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Stability in Terms of Two Measures for Matrix Differential Equations and Graph Differential Equations

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Abstract: In this paper, an attempt has been made to study the qualitative theory of MDEs and its associated GDEs using the Lyapunov function and the concepts of stability in terms of two measures. The theory is well supported with examples. Further, a comparison method wherein the Lyapunov function is used to simplify the complicated MDE is given.

Keywords: *matrix differential equations; graph differential equations; stability in two measures.*

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1 Introduction

Any natural or manmade systems involve interactions between its constituients, which can be considered as interconnections between them. These interconnections form a network, which can be expressed by a graph [12, 2]. Also, graphs arise naturally when one models organizational structures in social sciences [10]. It has been observed that while many social phenomena change with respect to time, modeling them using static graphs has limited the study. Thus a dynamic graph, a graph that changes with time was introduced [12]. This also led to the concept of a rate of change of a graph with respect to time and a graph differential equation [12]. These concepts were introduced and successfully utilized to study the stability of complex dynamic systems through its associated adjacency matrix [12].

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In [13] the author and her group have utilized the concepts defined in [12] including a graph linear space and its associated matrix linear space. Observing that the study of graph differential equations (GDEs) falls into the realm of differential equations in abstract spaces, the author and her group planned to study GDEs through the associated matrix differential equations (MDEs). This approach appeared to be more reasonable and practical for the study of GDEs. Hence in [13], a weighted directed simple graph was considered as a basic element and existence and uniqueness results were obtained by using monotone iterative technique for the MDE. It is interesting to note that in 2008 a comparison principle for matrix differential equations was developed by Martynyuk [8]. It was realized that simple graphs have no loops and hence in terms of applications a simple graph is not a correct representative of a social structure. This led to the definition of a pseudo simple graph in [14]. Also in [14] a proposition was made that the non linearity of a prey predator model can be preserved using graphs. In [3, 11, 13, 15] many results have been obtained for MDEs and its associated GDEs in terms of iterative techniques and basic theory. With the basic theory well placed the question of studying the qualitative theory of MDEs and its associated GDEs came to the fore. In this direction there is a paper dealing with the stability of dynamic graphs on time scales [2].

The Lyapunov second method, with its advantage of not requiring the knowledge of solutions, has gained increasing significance and gave impetus for developments in the stability theory of differential equations [5]. It is now recognized that the Lyapunov function can be considered to define a generalized distance and can be employed to study various qualitative and quantitative properties of dynamic systems. Further, Lyapunov function serves as a vehicle to transform a given completed differential system into a relatively simpler system and as a result, it is enough to study the properties of solutions of the simpler system.

It was observed that at times a single Lyapunov function might not cater to the needs of a problem and hence a vector Lyapunov function [6] was introduced. In another direction new concepts of stability were defined to be on par with the real world situations. Concepts like partial stability, eventual stability and practical stability were introduced. This posed the question of the possibility of unification of all the definitions. As an answer the concept of stability in terms of two measures [7] was introduced. At this stage, it is appropriate to mention that the study of stability of physical applications is quite appealing. In this context we refer to the following two papers dealing with stability for real world problems [9] and mechanical systems with swiching linear force fields [1].

In this paper, an attempt has been made to study the qualitative theory of MDEs and its associated GDEs using the Lyapunov function and the concept of stability in terms of two measures. The theory is well supported with examples. Further, a comparison method wherein the Lyapunov function is used to simplify the complicated MDE is given.

2 Preliminaries

In this section, we introduce all the necessary notation and results that have been developed in earlier works.

Definition 2.1 Pseudo simple graph: A simple graph having loops is called a pseudo simple graph.

Let $v_1, v_2, ...v_N$, be N vertice, where N is any positive integer. Let D_N be the set of all weighted directed pseudo simple graphs D=(V, E). Then $(D_N, +, .)$ is a linear space with respect to the operations + and . defined in [12, 13].

Let the set of all matrices be $\mathbb{R}^{N\times N}$. Then $(\mathbb{R}^{N\times N}, +, .)$ is a matrix linear space where '+' denotes matrix addition and '.' denotes multiplication of a matrix by a scalar.

Definition 2.2 Continuous and differentiable matrix function:
(1) A matrix function $E : J \to \mathbb{R}^{N\times N}$ defined by $E(t) = (e_{ij}(t))_{N\times N}$ is said to be continuous if and only if each entry $e_{ij}(t)$ is continuous for all $i, j = 1, 2, \ldots, N$ where $e_{ij} : J \to \mathbb{R}$.
(2) A continuous matrix function $E(t)$ is said to be differentiable if and only if each entry $e_{ij}(t)$ is differentiable for all $i, j = 1, 2, \ldots, N$. The derivative of $E(t)$ (if it exists) is denoted by $E'(t)$ and is given by $E'(t) = (e'_{ij}(t))_{N\times N}$.

Definition 2.3 Continuous and differentiable graph function: Let $D : J \to D_N$ be a graph function and $E : J \to \mathbb{R}^{N\times N}$ be its associated adjacency matrix function. Then

(1) $D(t)$ is said to be continuous if and only if $E(t)$ is continuous.
(2) $D(t)$ is said to be differentiable if and only if $E(t)$ is differentiable.

