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Boundedness and Square Integrability of Solutions of Nonlinear Fourth Order Differential Equations

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Abstract: Sufficient conditions for the boundedness of the solutions to a certain nonlinear fourth order differential equation are given by means of the Lyapunov's second method. We also give criteria for square integrability of solutions and their derivatives. Example is given to illustrate our results.

Keywords: *boundedness; stability; Lyapunov function; fourth-order differential equations; L^2 solutions; square integrable.*

Mathematics Subject Classification (2010): 34D20, 34C11.

1 Introduction

Higher-order nonlinear differential equations are frequently encountered in mathematical models of most dynamic processes in electromechanical systems in physics and engineering. The notions of stability and boundedness of solutions are fundamental in the theory and application of differential equations. In this way, both concepts lead to the real world applications. Many results relative to stability, boundedness, square integrability of solutions to differentiel equations have been obtained. See for instance ([1]– [42]). In discussing stability and boundedness of a nonlinear differential system, Lyapunov's direct method perhaps is the most effective method. Numerous methods have been proposed in the literature to derive suitable Lyapunov functions, but finding a proper Lyapunov's function in general is a big challenge.

The study of fourth order nonlinear differential equations has attracted the interest of many researchers. Many results concerning the stability and boundedness of solutions of fourth order differential equations have been obtained in view of various methods, especially, Lyapunov's method, see, the book of Reissig et al. [28] as a survey and the

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papers of Adesina and Ogundare [2], Cartwright [6], Chukwu [9], Abou-El-Ela and Sadek [1], Ezeilo [12], [14] Ezeilo and Tejumola [15], Harrow [17], Hu [18], Tejumola [30], Tunç [35], [36], [37], [38], Wu and Xiong [42], Vlček [41] and the references cited therein.

In 1956, Cartwright [6] investigated the asymptotic stability of zero solution of various linear and nonlinear fourth order differential equations. In [6], she considered the following differential equations

$$x'''' + a_1 x''' + a_2 x'' + a_3 x' + f(x) = 0, \tag{1}$$

$$x'''' + a_1 x''' + \psi(x')x'' + a_3 x' + a_4 x = 0, \tag{2}$$

$$x'''' + a_1 x''' + a_2 x'' + \psi(x)x' + f(x) = 0. \tag{3}$$

In [22] and [23], Omeike by using the Cauchy formula for the particular solution of nonlinear differential equations, has proved that every solution of the equations

$$\begin{aligned} x'''' + ax''' + bx'' + cx' + h(x) &= p(t), & (4)\\ x'''' + ax''' + \psi(x'') + g(x') + h(x) &= p(t), & (5) \end{aligned}$$

and its derivatives up to order three are bounded.

In [31], and [39] Tunç established sufficient conditions for the asymptotic stability of the zero solution of the equations and the boundedness of the following equations

$$\begin{aligned} x'''' + a_1 x''' + \psi(x,x')x'' + a_4 x' + h(x) &= 0, & (6)\\ x'''' + a_1 x''' + \psi(x,x')x'' + g(x') + a_4 x &= 0, & (7)\\ x'''' + ax''' + \psi(x,x',x'') + g(x,x') + h(x) &= p(t). & (8) \end{aligned}$$

The solution which is in $L^2[0,\infty)$ for higher order nonlinear differential equations was also of great interest, but it should be noted that only a few results are related to the fourth order nonlinear differential equations. Namely, in 1989, Andres and Vlček [3], established some sufficient conditions, when all the solutions of (4) are in $L^2[0,\infty)$.

In this paper, we develop the conditions under which all the solutions of the following equation (9) are bounded and are square integrable

$$\begin{aligned} x'''' + a\,(t)\Big(p\big(x(t)\big)x''(t)\Big)' + b\,(t)\Big(q\big(x(t)\big)x'(t)\Big)' + c\,(t)\,f\big(x(t)\big)x'(t) + d\,(t)\,h\big(x(t)\big)\\ = e(t), \qquad (9) \end{aligned}$$

where the primes in (9) denote differentiation with respect to t; the functions $a, b, c, d,$ are continuously differentiable functions. The functions $f, h, p, q,$ and e are continuous functions depending only on the arguments shown. It is also supposed that the derivatives, $p'(x)$, $q'(x)$, $f'(x)$ and $h'(x)$ exist and are continuous.

Equation (9) is equivalent to the system

$$\begin{cases} x' = y\\ y' = z\\ z' = w\\ w' = -a(t)p\,(x)\,w - \Big(b(t)q\,(x) + a(t)\theta_1\Big)z - \Big(b(t)\theta_2 + c\,(t)\,f\,(x)\Big)y - d\,(t)\,h\,(x) + e(t), \end{cases} \tag{10}$$

such that
$$\theta_1(t) = p'(x(t))\, x'(t), \quad \theta_2(t) = q'(x(t))\, x'(t).$$
The continuity of the functions $a, b, c, d, e, p, q, f, p', q', f'$ and h guarantees the existence of the solutions of (9) (see [11], p. 15). It is assumed that the right hand side of the system (10) satisfies a Lipschitz condition in $x(t), y(t), z(t)$, and $w(t)$. This assumption guarantees the uniqueness of solutions of (9) ([11], p. 15). The present work was motivated by the papers [3], [23], [31], [39] and the papers mentioned above, where the boundedness and square integrability of solutions for a fourth order nonlinear differential equation was studied. Using Lyapunov's method, we show that every solution $x(t)$ of equation (9) and its derivatives are bounded and square integrable.

