e^{-(h+\beta-\frac{\sigma}{G_\delta})(q-Y_0)} \\
&\qquad \times \int_{z=0}^{q-Y_0} e^{(r-\frac{\sigma}{G_\delta})z} I_0\big(2\sqrt{\xi_2(p-X_0)z}\big)dz \Big\} \mathbf{1}_{(X_0,\infty)}(p)\mathbf{1}_{(Y_0,\infty)}(q).
\end{aligned} \tag{3.68}$$

***(ii)* Case $\boldsymbol{\alpha = 0}$ and $\boldsymbol{\delta \neq \lambda_A}$.** Using (3.63)–(3.66) in (3.58) and then (3.5) we have

$$\begin{aligned}
&\mathcal{LC}_{xy}^{-1}\big(\mathit{\Gamma}_0^1 - \mathit{\Gamma}_0 \frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}\big)(p,q) \\
&\quad = e^{-\beta Y_0} \Big\{ \frac{\lambda_A \delta}{\Lambda(\delta+\theta+\lambda_B)} \cdot e^{-\xi_1(p-X_0)} e^{-(h+\beta-r)(q-Y_0)} I_0\big(2\sqrt{\xi_2(p-X_0)(q-Y_0)}\big) \\
&\qquad + \frac{\lambda_A \delta (h+\beta)^2}{\Lambda} \cdot \frac{1}{(\delta+\theta)(h+\beta)+\lambda_B \beta} \cdot e^{-\xi_1(p-X_0)} \\
&\qquad \times \int_{z=0}^{q-Y_0} e^{-(h+\beta-r)z} I_0\big(2\sqrt{\xi_2(p-X_0)z}\big)dz \\
&\qquad + \frac{-\lambda_A \lambda_B^2 h^2 \delta}{\Lambda(\delta+\theta+\lambda_B)^2} \cdot \frac{1}{(\delta+\theta)(h+\beta)+\lambda_B\beta} \cdot e^{-\xi_1(p-X_0)} \cdot e^{-(h+\beta-\frac{\lambda_B h}{\delta+\theta+\lambda_B})(q-Y_0)} \\
&\qquad \times \int_{z=0}^{q-Y_0} e^{(r-\frac{\lambda_B h}{\delta+\theta+\lambda_B})z} I_0\big(2\sqrt{\xi_2(p-X_0)z}\big)dz \Big\} \mathbf{1}_{(X_0,\infty)}(p)\mathbf{1}_{(Y_0,\infty)}(q).
\end{aligned} \tag{3.69}$$

***(iii)* Case $\boldsymbol{\alpha = 0}$ and $\boldsymbol{\delta = \lambda_A}$.** Using (3.63)–(3.66) in (3.59) then (3.5) we get

$$\begin{aligned}
&\mathcal{LC}_{xy}^{-1}\left(\mathit{\Gamma}_0^1 - \mathit{\Gamma}_0 \frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}\right)(p,q) \\
&\quad = e^{-\beta Y_0} \Big\{ \frac{\lambda_A^2}{\Lambda^2} \cdot e^{-\xi_1(p-X_0)} e^{-(h+\beta-r)(q-Y_0)} I_0\big(2\sqrt{\xi_2(p-X_0)(q-Y_0)}\big) \\
&\qquad + \frac{\lambda_A^2 (h+\beta)^2}{\Lambda} \cdot \frac{1}{\Lambda(h+\beta)-\lambda_B h} \cdot e^{-\xi_1(p-X_0)} \\
&\qquad \times \int_{z=0}^{q-Y_0} e^{-(h+\beta-r)z} I_0 \ bigl(2\sqrt{\xi_2(p-X_0)z}\big)dz \\
&\qquad + \frac{-\lambda_A^2 \lambda_B^2 h^2}{\Lambda^3} \cdot \frac{1}{\Lambda(h+\beta)-\lambda_B h} \sqrt{\frac{q-Y_0}{\xi_2(p-X_0)}} \cdot e^{-\xi_1(p-X_0)} e^{-(h+\beta-r)(q-Y_0)} \\
&\qquad \times I_1\big(2\sqrt{\xi_2(p-X_0)(q-Y_0)}\big) \Big\} \mathbf{1}_{(X_0,\infty)}(p)\mathbf{1}_{(Y_0,\infty)}(q),
\end{aligned} \tag{3.70}$$

with the abbreviations:

$$\psi = e^{-\alpha X_0 - \beta Y_0}, \quad \Lambda = \theta + \lambda_A + \lambda_B, \quad \sigma = \lambda_B h(g+\alpha), \quad \xi_1 = g + \alpha - \frac{\lambda_A g}{\Lambda}, \tag{3.71}$$

$$\xi_2 = \frac{\lambda_A \lambda_B h g}{\Lambda^2}, \quad r = \frac{\lambda_B h}{\Lambda}, \quad G_\delta = (\delta + \Lambda)(g+\alpha) - \lambda_A g. \tag{3.72}$$

4 Marginal Functionals

Our next goal is to get the marginal transforms. This can be directly obtained from the version of $\Phi_{\mu\nu}(\alpha, \beta, \theta)$ in (3.68)–(3.70).

Case 1. With $\beta = \theta = 0$ we have the marginal Laplace–Stieltjes transform of the amount of casualties to player A at the A's ruin (which is the exit of the game):

$$\begin{aligned}
\Phi_{\mu\nu}(\alpha,0,0) &:= E\Big[e^{-\alpha A_\mu}\mathbf{1}_{\{\mu<\nu\}}\Big] \\
&= \Bigg\{ \frac{\lambda_A g\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)} \cdot \frac{1}{\alpha + \frac{(\delta+\lambda_B)g}{\delta+\lambda_A+\lambda_B}} \cdot e^{-\alpha M} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} \\
&\quad \times e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})(N-Y_0)} I_0\left(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\right) + \frac{\lambda_A hg\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)} \\
&\quad \times \frac{1}{\alpha+\frac{g\delta}{\delta+\lambda_A}} \cdot e^{-\alpha M} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} \int_{z=0}^{N-Y_0} e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})z} \\
&\quad \times I_0\left(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\right) dz + \int_{z=0}^{N-Y_0} \Bigg[\frac{-\lambda_A hg\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)} \cdot \frac{1}{\alpha+\frac{g\delta}{\delta+\lambda_A}} \\
&\quad + \frac{\lambda_A hg\delta(\delta+\lambda_A+2\lambda_B)}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)^2} \cdot \frac{1}{\alpha+\frac{(\delta+\lambda_B)g}{\delta+\lambda_A+\lambda_B}} \\
&\quad + \frac{\lambda_A^2\lambda_B hg^2\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)^3}\left(\frac{1}{\alpha+\frac{(\delta+\lambda_B)g}{\delta+\lambda_A+\lambda_B}}\right)^2 \Bigg] e^{-\alpha M} \\
&\quad \times e^{(\frac{\lambda_A\lambda_B hg}{(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{1}{\alpha+\frac{(\delta+\lambda_B)g}{\delta+\lambda_A+\lambda_B}})(N-Y_0-z)} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} e^{-(\frac{(\delta+\lambda_A)h}{\delta+\lambda_A+\lambda_B})(N-Y_0)} \\
&\quad \times e^{(\frac{\lambda_B h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)})z} I_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\Big) dz \Bigg\} \mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N).
\end{aligned} \tag{4.1}$$

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References

[1] Basar, T.S. and Olsder, G.J. *Dynamic Noncooperative Game Theory.* Academic Press, Orlando, 1982.

[2] Bateman, H. and Erdélyi, A. *Tables of Integral Transforms.* Vol. I. McGraw-Hill, 1954.

[3] Bateman, H. and Erdélyi, A. *Higher Transcedental Functions.* Vol. 2. McGraw-Hill, 1953.

[4] Dshalalow, J.H. and Ke, H-J. Layers of noncooperative games. *Nonlinear Analysis, Series A, Theory and Methods* **71** (2009) 283–291.

[5] Dshalalow, J.H. Random walk analysis in antagonistic stochastic games. *Stochastic Analysis and Applications* **26** (2008) 738–783.

[6] Dshalalow, J.H., Huang, W., Ke, H-J. and Treerattrakoon, A. On antagonistic games with constant initial condition. Marginal functionals. *Nonlinear Dynamics and System Theory* **16** (3) (2016) (to appear).

[7] Fishburn, P.C. Non-cooperative stochastic dominance games. *International Journal of Game Theory* **7** (1) (1978) 51–61.

[8] Konstantinov, R.V. and Polovinkin, E.S. Mathematical simulation of a dynamic game in the enterprise competition problem. *Cybernetics and Systems Analysis* **40** (5) (2004) 720–725.

[9] Kyprianou, A.E. and Pistorius, M.R. Perpetual options and Canadization through fluctuation theory. *Ann. Appl. Prob.* **13** (3) (2003) 1077–1098.

[10] Redner, S. *A Guide to First-Passage Processes*. Cambridge University Press, Cambridge, 2001.

[11] Shashikin, V.N. Antagonistic game with interval payoff functions. *Cybernetics and Systems Analysis* **40** (4) (2004) 556–564.

[12] Shima, T. Capture a conditions in a Pursuit–Evasion game between players with biproper dynamics. *Journal of Optimization Theory and Applications* **126** (3) (2005) 503–528.

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Existence of Solutions to a New Class of Abstract Non-Instantaneous Impulsive Fractional Integro-Differential Equations

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Abstract: In this paper we prove the sufficient conditions for the existence and uniqueness of piecewise continuous mild solutions to fractional integro-differential equations in a Banach space with non instantaneous impulses. The results are established by using the theory of sectorial operators and the fixed point theorem. We discuss an example to illustrate the analytical results obtained.

Keywords: *sectorial operator; solution operator; non-instantaneous impulses; Krasnoselskii's fixed point theorem.*

Mathematics Subject Classification (2010): 34G20, 34K30, 34K40, 47N20.

1 Introduction

Let $(X, \|\cdot\|)$ be a complex Banach space. The objective of this paper is to study the solutions to a new class of abstract integro-differential equations of fractional order with non-instantaneous impulses in X :

$$\left.\begin{aligned} {}^cD_t^{\alpha}[u(t)+\varphi(t,u(t))] &= Au(t) + J_t^{1-\alpha} f(t,u(t)), \\ & \quad t \in (s_i, t_{i+1}],\ i = 0,1,\cdots,N,\ 0<\alpha<1, \\ u(t) &= g_i(t,u(t)), \quad t \in (t_i, s_i], \quad i = 1,2,\cdots,N, \\ u(0) &= u_0, \end{aligned}\right\} \tag{1}$$

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where $A : D(A) \subset X \to X$ is a sectorial operator on $(X, \|.\|)$, $u_0 \in X$, $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_N \leq s_N \leq t_{N+1} = T_0$ are pre-fixed numbers, $g_i \in C((t_i, s_i] \times X; X)$ and $\varphi : [0, T_0] \times X \to X$, $f : [0, T_0] \times X \to X$ are suitably defined functions. The fractional derivative ${}^cD_t^\alpha$ is to be understood in Caputo sense and J_t^α denotes the Riemann-Liouville integral of order α. This paper is concerned with impulsive differential equations of fractional order, where an impulsive action starts suddenly at the points t_i and their action stays active on the interval $[t_i, s_i]$.

Fractional differential equations arise as models in many fields of engineering and science such as electrochemistry, electro-magnetics, electrical circuits control theory, visco-elasticity, porous media, neuron modelling etc. [5, 9, 13, 15, 16, 18–20, 22]. The plentiful occurrence and applications of fractional differential equations motivate the rapid developments and gained much attention in the recent years and have been studied extensively in [2–4, 6, 7, 14, 23–27, 29, 30]. But systems with non-instantaneous impulses do exist [10, 11]. For example, one can consider the hemodynamical equilibrium of a person in which impulses are non-instantaneous [10]. Such systems for the fractional differential equations are less studied. Recently, Hernández and O'regan introduced and investigated the existence of mild and classical solutions to a new class of abstract differential equations with non-instantaneous impulses in X:

$$\begin{aligned} &u'(t) = Au(t) + f(t, u(t)), \qquad t \in (s_i, t_{i+1}], \quad i = 0, 1, \cdots, N, \\ &u(t) = g_i(t, u(t)), \qquad t \in (t_i, s_i], \quad i = 1, 2, \cdots, N, \\ &u(0) = u_0. \end{aligned} \tag{2}$$

The operator A generates an infinitesimal C_0-semigroup of bounded linear operators $(X, \|.\|)$, the functions $g_i \in C((t_i, s_i] \times X; X)$ for each $i = 1, 2, \cdots, N$ and $f : [0, T_0] \times X \to X$ is a suitable function. The results are established by fixed point theorem with appropriate g_i and f [10].

Kumar et al [12] had extended the work in [10] to the following problem in a Banach space X :

$$\left.\begin{aligned} {}^cD_t^\alpha u(t) + Au(t) &= f(t, u(t), u(g(t))), \\ &\quad t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, N,\ 0 < \alpha < 1, \\ u(t) &= g_i(t, u(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, N, \\ u(0) &= u_0, \end{aligned}\right\} \tag{3}$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative of order α, $-A$ generates an analytic semigroup. The sufficient conditions are obtained if f and h_i are Lipschitz continuous in the second variable appropriately. For more details, we refer to [12].

With the strong motivation from Hernández and O'regan [10]; and Kumar et al. [12], we establish the existence and uniqueness of piecewise continuous mild solution to the class of fractional integro-differential equations (1), where the impulses are non-instantaneous. The main results are new and complement to the existing ones that generalize some results of [10, 12, 23] to the fractional integro-differential equations.

The paper is organized as follows. We collect the basic notations, definitions, lemmas and theorems in Section 2. We prove the existence as well as uniqueness of solution of (1) in Section 3. We provide an example in Section 4 as an application of the analytical results obtained.

2 Preliminaries and Assumptions

In this section, we will introduce some basic definitions, notations and lemmas that are useful throughout this paper. For more details, we refer to [13, 15–20]. For the Banach space X, we denote the Banach space of all bounded linear operator from X into X by $L(X)$. We denote a ball in X of radius r centered at y as $B_r(y, X)$. The set of all m^{th} order continuously differentiable functions from $J(J \subset \mathbb{R})$ into X is denoted by $C^m(J, X)$ for $m \in \mathbb{N}$. We begin with the following definition of sectorial operator.

Definition 2.1 A closed linear operator A is said to be sectorial of type ω if there exist constants $\omega \in \mathbb{R}$, $\theta \in [\frac{\pi}{2}, \pi]$, and $M > 0$ such that

(a) $\rho(A) \subset \Sigma_{\theta,\omega} = \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| < \theta,\ \lambda \neq \omega\}$,

(b) $\|R(\lambda, A)\|_{L(X)} \leq \frac{M}{|\lambda-\omega|}$, $\lambda \in \Sigma_{\theta,\omega}$.

Definition 2.2 For $f \in L^1((0, T), X)$ and $\alpha \geq 0$, we define the Reimann-Liouville integral of order α of f as

$$J_t^\alpha f(t) = (f * \Theta_\alpha)(t) = \frac{1}{\Gamma\alpha} \int_0^t (t-s)^{\alpha-1} f(s) ds,\ t > 0,\ \alpha > 0, \tag{1}$$

where $J_t^0 f(t) = f(t)$ and

$$\Theta_\alpha(t) = \begin{cases} \dfrac{1}{\Gamma\alpha} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and $\Theta_0(t) = 0$.

Definition 2.3 If $f \in C^{m-1}((0, T), X)$ and $(\Theta_{m-\alpha} * f) \in W^{m,1}((0, T), X)$, $0 \leq m - 1 < \alpha < m$, $m \in \mathbb{N}$, then the the Caputo fractional derivative of order α of f is defined as

$${}^cD_t^\alpha f(t) = D_t^m J_t^{m-\alpha} \Big(f(t) - \sum_0^{m-1} f^i(0) \Theta_{i+1}(t) \Big), \tag{2}$$

where $D_t^m = \frac{d^m}{dt^m}$ and

$$W^{m,1}((0, T); X) = \Big\{ f \in X : f^m \in L^1((0, T); X)\ f(t) = \sum_{j=0}^{m-1} f^j(0) \frac{t^j}{j!} + \frac{t^{m-1}}{(m-1)!} * f^m(t) \Big\}.$$

We note the following properties of J_t^α

Lemma 2.1 [28, Proposition 2.4] *For $\alpha, \beta > 0$, we have*

(i) $J_t^\alpha J_t^\beta f(t) = J_t^{\alpha+\beta} f(t)$ *for all* $f \in L^1(J; X)$*;*

(ii) $J_t^\alpha (f * g) = J_t^\alpha f * g$ *for all* $f, g \in L^p(J; X)(1 \leq p < +\infty)$*;*

(iii) The Caputo fractional derivative ${}^cD_t^\alpha$ *is a left inverse of* J_t^α*:*

$${}^cD_t^\alpha J_t^\alpha f = f, \quad \textit{for all } f \in L^1(J; X),$$

but in general not a right inverse, in fact, for all $f(t) \in C^{m-1}(J;X)$ *with* $\Theta_{m-\alpha} * f \in W^{m,1}(J,X) (m \in \mathbb{N},\ 0 \leq m-1 < \alpha < m)$*, one has*

$$J_t^\alpha({}^cD_t^\alpha) f(t) = f(t) - \sum_{i=0}^{m-1} f^{(i)}(0)\Theta_{i+1}(t). \tag{3}$$

We consider the following Cauchy problem

$$\begin{aligned} {}^cD_t^\alpha u(t) + \lambda u(t) &= 0, \qquad t > 0, \\ u(0) &= u_0, \quad 0 < \alpha < 1. \end{aligned} \tag{4}$$

Then the solution of (4) is $u(t) = S(t)u_0$, where $S(t) = E_{\alpha,1}(-\lambda t^\alpha) = E_\alpha(-\lambda t^\alpha)$ [8], where $E_{\alpha,\beta}$ is the generalized Mittag-Leffler function. The generalized Mittag-Leffler function $E_{\alpha,\beta}$ is defined as

$$E_{\alpha,\beta} := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i}\int_\chi \frac{\lambda^{\alpha-\beta}e^\lambda}{\lambda^\alpha - z} d\lambda \quad \text{for} \quad \alpha, \beta > 0, z \in \mathbb{C},$$

where χ is a contour that starts and ends at $-\infty$ and encircles the disc $|\lambda| \leq |z|^{1/\alpha}$ counterclockwise.