Consider the initial value problem

$$D' = G(t, D), \qquad D(t_0) = D_0, \tag{2.1}$$

where $G \in C[J \times D_N, D_N]$ and $J = [t_0, T]$. The derivative of a graph function D denoted by D' is the graph function whose edges have weight functions that are derivatives of the weight functions of the corresponding edges of D.

The integral of a graph function D denoted by $\int D\ dt$ is the graph function whose edges have weight functions that are integrals of the weight functions of the corresponding edges of D. With the above definitions the initial value problem (IVP) of GDE (2.1) can be written as the graph integral equation

$$D(t) = D_0 + \int_{t_0}^{t} G(s, D(s))ds. \tag{2.2}$$

Now using the isomorphism between graphs and matrices we observe that the graph function $G(t, D)$ will be isomorphic to some matrix function $F(t, E)$, and corresponding to (2.1) and (2.2), we can consider the IVP of matrix differential equation

$$E' = F(t, E), \qquad E(t_0) = E_0, \tag{2.3}$$

and the matrix integral equation

$$E(t) = E_0 + \int_{t_0}^{t} F(s, E(s))ds, \tag{2.4}$$

where E_0 is the adjacency matrix of D_0.

In the following sections, we study stability results for the MDE and using the isomorphism that exists between graphs and matrices, we obtain similar results for the corresponding GDE. In order to do so we begin with the following definitions.

Definition 2.4 Stability: Consider the differential system

$$E' = F(t, E), \qquad E(t_0) = E_0, \qquad t \geq t_0, \tag{2.5}$$

where $F \in [R_+ \times \mathbb{R}^{N\times N}, \mathbb{R}^{N\times N}]$. Suppose that the function F is smooth enough to guarantee existence, uniqueness and continuous dependence of solutions $E(t) = E(t, t_0, E_0)$ of (2.5). Before proceeding further, we introduce the following classes of functions which are needed in our work

$$K = \{a \in C[R_+, R_+] : a(u) \ is \ \ strictly \ increasing \ in \ u \ and \ a(0) = 0\},$$

$$L = \{\sigma \in C[R_+, R_+] : \sigma(u) \ is \ \ strictly \ decreasing \ in \ u \ and \ \ lim_{u\to\infty}\sigma(u) = 0\},$$

$$KL = \{a \in C[R_+^2, R_+] : a(t, s) \in K \ for \ each \ s \ and \ a(t, s) \in L \ for \ each \ t\},$$

$$CK = \{a \in C[R_+^2, R_+] : a(t, s) \in K \ for \ each \ t\},$$

$$\Gamma = \{h \in C[R_+^2 \times \ \mathbb{R}^{N\times N}, R_+] : inf_{\{t,E\}}h(t, E) = 0\},$$

$$\Gamma_0 = \{h \in \Gamma \ inf \ h(t, E) = 0 \ for \ each \ t \in R_+\}.$$

We are ready to define various stability concepts for the system (2.3) in terms of two measures $h_0, h \in \Gamma$.

Definition 2.5 The differential system (2.3) is said to be

(S_1) (h_0, h) -equi-stable if, for each $\epsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that $h_0(t_0, E_0) < \delta$ implies $h(t, E(t)) < \epsilon$, $t \geq t_0$ where $E(t) = E(t, t_0, E_0)$ is any solution of the system (2.5)

(S_2) (h_0, h)-uniformly stable if the δ in (S_1) is independent of t_0;

(S_3) (h_0, h)-equi-attractive-uniformly stable, if for each $\epsilon > 0$ and $t_0 \in R_+$ there exist positive constants $\delta_0 = \delta(t_0)$ and $T = T(t_0, \epsilon)$ such that $h_0(t_0, E_0) < \delta_0$ implies that $h(t, E(t)) < \epsilon, t \geq t_0 + T$;

(S_4) (h_0, h)-uniformly attractive, if (S_3) holds with δ_0 and T being independent of t_0;

(S_5) (h_0, h)-equi-asymptotically stable if (S_1) and (S_3) hold simultaneously;

(S_6) (h_0, h)-uniformly-asymptotically stable if (S_2) and (S_4) hold together;

(S_7) (h_0, h)-equi attractive in the large if for each $\epsilon > 0$ and $\alpha > 0$ and $t_0 \in R_+$, there exists a positive number $T = T(t_0, \epsilon, \alpha)$ such that $h_0(t_0, E_0) < \alpha$ implies $h(t, E(t)) < \epsilon, \ t \geq t_0 + T$;

(S_8) (h_0, h)-uniformly attractive in the large if the constant T in (S_7) is independent of t_0;

(S_9) (h_0, h)-unstable if (S_1) fails to hold.

In order to understand the generality of the above stability definitions refer to [P. 5,6 of [7]] where examples are given.

Next, we need the following definitions.

Definition 2.6 Let $h_0, h \in \Gamma$. Then we say that

(i) h_0 is finer than h if there exist a $\rho > 0$ and a function $\phi \in CK$ such that $h_0(t, E) < \rho$ implies $h(t, E) \leq \phi(t, h_0(t, E))$;

(ii) h_0 is uniformly finer than h if in (i) ϕ is independent of t;

(iii) h_0 is asymptotically finer than h if there exist a $\rho > 0$ and a function KL such that $h_0(t, E) < \rho$ implies $h(t, E) \leq \phi(h_0(t, E), t)$.