2 Assumptions and Main Results

First, we state some assumptions on the functions that appeared in (9). Suppose that there are positive constants $a_0, b_0, c_0, d_0, f_0, p_0, q_0, a_1, b_1, c_1, d_1, f_1, p_1, q_1, m, M, \delta$, and η_1, such that the following conditions are satisfied

i) $0 < a_0 \le a(t) \le a_1$; $0 < b_0 \le b(t) \le b_1$; $0 < c_0 \le c(t) \le c_1$; $0 < d_0 \le d(t) \le d_1$ for $t \ge 0$.

ii) $0 < f_0 \le f(x) \le f_1$; $0 < p_0 \le p(x) \le p_1$; $0 < q_0 \le q(x) \le q_1$ for $x \in \mathbb{R}$ and $0 < m < \min\{f_0, p_0, 1\}$, $M > \max\{f_1, p_1, 1\}$.

iii) $\dfrac{h(x)}{x} \ge \delta > 0$ (for $x \ne 0$); $h(0) = 0$.

iv) $\displaystyle\int_0^{+\infty} \left(|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|\right) dt < \eta_1$.

The following lemma will be useful in the proof of the next theorem.

Lemma 2.1 *[20] Let $h(0) = 0$, $xh(x) > 0$ $(x \ne 0)$ and $\delta(t) - h'(x) \ge 0$ $(\delta(t) > 0)$, then*
$$2\delta(t) H(x) \ge h^2(x), \quad \text{where} \quad H(x) = \int_0^x h(s)ds.$$

Theorem 2.1 *In addition to conditions (i)-(iv) being satisfied, suppose that there are positive constants $h_0, \delta_0, \delta_1, \eta_2$ and η_3 such that the following conditions hold*

H1) $h_0 - \dfrac{a_0 m \delta_0}{d_1} \le h'(x) \le \dfrac{h_0}{2}$ *for* $x \in \mathbb{R}$.

H2) $\delta_1 = \dfrac{d_1 h_0 a_1 M}{c_0 m} + \dfrac{c_1 M + \delta_0}{a_0 m} < b_0 q_0$.

H3) $\displaystyle\int_{-\infty}^{+\infty} \left(|p'(s)| + |q'(s)| + |f'(s)|\right) ds < \eta_2$.

H4) $\displaystyle\int_0^{+\infty} |e(t)|\, dt < \eta_3$.

Then any solution $x(t)$ of (9) and its derivatives $x'(t), x''(t)$ and $x'''(t)$ are bounded and satisfy
$$\int_0^{\infty} \left(x^2(s) + x'^2(s) + x''^2(s) + x'''^2(s)\right) ds < \infty.$$

Remark 2.1 Equation (9) can be rewritten as

$$x''''(t) + a(t)p(x)x''' + \varphi_1(t,x,x')x'' + \varphi_2(t,x,x')x' + d(t)h(x) = e(t),$$

where

$$\varphi_1(t,x,x') = b(t)q(x) + \frac{1}{2}a(t)p'(x)x', \quad and \quad \varphi_2(t,x,x') = b(t)q'(x)x' + c(t)f(x).$$

If we apply Tunç theorem [39] to show that every solution x(t) of (9) is bounded, we must take $\psi(x,x',x'') = \varphi_1(t,x,x')x''$ and $g(x,x') = \varphi_2(t,x,x')x'$ then the boundedness of $\frac{\psi(x,y,z)}{z}$ and $\frac{g(x,y)}{y}$ is needed. However in our theorem this latter condition is not required since we just need to deal with the boundedness of $a(t), b(t), p(x)$, and $q(x)$.

Proof. **Boundedness of solutions**.
First we proof the boundedness of solutions. The proof of this theorem depends on properties of the continuously differentiable function $W = W(t,x,y,z,w)$ defined as

$$W = e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} V, \tag{11}$$

where

$$\gamma(t) = |a'(t)| + |b'(t)| + |c'(t)| + |d'(t)| + |\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|,$$
$$\theta_3(t) = f'(x(t))\,x'(t)$$

and

$$\begin{aligned} 2V \;=\;& 2\beta d(t)H(x) + c(t)f(x)y^2 + \alpha b(t)q(x)z^2 + a(t)p(x)z^2 + 2\beta a(t)p(x)yz \\ &+ [\beta b(t)q(x) - \alpha h_0 d(t)]y^2 - \beta z^2 + \alpha w^2 + 2d(t)h(x)y + 2\alpha d(t)h(x)z \\ &+ 2\alpha c(t)f(x)yz + 2\beta yw + 2zw, \end{aligned}$$