Replacing λ by $-A$, we rewrite $S(t)$ as

$$S(t) = \frac{1}{2\pi i}\int_{B_\gamma} e^{\lambda t}\lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda,$$

where B_γ denotes the Bromwich path. Moreover, if A is a sectorial operator of type ω then A is the generator of a solution operator given by

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\Upsilon e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} d\lambda,$$

where Υ is suitable path lying on $\Sigma_{\theta,\omega}$. For more details, we refer the reader to [3, 6, 15, 16, 23, 27–29].

We consider the following Cauchy problem

$$\left.\begin{aligned} {}^cD_t^\alpha[u(t) + \Phi(t)] = Au(t) + J_t^{1-\alpha} f(t), \quad 0 < \alpha < 1, \\ u(0) = u_0 \in X, \end{aligned}\right\} \tag{5}$$

where $f : [0, \infty) \to X$ and A is a sectorial operator. The solution of (5) is given by the following theorem.

Theorem 2.1 *If* f *and* Φ *satisfy the uniform Hölder condition with exponent* $\beta \in (0, 1]$ *and* A *is a sectorial operator, then the unique solution of the Cauchy problem (5) is given by*

$$u(t) = S_\alpha(t)[u_0 + \Phi(0)] - \Phi(t) - \int_0^t T_\alpha(t-s)\Phi(s)ds + \int_0^t S_\alpha(t-s)f(s)ds,$$

where

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha, A)d\lambda, \quad T_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda^\alpha, A)d\lambda,$$

for a suitable path Γ *lying on* $\Sigma_{\theta,\omega}$.

Proof. Applying the Riemann-Liouville fractional integral operator J_t^α to both sides of equation (5), we get

$$J_t^\alpha({}^cD_t^\alpha)[u(t)+\Phi(t)] \quad = \quad J_t^\alpha Au(t) + J_t^1 f(t).$$

Using (1) and (3), we get

$$\begin{aligned} u(t)+\Phi(t) &= [u_0+\Phi(0)] + \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}Au(s)ds + \int_0^t f(s)ds \\ &= [u_0+\Phi(0)] + \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}[Au(s)+\Phi(s)]ds \\ &\quad -\frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}\Phi(s)ds + \int_0^t f(s)ds. \end{aligned} \tag{6}$$

Applying the Laplace transform to equation (6), we get

$$(\mathcal{L}(u+\Phi))(\lambda) = \frac{1}{\lambda}[u_0+\Phi(0)] + \frac{1}{\lambda^\alpha}A(\mathcal{L}(u+\Phi)(\lambda) - \frac{1}{\lambda^\alpha}(\mathcal{L}\Phi)(\lambda) + \frac{1}{\lambda}(\mathcal{L}f)(\lambda).$$

Since $(\lambda^\alpha I - A)^{-1}$ exists, i.e., $\lambda^\alpha \in \rho(A)$, we obtain

$$(\mathcal{L}(u+\Phi))(\lambda) = (\lambda^\alpha I - A)^{-1}\Big[\lambda^{\alpha-1}(u_0+\Phi(0)) - (\mathcal{L}\Phi)(\lambda) + \lambda^{\alpha-1}(\mathcal{L}f)(\lambda)\Big].$$

Applying the inverse Laplace transform, we get

$$u(t) = S_\alpha(t)[u_0+\Phi(0)] - \Phi(t) - \int_0^t T_\alpha(t-s)\Phi(s)ds + \int_0^t S_\alpha(t-s)f(s)ds.$$

□

We define the set $\mathcal{PC}(X)$ for the solution space as follows

$$\mathcal{PC}(X) = \{u : [0,T_0] \to X : u(\cdot) \text{ is continuous at } t \neq t_i, u(t_k^-) = u(t_k), u(t_k^+) \\ \text{exists for all } i = 1,2,\cdots,N\}.$$

We note that $\mathcal{PC}(X)$ is a Banach space endowed with the supremum norm

$$\|u\|_{\mathcal{PC}} := \sup_{t\in[0,T_0]} \|u(t)\|.$$

Now, we define the functions $\widetilde{u}_i \in C([t_i,t_{i+1}];X)$ given by

$$\widetilde{u}_i(t) = \begin{cases} u(t), & \text{for } t \in (t_i,t_{i+1}], \\ u(t_i^+), & \text{for } t = t_i. \end{cases}$$

For a ball $B_r \subseteq \mathcal{PC}(X)$, we define

$$\widetilde{B}_i = \{\widetilde{u}_i : u \in B_r\}.$$

The following Arzela-Ascoli type lemma will be used to establish the main result.

Lemma 2.2 [10, Lemma 1.1] *A set $B_r \subseteq \mathcal{PC}(X)$ is relatively compact in $\mathcal{PC}(X)$ if and only if $\widetilde{B}_i$ is relatively compact in $C([t_i,t_{i+1}];X])$ for every $i = 0,1,2,\cdots,N$.*

Definition 2.4 A function $u \in \mathcal{PC}(X)$ is said to be a mild solution of the problem (1) if $u(0) = u_0$, $u(t) = g_i(t, u(t))$ for all $t \in (t_i, s_i]$ and each $i = 1, \cdots, N$, and

$$\begin{aligned} u(t) &= S_\alpha(t)[u_0 + \varphi(0, u_0)] - \varphi(t, u(t)) - \int_0^t T_\alpha(t-s)\varphi(s, u(s))ds \\ &\quad + \int_0^t S_\alpha(t-s) f(s, u(s))ds, \text{for all } \ t \in [0, t_1], \end{aligned}$$

and

$$\begin{aligned} u(t) &= S_\alpha(t-s_i) g_i(s_i, u(s_i)) - \varphi(t, u(t)) - \int_{s_i}^t T_\alpha(t-s)\varphi(s, u(s))ds \\ &+ \int_{s_i}^t S_\alpha(t-s) f(s, u(s))ds, \text{for all } \ t \in [s_i, t_{i+1}] \ \ i = 1, \cdots, N. \end{aligned}$$

3 The Main Results

In this section, we prove the existence of solution to problem (1). The idea of the proof is based on [10,23]. We need the following hypothesis on f, φ and g_i. Let V be an open subset of X. For each $v \in V$, there is a ball $B(v, r)$ such that $B(v, r) \subset V$ for $r > 0$.

(H1) There exist constants $L_f > 0$, $L_\varphi > 0$ such that the nonlinear maps $f, \varphi : [0, T_0] \times V \to X$, will satisfy the following conditions,

$$\|f(t, u) - f(t, u_1)\| \leq L_f \|u - u_1\|, \tag{1}$$

$$\|\varphi(t, u) - \varphi(t, u_1)\| \leq L_\varphi \|u - u_1\|, \tag{2}$$

for all $u, u_1 \in V$ and $t > 0$.

(H2) The functions $g_i : [t_i, s_i] \times X \to X$ are continuous and there are positive constants L_{g_i} such that

$$\|g_i(t, x) - g_i(t, y)\| \leq L_{g_i} \|x - y\|,$$

for all $x, y \in X$, $t \in [t_i, s_i]$ and each $i = 0, 1, \cdots, N$.

(H3) The solution operators $S_\alpha, T_\alpha : \mathbb{R}_+ \to L(X)$ are bounded i.e., there exist constants $\mathcal{M}_1$ and $\mathcal{M}_2$ such that

$$\|S_\alpha(t)\|_{L(X)} \leq \mathcal{M}_1, \ \|T_\alpha(t)\|_{L(X)} \leq \mathcal{M}_2 \ \text{ for } \quad t > 0.$$

And the operators $(S_\alpha(t))_{t \geq 0}$, $(\overline{T_\alpha(t)})_{t \geq 0}$ are compact, where $(\overline{T_\alpha(t)}) = t^{1-\alpha} T_\alpha(t)$.

Theorem 3.1 *Let $u_0 \in X$. Also let the assumptions (H1)-(H2) hold such that*

$$L = \max\{\mathcal{M}_1(L_{g_i} + L_f \ T_0) + L_\varphi(1 + \mathcal{M}_2 T_0), L_{g_i} : \ i = 1, \cdots, N\} < 1. \tag{3}$$

Then there exists a unique mild solution $u \in \mathcal{PC}(X)$ of the problem (1).

Proof. We define a map $\digamma : \mathcal{PC}(X) \to \mathcal{PC}(X)$, given by $\digamma u(0) = u_0$, $\digamma u(t) = g_i(t, u(t))$ for each $t \in (t_i, s_i]$, $i = 1, \cdots, N$ and

$$\begin{aligned} \digamma u(t) &= S_\alpha(t)[u_0 + \varphi(0, u_0)] - \varphi(t, u(t)) - \int_0^t T_\alpha(t-s)\varphi(s, u(s))ds \\ &\quad + \int_0^t S_\alpha(t-s) f(s, u(s))ds, \quad \text{for all } \ t \in [0, t_1], \end{aligned}$$

and

$$\begin{aligned} \digamma u(t) &= S_\alpha(t-s_i)g_i(s_i, u(s_i)) - \varphi(t, u(t)) - \int_{s_i}^t T_\alpha(t-s)\varphi(s, u(s))ds \\ &\quad + \int_{s_i}^t S_\alpha(t-s) f(s, u(s))ds, \quad \text{for all } \ t \in [s_i, t_{i+1}] \text{ and } \quad i = 1, \cdots, N. \end{aligned}$$

Then $\digamma$ is well defined. Next we show that $\digamma$ is a contraction map on $\mathcal{PC}(X)$. For $u, v \in \mathcal{PC}(X)$, $i = 1, \cdots, N$ and $t \in [s_i, t_{i+1}]$, we have

$$\begin{aligned} \|\digamma u(t) - \digamma v(t)\| &\leq \|S_\alpha(t-s_i)\| \, \|g_i(s_i, u(s_i)) - g_i(s_i, v(s_i))\| \\ &\quad + \|\varphi(t, u(t)) - \varphi(t, v(t))\| \\ &\quad + \int_{s_i}^t \|T_\alpha(t-s)\| \, \|\varphi(s, u(s)) - \varphi(s, v(s))\| ds \\ &\quad + \int_{s_i}^t \|S_\alpha(t-s)\| \, \|f(s, u(s)) - f(s, v(s))\| ds \\ &\leq [\mathcal{M}_1(L_{g_i} + L_f T_0) + L_\varphi(1 + \mathcal{M}_2 T_0)] \|u - v\|_{\mathcal{PC}(X)}. \end{aligned}$$

Thus we obtain

$$\|\digamma u - \digamma v\|_{C([s_i, t_{i+1}]; X)} \leq [\mathcal{M}_1(L_{g_i} + L_f \ T_0) + L_\varphi(1 + \mathcal{M}_2 T_0)] \|u - v\|_{\mathcal{PC}(X)}. \quad (4)$$

Similarly, we obtain

$$\|\digamma u - \digamma v\|_{C([0, t_1]; X)} \leq (\mathcal{M}_1 L_f T_0 + L_\varphi(1 + \mathcal{M}_2 T_0) \|u - v\|_{\mathcal{PC}(X)}, \quad (5)$$

$$\|\digamma u - \digamma v\|_{C((t_i, s_i]; X)} \leq L_{g_i} \|u - v\|_{\mathcal{PC}(X)} \quad i = 1, 2, 3, ...N. \quad (6)$$

It follows from (4)-(6) that

$$\|\digamma u - \digamma v\|_{\mathcal{PC}(X)} \leq L \|u - v\|_{\mathcal{PC}(X)}. \quad (7)$$

By the assumption (3), the map $\digamma(\cdot)$ is a contraction and hence there exists a unique mild solution of (1). □

By a ball B_r with center at 0 and radius r, we mean the set $B_r(0, \mathcal{PC}(X)) = \{u \in \mathcal{PC}(X) : \|u\|_{\mathcal{PC}} \leq r\}$. We define

$$N_f = \sup_{s \in [s_i, t_{i+1}], v \in B_r(0, \mathcal{PC}(X))} \|f(s, v(s))\| \ \ N_\varphi = \sup_{s \in [s_i, t_{i+1}], v \in B_r(0, \mathcal{PC}(X))} \|\varphi(s, v(s))\|.$$

Theorem 3.1 can be proved with a weaker assumptions on f. We prove the theorem for the existence of mild solution to problem (1) with the following hypothesis.

(H1)´ There exists constant $L_\varphi > 0$ such that the nonlinear maps $\varphi : [0, T_0] \times V \to X$, will satisfy

$$\|\varphi(t, u) - \varphi(t, u_1)\| \leq L_\varphi \|u - u_1\|, \tag{8}$$

for all $u, u_1 \in V$ and $t > 0$.

Theorem 3.2 *Let $f : [0, T_0] \times X \to X$ be a continuous function that maps a bounded set into bounded set and $\varphi(\cdot, 0)$, $g_i(\cdot, 0)$ are bounded for each $u_0 \in X$. Let $r > 1$ and $0 < \delta < 1$ be two numbers such that*

$$\mathcal{M}_1 \|[u_0 + \varphi(0, u_0)]\| + (1 + \mathcal{M}_1) \max_{i=1,\cdots,N} \|g_i(\cdot, 0)\| \leq (1 - \delta) r, \tag{9}$$

$$\max_{i=1,\cdots,N} \Big\{ N_\varphi + L_{g_i}(1 + \mathcal{M}_1)\|u\|_{\mathcal{PC}} + (\mathcal{M}_2 N_\varphi + \mathcal{M}_1 N_f) T_0 + (1 + \mathcal{M}_1)\|g_i(t, 0)\| \Big\} \leq \delta r, \tag{10}$$

$$\Big(\mathcal{M}_1 \sup_{s \in [0, t_1], v \in B_r(0, \mathcal{PC}(X))} \|f(s, v(s))\| + \mathcal{M}_2 \sup_{s \in [0, t_1], v \in B_r(0, \mathcal{PC}(X))} \|\varphi(s, v(s))\| \Big) T_0 \leq \delta r, \tag{11}$$

Also, we assume that

$$(1 + \mathcal{M}_1) L_{g_i} + L_\varphi (1 + \mathcal{M}_2 T_0) < 1. \tag{12}$$

*If assumptions **(H1)´**, **(H2)** and **(H3)** hold, then there exists a mild solution $u \in \mathcal{PC}(X)$ to problem (1).*

Proof. We decompose $\digamma$ as

$$\digamma = \digamma_1 + \digamma_2,$$

where $\digamma_1 = \sum_{i=0}^N \digamma_i^1$, $\digamma_2 = \sum_{i=0}^N \digamma_i^2$ and $\digamma_i^k : \mathcal{PC}(X) \to \mathcal{PC}(X)$, $i = 0, 1, \cdots, N$, $k = 1, 2$. The map $\digamma_i^k$ is given by

$$(\digamma_i^1 u)(t) = \begin{cases} g_i(t, u(t)), & \text{for } t \in (t_i, s_i],\ i \geq 1, \\[2ex] S_\alpha(t - s_i) g_i(s_i, u(s_i)) - \varphi(t, u(t)) \\ -\int_{s_i}^t T_\alpha(t - s)\varphi(s, u(s))ds, & \text{for } t \in (s_i, t_{i+1}],\ i \geq 1, \\[2ex] 0, & \text{for } t \notin (t_i, t_{i+1}],\ i \geq 0, \\[2ex] S_\alpha(t)[u_0 + \varphi(0, u_0)] - \varphi(t, u(t)) \\ -\int_0^t T_\alpha(t - s)\varphi(s, u(s))ds, & \text{for } t \in [0, t_1],\ i = 0, \end{cases}$$

$$(\digamma_i^2 u)(t) = \begin{cases} \int_{s_i}^t S_\alpha(t - s) f(s, u(s))ds, & \text{for } t \in [s_i, t_{i+1}], i \geq 0, \\[2ex] 0, & \text{for } t \notin [s_i, t_{i+1}], i \geq 0. \end{cases}$$

The proof is divided into four steps.

Step 1. We begin by showing $\mathcal{F}B_r(0,\mathcal{PC}(X)) \subset B_r(0,\mathcal{PC}(X))$. Let $u \in B_r(0,\mathcal{PC}(X))$. For $i \geq 1$ and $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned}
&\|(\mathcal{F}_1 u)(t) + (\mathcal{F}_2 u)(t)\| \\
&\leq \|g_i(t,u(t)) - g_i(t,0)\| + \|g_i(t,0)\| + \|\varphi(t,u(t))\| \\
&\quad + \|S_\alpha(t-s_i)\|\ \|g_i(s_i,u(s_i)) - g_i(s_i,0)\| + \|S_\alpha(t-s_i)\|\|g_i(s_i,0)\| \\
&\quad + \int_{s_i}^t \|T_\alpha(t-s)\|\ \|\varphi(s,u(s))\|ds + \int_{s_i}^t \|S_\alpha(t-s)\|\ \|f(s,u(s))\|ds \\
&\leq N_\varphi + L_{g_i}\|u(t)\| + \|g_i(t,0)\| + \mathcal{M}_1 L_{g_i}\|u(t)\| + \mathcal{M}_1\|g_i(t,0)\| \\
&\quad + \mathcal{M}_2 N_\varphi (t-s_i) + \mathcal{M}_1 N_f (t-s_i) \\
&\leq N_\varphi + L_{g_i}(1+\mathcal{M}_1)\|u\|_{\mathcal{PC}} + (1+\mathcal{M}_1)\|g_i(t,0)\| \\
&\quad + (\mathcal{M}_2 N_\varphi + \mathcal{M}_1 N_f)T_0,
\end{aligned}$$

It follows from assumption (10) that

$$\|\mathcal{F}_1 u + \mathcal{F}_1 u\|_{\mathcal{PC}} \leq r \quad \forall i \geq 1.$$

Similarly, for each $t \in [0, t_1]$, we have

$$\begin{aligned}
&\|(\mathcal{F}_1 u)(t) + (\mathcal{F}_2 u)(t)\| \\
&\leq \|S_\alpha(t)\|\ \|u_0 + \varphi(0,u_0)\| + \int_0^t \|T_\alpha(t-s)\|\|\varphi(s,u(s))\|ds \\
&\qquad + \int_0^t \|S_\alpha(t-s)\|\|f(s,u(s))\|ds + \|\varphi(t,u(t))\| \\
&\leq N_\varphi + \mathcal{M}_1\|[u_0 + \varphi(0,u_0)]\| + (\mathcal{M}_2 N_\varphi + \mathcal{M}_1 N_f)T_0.
\end{aligned}$$

Using (9) and (10), we can conclude that

$$\|\mathcal{F}_1 u + \mathcal{F}_2 u\|_{\mathcal{PC}} \leq r.$$

Thus, we have $\mathcal{F}_1 u + \mathcal{F}_2 u \in B_r(0,\mathcal{PC}(X))$.