Definition 2.7 Let $V \in C[R_+ \times \mathbb{R}^{N\times N}, R_+]$ then V is said to be
(i) h-positive definite if there exist a $\rho > 0$ and a function $b \in K$ such that $b(h(t,E)) \leq V(t,E)$ whenever $h(t,E) \leq \rho$;
(ii) h- decrescent if there exist a $\rho > 0$ and a function $a \in K$ such that $V(t,E) \leq a(h(t,E))$ whenever $h(t,E) < \rho$;
(iii) h-weakly decrescent if there exist a $\rho > 0$ and a function $a \in CK$ such that $V_0(t,E) \leq a(t, h(t,E))$ whenever $h(t,E) < \rho$;
(iv) h-asymptotically decrescent if there exist a $\rho > 0$ and a function $a \in KL$ such that $V(t,E) \leq a(h(t,E),t)$ whenever $h(t,E) < \rho$.

For any function $V \in C[R_+ \times \mathbb{R}^{N\times N}, R_+]$ we define the function

$$D^+V(t,E) = \lim_{\delta\to 0^+} = \sup \frac{1}{\delta}[V(t+\delta, E+\delta F(t,E)) - V(t,E))] \tag{2.6}$$

for $(t,E) \in R_+ \times \mathbb{R}^{N\times N}$.

Let $E(t)$ be a solution of (2.3) existing on $[t_0,\infty)$ and $V(t,E)$ be locally Lipschitzian in E . Then, given $t \geq t_0$, there exists a neighbourhood U of $(t, E(t))$ and an $L > 0$ such that $|V(\tau,\zeta) - V(\tau,\eta)| \leq L||\zeta - \eta||$ for $(\tau,\zeta), (\tau,\eta) \in U$.

3 Lyapunov Theorems in Two Measures

In this section we propose to state and prove the theorems due to Lyapunov in terms of two measures for GDEs through its associated MDEs. Though the two theorems of Lyapunov deal with uniform stability and uniform asymptotic stability, we begin with a result on equi stability. We weaken the condition of differentiability of the Lyapunov function by assuming continuity and that it possesses a Dini derivative. We consider the IVP of MDE given by

$$E' = F(t,E), \quad E(t_0) = E_0, \quad t \geq t_0, \tag{3.1}$$

where $F \in C[\mathbb{R}_+ \times \mathbb{R}^{N\times N}, \mathbb{R}^{N\times N}]$.

Theorem 3.1 *Assume that*
(H1) $V \in C[\mathbb{R}_+ \times \mathbb{R}^{N\times N}, \mathbb{R}_+]$, $h \in \Gamma$, $V(t,E)$ *is locally Lipschitzian in E and h-positive definite;*
(H2) $D^+V(t,E) \leq 0$, $(t,E) \in S(h,\rho) = \{(t,E) \in \mathbb{R}_+ \times \mathbb{R}^{N\times N},\ h(t,E) < \rho,\ \rho > 0\}$*;*
(H3) $h_0 \in \Gamma$, h_0 *is finer than* h *and V(t,E) is* h_0 *weakly decrescent. Then the system (3.1) is* $(h_0,h)-$ *equi stable.*

Proof. From the hypothesis (H1), V is h-positive definite, hence there exist a positive constant $\rho_0 \in (0,\rho)$ and a function $b \in K$ such that

$$b(h(t,E)) \leq V(t,E) \quad whenever \quad h(t,E) \leq \rho_0. \tag{3.2}$$

By hypothesis (H2), V(t,E) is h_0- weakly decrescent, therefore for $t_0 \in \mathbb{R}_+$, $E_0 \in \mathbb{R}^{N\times N}$, there exist a constant $\delta_0 = \delta(t_0) > 0$ and a function $a \in K$ such that $h_0(t_0,E_0) < \delta_0$ implies

$$V(t_0,E_0) \leq a(t_0, h_0(t_0,E_0)). \tag{3.3}$$

Further, the fact that h_0 is finer than h implies that there exist a constant $\delta_1 = \delta_1(t_0) > 0$ and a function $\psi \in CK$ such that

$$h(t_0, E_0) \leq \psi(t_0, h(t_0, E_0)) \quad whenever \quad h(t_0, E_0) < \delta_1, \tag{3.4}$$

where δ_1 is chosen so that $(t_0, \delta_1) < \rho_0$. Let $\epsilon \in (0, \rho_0)$ and $t_0 \in \psi_+$ be given. Since $a \in CK$, there exists a $\delta_2 = \delta_2(t_0, \epsilon) > 0$ that is continuous in t_0 such that

$$a(t_0, \delta_2) < b(\epsilon). \tag{3.5}$$

Choose $\delta(t_0) = min\{\delta_0, \delta_1, \delta_2\}$. Then, using the fact that $h(t_0, E_0) < \delta_0$ and the relations from (3.2) to (3.5) we get

$$b(h(t_0, E_0)) \leq V(t_0, E_0) \leq a(t_0, h_0(t_0, E_0)) < b(\epsilon), \tag{3.6}$$

which in turn yields that $h(t_0, E_0) < \epsilon$. We claim that for every solution $E(t) = E(t, t_0, E_0)$ of (3.1) satisfying $h(t_0, E_0) < \delta$, we have

$$h(t, E(t)) < \epsilon, \quad t \geq t_0. \tag{3.7}$$

If this is not true, there exists a $t_1 > t_0$ such that

$$h(t_1, E(t_1)) = \epsilon \;\; and \;\; h(t, E(t)) < \epsilon, \;\; t \in [t_0, t_1], \tag{3.8}$$

for some solution $E(t, t_0, E_0)$ of (3.1). Set $m(t) = V(t, E(t)), \quad for \ t \in [t_0, t_1]$ and using the fact that V is Lipschitzian in E and the definition of $D^+V(t, E)$ we arrive at