Step 2. In this step, we prove that $\mathcal{F}_1 = \sum_{i=0}^N \mathcal{F}_i^1$ is a contraction on $B_r(0,\mathcal{PC}(X))$. Let $t \in (t_i, t_{i+1}]$ and $u, v \in B_r(0,\mathcal{PC}(X))$. For $i = 1, \cdots, N$, we have

$$\|(\mathcal{F}_i^1 u)(t) - (\mathcal{F}_i^1 v)(t)\| \leq \Big[(1+\mathcal{M}_1)L_{g_i} + L_\varphi(1+\mathcal{M}_2 T_0)\Big]\|u-v\|_{C((t_i,t_{i+1}],X)}.$$

Thus

$$\left\|\sum_{i=0}^N \mathcal{F}_i^1 u - \sum_{i=0}^N \mathcal{F}_i^1 v\right\|_{\mathcal{PC}} \leq \Big[(1+\mathcal{M}_1)L_{g_i} + L_\varphi(1+\mathcal{M}_2 T_0)\Big]\|u-v\|_{\mathcal{PC}}.$$

It is clear from (12) that $\mathcal{F}_1$ is a contraction on $B_r(0,\mathcal{PC}(X))$.

Step 3. We prove that the set $\{\mathcal{F}_2 u : u \in B_r\}$ is relatively compact i.e., the set $\{(\mathcal{F}_2 u)(t) : u \in B_r\}$ is uniformly bounded, equicontinuous and for any $t \in [0, T_0]$.

The continuity of f implies that $\mathcal{F}_i^2$ is continuous for each $i = 0, 1, \cdots, N$ and $t \in [s_i, t_{i+1}]$. Thus $\mathcal{F}_2 = \sum_{i=0}^N \mathcal{F}_i^2$ is continuous and we have the following estimates

$$\|(\mathcal{F}_i^2 u)(t)\| \leq \mathcal{M}_1 N_f T_0, \quad \text{for } i = 0, 1, \cdots, N$$

for any $u \in B_r(0, \mathcal{PC}(X))$. Therefore, $\{\digamma_2 u : u \in B_r\}$ is uniformly bounded on B_r. Next, we prove that the set $\bigcup \digamma_i^2 B_r(0, \mathcal{PC}(X))(t)$ for $t \in [s_i, t_{i+1}],\ i = 0, 1, \cdots, N$, is relatively compact in X, where

$$\digamma_i^2 B_r(0, \mathcal{PC}(X))(t) = \{(\digamma_i^2 u)(t) : B_r(0, \mathcal{PC}(X))\}.$$

Applying mean value theorem for Bochner integral [17] and Young inequality, we have

$$(\digamma_0^2 u)(t) \subset \frac{t^{1+\alpha}}{\alpha} \overline{co\{\overline{S_\alpha(t-s)} f(s, u(s)) : s \in [0, t_1], u \in B_r\}}.$$

Similarly, for $t \in (s_i, t_{i+1}],\ i = 1, \cdots, N$, we obtain

$$(\digamma_i^2 u)(t) \subset \frac{(t-s_i)^{1+\alpha}}{\alpha} \overline{co\{\overline{S_\alpha(t-s)} f(s, u(s)) : s \in [s_i, t_{1+i}], u \in B_r\}}.$$

It follows from assumption (**H3**) that $\{(\digamma_i^2 u)(t)\}$ is a compact subset of X, for $t \in I,\ u \in B_r$. So, $\digamma_2$ is compact.

Step 4. In this step, we prove that the set of functions $[\digamma_i^2 B_r(\widetilde{0, \mathcal{PC}(X)})]_i, \quad i = 0, 1, \cdots, N$ is an equicontinuous subset of $C([t_i, t_{i+1}], X)$.

Clearly, $[\digamma_i^2 B_r(\widetilde{0, \mathcal{PC}(X)})]_i$ is equicontinuous on $[t_i, s_i]$, for each $i = 0, 1, \cdots, N$. Let $t_1, t_2 \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, N,$ with $t_1 < t_2$ and $u \in B_r(0, \mathcal{PC}(X))$, we get

$$\begin{aligned} \|\widetilde{\digamma_i^2} u(t_2) - \widetilde{\digamma_i^2} u(t_1)\| \quad \leq \quad & \int_{t_1}^{t_2} \|S_\alpha(t_2 - s)\| \, \|f(s, u(s))\| ds \\ & + \int_{s_i}^{t_1} \|S_\alpha(t_2 - s) - S_\alpha(t_1 - s)\| \|f(s, u(s))\| ds. \end{aligned} \tag{13}$$

For the first term on the right hand side of (13), we have

$$\int_{t_1}^{t_2} \|S_\alpha(t_2 - s)\| \|\varphi(s, u(s))\| ds \leq \mathcal{M}_1 N_f s(t_2 - t_1). \tag{14}$$

For $t_1 = s_i$, it is easy to see that the second term on the right hand side of (13) will be zero. If $t_1 > s_i$ and $\nu > 0$ be sufficiently small, we have

$$\begin{aligned} & \int_{s_i}^{t_1 - \nu} \|[S_\alpha(t_2 - s) - S_\alpha(t_1 - s)]\| \|f(s, u(s))\| ds \\ & \qquad + \int_{t_1 - \nu}^{t_1} \|[S_\alpha(t_2 - s) - S_\alpha(t_1 - s)]\| \|f(s, u(s))\| ds \\ \leq \quad & N_f \sup_{s \in [s_i, t_1 - \nu]} \|S_\alpha(t_2 - s) - S_\alpha(t_1 - s)\| (t_1 - \nu) + 2\mathcal{M}_1 N_f \nu. \end{aligned} \tag{15}$$

It follows from (14) and (15) that

$$\|\widetilde{\digamma_i^2} u(t_2) - \widetilde{\digamma_i^2} u(t_1)\|$$

tends to zero as $t_2 \to t_1$ and $\nu \to 0$ for any $u \in B_r(0, \mathcal{PC}(X))$. This means that $[\digamma_i^2 B_r(\widetilde{0, \mathcal{PC}(X)})]_i$ is equicontinuous. Thus $[\digamma_i^2 B_r(\widetilde{0, \mathcal{PC}(X)})]_i$ is an equicontinuous subset of $C([t_i, t_{i+1}], X)$.

By Ascoli-Arzela theorem, $\{\digamma_2 u : u \in B_r\}$ is relatively compact. Hence $\digamma_2$ is a completely continuous operator. So by Krasnoselskii's fixed point theorem [1], $\digamma$ has a fixed point. This completes the proof of the existence of a mild solution. □

4 Application

We discuss the following problem to illustrate the results. We consider the following system with noninstantaneous impulse for fractional partial differential equations in $L^2([0,\pi])$,

$$\left.\begin{array}{rcl} {}^cD_t^\alpha[u(t,x)+\partial_x G(t,x,u(t,x))] &=& \dfrac{\partial^2}{\partial x^2}u(t,x) \\ && +\frac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t (t-s)^{-\alpha}F(s,x,u(s,x))ds, \\ && \quad (t,x)\in\bigcup_{i=1}^N[s_i,t_{i+1}]\times(0,\pi), \\ u(t,0) &=& u(t,\pi)=0, \qquad t\in[0,T_0], \\ u(0,x) &=& u_0(x), \qquad x\in(0,\pi), \\ u(t,x) &=& H_i(t,x,u(t,x)),\ x\in(0,\pi), t\in(t_i,s_i], \end{array}\right\} \quad (1)$$

where $0=t_0=s_0<t_1\le s_1<\cdots<t_N\le s_N<t_{N+1}=T_0$. Here T_0 is a fixed real number, $u_0\in X$, $F\in([0,T_0]\times[0,\pi]\times\mathbb{R},\mathbb{R})$ and $H_i\in C((t_i,s_i]\times[0,\pi]\times\mathbb{R},\mathbb{R})$ for all $i=1,\cdots,N$.

Let $X=L^2([0,\pi])$ and $Au=\frac{\partial^2}{\partial x^2}u$ with

$$D(A)=\{u\in X:\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}\in X, u(0)=u(\pi)=0\}.$$

Then the operator $A:D(A)\subset X\to X$ is the infinitesimal generator of a solution operator $\{S_\alpha(t)\}_{t\ge 0}$ [3, see Theorem 3.1].

The system (1) can be formulated in the abstract form (1), where $u(t)=u(t,.)$, i.e., $u(t)(x)=u(t,x)$ and the functions $f:[0,T_0]\times X\to X$ and $g_i:(t_i,s_i]\times X\to X$ are given by

$$\begin{array}{rcl} f(t,u(t))(x) &=& F(t,x,u(t,x)), \\ \varphi(t,u(t))(x) &=& \partial_x G(t,x,u(t,x)), \\ g_i(t,u(t))(x) &=& H_i(t,x,u(t,x)). \end{array}$$

For $t\in[0,T_0],\ \ u\in X,\ x\in(0,\pi)$, we define f as

$$f(t,u(t))(x) \;=\; \frac{2e^{-t}|u(t,x)|}{(a+2e^t)(1+2|u(t,x)|)},\quad a>-1.$$

Then $f:[0,T_0]\times X\to X$ is continuous function and satisfies

$$\|f(t,u_1)-f(t,u_2)\|\le L_f\ \|u_1-u_2\|,$$

for $u_1,u_2\in X$ and $L_f=\frac{2}{a+2}$.

If we define g_i as follows

$$\begin{array}{rcl} g_i(t,u(t))(x) &=& \dfrac{(\cos(e^t)+\sin(e^{-t}))|u(t,x)|}{4(1+|u(t,x)|)}, \\ && t\in[t_i,s_i],\ \ u\in X,\ x\in(0,\pi), \end{array}$$

then $g_i:[t_i,s_i]\times X\to X$ is continuous function and satisfies

$$\|g_i(t,u_1)-g_i(t,u_2)\|\le L_{g_i}\|u_1-u_2\|,$$

for $u_1,u_2\in X$ and $L_{g_i}=\frac{1}{2}$. Hence the assumptions in Theorem 3.1 are satisfied [25]. Thus we have the following theorem for the existence.

Theorem 4.1 *If φ is chosen such that*

$$\|\varphi(t,u)-\varphi(t,v)\| \le L_\phi\|u-v\|, \quad t\in[0,T_0], \quad u,v\in X$$

and

$$L=\max\{\mathcal{M}_1(1/2+\frac{2}{a+2}\ T_0)+L_\varphi(1+\mathcal{M}_2T_0),1/2:\ i=1,\cdots,N\}<1,$$

then problem (1) has a unique piecewise continuous mild solution.

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References

[1] Burton, T. A. A fixed-point theorem of Krasnoselskii. *Appl. Math. Lett.* **11** (1) (1998) 85–88.

[2] Ahmad, B. and Nieto, J. J. A study of impulsive fractional differential inclusions with anti-periodic boundary conditions. *Fract. Differ. Calc.* **2** (1) (2012) 1–15.

[3] Bazhlekova, E. *Fractional evolution equations in Banach spaces.* Ph. D. Thesis, Eindhoven University of Technology, 2001.

[4] Cao, J. and Chen, H. Impulsive fractional differential equations with nonlinear boundary conditions. *Math. Comput. Modelling* **55** (3-4) (2012) 303–311.

[5] Chadha, A. Mild Solution for Impulsive Neutral Integro-Differential Equation of Sobolev Type with Infinite Delay. *Nonlinear Dynamics and System Theory* **15** (3) (2015) 272–289.

[6] Dabas, J. and Chauhan, A. Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equation with infinite delay. *Math. Comput. Modelling* **57** (3-4) (2013) 754–763.

[7] Debbouche, A. and Baleanu, D. Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems. *Comput. Math. Appl.* **62** (3) (2011) 1442–1450.

[8] Džrbašjan, M. M. and Nersesjan, A. B. Fractional derivatives and the Cauchy problem for differential equations of fractional order. *Izv. Akad. Nauk Armjan. SSR Ser. Mat.* **3** (1) (1968) 3–29.

[9] Gupta, V. and Dabas, J. Results for a Fractional Integro-Differential Equation with Nonlocal Boundary Conditions and Fractional Impulsive Conditions. *Nonlinear Dynamics and System Theory* **15** (4) (2015) 370–382.

[10] Hernández, E. and O'Regan, D. On a new class of abstract impulsive differential equations. *Proc. Amer. Math. Soc.* **141** (5) (2013) 1641–1649.

[11] Hernández, E. and O'Regan, D. Existence results for a class of abstract impulsive differential equations. *Bull. Aust. Math. Soc.* **87** (3) (2013) 366–385.

[12] Kumar, P., Pandey, D. N. and Bahuguna, D. On a new class of abstract impulsive functional differential equations of fractional order. *J. Nonlinear Sci. Appl.* **7** (2) (2014) 102–114.

[13] Lakshmikantham, V., Leela, S. and Devi, J. V. *Theory of Fractional Dynamic Systems.* Cambridge Scientific Publishers, 2009.

[14] Mahto, L., Abbas, S. and Favini, A. Analysis of Caputo impulsive fractional order differential equations with applications. *Int. J. Differ. Equ.* (2013) 1–11.

[15] Mainardi, F. On the initial value problem for the fractional diffusion-wave equation. Waves and stability in continuous media (Bologna, 1993), 246–251, *Ser. Adv. Math. Appl.* Sci., **23**, World Sci. Publ., River Edge, NJ, 1994.

[16] Mainardi, F. On a special function arising in the time fractional diffusion-wave equation. *Transform methods and special functions* (1994) 171–183, Science Culture Technology, Singopore, 1995.

[17] Martin, R.H. *Nonlinear Operators and Differential Equations in Banach Spaces.* Robert E. Krieger Publ. Co., Florida, 1987.

[18] Miller, K. S. and Ross, B. *An introduction to the Fractional Calculus and Differential Equations.* John Wiley, 1993.

[19] Michalski, M. W. *Derivatives of Non-integer Order and Their Applications.* Dissertationes mathematicae, Inst Math Polish Acad Sci, 1993.

[20] Podlubny, I. *Fractional Differential Equations.* Academic Press, 1999.

[21] Pollard, H. The representation of $e^{-x^{\lambda}}$ as a Laplace integral. *Bull. Amer. Math. Soc* **52** (1946) 908–910.

[22] Shukla, A., Sukavanam, N. and Pandey, D.N. Approximate Controllability of Semilinear Stochastic Control System with Nonlocal Conditions. *Nonlinear Dynamics and System Theory* **15** (3) (2015) 321–333.

[23] Shu, X. B., Lai, Y. and Chen, Y. The existence of mild solutions for impulsive fractional partial differential equations. *Nonlinear Anal.* **74** (5) (2011) 2003–2011.

[24] Wang, J. R., Wei, W. and Yang, Y. On some impulsive fractional differential equations in Banach spaces. *Opuscula Math.* **30** (4) (2010) 507–525.

[25] Wang, J., Fečkan, M. and Zhou, Y. On the new concept of solutions and existence results for impulsive fractional evolution equations. *Dyn. Partial Differ. Equ.* **8** (4) (2011) 345–361.

[26] Wang, J. R., Zhou, Y. and Fečkan, M. Nonlinear impulsive problems for fractional differential equations and Ulam stability. *Comput. Math. Appl.* **64** (10) (2012) 3389–3405.

[27] Wang, J. R., Zhou, Y. and Fečkan, M. On recent developments in the theory of boundary value problems for impulsive fractional differential equations. *Comput. Math. Appl.* **64** (10) (2012) 3008–3020.

[28] Wang, R. N., Chen, D. H. and Xiao, T. J. Abstract fractional Cauchy problems with almost sectorial operators, *J. Differential Equations* **252** (2012) 202–235.

[29] Yan, Z. and Zhang, H. Existence of solutions to impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay. *Electron. J. Differential Equations* (81) (2013) 1–21.

[30] Zhenhai, Z. and Li, X. Existence and uniqueness of solutions for the nonlinear impulsive fractional differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **18** (6) (2013) 1362–1373.

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A Predator-Prey System with Herd Behaviour and Strong Allee Effect

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Abstract: In this paper, we have studied the dynamical behaviours of a predator-prey system. The prey exhibits herd behaviour, and is also subject to strong Allee effect. Positivity and boundedness of the system are discussed. Some criteria for the extinction of prey and predator populations are derived. Stability analysis of the equilibrium points is presented. A criterion for Hopf bifurcation is derived. Numerical simulations are carried out to validate our analytical findings. Implications of our analytical and numerical findings are discussed critically.

Keywords: *Prey-predator system; Allee effect; stability; Hopf bifurcation.*

Mathematics Subject Classification (2010): 34C60, 92B05.

1 Introduction

It is a fact that species does not survive alone. Individuals of one species are usually biologically associated to members of other. Their interactions take several forms, depending on whether the influences are beneficial or detrimental. Among these interactions, predator-prey relationship is considered to be an extremely important one. It is true that the preys always try to develop the methods of evasion to avoid being eaten. However, it is certainly not true that a predator-prey relationship is always harmful for the preys,

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it might be beneficial to both. Further, such a relationship often plays an important role to keep ecological balance in nature. Mathematical modelling of predator-prey interaction was started in 1920s. Interestingly, the first predator-prey model in the history of theoretical ecology was developed independently by Alfred James Lotka (a US physical chemist) and Vito Volterra (an Italian mathematician) [29,40]. Subsequently, this model has been used as a machine to introduce numerous mathematical and practical concepts in theoretical ecology. Many refinements of the Lotka-Volterra model have also been made to overcome the shortcomings of the model and to get better insights of predator-prey interactions. If we summarize the basic considerations behind such modelling, it would be evident that the most crucial elements of predator-prey models are the choices of *growth function* of the prey and *functional response* of the predator.

So far as the growth of the prey is concerned, many modellers have considered logistic growth function to be a logically acceptable function. The function was introduced in 1838 by the Belgian mathematician Pierre Francois Verhulst [39]. If $X(T)$ denotes the population density at time T, then the logistic growth equation is given by

$$\frac{dX}{dT} = rX\left(1 - \frac{X}{K}\right), \tag{1}$$

where r is the intrinsic per capita growth rate and K is the carrying capacity of the environment. The logic behind this is very simple. As the resources (e.g., space, food, essential nutrients) are limited, every population grows into a saturated phase from which it cannot grow further; the ecological habitat of the population can carry just so much of it and no more. Therefore, the per capita growth rate is a decreasing function of the size of the population, and reaches zero as the population achieves a size that can be maintained; further, any population reaching a size that is above this value will experience a negative growth rate. However, there are many evidences where the reverse holds true in low population density [9,18–20,31,34]. This phenomenon of positive density dependence of population growth at low densities is known as the Allee effect [19,37].

Warder Clyde Allee, the US behavioral scientist after whom the phenomenon is named, was the pioneer to describe this concept (although Allee never used the term 'Allee effect') [2–4]. The term 'Allee effect' was introduced by Odum [33]. Since the late eighties of the 20th century, the concept gained importance but there was a necessity of clear-cut definitions and clarification of concepts. In 1999, three important reviews gave these much needed definitions and clarifications, which are used even today [18,36,37]. There might be countless reasons for the Allee effect, such as difficulty in mate finding, reduced antipredator vigilance, problem of environmental conditioning, reduced defense against predators, and many others (for thorough reviews, see [9,19]). The Allee effect can be divided into two main types, depending on how strong the per capita growth rate is depleted at low population densities. These two types are called the strong Allee effect [26,38,42,43] or critical depensation [14,15,28], and the weak Allee effect [37,41] or noncritical depensation [14,15,28]. Usually, the Allee effect is modelled by a growth equation of the form

$$\frac{dX}{dT} = rX\left(1 - \frac{X}{K}\right)\left(\frac{X}{K_0} - 1\right), \tag{2}$$

where $X(T)$ denotes the population density at time T, r is the intrinsic per capita growth rate, and K is the carrying capacity of the environment. Here $0 < K_0 << K$. When $K_0 > 0$ and the population size is below the threshold level K_0, then the population growth rate decreases [10,16,21,26], and the population goes to extinction. In this case, the equation describes the *strong Allee effect* [38,42,43]. On the contrary, the description of *weak Allee effect* is also available (see [22,42]). In this paper, we are concerned with strong Allee effect. The above growth is often said to have a *multiplicative Allee effect*. There is another mathematical form of the growth function featuring the *additive Allee effect*. In this paper, we are not interested in additive Allee effect (interested readers might see the works of Aguirre et al. [5,6]). A comparison of the logistic growth function of (1) and the function representing the Allee effect in equation (2) is depicted in Figure 1.

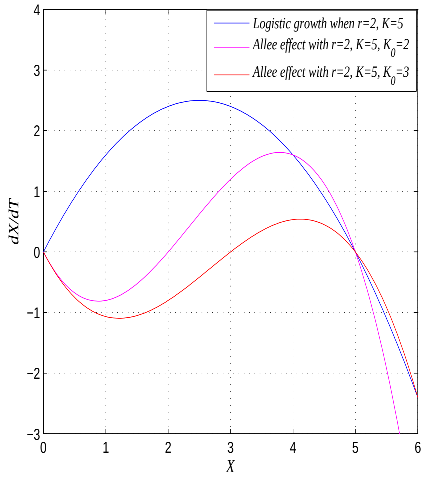

Figure 1: Comparison of the logistic growth function of (1) and the function representing the Allee effect in equation (2), when $r = 2$ and $K = 5$. The blue curve is the logistic growth curve. The magenta curve and the red curve are the graphs of the function on the right hand side of (2) when $K_0 = 2$ and $K_0 = 3$, respectively.

Let us now turn our attention from the individual growth of the prey to the interaction of the prey and its predator. The function that describes the number of prey consumed per predator per unit time for given quantities of prey and predator is known as the *functional response* or *trophic function*. Depending upon the behaviour of populations, more suitable functional responses have been developed as a quantification of the relative responsiveness of the predation rate to change in prey density at various populations of prey. In this connection, Holling family of functional responses are the most focused [24, 25]. The Holling type-I functional response (or the Lotka-Volterra functional response) is given by $F(X) = \alpha X$, where $X(T)$ is the prey density at time T and $\alpha > 0$ is a constant. In particular, the Holling type-II functional response has become extremely popular, and served as basis for a very large literature in predator-prey theory (see [30, 32, 35], and references therein). The type-II functional response includes the fact that a single individual can feed only until the stomach is not full, and so a saturation function would be better to describe the intake of food. This is similar to the concept of the *law of diminishing returns* borrowed from operations research, via a hyperbola rising up to an asymptotic value. In other words, the functional response would be of the following form

$$F(X) = \frac{\alpha X}{1 + T_h \alpha X}, \tag{3}$$

where $X(T)$ is the prey density at time T, α is the search efficiency of the predator for prey, T_h is the average handling time for each prey.

If a population is vulnerable to the Allee effect, there might be an important role of herd behaviour of the population. Very recently, Angulo et al. [7] have suggested that group behaviour diminishes extinction risks caused by the Allee effects. Now, when a population lives forming groups, then all members of a group do not interact at a time. There are many reasons for this herd behaviour, such as searching for food resources, defending the predators, etc. As a consequence, it is necessary to search for suitable form of functional response to describe this social behaviour. Only a few works have so far tried to enlighten this area. These works demonstrated an ingenious idea that suitable powers of the state variables can account for the social behaviour of the populations. For example, to explore the consequence of forming spatial group of fixed shape by predators, Cosner et al. [17] introduced the idea that the square root of the predator variable is to be used in the function describing the encounter rate in two-dimensional systems. Similarly, for three-dimensional systems, the two-third power of the predator in the encounter rate would better describe such group behaviour by predators. Unfortunately, such an idea has not been used by the researchers for about a decade. The work of Chattopadhyay et al. [13] may be regarded as a strong recognition of this concept. Then came the most innovative works of Ajraldi et al. [1] and Braza [12], which gave such modelling a new dimension. We recall their central ideas in the next paragraph.

Let X be the density of a population that gathers in herds, and suppose that herd occupies an area A. The number of individuals staying at outermost positions in the herd is proportional to the length of the perimeter of the patch where the herd is located. Clearly, its length is proportional to $\sqrt{A}$. Since X is distributed over a two-dimensional domain, $\sqrt{X}$ would therefore count the individuals at the edge of the patch. Thus, when attack of a predator on this population is to be modelled, the functional response should be in terms of square root of prey population. This is the main idea of Ajraldi et al. [1]. Braza [12] has placed a strong emphasis on this concept, and he has introduced a new functional response, where the prey density in (3) is replaced by its square root. That is, the functional response takes the form

$$F_1(X) = \frac{\alpha\sqrt{X}}{1 + T_h\alpha\sqrt{X}}. \tag{4}$$

It is already mentioned that if a population is susceptible to the Allee effect, then living in herds might be beneficial for it [7]. Now, if there is a predator, such behaviour plays a key role so far as the vigilance and predation risk is concerned [31]. The dynamics of predator-prey systems with herd behaviour of the prey has got the attention of theoretical ecologists very recently, but in all the cases it is assumed that the prey has a logistic growth (see [11] and references therein). It would be of utmost importance to consider predator-prey systems with herd behaviour and the Allee effect. There should be no denying that such considerations would be very interesting from both theoretical and practical point of view. The dynamics of such models has so far not been studied in literature. Our endeavour might accomplish such a necessity.

The rest of the paper is organized as follows. In Section 2, we present the mathematical model with basic considerations. Boundedness and positivity of the solutions of the model are established in Section 3. Some results on the extinction of prey and predator are derived in Section 4. Section 5 deals with all the possible equilibrium points of the model and their stability analysis. A criterion for Hopf bifurcation is derived in Section

6. To illustrate our analytical findings, computer simulations of variety of solutions of the system are performed; and the results are presented in Section 7. Section 8 contains the general discussion of the paper and biological significance of our analytical findings.

2 The Mathematical Model

At time T, let $X(T)$ denote the density of the prey, and $Y(T)$ denote the density of the predator. We assume that the preys live in herds. We also consider a multiplicative Allee effect in prey population growth. These considerations motivate us to introduce the following predator-prey system within the framework of the following set of nonlinear ordinary differential equations:

$$\begin{aligned} \frac{dX}{dT} &= rX\left(1-\frac{X}{K}\right)\left(\frac{X}{K_0}-1\right)-\frac{\alpha\sqrt{X}Y}{1+T_h\alpha\sqrt{X}}, \quad X(0)>0, \\ \frac{dY}{dT} &= -\delta Y+\frac{\beta\alpha\sqrt{X}Y}{1+T_h\alpha\sqrt{X}}, \quad Y(0)>0. \end{aligned} \tag{5}$$

The parameter r is the intrinsic growth rate of the prey, K is the carring capacity of the prey, δ represents the death rate of the predator. We assume a strong Allee effect on the prey. The parameter K_0 with $0 < K_0 << K$ is the prey population Allee threshold in the absence of predators. As the prey exhibits herd behaviour, here we have used the modified functional response (4) (suggested by Braza [12]) to represent the interaction between prey and predator. So α, T_h, β stand for the search efficiency of the predator for prey, the average handling time for each prey, and the biomass conversion rate, respectively. It is an obvious assumption that all the parameters are positive.

To reduce the number of parameters in the system (5), we use the following scaling

$$x=\frac{X}{K}, \quad y=\frac{Y}{K}, \quad \text{and} \quad t=r\frac{K}{K_0}T.$$

Then the system (5) takes the following form (after some simplifications)

$$\begin{aligned} \frac{dx}{dt} &= x(1-x)(x-m)-\frac{b\sqrt{x}y}{1+a\sqrt{x}}, \quad x(0)>0, \\ \frac{dy}{dt} &= -dy+\frac{c\sqrt{x}y}{1+a\sqrt{x}}, \quad y(0)>0, \end{aligned} \tag{6}$$

where

$$m=\frac{K_0}{K}, \quad a=T_h\alpha\sqrt{K}, \quad b=\frac{\alpha K_0}{r\sqrt{K}}, \quad c=\frac{\beta\alpha K_0}{r\sqrt{K}}, \quad d=\frac{\delta K_0}{rK}.$$

3 Positivity and Boundedness

Positivity and boundedness of a model guarantee that the model is biologically well behaved. For positivity of the system (6), we have the following theorem.

Theorem 3.1 *All solutions of the system* (6) *that start in* $\mathbb{R}_+^2$ *remain positive forever.*

The proof is simple and therefore it is omitted. The following theorem ensures the boundedness of the system (6).

Theorem 3.2 *All solutions of the system* (6) *that start in* $\mathbb{R}^2_+$ *are uniformly bounded.*

Proof. Let $(x(t), y(t))$ be any solution of the system (6). We consider two possible cases separately.

Case-I. Let $x(0) \leq 1$. We claim that $x(t) \leq 1$ for all $t \geq 0$.
If possible, assume that our claim is not true. Then it is possible to find two positive real numbers t' and t'' such that $x(t') = 1$ and $x(t) > 1$ for all $t \in (t', t'')$.
Now, for all $t \in (t', t'')$, we have from the first equation of(6)

$$x(t) = x(0) \exp\left(\int_0^t \phi(x(s), y(s))ds\right),$$

where $\phi(x(t), y(t)) = (1 - x(t))(x(t) - m) - b\frac{y(t)}{\sqrt{x(t)}(1+a\sqrt{x(t)})}$.
This implies that

$$\begin{aligned} x(t) &= x(0)\left[\exp\left(\int_0^{t'} \phi(x(s), y(s))ds\right)\right]\left[\exp\left(\int_{t'}^t \phi(x(s), y(s))ds\right)\right] \\ &= x(t') \exp\left(\int_{t'}^t \phi(x(s), y(s))ds\right), \quad \text{for all } t \in (t', t''). \end{aligned}$$

Since $m < 1$, we have $\phi(x(t), y(t)) < 0$ for all $t \in (t', t'')$. Consequently, we have

$$x(t) < x(t'), \quad \text{where } x(t') = 1.$$

This is contrary to the assumption that $x(t) > 1$ for all $t \in (t', t'')$. Thus our claim is true.

Case-II. Let $x(0) > 1$. We claim that $\limsup_{t\to\infty} x(t) \leq 1$.
If possible, assume that this claim is false. Then $x(t) > 1$ for all $t > 0$. So $\phi(x(t), y(t)) < 0$ (where ϕ has the same expression as in Case-I); and consequently, we have from the first equation of(6) that

$$x(t) = x(0) \exp\left(\int_0^t \phi(x(s), y(s))ds\right) < x(0).$$

Also from the first equation of (6), we obtain

$$\frac{dx}{dt} < (x(0) - m)x(1 - x), \quad \text{where } x(0) - m > 0.$$

This implies that $\limsup_{t\to\infty} x(t) \leq 1$, which is contradictory to our assumption. Therefore our claim is true.

From the above two cases, we have $\limsup_{t\to\infty} x(t) \leq 1$.

Let $W = cx + by$. Then, for large t, we have

$$\begin{aligned} \frac{dW}{dt} &= cx(1 - x)(x - m) - bdy \\ &= cx\{(1 + m)x - m - x^2\} - bdy \\ &\leq c(1 + m)x - bdy \\ &\leq 2c(1 + m) - \lambda W, \quad \text{where} \quad \lambda = \min\{(1 + m), d\}. \end{aligned}$$

Therefore,

$$\frac{dW}{dt} + \lambda W \leq 2c(1+m).$$

Applying the theory of differential inequalities, we obtain

$$0 \leq W(x,y) \leq \frac{2c(1+m)}{\lambda} + \frac{W(x(0),y(0))}{e^{\lambda t}}.$$

and for $t \to \infty$,

$$0 \leq W \leq \frac{2c(1+m)}{\lambda}.$$

Thus, all the solutions of (2.2) enter into the region

$$B = \left\{(x,y) : 0 \leq W \leq \frac{2c(1+m)}{\lambda} + \epsilon, \quad \text{for any} \quad \epsilon > 0\right\}.$$

Hence the theorem is proved. □

4 Extinction Scenarios

In this section, we find some conditions for extinction of the prey or predator. Here we use the symbols $\overline{x}$ and $\underline{y}$ to represent $\limsup_{t\to\infty} x(t)$ and $\liminf_{t\to\infty} y(t)$, respectively. We frequently use the fact that $\overline{x} \leq 1$, which is proved in Theorem 3.2.

The first two theorems of this section are on the extinction of the prey species. It is quite obvious that if, after certain time, the prey population density lies below the Allee threshold (moreover there is attack of predator), then it is really impossible for the prey to survive. This fact is represented in mathematical terms in the following theorem.

Theorem 4.1 *If $\overline{x} < m$, then $\lim_{t\to\infty} x(t) = 0$.*

Proof. If possible, let $\lim_{t\to\infty} x(t) = \mu > 0$. The definition of $\overline{x}$ implies that for any ϵ satisfying $0 < \epsilon < m - \overline{x}$, there exists $t_\epsilon > 0$ such that $x(t) < \overline{x} + \epsilon$ for $t > t_\epsilon$.
Then, for $t > t_\epsilon$, we have from the first equation of (6) that

$$\begin{aligned} x(t) &= x(0)\exp\left[\int_0^t \left\{(1-x(s))(x(s)-m) - \frac{b\sqrt{x(s)}y(s)}{x(s)(1+a\sqrt{x(s)})}\right) ds\right] \\ &< x(0)\exp\left[\int_0^t (\overline{x}+\epsilon-m)\, ds\right] \\ &< x(0)\exp\{-(m-\overline{x}-\epsilon)t\} \to 0 \text{ as } t\to\infty, \end{aligned}$$

which is a contradiction. This proves the theorem. □

If the condition of the above theorem is satisfied, then the predator has no vital role in leading the prey to extinction, because the Allee effect is enough to do this (of course, the predator might expedite the process of extinction of the prey). The following theorem shows that the predator might also play a key role to prompt the prey to die out.