$D^+m(t) \leq 0$, which implies by Lemma 1.1 [4] , that $m(t)$ is nonincreasing in $[t_0, t_1]$, that is $V(t, E(t))$ is nonincreasing in $[t_0, t_1]$, which yields $V(t_1, E(t_1)) \leq V(t_0, E(t_0))$. On combining the relations from (3.5) to (3.8), we obtain

$$b(\epsilon) = V(t_1, E(t_1)) \leq V(t_0, E(t_0)) \leq a(t_0, h_0(t_0, E_0(t_0))) < b(\epsilon) \tag{3.9}$$

which is a contradiction. Hence (3.7) holds, which means that $E(t) < \epsilon$ for all $t \geq t_0$. The proof is complete.

Theorem 3.2 *Assume that the hypotheses (H1) and (H2) of Theorem 2.1 hold. Further assume that $h_0 \in \Gamma$, h_0 is uniformly finer than h, and $V(t, E)$ is h_0- decrescent. Then the system (3.1) is $(h_0, h)-$ uniformly stable.*

Proof. Since h_0 is uniformly finer than h and $V(t, E)$ is h_0- decrescent, there exist functions $a \in K$ and $\psi \in K$ such that

$$h(t_0, E_0) \leq \psi(h_0(\epsilon)), \tag{3.10}$$

$$V(t_0, E_0) \leq a(h_0(\epsilon)). \tag{3.11}$$

Working along the lines of the proof of Theorem 3.1, the relations (3.2), (3.5), (3.9) together with the relations (3.10) and (3.11) yield the uniform stability of system (3.1). The proof is complete.

Theorem 3.3 *Assume that*

(i) h_0, $h \in \Gamma$ *and* h_0 *is uniformly finer than h;*

(ii) $V \in C[\mathbb{R}_+ \times \mathbb{R}^{N \times N}, \mathbb{R}_+]$, $V(t, E)$ *is locally Lipschitzian in E, h-positive definite,* h_0- *decrescent and*

$$D^+V(t, E) \leq -c(h_0(t, E)), \quad (t, E) \in S(h, \rho), \quad c \in K. \tag{3.12}$$

Then the system (3.1) *is* (h_0, h)*-uniformly asymptotically stable.*

Proof. Since $V(t,E)$ is h-positively definite and h_0-decrescent, there exist constants $\rho_0,\ \delta_0$ with $0 \le \rho_0 \le \rho,\ \ \delta_0 > 0$ and functions $a, b \in K$ such that

$$b(h(t,E)) \le V(t,E), \quad (t,E) \in S(h,\rho_0) \tag{3.13}$$

and

$$V(t,E) \le a(h_0(t,\epsilon)), \quad whenever \quad h_0(t,E) < \delta_0. \tag{3.14}$$

Since the hypothesis of Theorem 3.2 is satisfied, the system (3.1) is (h_0,h)-uniformly stable. Thus setting $\epsilon = \rho_0$, there exists a $\delta_1 = \delta_1(\rho_0) > 0$ such that $h_0(t_0,E_0) < \delta$ implies $h(t,E(t)) < \rho_0,\ \ t \ge t_0$, where $E(t) = E(t,t_0,E_0)$ is any solution of the system (3.1).

Let $0 < \epsilon < \rho_0$. Then the (h_0,h) uniform stability of the system (3.1) yields a $\delta = \delta(\epsilon)$ such that $h_0(t_0,E_0) < \delta$ implies $h(t,E(t)) < \epsilon,\ \ t \ge t_0$. Taking $\overline{\delta} = min\{\delta_0,\delta_1\}$, we assume that $h_0(t_0,E_0) < \overline{\delta}$, and choose $T = T(\epsilon) = a(\overline{\delta})/c(\delta) + 1$.

To show that the system (3.1) is (h_0,h)-uniformly stable, it is enough to show that there exists a $t \in [t_0, t_0+T]$ such that $h_0(\overline{t}, E(\overline{t})) < \delta$. If the above relation does not hold, then there exists a solution $E(t) = E(t,t_0,E_0)$ of the system (3.1) with $h_0(t_0,E_0) < \overline{\delta}$ such that

$$h(t,E(t)) \ge \delta, \quad t \in [t_0, t_0+T]. \tag{3.15}$$

Let $m(t) = V(t,E(t))$. Then, since $V(t,E)$ is locally Lipschitzian in E, taking Dini derivative we get $D^+m(t) \le D^+V(t,E(t)) \le -c[h_0(t,E(t))],\ \ t \ge t_0$, which yields
$m(t_0+T) - m(t_0) \le -\int_{t_0}^{t_0+T} c(h_0(s,E(s)))ds$. Thus
$\int_{t_0}^{t_0+T}(h_0(s,E(s)))ds \le m(t_0) - m(t_0+T) \le V(t,E(t_0)) \le a(h_0(t_0,E(t_0))) < a(\overline{\delta})$.