Theorem 4.2 *If $\underline{y} > \frac{2\sqrt{2}}{b}(1+a\sqrt{2})(1-m)$, then $\lim_{t\to\infty} x(t) = 0$.*

Proof. If possible, let $\lim_{t\to\infty} x(t) = \mu > 0$. Since $\overline{x} \leq 1$, for any $0 < \epsilon < 1 - m$, there exists $t_\epsilon > 0$ such that $x(t) < 1 + \epsilon$ for $t > t_\epsilon$.
From the definition of $\underline{y}$, it follows that, for any $0 < \epsilon' < \underline{y} - \frac{2\sqrt{2}}{b}(1 + a\sqrt{2})(2 - m)$, there exists $t_{\epsilon'} > 0$ such that $y(t) > \underline{y} - \epsilon'$ for $t > t_{\epsilon'}$.
Then, for $t > \max\{t_\epsilon, t_{\epsilon'}\}$, we have from the first equation of (6) that

$$\begin{aligned}
\frac{dx}{dt} &< x(1+\epsilon-m) - \frac{b\sqrt{x}y}{1+a\sqrt{1+\epsilon}}, \\
&< x(1+\epsilon-m) - \frac{bxy}{\sqrt{1+\epsilon}(1+a\sqrt{1+\epsilon})}, \quad \text{as } x < \sqrt{1+\epsilon}\sqrt{x}, \\
&< x\left\{2(1-m) - \frac{b(\underline{y}-\epsilon')}{\sqrt{2}(1+a\sqrt{2})}\right\}, \\
&< -\frac{bx}{\sqrt{2}(1+a\sqrt{2})}\left\{\underline{y} - \epsilon' - \frac{2\sqrt{2}}{b}(1+a\sqrt{2})(1-m)\right\} < 0,
\end{aligned}$$

which implies that $\lim_{t\to\infty} x(t) = 0$, a contradiction.
Hence the theorem is established. □

A very simple criterion for the extinction of the predator is given in the following theorem.

Theorem 4.3 *If $d > c$, then* $\lim_{t\to\infty} y(t) = 0$.

Proof. Since $\overline{x} \leq 1$, for any $0 < \epsilon < \frac{d^2}{c^2} - 1$, there exists $t_\epsilon > 0$ such that $x(t) < 1 + \epsilon$ for $t > t_\epsilon$. For $t > t_\epsilon$, we have from the second equation of (6) that

$$\begin{aligned}
\frac{dy}{dt} &= y\left(-d + \frac{c\sqrt{x}}{1+a\sqrt{x}}\right) \\
&< y\left(-d + c\sqrt{x}\right) < y\left(-d + c\sqrt{1+\epsilon}\right) \\
&< -cy\left(\frac{d}{c} - \sqrt{1+\epsilon}\right) < 0.
\end{aligned}$$

Therefore, $\lim_{t\to\infty} y(t) = 0$. □

Remark 4.1 We notice that if the predator is aggressive (characterized by the high value of b) or the Allee effect is very strong ($m \approx 1$), then the condition of Theorem 4.2 might be satisfied automatically. On the other hand, if the maximal benifit of the predator (in interaction with the prey) fails to overcome its loss due to death, then the predator will be ultimately washed out from the system.

5 Equilibria and Their Stability

In this section, we find the equilibrium points of the system (6) and study their stability. The nullclines are shown in Figure 2. The following lemma gives the equilibrium points with the conditions of their existence.

Lemma 5.1 *The trivial equilibrium $E_0(0,0)$ of the system* (6) *always exists. There are two axial (predator-free) equilibrium points $E_1(1,0)$ and $E_2(m,0)$, each of which also exists unconditionally. The interior or coexistence equilibrium $E^*(x^*, y^*)$ exists if and*

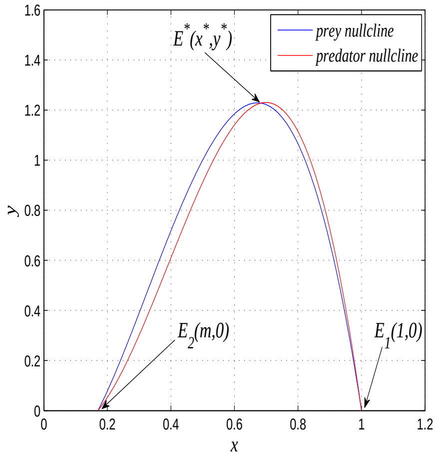

Figure 2: Nullclines of the system (6) for $a = 0.89,\ b = 0.19,\ c = 0.21,\ d = 0.1,\ m = 0.17$.

only if $(c - ad)\sqrt{m} < d < (c - ad)$. *When* $E^*(x^*, y^*)$ *exists, the expressions for* x^* *and* y^* *are given by*

$$x^* = \frac{d^2}{(c - ad)^2}, \quad y^* = \frac{cx^*(1 - x^*)(x^* - m)}{bd}.$$

It is not possible to linearize the system (6) about the trivial equilibrium. Therefore, local stability of $E_0(0, 0)$ cannot be studied. However, results of the previous section could provide some results on global stability of $E_0(0, 0)$. For example, if the conditions of Theorem 4.1 and Theorem 4.3 are satisfied simultaneously, then $E_0(0, 0)$ is globally stable. Also $E_0(0, 0)$ is globally stable if the conditions of Theorem 4.2 and Theorem 4.3 are satisfied. We are not interested to restate those results here.

The Jacobian matrix $J(E_1)$ at $E_1(1, 0)$ is given by

$$J(E_1) = \begin{bmatrix} m - 1 & -\frac{b}{1+a} \\ 0 & -d + \frac{c}{1+a} \end{bmatrix}.$$

Clearly, the eigenvalues of $J(E_1)$ are $m - 1$ and $-d + \frac{c}{1+a}$. Since $m < 1$, the first eigenvalue $m - 1$ is always negative. The second one will also be negative if and only if $c < d(1 + a)$. Thus we have the following theorem on stability of $E_1(1, 0)$.

Theorem 5.1 *The equilibrium* $E_1(1, 0)$ *is locally asymptotically stable if and only if* $c < d(1 + a)$.

Remark 5.1 We notice that the existence of E^* destabilizes E_1.

The Jacobian matrix $J(E_2)$ at $E_2(m, 0)$ is given by

$$J(E_2) = \begin{bmatrix} m(1 - m) & -\frac{b\sqrt{m}}{1+a\sqrt{m}} \\ 0 & -d + \frac{c\sqrt{m}}{1+a\sqrt{m}} \end{bmatrix}.$$

The eigenvalues of $J(E_2)$ are $m(1-m)$ and $-d+\frac{c\sqrt{m}}{1+a\sqrt{m}}$. The first eigenvalue is always positive, as $0<m<1$. The second one will be negative if and only if $c\sqrt{m}<d(1+a\sqrt{m})$. Therefore, we have the following theorem.

Theorem 5.2 *The equilibrium $E_2(m,0)$ is always unstable. It is a saddle point if and only if $c\sqrt{m}<d(1+a\sqrt{m})$.*

Finally, we consider the stability issue of the most important equilibrium $E^*(x^*,y^*)$. We have the following Jacobian matrix at $E^*(x^*,y^*)$:

$$J(E^*)=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{bmatrix},$$

where

$$\begin{aligned}
a_{11} &= (1-2x^*)(x^*-m)+x^*(1-x^*)-\frac{by^*}{2\sqrt{x^*}(1+a\sqrt{x^*})^2},\\
a_{12} &= -\frac{b\sqrt{x^*}}{1+a\sqrt{x^*}}, \quad a_{21}=\frac{cy^*}{2\sqrt{x^*}(1+a\sqrt{x^*})^2}.
\end{aligned}$$

The characteristic equation of $J(E_3)$ is

$$\lambda^2+P\lambda+Q=0,$$

where $P=-\mathrm{tr}\ J(E^*)=-a_{11}$, $Q=\det J(E^*)=-a_{12}a_{21}>0$. A little algebraic manipulation yields

$$P=\frac{ABm-d^2\{2c(A-d^2)+B\}}{2cA^2},$$

where $A=(c-ad)^2$ and $B=(c+ad)A-(3c+ad)d^2$. Then we have the following theorem guaranteeing the stability of $E^*(x^*,y^*)$.

Theorem 5.3 *The necessary and sufficient condition for local asymptotic stability of the interior equilibrium $E^*(x^*,y^*)$ is that $\Delta=ABm-d^2\{2c(A-d^2)+B\}>0$.*

It would be interesting if we can establish some sort of global behaviour of the interior equilibrium. Let $\Omega=\{(x,y)\in\mathbb{R}^2: 0<x<1,\ y>0\}$. Clearly, $E^*(x^*,y^*)\in\Omega$. Then we have the following theorem.

Theorem 5.4 *If $E^*(x^*,y^*)$ is locally asymptotically stable with $d>c+m+2$, then E^* attracts all solutions of the system* (6) *lying in Ω.*

Proof. Let us write the first equation of the system (6) as $\frac{dx}{dt}=P(x,y)$, and the second equation as $\frac{dy}{dt}=Q(x,y)$. Then, for all $(x,y)\in\Omega$, we notice that

$$\begin{aligned}
\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y} &= (1-2x)(x-m)+x(1-x)-\frac{by}{2\sqrt{x}(1+a\sqrt{x})^2}-d+\frac{c\sqrt{x}}{1+a\sqrt{x}}\\
&\le 2x+2mx-m-d+c\sqrt{x}\\
&\le 2+m-d+c<0.
\end{aligned}$$

Therefore, by Bendixson's criterion, there is no periodic orbit in Ω. Hence the theorem follows from the Poincaré-Bendixson theorem. □

6 Hopf Bifurcation

In this section, we provide conditions for the occurrence of a simple Hopf bifurcation near the interior equilibrium $E^*(x^*, y^*)$. We use the Hopf bifurcation theorem [8, 23, 32] for this purpose.

Theorem 6.1 *If the equilibrium point $E^*(x^*, y^*)$ exists, then Hopf bifurcation occurs at $m = m^* = \frac{d^2\{2c(A-d^2)+B\}}{AB}$, provided m^* is positive.*

Proof. We notice that
(i) $[\text{tr } J(E^*)]_{m=m^*} = 0$,
(ii) $[\det J(E^*)]_{m=m^*} > 0$,
(iii) when $m = m^*$ the characteristic equation is $\lambda^2 + \det J(E^*) = 0$, whose roots are purely imaginary,
(iv) $[(d/dm)(\text{tr } J(E^*))]_{m=m^*} = -\frac{B}{2cA} \neq 0$.

Hence all the conditions of the Hopf-bifurcation theorem are satisfied, and the theorem follows. □

7 Numerical Simulation

In this section, we present computer simulations of some solutions of the system (6). These simulations are performed to validate the analytical findings of the last two sections.

First, we take the parameters of the system (6) as $m = 0.2$, $b = 0.19$, $a = 0.89$, $d = 0.1$ and $c = 0.17$. Then $c < d(1 + a)$, and consequently by Theorem 5.1, $E_1(1, 0)$ is locally asymptotically stable. Figure 3 illustrates this. Clearly, x approaches 1 and y dies out in finite time.

Next we consider the stability of the interior equilibrium. We take the parameter values as $m = 0.17$, $b = 0.19$, $a = 0.89$, $d = 0.1$ and $c = 0.21$. Then $\Delta = 0.0000016314 > 0$. Therefore, by Theorem 5.3, the interior equilibrium point $E^*(x^*, y^*) = (0.6830, 1.2276)$ is locally asymptotically stable. The corresponding phase portrait for different choices of $(x(0), y(0))$ is depicted in Figure 4. Clearly the trajectories are stable spirals converging to E^*. Figure 5 shows the behaviour of x and y with time, when $(x(0), y(0)) = (0.85, 1.2)$, and it is evident that (x, y) approaches to (x^*, y^*) in finite time.

If we gradually increase the value of m, keeping other parameters fixed, then following Theorem 6.1, we have a critical value $m^* = 0.2096$ such that E^* loses its stability as m passes through m^*. For $m = 0.22 > m^*$, we verify that $E^*(0.6830, 1.1080)$ is unstable ($\Delta = -0.0000004274 < 0$) and there is a periodic orbit near E^* (see Figure 6). The oscillations of x and y in time are shown in Figure 7.

A bifurcation diagram is shown in Figure 8. As the parameter m passes through the bifurcation value $m^* = 0.2096$, there is a change in stability behaviour.

8 Concluding Remarks

Recently it has been suggested by researchers that herd behaviour of populations could act as a buffer against population extinction due to the Allee effect (see [7]). Modelling of the Allee effect has been done by many researchers. Nowadays there has been a growing concern about modelling of herd behaviour of populations. In this paper, we

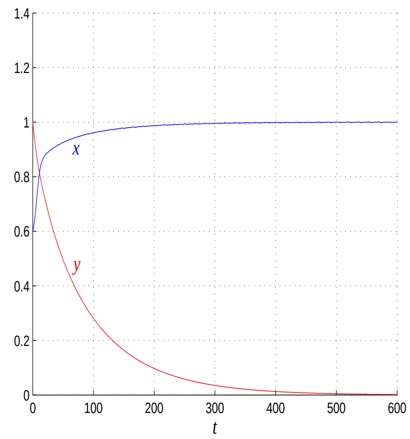

Figure 3: Behaviour of the system (6) with time when $m = 0.2$, $b = 0.19$, $a = 0.89$, $d = 0.1$ and $c = 0.17$.

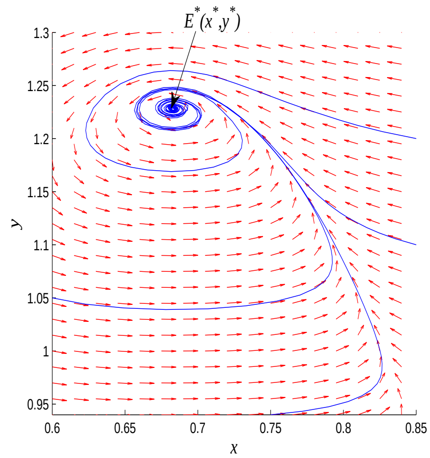

Figure 4: Here $m = 0.17$, $b = 0.19$, $a = 0.89$, $d = 0.1$ and $c = 0.21$. Phase portrait of the system (6) for different choices of $x(0)$ and $y(0)$ showing stable spirals converging to $E^*(0.6830, 1.2276)$.

have considered a predator-prey model where the prey shows herd behaviour and also susceptible to the Allee effect. The number of parameters of the model has been reduced by suitable scaling. Then the dynamical behaviours of the resulting model (6) is studied. It is shown (in Theorem 3.1 and Theorem 3.2) that the solutions of the system (6) remain non-negative forever, and they are uniformly bounded. These, in turn, imply that the system is biologically well-behaved. We have derived some results on extinction of prey and predator. It is seen that if there is a very strong the Allee effect, then it is almost impossible for the prey to survive. Also, an aggressive predator might cause extinction

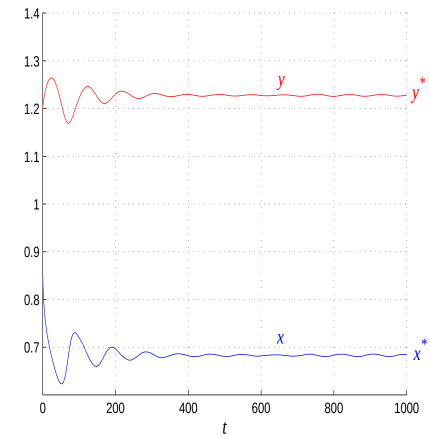

Figure 5: Here the values of the parameters are as in Figure 4. When $(x(0), y(0)) = (0.85, 1.2)$, both the populations converge to their equilibrium-state values in finite time. The blue curve represents x and the red one represents y.

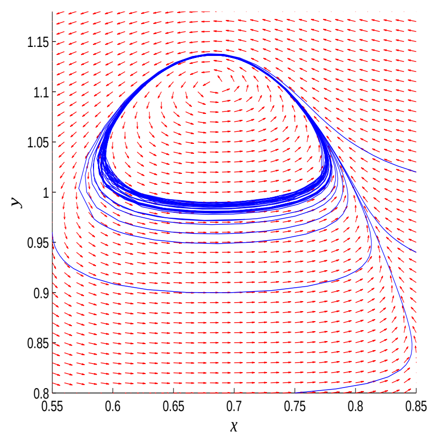

Figure 6: Here all the parameters are same as in Figure 4 except $m = 0.22 > m^*$. Phase portrait of the system (6) showing a periodic orbit near $E^*(0.6830, 1.1080)$.

of the prey, and this ultimately backfires (because the predator dies out in starvation, which is clear from the second equation of (6)). It is also established that if the maximal benefit of the predator (in interaction with the prey) fails to overcome its loss due to death, then the predator would ultimately be washed out.

It has long been recognized that most of the studies of continuous time deterministic models reveal two basic patterns: approach to an equilibrium or to a limit cycle. The basic rationale behind such type of analysis is perhaps that these two patterns are very common in many predator-prey systems we observe in nature. From this viewpoint, we

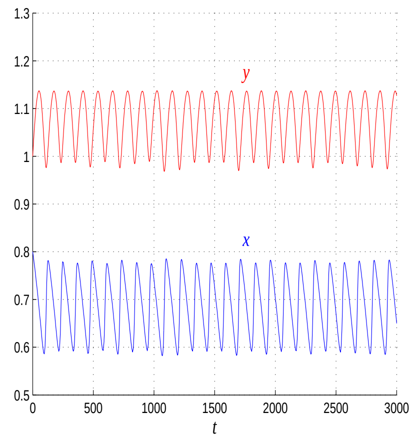

Figure 7: Here all the parameters are same as in Figure 6. It shows oscillations of the x and y in time. The blue curve represents x and the red one represents y.

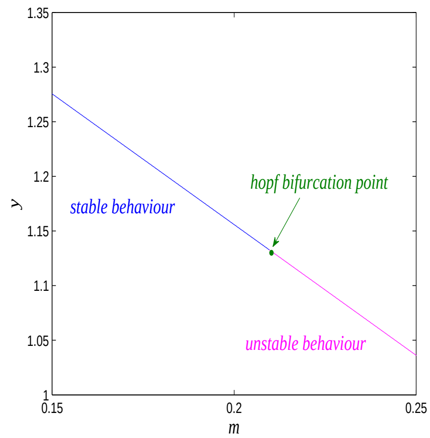

Figure 8: A bifurcation diagram with m as bifurcation parameter, when $b = 0.19$, $a = 0.89$, $d = 0.1$ and $c = 0.21$. The blue curve depicts stable behavior and the magenta curve depict unstable behavior.

have presented the stability analysis of the equilibrium points, and bifurcation analysis of the most important interior equilibrium point. The criterion for existence of the interior equilibrium suggests that an aggressive predator with moderate death rate might give a guarantee for the coexistence equilibrium to be feasible. Also, the existence of the interior equilibrium destabilizes E_1. The Allee effect has a negative effect on the fitness of the predator (see Figure 8). Further, the bifurcation analysis presented here shows that the Allee effect could have a regulatory impact on the whole system.

References

[1] Ajraldi, V., Pittavino, M. and Venturino, E. Modelling herd behavior in population systems. *Nonlinear Anal. RWA* **12** (2011) 2319–2338.

[2] Allee, W. Animal aggregations. *Quart. Rev. Biol.* **2** (1927a) 367–398.

[3] Allee, W. Studies in animal aggregations: some physiological effects of aggregation on the brittle starfish, Ophioderma brevispina. *J. Exp. Zool.* **48** (1927b) 475–495.

[4] Allee, W. Studies in animal aggregations: mass protection against colloidal silver among goldfishes. *J. Exp. Zool.* **61** (1932) 185–207.

[5] Aguirre, P. González-Olivares, E. and Sáez, E. Two limit cycles in a Leslie-Gower predator-prey model with additive Allee effect. *Nonlinear Anal. RWA* **10** (2009a) 1401–1416.

[6] Aguirre, P. González-Olivares, E. and Sáez, E. Three limit cycles in a Leslie-Gower predator-prey model with additive Allee effect. *SIAM J. Appl. Math.* **69** (2009b) 1244–1262.

[7] Angulo, E., Rasmussen, G.S.A., Macdonald, D.W., and Courchamp, F. Do social groups prevent Allee effect related extinctions?: The case of wild dogs. *Frontiers in Zoology* **10**:11 (2013) doi:10.1186/1742-9994-10-11.

[8] Arrowsmith, D.K. and Place, C.M. *Dynamical Systems: Differential Equations, Maps, and Chaotic Behaviour.* Chapman & Hall, London, 1992.

[9] Berec, L., Angulo, E. and Courchamp, F. Multiple allee effects and population management. *Trends Ecol. Evol.* **22** (2006) 185–191.

[10] Bazykin, A.D., Berezovskaya, F.S., Isaev, A.S. and Khlebopros, R.G. Dynamics of forest insect density: bifurcation approach. *J. Theor. Biol.* **186** (1997) 267–278.

[11] Bera, S.P., Maiti, A. and Samanta, G.P. Modelling herd behavior of prey: analysis of a prey-predator model. *World Journal of Modelling and Simulation* **11** (2015) 3–14.

[12] Braza, A.P. Predator-prey dynamics with square root functional responses. *Nonlinear Anal. RWA* **13** (2012) 1837–1843.

[13] Chattopadhyay, J., Chatterjee, S. and Venturino, E. Patchy agglomeration as a transition from monospecies to recurrent plankton blooms. *J. Theor. Biol.* **253** (2008) 289–295.

[14] Clark, C.W. *Mathematical Bioeconomic: The Optimal Mmanagement of Renewable Resources.* Wiley, New York, 1990.

[15] Clark, C.W. *The Worldwide Crisis in Fisheries: Economic Models and Human Behavior.* Cambridge University Press, Cambridge, 2007.

[16] Conway, E.D. and Smoller, J.A. Global analysis of a system of predator-prey equations. *SIAM J. Appl. Math.* **46** (1986) 630–642.

[17] Cosner, C., DeAngelis, D.L., Ault, J.S. and Olson D.E. Effects of spatial grouping on the functional response of predators. *Theor. Pop. Biol.* **56** (1999) 65–75.

[18] Courchamp, F., Clutton-Brock, T. and Grenfell, B. Inverse dependence and the Allee effect. *Trends Ecol. Evol.* **14** (1999) 405–410.

[19] Courchamp, F., Berec, L. and Gascoigne, J. *Allee Effects in Ecology and Conservation.* Oxford University Press, Oxford, 2008.

[20] Clutton-Brock, T., Gaynor, D., McIlrath, G.M., Maccoll, A.D.C., Kansky, R., Chadwick, P., Manser, M., Skinner, J.D. and Brotherton, P.N.M. Predation, group size and mortality in a cooperative mongoose, Suricata suricatta. *J. Anim. Ecol.* **68** (1999) 672–683.

[21] Flores, J.D., Mena-Lorca, J., González-Yañez, B. and González-Olivares, E. Consequences of depensation in a Smiths bioeconomic model for open-access fishery. In: *Proceedings of international symposium on mathematical and computational biology* (Eds.: R. Mondaini and R. Dilao). E-papers Servios Editoriais Ltda, 2007, 219–232.

[22] González-Olivares, E. and Rojas-Palma, A. Multiple Limit Cycles in a Gause type predator-prey model with Holling type III functional response and Allee effect on prey. *Bull. Math. Biol.* **73** (2011) 1378–1397.

[23] Hassard, B.D., Kazarinoff, N.D. and Wan, Y.H. *Theory and Application of Hopf-bifurcation.* Cambridge University Press, Cambridge, 1981.

[24] Holling, C.S. The components of predation as revealed by a study of small mammal predation of the European pine sawfly. *Can. Entomol.* **91** (1959a) 293–320.

[25] Holling, C.S. Some characteristics of simple types of predation and parasitism. *Can. Entomol.* **91** (1959b) 385–398.

[26] Kot, M. *Elements of Mathematical Biology.* Cambridge University Press, Cambridge, 2001.

[27] Kumar, R. and Freedman, H.I. Mathematical analysis in a model of obligate mutualism with food chain populations. *Nonlinear Dynamics and Systems Theory* **2** (1) (2002) 25–44.

[28] Liermann, M. and Hilborn, R. Depensation: evidence, models and implications. *Fish Fish.* **2** (2001) 33–58.

[29] Lotka, A. *Elements of Physical Biology.* Williams and Wilkins, Baltimore, 1925.

[30] Maiti, A. and Samanta, G.P. Deterministic and stochastic analysis of a prey-dependent predator-prey system. *Internat. J. Math. Ed. Sci. Tech.* **36** (2005) 65–83.

[31] Mooring, M.S., Fitzpatrick, T.A., Nishihira, T.T. and Reisig, D.D. Vigilance, predation risk, and the Allee effect in desert bighorn sheep. *J. Wildlife Management* **68** (2004) 519–532.

[32] Murray, J.D. *Mathematical Biology.* Springer-Verlag, New York, 1993.

[33] Odum, E. *Fundamentals of Ecology.* Saunders, Philadelphia, 1953.

[34] Rinella, D.J. Wipfli, M.S., Stricker, C.A., Heintz, R.A. and Rinella, M.J. Pacific salmon (Oncorhynchus sp.) runs and consumer fitness: growth and energy storage in stream-dwelling salmonids increase with salmon spawner density. *Canad. J. Fish. Aquat. Sci.* **69** (2012) 73–84.

[35] Ruan, S. and Xiao, D. Global analysis in a predator-prey system with nonmonotonic functional response. *SIAM J. Appl. Math.* **61** (2001) 1445–1472.

[36] Stephens, P.A. and Sutherland, W.J. Consequences of the Allee effect for behaviour, ecology and conservation. *Trends Ecol. Evol.* **14** (1999) 401–405.

[37] Stephens, P.A., Sutherland, W.J. and Freckleton, R.P. What is the Allee effect? *Oikos* **87** (1999) 185–190.

[38] van Voorn, G.A.K., Hemerik, L., Boer, M.P. and Kooi, B.W. Heteroclinic orbits indicate overexploitation in predator-prey systems with a strong Allee effect. *Math. Biosci.* **209** (2007) 451–469.

[39] Verhulst, P.F. Notice sur la loi que la population poursuit dans son accroissement. *Correspondance mathématique et physique* **10** (1838) 113–121.

[40] Volterra, V. Variazioni e fluttuazioni del numero di individui in specie animali conviventi. *Mem. Accd. Lincei.* **2** (1926) 31–113.

[41] Wang, G., Liang, X.G. and Wang, F.Z. The competitive dynamics of populations subject to an Allee effect. *Ecol. Model.* **124** (1999) 183–192.

[42] Wang, M.H. and Kot, M. Speeds of invasion in a model with strong or weak Allee effects. *Math. Biosci.* **171** (2001) 83–97.

[43] Wang, J., Shi, J. and Wei, J. Predator-prey system with strong Allee effect in prey. *J. Math. Biol.* **62** (2011) 291–331.

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Multivalued Homogeneous Neumann Problem Involving Diffuse Measure Data and Variable Exponent

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Abstract: We study a nonlinear elliptic problem with homogeneous Neumann boundary condition, governed by a general Leray-Lions operator with variable exponents and Radon measure data which does not charge the sets of zero $p(.)$-capacity. We prove an existence and uniqueness result of weak solution.

Keywords: *Neumann boundary condition; diffuse measure; biting lemma of Chacon; maximal monotone graph; Radon measure data; weak solution; entropic solution; Leray-Lions operator.*

Mathematics Subject Classification (2010): 35J20, 35J25, 35D30, 35B38, 35J60.

1 Introduction and Main Results

Our aim is to study the existence and uniqueness of a solution for nonlinear homogeneous Neumann boundary value problem of the form

$$N(\beta,\mu)\begin{cases} -\nabla\cdot a(x,\nabla u)+\beta(u)\ni\mu & \text{in } \Omega,\\ a(x,\nabla u).\eta=0 & \text{on } \partial\Omega,\end{cases}$$

where η is the unit outward normal vector on $\partial\Omega$, β is a maximal monotone graph on $\mathbb{R}$ such that $0\in\beta(0)$, a is a Leray-Lions operator, μ is a diffuse measure such that $\mu=\mu\lfloor\Omega$

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and $\Omega \subset \mathbb{R}^N$ is a smooth open bounded domain ($N \geq 1$). We set $\overline{\mathrm{dom}(\beta)} = [m, M] \subset \mathbb{R}$ with $m \leq 0 \leq M$.

Recall that a Leray-Lions operator which involves variable exponents is a Carathéodory function $a(x, \xi) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ (i.e. $a(x, \xi)$ is continuous in ξ for a.e. $x \in \Omega$ and measurable in x for every $\xi \in \mathbb{R}^N$) such that:

• There exists a positive constant C_1 such that

$$|a(x, \xi)| \leq C_1(j(x) + |\xi|^{p(x)-1}) \tag{1}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$ where j is a nonnegative function in $L^{p'(.)}(\Omega)$, with $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$.

• The following inequalities hold

$$(a(x, \xi) - a(x, \eta)).(\xi - \eta) > 0 \tag{2}$$

for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$, with $\xi \neq \eta$, and there exists $C > 0$ such that

$$\frac{1}{C}|\xi|^{p(x)} \leq a(x, \xi).\xi, \tag{3}$$

for almost every $x \in \Omega$, and for every $\xi \in \mathbb{R}^N$.

In this paper, we make the following assumption on the variable exponent:

$$p(.) : \overline{\Omega} \to \mathbb{R} \text{ is a continuous function such that } 1 < p_- \leq p_+ < +\infty, \tag{4}$$

where $p_- := \operatorname{ess} \inf\limits_{x \in \Omega} p(x)$ and $p_+ := \operatorname{ess} \sup\limits_{x \in \Omega} p(x)$.

We denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure of $\mathbb{R}^N$ and by $\mathcal{M}_b(X)$ the space of bounded Radon measure in X, equipped with its standard norm $||.||_{\mathcal{M}_b(X)}$. Given $\nu \in \mathcal{M}_b(X)$, we say that ν is diffuse with respect to the capacity $W^{1,p(.)}(X)(p(.)-$capacity for short) if $\nu(B) = 0$ for every set B such that $Cap_{p(.)}(B, X) = 0$, where the Sobolev $p(.)-$capacity of B is defined by

$$Cap_{p(.)}(B, X) = \inf_{u \in S_{p(.)}(B)} \int_X \left(|u|^{p(x)} + |\nabla u|^{p(x)} \right) dx,$$

with

$$S_{p(.)}(B) = \{u \in W_0^{1,p(.)}(X) : u \geq 1 \text{ in an open set containing } B \text{ and } \ u \geq 0 \text{ in } X\}.$$

In the case $S_{p(.)}(B) = \emptyset$, we set $Cap_{p(.)}(B, X) = +\infty$.

The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(.)}(X)$.

Elliptic problems with measures data in the context of constant exponent was studied by many authors (see [4–6,10,12]). The multivalued case for Dirichlet boundary condition with constant exponent was studied by some authors among whose papers one can cite the most recent one by Igbida *et als* [14]. The study of multivalued elliptic problems with measure data in the context of variable exponent was carried out for the first time by Nyanquini *et als* [16] under homogeneous Dirichlet Boundary condition. In [16], the authors first proved a decomposition theorem for the measure data (more precisely, as a sum of a function in $L^1(\Omega)$ and of a measure in $W^{-1,p'(.)}(\Omega)$) and used it to prove, following [14], a result on existence and uniqueness of entropy solution of the problem considered.

In this paper, we consider Neumann homogeneous boundary condition. Since the boundary condition is the Neumann condition, we cannot work with the common space $W_0^{1,p(.)}(\Omega)$ in which, we can use the Poincaré inequality but also, when one uses the integration by parts formula, the term which appears at the boundary due to the part of the measure in $W^{-1,p'(.)}(\Omega)$, vanishes. We have to work in the space $W^{1,p(.)}(\Omega)$. The first main difficulty which appears in this case is that for the proof of some a priori estimates, the famous Poincaré inequality doesn't apply, and neither do the Poincaré-Wirtinger inequality and the Poincaré-Sobolev inequality (since we have homogeneous Neumann condition). A second main difficulty is that, when one uses the integration by parts formula in the Yosida approximated problem (see problem $N(\beta_\epsilon, \mu_\epsilon)$ below), a term which cannot vanish appears at the boundary, for the part of the measure data which is in $W^{-1,p'(.)}(\Omega)$. In order to treat this difficulty, we consider a smooth domain Ω in order to work with the space $W_0^{1,\widetilde{p}(.)}(U_\Omega)$, where $\widetilde{p}(.) : U_\Omega \to (1, \infty)$ is continuous such that $\widetilde{p}(x) = p(x)$ for all $x \in \overline{\Omega}$, and to go back later to the space $W^{1,p(.)}(\Omega)$. More precisely, Ω is assumed to be a bounded domain in $\mathbb{R}^N$ with a boundary $\partial\Omega$ of class C^1. Then, Ω is an extension domain (see [8]), so we can fix an open bounded subset U_Ω of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U_\Omega$, and there exists a bounded linear operator

$$E : W^{1,p(.)}(\Omega) \to W_0^{1,\widetilde{p}(.)}(U_\Omega),$$

for which
(i) $E(u) = u$ a.e. in Ω for each $u \in W^{1,p(.)}(\Omega)$,
(ii) $\|E(u)\|_{W_0^{1,\widetilde{p}(.)}(U_\Omega)} \leq C\|u\|_{W^{1,p(.)}(\Omega)}$, where C is a constant depending only on Ω.

We define

$$\mathfrak{M}_b^{p(.)}(\Omega) := \{\mu \in \mathcal{M}_b^{\widetilde{p}(.)}(U_\Omega) : \ \mu \text{ is concentrated on } \Omega\}.$$

This definition is independent of the open set U_Ω. Note that for $u \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and $\mu \in \mathfrak{M}_b^{p(.)}(\Omega)$, we have

$$\langle \mu, E(u) \rangle = \int_\Omega u \, d\mu.$$

On the other hand, as μ is diffuse (cf. Theorem 3.1 below), there exist $f \in L^1(U_\Omega)$ and $F \in (L^{\widetilde{p}'(.)}(U_\Omega))^N$ such that $\mu = f-\operatorname{div}(F)$ in $\mathcal{D}'(U_\Omega)$. Therefore, we can also write

$$\langle \mu, E(u) \rangle = \int_{U_\Omega} f E(u) \, dx + \int_{U_\Omega} F.\nabla E(u) \, dx.$$

Now, define the following spaces which are similar to that introduced in [1, 3] (see also [7]). We note

$$\mathcal{T}^{1,p(.)}(\Omega) := \big\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable}; \quad T_k(u) \in W^{1,p(.)}(\Omega) \text{ for all } k > 0\big\}.$$

As in [3], we can prove that for $u \in \mathcal{T}^{1,p(.)}(\Omega)$, there exists a unique measurable function $w : \Omega \longrightarrow \mathbb{R}$ such that $\nabla T_k(u) = w\chi_{\{|u|<k\}}$ $\forall k > 0$. This function w will be denoted by ∇u.

We define $\mathcal{T}_{\mathcal{H}}^{1,p(.)}(\Omega)$ (see [7]) as the set of functions $u \in \mathcal{T}^{1,p(.)}(\Omega)$ such that there exists a sequence $(u_\delta)_\delta \subset W^{1,p(.)}(\Omega)$ satisfying the following conditions:

(i) $u_\delta \longrightarrow u$ a.e. in Ω as $\delta \to 0$.
(ii) $\nabla T_k(u_\delta) \longrightarrow \nabla T_k(u)$ in $L^1(\Omega)$ for any $k > 0$ as $\delta \to 0$.
The symbol $\mathcal{H}$ in the notation is related to the fact that we consider here homogeneous Neumann boundary condition.

Our main results are the following theorems.

Theorem 1.1 *For any $\mu \in \mathfrak{M}_b^{p(\cdot)}(\Omega)$, the problem $N(\beta,\mu)$ has at least one solution (u, w, ν) in the sense that*

$$(u, w, \nu) \in W^{1,p(\cdot)}(\Omega) \times L^1(\Omega) \times \mathcal{M}_b^{p(\cdot)}(\Omega)$$

such that
(i) $u \in dom(\beta)$ $\mathcal{L}^N - a.e.$ in Ω,
(ii) $w \in \beta(u)$ $\mathcal{L}^N - a.e.$ in Ω,
(iii) $\nu \perp \mathcal{L}^N$, ν^+ is concentrated on $[u = M]$, ν^- is concentrated on $[u = m]$,
(iv) for any $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x, \nabla u).\nabla\varphi \, dx + \int_\Omega w\varphi \, dx + \int_\Omega \varphi \, d\nu = \int_\Omega \varphi \, d\mu. \tag{5}$$

The uniqueness of the solution is given in the following theorem.