On the other hand,
$\int_{t_0}^{t_0+T} c(h_0(s,E(s)))ds \ge c(\delta)T = c(\delta).a(\delta^*)/c(\delta) + 1 = a(\widehat{\delta + 1}) > a(\delta^*)$,
which is a contradiction. Thus, the proof of the theorem is complete.

Now we proceed to consider the IVP of GDE given by

$$D' = G(t,E), \qquad D(t_0) = D_0, \tag{3.16}$$

where $G \in C[\mathbb{R}_+ \times D_N, D_N]$. In order to study the stability properties of the system (3.16), we use the existence of an isomorphism between graphs and matrices and state and prove the following theorem.

Theorem 3.4 *Assume that there exists a function $F(t,E)$ isomorphic to $G(t,D)$ in GDE* (3.16) *such that $F \in C[\mathbb{R}_+ \times \mathbb{R}^{N\times N}, \mathbb{R}_+]$. Further, assume that there exists a function $V \in C[\mathbb{R}_+ \times \mathbb{R}^{N\times N}, \mathbb{R}_+]$ satisfying the hypothesis of Theorem 3.1. Then the system* (3.16) *is equistable.*

Proof. Since F is isomorphic to G and the existence of continuous function F is given, we consider the IVP for MDE (3.1). As the hypothesis of Theorem 3.1 is satisfied, we have that the system (3.1) is equistable. Now by virtue of the existence of isomorphism between graphs and matrices, we observe that the Lyapunov function V also caters to the GDE (3.16) and hence the system (3.16) is equistable.

Similar results parallel to Theorem 3.2 and Theorem 3.3 can be established for the IVP of the GDE (3.16).

4 Examples

In this section, we proceeed to give examples to each of the theorems in the previous section. We consider a graph differential equation of a system having two vertices and weighted edge functions. Note that we have taken the examples in 7 and extended them suitably to cater to our need.

Example 4.1 Consider a graph differential equation given by two vertices V_1 and V_2 and whose derivatives of weighted edges are given by the following equations

$$\begin{cases} e'_{11} = -e_{12}\ e^t, \\ e'_{12} = -\ \dfrac{1}{2}\ e_{12} + e_{11} - e_{21} + \dfrac{1}{2}\ e_{22}, \\ e'_{21} = (e_{11} - e_{21})\ e^t, \\ e'_{22} = -\ \dfrac{1}{2}\ (e_{12} + e_{22})e^t. \end{cases} \tag{4.1}$$

Using the isomorphism between the graphs and the matrices, the fore mentioned graph differential equation can be written as the matrix differential equation given by

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}' = \begin{bmatrix} -e^t x_2 & -\frac{1}{2}\ x_2 + x_1 - x_3 + \frac{1}{2}\ x_4 \\ (x_1 - x_3)e^t & -\frac{1}{2}(x_2 + x_4)e^t \end{bmatrix}, \tag{4.2}$$

where $x_1,\ x_2,\ x_3,\ x_4$ represent the weighted edges $e_{11},\ e_{13},\ e_{13},\ e_{14}$ respectively. Thus

$$E = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

Now we define the Lyapunov function $V(t, E) = (x_2^2 + x_4^2)e^t + (x_1 - x_3)^2$ and

$$h(t, E) = \sqrt{x_1^2 + x_4^2}, \quad h_0(t, E) = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}.$$

Then clearly

$$[h(t, E)]^2 \leq V(t, E) \leq [h_0(t, E)]^2,\ D^+V(t, E) \leq -2(x_1 - x_3)^2\ e^t \leq 0,\ (t, E) \in \mathbb{R}_+ \times \mathbb{R}^{2\times 2}.$$

Hence by Theorem 3.1, the matrix differential equation (4.2) equistable, which in turn yields on using Theorem 3.4, that the graph differential equation (4.1) is also equistable.

Example 4.2 Consider a graph differential equation associated with two vertices V_1 and V_2 and weighted edge function $e_{i,j}(t),\ \ i, j = 1, 2$ given by the following equations

$$\begin{cases} e'_{11} = -e_{22}, \\ e'_{12} = -\ e_{21} + (1 - e_{12}^2 - e_{21}^2)\ e_{12}\ e^{-t}, \\ e'_{21} = e_{12} + (1 - e_{12}^2 - e_{21}^2)\ e_{21}\ \sin^2 x, \\ e'_{22} = e_{11}. \end{cases} \tag{4.3}$$

Associated with the above graph differential equation (4.3), we can write the matrix differential equation, where $x_1,\ x_2,\ x_3,\ x_4$ represent the $e_{11},\ e_{12},\ e_{21},\ e_{22}$ respectively as

$$E' = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}' = \begin{bmatrix} -x_4 & -x_3 + (1 - x_2^2 - x_3^2)x_2 e^{-t} \\ x_2 + (1 - x^2 - x_3^2)\ x_3 \sin^2 x_2 & x_1 \end{bmatrix}, \tag{4.4}$$

where $E \in \mathbb{R}^{2\times 2}$. Let $V(E) = (x_1^2 + x_4^2 - 1)^2 + (x_2^2 + x_3^2 - 1)^2$ and

$$h(E) = \sqrt{(x_2^2 + x_3^2 - 1)^2}, \quad h_0(E) = \sqrt{(x_1^2 + x_4^2 - 1)^2 + (x_2^2 + x_3^2 - 1)^2}.$$

Then clearly $[h(E)]^2 \le V(E) \le [h_0(E)]^2$, $E \in \mathbb{R}^{2\times 2}$ and

$$D^+V(E) = (-4)\ (x_2^2 + x_3^2 - 1)^2\ (x_2^2 e^{-t} + x_3^2 \sin^2 x) \le 0,\ (t, E) \in \mathbb{R}_+ \times \mathbb{R}^{2\times 2}.$$

The $(h_0,\ h)-$uniform stability follows from Theorem 3.2. Observe that

$$\begin{bmatrix} x_1(t) & x_2(t) \\ x_3(t) & x_4(t) \end{bmatrix} = \begin{bmatrix} \sin t & \cos t \\ \sin t & \cos t \end{bmatrix}$$

and has components $(x_1(t),\ x_4(t)) = (\cos t,\ \sin t)$ and $(x_2(t),\ x_3(t)) = (\sin t,\ \cos t)$ which are periodic, hence the system in pairs $(x_1(t),\ x_4(t))$ and $(x_2(t),\ x_3(t))$ is uniformly orbitally stable. It now follows that the considered graph differential equation is also $(h_0 - h)$-uniformly stable.

The following example will illustrate Theorem 3.3.

Consider a graph having two vertices V_1 and V_2. Suppose a graph differential equation is defined on this graph, where the edges satisfy the relations

$$\begin{cases} e_{11}' = 2e_{12} - e_{11}\ e^t - e_{22}, \\ e_{12}' = -e_{12}(1 + \sin^2\ e_{21}) - 2e_{11}e^{-t} - e_{22}, \\ e_{21}' = -e_{12}\ e^{-t} + e_{11}\ \cos t + e_{21}\ \sin t, \\ e_{22}' = -(e_{12} + e_{11})e^{-t} - e_{22}. \end{cases} \tag{4.5}$$

Then we construct the adjacency matrix by replacing $e_{11},\ e_{12},\ e_{21},\ e_{22}$ by $x_1,\ x_2,\ x_3,\ x_4$ respectively and obtain the matrix differential equation

$$E' = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}' = \begin{bmatrix} 2x_2 - x_1\ e^t - x_4 & -x_2(1 + \sin^2 x_3) - 2x^2 e^{-t} + x_4) \\ -x_2 e^{-t} + x_1 \cos t + x_3 \sin t & -x_2 + x_1 e^{-t} - x_4 \end{bmatrix}, \tag{4.6}$$

where $E \in \mathbb{R}^{2\times 2}$. Define

$$A = \{\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in \mathbb{R}^{2\times 2} : x_1 = x_2 = x_4 = 0\},\quad B = \{\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in \mathbb{R}^{2\times 2} : x_1 = x_4 = 0\}$$

and $V(t, E) = x_1^2 + x_2^2 e^{-t} + x_4^2$. For $E_1 = (c_{ij})_{2\times 2}$ and $E_2 = (d_{ij})_{2\times 2}$, we define

$$d(E_1,\ E_2) = \sqrt{\sum_{i,j=1}^{2} (c_{ij} - d_{ij})^2}$$

and consider $h(t, E) = d(E, B)$ and $h_0(t, E) = d(E, A)$. Then

$$h_0(t, E) = \sqrt{x_1^2 + x_2^2 + x_4^2}, \quad h(t, E) = \sqrt{x_1^2 + x_4^2}$$

which yield $A \subset B$ and

$$[h(t, E)]^2 \le V(t, E) \le [h_0(t, E)]^2.$$

Also

$$D^+V(t, E) \le (-2)[h_0(t, E)]^2.$$

An application of Theorem 3.3 yields that the matrix differential equation (4.6) is $(h_0 - h)$ uniformly asymptotically stable. From which we can make the same conclusion for the graph differential equation (4.5) using the isomorphism between matrices and graphs.

5 Comparison Technique

It is well known that a Lyapunov function can be considered as a vehicle to transform a given complicated differential system into a relatively simpler scalar differential equation. Thus using the concept of a Lyapunov function and theory of differential inequalities we obtain a very general comparison principle in terms of two measures. In order to achive our goal we need the following results from [4,15].

Consider the scalar differential equation given by

$$u' = g(t,u), \quad u(t_0) = u_0 \geq 0, \tag{5.1}$$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$ and $g(t_0) = 0$.