Theorem 1.2 *Let (u_1, w_1, ν_1) and (u_2, w_2, ν_2) be two solutions of $N(\beta,\mu)$. Then*

$$\begin{cases} u_1 - u_2 = c \;\; a.e. \;\; in \; \Omega, \\ w_1 = w_2 \;\; a.e. \;\; in \; \Omega, \\ \nu_1 = \nu_2. \end{cases} \tag{6}$$

Moreover,

$$\nu^+ \leq \mu_s \;\lfloor[u = M] \tag{7}$$

and

$$\nu^- \leq -\mu_s \;\lfloor[u = m]. \tag{8}$$

2 Preliminary

As the exponent $p(.)$ appearing in (1) and (3) depends on the variable x, we must work with Lebesgue and Sobolev spaces with variable exponents. We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable function $u : \Omega \longrightarrow \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} \, dx$$

is finite. If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$|u|_{p(.)} := \inf \{\lambda > 0 : \rho_{p(.)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(\cdot)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Moreover, if $1 < p_- \leq p_+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)}+\dfrac{1}{p'(x)}=1$. Finally, we have the Hölder type inequality:

$$\Big|\int_\Omega uv\,dx\Big| \leq \Big(\frac{1}{p_-}+\frac{1}{(p')_-}\Big)|u|_{p(\cdot)}|v|_{p'(\cdot)}, \tag{9}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

Now, let

$$W^{1,p(\cdot)}(\Omega) := \Big\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\Big\},$$

which is a Banach space equipped with the following norm

$$||u||_{1,p(.)} = |u|_{p(.)} + |(|\nabla u|)|_{p(.)}.$$

The space $\Big(W^{1,p(\cdot)}(\Omega), ||u||_{1,p(.)}\Big)$ is a separable and reflexive Banach space. For the interested reader, more details about Lebesgue and Sobolev spaces with variable exponent can be found in [11, 15].

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result (cf. [13]):

Lemma 2.1 *If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p_+ < +\infty$, then the following properties hold:*
i) $|u|_{p(.)} > 1 \Longrightarrow |u|^{p_-}_{p(.)} \leq \rho_{p(.)}(u) \leq |u|^{p_+}_{p(.)}$;
ii) $|u|_{p(.)} < 1 \Longrightarrow |u|^{p_+}_{p(.)} \leq \rho_{p(.)}(u) \leq |u|^{p_-}_{p(.)}$;
iii) $|u|_{p(.)} < 1$ *(respectively* $= 1; > 1$*)* $\Longleftrightarrow \rho_{p(.)}(u) < 1$ *(respectively* $= 1; > 1$*)*;
iv) $|u_n|_{p(.)} \longrightarrow 0$ *(respectively* $\longrightarrow +\infty$*)* $\Longleftrightarrow \rho_{p(.)}(u_n) \longrightarrow 0$ *(respectively* $\longrightarrow +\infty$*)*;
v) $\rho_{p(.)}\big(u/|u|_{p(.)}\big) = 1$.

For a measurable function $u : \Omega \to \mathbb{R}$, we introduce the functional

$$\rho_{1,p(.)}(u) := \int_\Omega |u|^{p(x)}\,dx + \int_\Omega |\nabla u|^{p(x)}\,dx.$$

Then, we have the following lemma (see [17, 18]).

Lemma 2.2 *If $u_n, u \in W^{1,p(\cdot)}(\Omega)$ and $p_+ < +\infty$, then the following properties hold:*
(i) $\|u\|_{1,p(.)} > 1 \Longrightarrow \|u\|^{p_-}_{1,p(.)} \leq \rho_{1,p(.)}(u) \leq \|u\|^{p_+}_{1,p(.)}$;
(ii) $\|u\|_{1,p(.)} < 1 \Longrightarrow \|u\|^{p_+}_{1,p(.)} \leq \rho_{1,p(.)}(u) \leq \|u\|^{p_-}_{1,p(.)}$;
(iii) $\|u\|_{1,p(.)} < 1$ *(respectively* $= 1; > 1$*)* $\Longleftrightarrow \rho_{1,p(.)}(u) < 1$ *(respectively* $= 1; > 1$*)*;
(iv) $\|u_n\|_{1,p(.)} \longrightarrow 0$ *(respectively* $\longrightarrow +\infty$*)* $\Longleftrightarrow \rho_{1,p(.)}(u_n) \longrightarrow 0$ *(respectively* $\longrightarrow +\infty$*)*.

For any given $l, k > 0$, we define the function h_l by $h_l(r) = \min\big((l+1-|r|)^+, 1\big)$ and the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by $T_k(s) = \max\{-k, \min(k, s)\}$.

For any $l_0 > 0$, we consider a function h_0 such that
(i) $h_0 \in C^1_c(\mathbb{R})$, $h_0(r) \geq 0$, for all $r \in \mathbb{R}$,
(ii) $h_0(r) = 1$ if $|r| \leq l_0$ and $h_0(r) = 0$ if $|r| \geq l_0 + 1$.

Let γ be a maximal monotone operator defined on $\mathbb{R}$. We recall the definition of the main section γ_0 of γ:

$$\gamma_0(s) = \begin{cases} \text{the element of minimal absolute value of } \gamma(s), & \text{if } \gamma(s) \neq \phi, \\ +\infty, & \text{if } [s, +\infty) \cap D(\gamma) = \phi, \\ -\infty, & \text{if } (-\infty, s] \cap D(\gamma) = \phi. \end{cases}$$

We write for any $u : \Omega \to \mathbb{R}$ and $k \geq 0, \{|u| \leq k(< k, > k, \geq k, = k)\}$ for the set $\{x \in \Omega / |u(x)| \leq k(< k, > k, \geq k, = k)\}$.

To end this section, we give a useful convergence result.

Lemma 2.3 *(Lebesgue generalized convergence theorem) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions and f be a measurable function such that $f_n \to f$ a.e. in Ω. Let $(g_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$ such that for all $n \in \mathbb{N}$, $|f_n| \leq g_n$ a.e. in Ω and $g_n \to g$ in $L^1(\Omega)$. Then*

$$\int_\Omega f_n \, dx \to \int_\Omega f \, dx.$$

3 Decomposition of a Measure in $\mathcal{M}_b^{p(.)}(X)$

Let X be an open subset of $\mathbb{R}^N$. We have the following result.

Theorem 3.1 *Let $p(.) : \overline{X_1} \subset X \longrightarrow [1, +\infty]$ with $1 < p_- \leq p_+ < +\infty$ be a continuous function and $\mu \in \mathcal{M}_b(X)$. Then $\mu \in \mathcal{M}_b^{p(.)}(X)$ if and only if $\mu \in L^1(X) + W^{-1,p'(.)}(X)$.*

Proof. The proof of Theorem 3.1 is carried out in the same way as in [16], Theorem 1.2.

4 Proof of Theorem 1.1

For every $\epsilon > 0$, we consider the Yosida regularisation β_ϵ of β given by

$$\beta_\epsilon = \frac{1}{\epsilon}\big(I - (I + \epsilon\beta)^{-1}\big).$$

In accordance to [9], there exists a nonnegative, convex and l.s.c. function j defined on $\mathbb{R}$, such that $\beta = \partial j$. To regularize β, we consider

$$j_\epsilon(s) = \min_{r\in\mathbb{R}} \Big\{ \frac{1}{2\epsilon}|s - r|^2 + j(r) \Big\}, \ \forall s \in \mathbb{R}, \ \forall \epsilon > 0.$$

According to ([9], Proposition 2.11) we have

(i) $\mathrm{dom}(\beta) \subset \mathrm{dom}(j) \subset \overline{\mathrm{dom}(j)} \subset \overline{\mathrm{dom}(\beta)}$.

(ii) $j_\epsilon(s) = \frac{\epsilon}{2}\big|\beta_\epsilon(s)\big|^2 + j(J_\epsilon)$ where $J_\epsilon = (I + \epsilon\beta)^{-1}$,

(iii) j_ϵ is convex, Frechet-differentiable and $\beta_\epsilon = \partial j_\epsilon$,

(iv) $j_\epsilon \uparrow j$ as $\epsilon \downarrow 0$.

Note that β_ϵ is a nondecreasing and Lipschitz-continuous function.

Since $\mu \in \mathcal{M}_b^{\widetilde{p}(.)}(U_\Omega)$, recall that (cf. Theorem 3.1) $\mu = f - \mathrm{div}(F)$ in $\mathcal{D}'(U_\Omega)$ with $f \in L^1(U_\Omega)$ and $F \in (L^{\widetilde{p}'(.)}(U_\Omega))^N$ where U_Ω is the open bounded subset of $\mathbb{R}^N$ which extends Ω via the operator E.

We regularize μ as follows: $\forall \epsilon > 0, \ \forall x \in U_\Omega$ we define

$$f_\epsilon(x) = T_{\frac{1}{\epsilon}}(f(x))\chi_\Omega(x).$$

Let $(F_\epsilon)_{\epsilon>1} \subset C_0^\infty(U_\Omega)$ be a sequence such that $F_\epsilon \to F$ strongly in $(L^{\widetilde{p}'(.)}(U_\Omega))^N$. For any $\epsilon > 0$, we set $\tilde{F}_\epsilon = \chi_\Omega F_\epsilon$ and $\mu_\epsilon = f_\epsilon - \mathrm{div}(\tilde{F}_\epsilon)$. For any $\epsilon > 0$, one has

$\mu_\epsilon \in \mathfrak{M}_b^{p(\cdot)}(\Omega)$, $\mu_\epsilon \rightharpoonup \mu$ in $\mathcal{M}_b(U_\Omega)$ and $\mu_\epsilon \in L^\infty(\Omega)$. Furthermore, for any $k > 0$ and any $\xi \in \mathcal{T}^{1,p(\cdot)}(\Omega)$,

$$\left| \int_\Omega T_k(\xi)\, d\mu_\epsilon \right| \leq kC(\mu, \Omega).$$

Lemma 4.1 *The Yosida regularisation β_ϵ is a surjective operator.*

Proof. Since $\mathrm{dom}(\beta) \subset [m, M]$, we have $\forall\ r \in\ \mathbb{R},\ J_\epsilon(r) = \big(I + \epsilon\beta\big)^{-1}(r) \in [m, M]$. Consequently

$$\lim_{r\to+\infty} \beta_\epsilon(r) = \lim_{r\to+\infty} \frac{r - J_\epsilon(r)}{\epsilon} = +\infty$$

and

$$\lim_{r\to-\infty} \beta_\epsilon(r) = \lim_{r\to-\infty} \frac{r - J_\epsilon(r)}{\epsilon} = -\infty.$$

As β_ϵ is a maximal monotone graph, according to ([9], Corollaire 2.3), we conclude that β_ϵ is surjective.

Now, we consider the following approximating scheme problem

$$N(\beta_\epsilon, \mu_\epsilon) \begin{cases} -div\ a(x, \nabla u_\epsilon) + \beta_\epsilon(u_\epsilon) = \mu_\epsilon & \text{in } \Omega, \\ \\ a(x, \nabla u_\epsilon).\eta = 0 & \text{on } \partial\Omega. \end{cases}$$

We have the following results (see [16]).

Proposition 4.1
(i) There exists a unique weak solution u_ϵ for problem $N(\beta_\epsilon, \mu_\epsilon)$ in the sense that $u_\epsilon \in W^{1,p(\cdot)}(\Omega), \beta_\epsilon(u_\epsilon) \in L^\infty(\Omega)$ and $\forall \varphi \in W^{1,p(\cdot)}(\Omega)$,

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla\varphi\, dx + \int_\Omega \beta_\epsilon\big(u_\epsilon\big)\varphi\, dx = \int_\Omega \varphi d\,\mu_\epsilon. \tag{10}$$

(ii) Moreover, for any $k > 0$,

$$\int_\Omega |\nabla T_k(u_\epsilon)|^{p(x)} dx \ \leq\ kC(\mu, \Omega) \tag{11}$$

and

$$\int_\Omega \beta_\epsilon(u_\epsilon) T_k(u_\epsilon) dx \ \leq\ kC(\mu, \Omega), \tag{12}$$

where $C(\mu, \Omega)$ is a positive constant.

Proposition 4.2 *The sequences $\Big(\beta_\epsilon(u_\epsilon)\Big)_{\epsilon>0}$ and $\Big(\beta_\epsilon(T_k(u_\epsilon))\Big)_{\epsilon>0}$ are uniformly bounded in $L^1(\Omega)$.*

Proposition 4.3 *Let u_ϵ be a solution of $N(\beta_\epsilon, \mu_\epsilon)$, then*

$$meas\{|u_\epsilon| > k\} \ \leq\ \frac{C(\mu, \Omega)}{\min\Big(\beta_\epsilon(k), \big|\beta_\epsilon(-k)\big|\Big)} \quad \text{for } \ k > 0 \text{ large enough} \tag{13}$$

and

$$meas\{|\nabla u_\epsilon| > k\} \leq \frac{(k+1)C}{k^{p_-}} + \frac{C(\mu,\Omega)}{\min\Big(\beta_\epsilon(k), \big|\beta_\epsilon(-k)\big|\Big)} \quad \text{for } k>0 \text{ large enough}, \tag{14}$$

where C is a positive constant.

Proposition 4.4 *For all $k > 0, T_k(u_\epsilon) \to T_k(u)$ in $L^{p_-}(\Omega)$ and a.e. in Ω, as $\epsilon \to 0$. Moreover, $u : \Omega \to \mathbb{R}$ is such that $u \in dom(\beta)$ a.e. in Ω and $u_\epsilon \to u$ in measure and a.e. in Ω, as $\epsilon \to 0$.*

Proposition 4.5 *For any $k > 0$, as ϵ tends to 0, we have*
(i) $a(x, \nabla T_k(u_\epsilon)) \rightharpoonup a(x, \nabla T_k(u))$ weakly in $\Big(L^{p'(\cdot)}(\Omega)\Big)^N$.
(ii) $\nabla T_k(u_\epsilon) \longrightarrow \nabla T_k(u)$ a.e. in Ω.
(iii) $a(x, \nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon) \longrightarrow a(x, \nabla T_k(u)).\nabla T_k(u)$ a.e. in Ω and strongly in $L^1(\Omega)$.
(iv) $\nabla T_k(u_\epsilon) \longrightarrow \nabla T_k(u)$ strongly in $\big(L^{p(\cdot)}(\Omega)\big)^N$.

Proof. The proof can be carried out in the same way as the proof of Proposition 4.5 in [16]. The following lemmas are useful for the subsequent presentation.

Lemma 4.2 *For any $h \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$,*

$$\nabla[h(u_\epsilon)\varphi] \longrightarrow \nabla[h(u)\varphi] \text{ strongly in } (L^{p(\cdot)}(\Omega))^N \text{ as } \epsilon \to 0.$$

Proof. For any $h \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we have

$$\begin{aligned}&\nabla[h(u_\epsilon)\varphi] - \nabla[h(u)\varphi] = (h(u_\epsilon) - h(u))\nabla\varphi + h'(u_\epsilon)\varphi[\nabla u_\epsilon - \nabla u]\\ &+(h'(u_\epsilon) - h'(u))\varphi\nabla u := \psi_1^\epsilon + \psi_2^\epsilon + \psi_3^\epsilon.\end{aligned} \tag{15}$$

For the term ψ_1^ϵ, we consider $\rho_{p(.)}(\psi_1^\epsilon) = \displaystyle\int_\Omega |(h(u_\epsilon) - h(u))\nabla\varphi|^{p(x)}\, dx$.

Set $\Theta_1^\epsilon(x) = |(h(u_\epsilon) - h(u))\nabla\varphi|^{p(x)}$. We have $\Theta_1^\epsilon(x) \to 0$ a.e. $x \in \Omega$ as $\epsilon \to 0$ and $|\Theta_1^\epsilon(x)| \leq C(h, p_-, p_+)|\nabla\varphi|^{p(x)} \in L^1(\Omega)$. Then, by the Lebesgue dominated convergence theorem, we get that $\lim\limits_{\epsilon\to 0} \rho_{p(.)}(\psi_1^\epsilon) = 0$. Hence,

$$\|\psi_1^\epsilon\|_{L^{p(\cdot)}(\Omega)} \to 0 \text{ as } \epsilon \to 0. \tag{16}$$

For the term ψ_2^ϵ we consider $\rho_{p(.)}(\psi_2^\epsilon) = \displaystyle\int_\Omega |h'(u_\epsilon)\varphi(\nabla T_l(u_\epsilon) - \nabla T_l(u))|^{p(x)}\, dx$ for some $l > 0$ such that $\mathrm{supp}(h) \subset [-l, l]$.

Set $\Theta_2^\epsilon(x) = |h'(u_\epsilon)\varphi(\nabla T_l(u_\epsilon) - \nabla T_l(u))|^{p(x)}$. We have $\Theta_2^\epsilon(x) \to 0$ a.e. $x \in \Omega$ as $\epsilon \to 0$ and $|\Theta_2^\epsilon(x)| \leq C(h, p_-, p_+, \|\varphi\|_\infty)|\nabla T_l(u_\epsilon) - \nabla T_l(u)|^{p(x)}$. Since $\nabla T_l(u_\epsilon) \to \nabla T_l(u)$ strongly in $\big(L^{p(\cdot)}(\Omega)\big)^N$, we get $\rho_{p(.)}(\nabla T_l(u_\epsilon) - \nabla T_l(u)) \to 0$ as $\epsilon \to 0$, which is equivalent to, say

$$\lim_{\epsilon\to 0} \int_\Omega |\nabla T_l(u_\epsilon) - \nabla T_l(u)|^{p(x)}\, dx = 0.$$

Then $|\nabla T_l(u_\epsilon) - \nabla T_l(u)|^{p(\cdot)} \to 0$ strongly in $L^1(\Omega)$.