Definition 5.1 Let $r(t)$ be a solution of (5.1) existing on some interval $I = [t_0, t_0 + \alpha]$, $0 < \alpha < \infty$. Then $r(t)$ is said to be a maximal solution of (5.1) if for every solution $u(t) = u(t, t_0, u_0)$ of (5.1) existing on J, the following inequality holds

$$u(t) \leq r(t), \quad t \in J. \tag{5.2}$$

Lemma 5.1 *Let $g \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$ and $r(t) = r(t, t_0, u_0)$ be the maximal solution of* (5.1) *existing on J. Suppose that $m \in C[\mathbb{R}_+, \mathbb{R}_+]$ and $Dm(t) \leq g(t, m(t))$, $t \in J$, where D is any fixed Dini derivative. Then $m(t_0) \leq u_0$ implies $m(t) \leq r(t)$, $t \in J$.*

We now formulate a basic comparison theorem in terms of Lyapunov function V for MDE (3.1).

Theorem 5.1 *Let $V \in C[R_+ \times \mathbb{R}^{N\times N}, R_+]$ and $V(t,E)$ be locally Lipschitzian in E for each $t \in \mathbb{R}_+$. Assume further that*

$$D^+V(t,E) \leq g(t, V(t,E)), \quad (t,E) \in \mathbb{R}_+ \times \mathbb{R}^{N\times N}, \tag{5.3}$$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$. Let $r(t) = r(t, t_0, u_0)$ be the maximal solution of (5.1) existing on J. Then, for any solution $E(t) = E(t, t_0, E_0)$ of (3.1) existing on J, $V(t_0, E_0) \leq u_0$ implies

$$V(t, E(t)) \leq r(t), \quad t \in J. \tag{5.4}$$

Proof. Let $E(t) = E(t, t_0, E_0)$ be a solution of (3.1). Set $m(t) = V(t, E(t))$ such that $V(t_0, E_0) \leq u_0$. Using the fact that $V(t,E)$ is locally Lipschitzian in E, the definition of Dini derivative and the relation (5.3) we arrive at the inequality $D^+m(t) \leq g(t, V(t, m(t)))$, $m(t_0) \leq u_0, t \in J$ From Lemma 5.1, we conclude that $V(t, E(t)) \leq r(t), t \in J$, completing the proof.

For the sake of completeness, we define the stability concept for the trivial solution of the comparision equation (5.1). We give here the definition of equistability only.

Definition 5.2 Let $u(t, t_0, u_0)$ be any solution of (5.1). The trivial solution $u(t) \equiv 0$ of (5.1) is said to be equistable if for any $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ that is continuous in t_o for each ϵ such that $u_0 < \delta$ implies $u(t, t_0, u_0) < \epsilon, t \geq t_0$.

We will now state and prove the following theorem which gives sufficient conditions for the (h_0, h)-stability properties of the differential system.

Theorem 5.2 *Assume that*
(i) $h_0,\ h \in \Gamma$ *and* h_0 *is uniformly finer than* h*;*
(ii) $V \in C[\mathbb{R}_+ \times \mathbb{R}^{N\times N},\ \mathbb{R}_+],\ V(t,E)$ *is locally Lipshitzian in* E, V *is* $h-$ *positive definite and* h_0-*decrescent;*
(iii) $g \in C[\mathbb{R}_+ \times \mathbb{R},\ \mathbb{R}]$ *and* $g(t,0) \equiv 0$*;*
(iv) $D^+V(t,E) \le g(t,\ V(t,E)),\ (t,E) \in S(h,\rho)$, *where* $S(h,\rho) = \{(t,E) \in \mathbb{R}_+ \times \mathbb{R}^{N\times N} :\ h(t,E) < \rho,\ \rho > 0\}$.

Then, the stability properties of the trivial solution of (4.2) *imply the corresponding* $(h_0,\ h)-$ *stability properties of MDE* (3.1).

Proof. As the proofs of various stability properties are similar, we shall only prove the $(h_0,\ h)-$ equiasymptotic stability property of (3.1). In order to do so, we begin by proving $(h_0,\ h)-$ stability.

Since V is $h-$ positive definite, there exist a $\lambda \in (0,\rho]$ and a $b \in K$ such that

$$b(h(t,E)) \le V(t,E), \quad (t,E) \in S(h,\ \lambda). \tag{5.5}$$

Let $0 < \epsilon < \lambda$ and $t_0 \in \mathbb{R}_+$ be given and assume that the trivial solution of (5.1) is equistable. Then, given $b(\epsilon) > 0$ and $t_0 \in \mathbb{R}_+$, there exists a positive function $\delta_1 = \delta_1(t_0,\epsilon)$ such that

$$u_0 < \delta \ \ \textit{implies} \ \ u(t,t_0,u_0) < b(\epsilon), \quad t \ge t_0, \tag{5.6}$$

where $u(t,t_0,u_0)$ is any solution of (5.1). Set $u_0 = V(t_0,E_0)$. Using hypotheses (i) and (ii) (i.e., h_0 is finer than h and V is h_0- decrescent) we find that there exist a $\lambda_0 > 0$ and a function $a \in K$ such that for $(t_0,E_0) \in S(h_0,\ \lambda_0)$

$$h(t_0,E_0) < \lambda \ \ \textit{and} \ \ V(t_0,E_0) \le a(h(t_0,E_0)). \tag{5.7}$$

The above relation (5.7) along with the relation (5.5) yields

$$b(h(t_0,E_0)) \le V(t_0,E_0) \le a(h_0(t_0,E_0)), \quad (t_0,E_0) \in S(h_0,\ \lambda_0). \tag{5.8}$$

Next choose a positive $\delta = \delta(t_0,\epsilon)$ such that $\delta \in (0,\lambda_0],\ \ a(\delta) < \delta_1$ and let $h_0(t_0,E_0) < \delta$. Then from relations (5.8) we get, on using the fact that $\delta_1 < b(\epsilon),\ h(t_0,E_0) < b(\epsilon)$. Now for any solution $E(t) = E(t,t_0,E_0)$ claim that $h(t,E(t)) < \epsilon,\ \ t \ge t_0$, whenever $h(t_0,E_0)) < \delta$.

If possible, suppose our claim is incorrect. Then there exist a $t_1 > t_0$ and a solution $E(t)$ of (3.1) such that

$$h(t_1,E(t_1)) = \epsilon \ \textit{and} \ h(t,E(t)) < \epsilon, \quad t_0 \le t \le t_1, \tag{5.9}$$

since $h(t_0,E_0) < \epsilon$ whenever $h_0(t_0,E_0) < \delta$. From this we deduce that

$$h(t,E(t)) \in S(h,\ \lambda)$$

for $t_0 \le t \le t_1$ and thus by Theorem (5.1), we conclude

$$V(t,E(t)) \le r(t,t_0,u_0), \quad t_0 \le t \le t_1, \tag{5.10}$$

where $r(t,t_0,E_0)$ is the maximal solution of (5.1).

On using the relations (5.5), (5.6), (5.7) and (5.10) we arrive at

$$b(\epsilon) < V(t_1,E(t_1)) \le r(t,t_0,E_0) < b(\epsilon),$$

which is a contraduction, proving $h_0, h-$equistability of (3.1).

Next, we assume that the trivial solution of (5.1) is equiattractive. Since the equation (5.1) is (h_0, h)-stable, we set $\epsilon = \lambda$ which implies that

$$\widehat{\delta_0} = \delta(t_0,\ \lambda).$$

Let $0 < \eta < \lambda$. Then since the equation (5.1) is equiattractive, given $b(\eta) > 0$ and $t_0 \in \mathbb{R}_+$, there exist $\delta_1^* = \delta_1^*(t_0) > 0$ and $T = T(t_0, \eta) > 0$ such that

$$u_0 < \delta_1^* \textit{ implies } u(t, t_0, u_0) < b(\eta), \quad t \geq t_0 + T. \tag{5.11}$$

Choose $u_0 = V(t_0, E_0)$ and working as before, we find a $\delta_0^* = \delta_0^*(t_0) > 0$ such that $\delta_0^* \in (0, \lambda_0]$ and $a(\delta_0^*) < \delta_1^*$. Let $\delta_0 = min(\delta_0^*,\ \widehat{\delta_0})$ and $h(t_0, E(t_0)) < \delta_0$, which implies that $h(t, ...E(t)) < \lambda,\ t \geq t_0$, and hence the relation (5.10) holds for all $t \geq t_0$. Now suppose that the system (5.1) is not $(h_0, h)-$ equialtractive then there exists a sequence $\{t_k\},\ \ t_k \geq t_0 + T,\ t_k \to \infty$ as $k \to \infty$ such that $\eta_k < h(t_k,\ E(t_k))$, where $E(t)$ is any solution of (3.1) such that $h_0(t_0, E_0) < \delta_0$. Then using the above inequality along with relations (5.10) and (5.1), we obtain

$$b(\eta_k) < b(h(t_k, E(t_k))) \leq V(t_k, E(t_k)) < r(t, t_0, E_0) < b(\eta),$$

which is a contradiction. Hence the system (3.1) is $(h_0, h)-$ asymptotically stable and hence the proof.

Theorem 5.3 *Supose that the function $G \in C[\mathbb{R}_+ \times D_N, D_N]$ in* (3.16) *is isomorphic to a function $F \in C[\mathbb{R}_+ \times \mathbb{R}^{N\times N}, \mathbb{R}^{N\times N}]$. Let $E(t)$ be the solution associated with the system* (3.1) *corresponding to the F obtained above. If the hypothesis of Theorem* 5.2 *is satisfied then the trivial solution or the null graph of GDE* (3.16) *has all the stability properties that the associated MDE possesses.*

Proof. Corresponding to the given graph function $G(t, D)$, we construct the matrix function $F(t, E)$. Owing to the isomorphism that exists between graphs and matrices $F(t, E)$ is continuous. Now from hypothesis, $E(t)$ is any solution of MDE (3.1). Also since the hypothesis of Theorem 5.2 is satisfied, we obtain that the zero solution of MDE (3.1) possesses all the stability properties of the comparison equation (5.1). Hence by the isomorphism that exists between graphs and matrices, we have that the zero solution, a null graph function of the GDE (3.16) has all the stability properties that the comparison equation (5.1) possesses. The proof is complete.

6 Conclusion

In this paper we have considered a MDE in terms of two measures and studied its stability properties using the basic Lyapunov theorems and the comparison methods. Using the isomorphism that exists between the graphs and matrices, we have extended these results to study the stability properties in terms of two measures, for the GDEs. We have also given examples to verify the stability properties of graph differential equations and its associated matrix differential equations using suitable Lyapunov functions.

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