By the Lebesgue generalized convergence theorem, one has

$$\lim_{\epsilon\to 0} \int_\Omega \Theta_2^\epsilon(x)\, dx = \lim_{\epsilon\to 0} \rho_{p(.)}(\psi_2^\epsilon) = 0.$$

Hence,

$$\|\psi_2^\epsilon\|_{L^{p(.)}(\Omega)} \to 0 \text{ as } \epsilon \to 0. \tag{17}$$

For the term ψ_3^ϵ we consider $\rho_{p(.)}(\psi_3^\epsilon) = \int_\Omega |(h'(u_\epsilon) - h'(u))\varphi \nabla u|^{p(x)}\, dx$.

Set $\Theta_3^\epsilon(x) = |(h'(u_\epsilon) - h'(u))\varphi \nabla u|^{p(x)}$. We have $\Theta_3^\epsilon(x) \to 0$ a.e. $x \in \Omega$ as $\epsilon \to 0$ and $|\Theta_3^\epsilon(x)| \leq C(h, p_-, p_+, \|\varphi\|_\infty)|\nabla T_l(u)|^{p(x)} \in L^1(\Omega)$, with some $l > 0$ such that $\operatorname{supp}(h) \subset [-l, l]$. Then, by the Lebesgue dominated convergence theorem, we get $\lim_{\epsilon\to 0} \rho_{p(.)}(\psi_3^\epsilon) = 0$. Hence,

$$\|\psi_3^\epsilon\|_{L^{p(.)}(\Omega)} \to 0 \text{ as } \epsilon \to 0. \tag{18}$$

According to (16)-(18), we get $\left\|\psi_1^\epsilon + \psi_2^\epsilon + \psi_3^\epsilon\right\|_{L^{p(.)}(\Omega)} \to 0$ as $\epsilon \to 0$ and the lemma is proved.

Lemma 4.3 *For any $h \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$,* $\lim_{\epsilon\to 0} \int_\Omega h(u_\epsilon)\varphi\, d\mu_\epsilon = \int_\Omega h(u)\varphi d\mu.$

Proof. We have

$$\begin{aligned}
\int_\Omega h(u_\epsilon)\varphi\, d\mu_\epsilon &= \int_\Omega E\big(h(u_\epsilon)\varphi\big)\, d\mu_\epsilon = \big\langle \mu_\epsilon, E\big(h(u_\epsilon)\varphi\big)\big\rangle \\
&= \int_{U_\Omega} f_\epsilon E\big(h(u_\epsilon)\varphi\big)\, dx + \int_{U_\Omega} \tilde{F}_\epsilon.\nabla E\big(h(u_\epsilon)\varphi\big)\, dx \\
&= \int_{U_\Omega} \chi_\Omega T_{\frac{1}{\epsilon}}(f) E\big(h(u_\epsilon)\varphi\big)\, dx + \int_{U_\Omega} (\chi_\Omega F_\epsilon).\nabla E\big(h(u_\epsilon)\varphi\big)\, dx \\
&= \int_\Omega T_{\frac{1}{\epsilon}}(f) h(u_\epsilon)\varphi\, dx + \int_{U_\Omega} F_\epsilon.\nabla E\big(\chi_\Omega h(u_\epsilon)\varphi\big)\, dx.
\end{aligned} \tag{19}$$

By the Lebesgue dominated convergence theorem, we have for the first term of the right hand side of (19),

$$\lim_{\epsilon\to 0} \int_\Omega T_{\frac{1}{\epsilon}}(f) h(u_\epsilon)\varphi\, dx = \int_\Omega f h(u)\varphi\, dx. \tag{20}$$

Furthermore, the sequence $\Big(E\big(\chi_\Omega h(u_\epsilon)\varphi\big)\Big)_{\epsilon>0}$ is bounded in $W_0^{1,\widetilde{p}(.)}(U_\Omega)$. Indeed, $\big(\chi_\Omega h(u_\epsilon)\varphi\big)_{\epsilon>0}$ is bounded in $W^{1,p(.)}(\Omega)$ and we use the inequality

$$\|E(v)\|_{W_0^{1,\widetilde{p}(.)}(U_\Omega)} \leq C\|v\|_{W^{1,p(.)}(\Omega)},\ \forall v \in W^{1,p(.)}(\Omega).$$

We also have $E\big(\chi_\Omega h(u_\epsilon)\varphi\big) = \chi_\Omega h(u_\epsilon)\varphi$ a.e. in U_Ω and $\chi_\Omega h(u_\epsilon)\varphi \to \chi_\Omega h(u)\varphi$ a.e. in U_Ω as $\epsilon \to 0$. Hence $E\big(\chi_\Omega h(u_\epsilon)\varphi\big) \to E\big(\chi_\Omega h(u)\varphi\big)$ a.e. in U_Ω as $\epsilon \to 0$. Then,

$$\nabla E\big(\chi_\Omega h(u_\epsilon)\varphi\big) \rightharpoonup \nabla E\big(\chi_\Omega h(u)\varphi\big) \text{ in } \big(L^{\widetilde{p}(.)}(U_\Omega)\big)^N.$$

Finally, we get for the second term in the right hand side of (19)

$$\lim_{\epsilon\to 0} \int_{U_\Omega} F_\epsilon.\nabla E\big(\chi_\Omega h(u_\epsilon)\varphi\big)\, dx = \int_{U_\Omega} F.\nabla E\big(\chi_\Omega h(u)\varphi\big)\, dx. \tag{21}$$

Using (20) and (21), we get from (19),

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\varphi\,d\mu_\epsilon &= \int_\Omega fh(u)\varphi\,dx + \int_{U_\Omega} F.\nabla E\big(\chi_\Omega h(u)\varphi\big)\,dx\\ &= \int_{U_\Omega} fE\big(\chi_\Omega h(u)\varphi\big)\,dx + \int_{U_\Omega} F.\nabla E\big(\chi_\Omega h(u)\varphi\big)\,dx\\ &= \Big\langle \mu, E\big(\chi_\Omega h(u)\varphi\big)\Big\rangle = \int_{U_\Omega} E\big(\chi_\Omega h(u)\varphi\big)\,d\mu = \int_\Omega h(u)\varphi\,d\mu.\end{aligned}$$

We continue the proof of Theorem 1.1. So we need to pass to the limit in the second integral of (10). Since, for any $k>0$, $(h_k(u_\epsilon)\beta_\epsilon(u_\epsilon))_{\epsilon>0}$ is bounded in $L^1(\Omega)$, there exists $z_k\in\mathcal{M}_b(\Omega)$, such that

$$h_k(u_\epsilon)\beta_\epsilon(u_\epsilon) \overset{*}{\rightharpoonup} z_k \text{ in } \mathcal{M}_b(\Omega) \text{ as } \epsilon\to 0.$$

Moreover, for any $\varphi\in W^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$, we have

$$\int_\Omega \varphi\,dz_k = \int_\Omega \varphi h_k(u)\,d\mu - \int_\Omega a(x,\nabla u)\cdot\nabla(h_k(u)\varphi)dx,$$

which implies that $z_k\in\mathcal{M}_b^{p(.)}(\Omega)$ and, for any $k\le l$, $z_k=z_l$ on $[|T_k(u)|<k]$.

Let us consider the Radon measure z defined by

$$\begin{cases} z = z_k, \quad \text{on } [|T_k(u)|<k] \text{ for } k\in\mathbb{N}^*,\\ z=0 \quad \text{on } \displaystyle\bigcap_{k\in\mathbb{N}^*}[|T_k(u)|=k].\end{cases} \tag{22}$$

For any $h\in\mathcal{C}_c^1(\mathbb{R})$, $h(u)\in L^\infty(\Omega,d|z|)$ and

$$\int_\Omega h(u)\varphi\,dz = -\int_\Omega a(x,\nabla u)\cdot\nabla(h(u)\varphi)dx + \int_\Omega h(u)\varphi d\mu,$$

for any $\varphi\in W^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$. Indeed, let $k_0>0$ be such that $\mathrm{supp}(h)\subseteq[-k_0,k_0]$,

$$\begin{aligned}\int_\Omega h(u)\varphi\,dz &= \int_\Omega h(u)\varphi\,dz_{k_0} = -\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla(h(u_\epsilon)\varphi)dx + \lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\varphi d\mu_\epsilon\\ &= -\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla T_{k_0}(u_\epsilon))\cdot\nabla(h(u_\epsilon)\varphi)dx + \lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\varphi d\mu_\epsilon\\ &= -\int_\Omega a(x,\nabla u)\cdot\nabla(h(u)\varphi)dx + \lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\varphi d\mu_\epsilon\\ &= -\int_\Omega a(x,\nabla u)\cdot\nabla(h(u)\varphi)dx + \int_\Omega h(u)\varphi d\mu.\end{aligned} \tag{23}$$

Moreover, we have (see [16])

Lemma 4.4 *The Radon-Nikodym decomposition of the measure z given by (22) with respect to $\mathcal{L}^N$,*

$$z = w\,\mathcal{L}^N + \nu \quad \textit{with } \nu\perp\mathcal{L}^N, \tag{24}$$

satisfies the following properties:
(i) $w\in\beta(u)$ $\mathcal{L}^N - a.e.$ in Ω, $w\in L^1(\Omega)$,
(ii) $\nu\in\mathcal{M}_b^{p(.)}(\Omega)$, ν^+ is concentrated on $[u=M]$ and ν^- is concentrated on $[u=m]$.

To finish the proof of Theorem 1.1, we consider $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and $h \in C_c^1(\mathbb{R})$. Then, we take $h(u_\epsilon)\varphi$ as test function in (10). We get

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla[h(u_\epsilon)\varphi]dx + \int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\varphi dx = \int_\Omega h(u_\epsilon)\varphi d\mu_\epsilon. \tag{25}$$

By Lemma 4.3, we have for the term in the right hand side of (25),

$$\lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\varphi\, d\mu_\epsilon = \int_\Omega h(u)\varphi\, d\mu.$$

The first term of (25) can be written as

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla[h(u_\epsilon)\varphi]dx = \int_\Omega a(x, \nabla T_{l_0+1}(u_\epsilon)).\nabla[h_0(u_\epsilon)\varphi]dx,$$

for some $l_0 > 0$ so that, by Proposition 4.5-(i) and Lemma 4.2, we have

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_\Omega a(x, \nabla u_\epsilon).\nabla[h(u_\epsilon)\varphi]dx &= \lim_{\epsilon\to 0}\int_\Omega a(x, \nabla T_{l_0+1}(u_\epsilon)).\nabla[h_0(u_\epsilon)\varphi]dx \\
&= \int_\Omega a(x, \nabla T_{l_0+1}(u)).\nabla[h_0(u)\varphi]dx \\
&= \int_\Omega a(x, \nabla u).\nabla[h(u)\varphi]dx.
\end{aligned}$$

Due to the convergence of Lemma 4.2 and Proposition 4.5-(i) we have from (25)

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\varphi dx &= \int_\Omega h(u)\varphi d\mu - \int_\Omega a(x, \nabla u).\nabla[h(u)\varphi]dx. \\
&= \int_\Omega h(u)\varphi dz = \int_\Omega h(u)w\varphi dx + \int_\Omega h(u)\varphi d\nu.
\end{aligned}$$

Letting ϵ go to 0 in (25), we obtain

$$\int_\Omega a(x, \nabla u).\nabla[h(u)\varphi]dx + \int_\Omega h(u)w\varphi dx + \int_\Omega h(u)\varphi d\nu = \int_\Omega h(u)\varphi d\mu. \tag{26}$$

In (26), we take $h \in C_c^1(\mathbb{R})$ such that $[m, M] \subset \text{supp}(h) \subset [-l, l]$ and $h(s) = 1$ for all $s \in [m, M]$. As $u \in \text{dom}(\beta)$, then $h(u) = 1$ and it yields that (u, w, ν) is a solution of the problem $N(\beta, \mu)$. □

5 Proof of Theorem 1.2

Proof. For u_1, we choose $\varphi = u_1 - u_2$ as test function in (5) to get

$$\int_\Omega a(x, \nabla u_1).\nabla(u_1 - u_2)dx + \int_\Omega w_1(u_1 - u_2)dx \leq \int_\Omega (u_1 - u_2)d\mu.$$

Similarly we get for u_2,

$$\int_\Omega a(x, \nabla u_2).\nabla(u_2 - u_1)dx + \int_\Omega w_2(u_2 - u_1)dx \leq \int_\Omega (u_2 - u_1)d\mu.$$

Adding these two last inequalities yields

$$\int_\Omega \Big(a(x,\nabla u_1)-a(x,\nabla u_2)\Big).\nabla(u_1-u_2)dx+\int_\Omega (w_1-w_2)\,(u_1-u_2)dx. \tag{27}$$

From (27) it yields

$$\int_\Omega \Big(a(x,\nabla u_1)-a(x,\nabla u_2)\Big).\nabla(u_1-u_2)dx=0 \tag{28}$$

From (28), it follows that there exists a constant c such that $u_1-u_2=c$ a.e. in Ω. Now, let us see that $w_1=w_2$ a.e. in Ω and $\nu_1=\nu_2$. Indeed, for any $\varphi\in\mathcal{D}(\Omega)$, taking φ as a test function in (5) for the solutions (u_1,w_1,ν_1) and (u_1,w_2,ν_2), after substraction, we get

$$\int_\Omega (w_1-w_2)\varphi dx+\int_\Omega \varphi d(\nu_1-\nu_2)=0.$$

Hence

$$\int_\Omega w_1\varphi dx+\int_\Omega \varphi d\nu_1=\int_\Omega w_2\varphi dx+\int_\Omega \varphi d\nu_2.$$

Therefore

$$w_1\mathcal{L}^N+\nu_1=w_2\mathcal{L}^N+\nu_2.$$

Since the Radon-Nikodym decomposition of a measure is unique, we get $w_1=w_2$ a.e. in Ω and $\nu_1=\nu_2$.

To complete the proof of Theorem 1.2, it remains to show that (7) and (8) hold. To this aim, let us recall the following result.

Lemma 5.1 *Let $\eta\in W^{1,p(.)}(\Omega)$, $Z\in\mathcal{M}_b^{p(.)}(\Omega)$ and $\lambda\in\mathbb{R}$ be such that*

$$\begin{cases} \eta\le\lambda \ \text{a.e. in } \Omega \quad (\text{respectively} \quad \eta\ge\lambda), \\ \\ Z=-\mathrm{div}\ a(x,\nabla\eta) \ \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{29}$$

Then

$$\int_{[\eta=\lambda]}\xi dZ\ge 0 \quad \Big(\text{respectively} \quad \int_{[\eta=\lambda]}\xi dZ\le 0\Big),$$

for any $\xi\in C_c^1(\Omega)$, $\xi\ge 0$.

Proof of Lemma 5.1 The proof of this lemma follows the same steps of [2].

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References

[1] Andreu, F., Mazón, J.M., Segura de Léon, S. and Toledo, J. Quasi-linear elliptic and parabolic equations in L^1 with nonlinear boundary conditions. *Adv. Math. Sci. Appl.* **7** (1) (1997) 183–213.

[2] Andreu, F., Igbida, N. and Mazón, J.M. Obstacle problems for degenerate elliptic equation with nonhomogeneous nonlinear boundary conditions. *Math. Mod. and Meth in App. Sciences* **18** (11) (2008) 1869–1893.

[3] Bénilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M. and Vazquez, J. L. An L^1 theory of existence and uniqueness of nonlinear elliptic equations. *Ann Scuola Norm. Sup. Pisa* **22** (2) (1995) 240–273.

[4] Boccardo, L. and Gallouët, T. Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.* **87** (1989) 149–169.

[5] Boccardo, L. and Gallouët, T. Nonlinear elliptic equations with right hand side measures. *Comm. Partial Differential Equations* **17** (1992) 641–655.

[6] Boccardo, L., Gallouët, T. and Orsina, L. Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **13** (5) (1996) 539–551.

[7] Bonzi, B.K., Nyankini, I. and Ouaro, S. Existence and uniqueness of weak and entropy solutions for homogeneous Neumann boundary-value problems involving variable exponents. *Electron. J. Differ. Eq.* (2012) Paper No. 12, 19 p.

[8] Brézis, H. *Analyse Fonctionnelle: Théorie et Applications*. Paris, Masson, 1983.

[9] Brézis, H. *Opérateurs Maximaux Monotones et Semigroupes de Contractions Dans Les Espaces de Hilbert*. North Holland, Amsterdam, 1973.

[10] Dal Maso, G., Murat, F., Orsina, L. and Prignet, A. Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa. Cl. Sci.* **28** (4) (1999) 741–808.

[11] Diening, L., Harjulehto, P., Hästö, P. and Ruzicka, M. Lebesgue and Sobolev spaces with variable exponents. *Lecture Notes in Mathematics* 2017, (2011).

[12] Dolzmann, G., Hungerbühlër, N. and Müller, S. Non-linear elliptic systems with measure-valued right-hand side. *Math Z.* 226 **(4)**: 545–574 (1997).

[13] Fan, X. and Zhao, D. On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *J. Math. Anal. Appl.* **263** (2001) 424–446.

[14] Igbida, N., Ouaro, S. and Soma, S. Elliptic problem involving diffuse measures data. *J. Differ. Equations* **253** (12) (2012) 3159–3183.

[15] Kovacik, O. and Rakosnik, J. On spaces $L^{p(x)}$ and $W^{1,p(x)}$. *Czech. Math. J.* **41** (1991) 592–618.

[16] Nyanquini, I., Ouaro, S. and Soma, S. Entropy solution to nonlinear multivalued elliptic problem with variable exponents and measure data. *Ann. Univ. Craiova. Math. Inform.* **40** (2) (2013) 174–198.

[17] Wang, L., Fan, Y. and Ge, W. Existence and multiplicity of solutions for a Neumann problem involving the $p(x)-$Laplace operator. *Nonlinear Anal., Theory Methods Appl.* **71** (9) A (2009) 4259–4270.

[18] Yao, J. Solutions for Neumann boundary value problems involving $p(x)-$Laplace operators. *Nonlinear Anal., Theory Methods Appl.* **68** (5) A (2008) 1271–1283.