with $H(x) = \int_0^x h(s)ds$, $\alpha = \dfrac{1}{a_0 m} + \epsilon$, $\beta = \dfrac{d_1 h_0}{c_0 m} + \epsilon$, ϵ, and η are positive constants to be determined later in the proof. We rewrite $2V$ as

$$\begin{aligned} 2V = {}& a(t)p(x)\left[\frac{w}{a(t)p(x)} + z + \beta y\right]^2 + c(t)f(x)\left[\frac{d(t)h(x)}{c(t)f(x)} + y + \alpha z\right]^2 \\ &+ \frac{d^2(t)h^2(x)}{c(t)f(x)} + 2\epsilon d(t)H(x) + V_1 + V_2 + V_3, \end{aligned}$$

where

$$\begin{aligned} V_1 &= 2d(t)\int_0^x h(s)\left[\frac{d_1 h_0}{c_0 m} - 2\frac{d(t)}{c(t)f(x)}h'(s)\right]ds, \\ V_2 &= \left[\alpha b(t)q(x) - \beta - \alpha^2 c(t)f(x)\right]z^2, \\ V_3 &= \left[\beta b(t)q(x) - \alpha h_0 d(t) - \beta^2 a(t)p(x)\right]y^2 + \left[\alpha - \frac{1}{a(t)p(x)}\right]w^2. \end{aligned}$$

Now, we will prove that V is positive definite. Take

$$\epsilon < \min\left\{\frac{1}{a_0 m}\;,\; \frac{d_1 h_0}{c_0 m}\;,\; \frac{b_0 q_0 - \delta_1}{M(a_1 + c_1)}\right\}, \tag{12}$$

then

$$\frac{1}{a_0 m} < \alpha < \frac{2}{a_0 m}, \qquad \frac{d_1 h_0}{c_0 m} < \beta < 2\frac{d_1 h_0}{c_0 m}. \tag{13}$$

Using conditions (i)-(iii), (H1), (H2) and inequalities (12), (13) we get

$$\begin{aligned}
V_1 &\geq 4d(t)\frac{d_1}{c_0 m}\int_0^x h(s)\left[\frac{h_0}{2} - h'(s)\right] ds \geq 0, \\
V_2 &= \left(\alpha\left(b(t)\,q(x) - \beta a(t) - \alpha c(t) f(x)\right) + \beta\big(\alpha a(t) - 1\big)\right) z^2 \\
&\geq \alpha\left(b_0 q_0 - \frac{d_1 h_0 a_1}{c_0 m} - \frac{c_1 M}{a_0 m} - \epsilon(a_1 + c_1 M)\right) z^2 + \beta\big(\frac{1}{m} - 1\big) z^2 \\
&\geq \alpha\big(b_0 q_0 - \delta_1 - \epsilon M(a_1 + c_1)\big) z^2 \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
V_3 &\geq \beta\left(b_0 q_0 - \frac{\alpha}{\beta} h_0 d_1 - \beta a_1 M\right) y^2 + \left(\alpha - \frac{1}{a_0 m}\right) w^2 \\
&\geq \beta\left(b_0 q_0 - \frac{c_0}{a_0} - a_1 \frac{d_1 h_0 M}{c_0 m} - \epsilon(c_0 m + a_1 M)\right) y^2 + \epsilon w^2 \\
&\geq \beta\big(b_0 q_0 - \delta_1 - \epsilon M(c_1 + a_1)\big) y^2 + \epsilon w^2 \geq 0.
\end{aligned}$$

Hence, it is evident from the terms contained in the last inequalities, that there exists positive constant D_0 such that

$$2V \geq D_0\left(y^2 + z^2 + w^2 + H(x)\right). \tag{14}$$

By Lemma 2.1 and conditions (iii) and (H1) it follows that there is a positive constant D_1 such that

$$2V \geq D_1\left(x^2 + y^2 + z^2 + w^2\right). \tag{15}$$

Thus V is positive definite. From (i)-(iii), it is not difficult to see that there is a positive constant U_1 such that

$$V \leq U_1\left(x^2 + y^2 + z^2 + w^2\right).$$

By (H3), we have

$$\begin{aligned}
\int_0^t \big(|\theta_1(s)| + |\theta_2(s)| + |\theta_3(s)|\big) ds &= \int_{\alpha_1(t)}^{\alpha_2(t)} \big(|p'(u)| + |q'(u)| + |f'(u)|\big) du \\
&\leq \int_{-\infty}^{+\infty} \big(|p'(u)| + |q'(u)| + |f'(u)|\big) du < \eta_2 < \infty,
\end{aligned} \tag{16}$$

where $\alpha_1(t) = \min\{x(0), x(t)\}$, and $\alpha_2(t) = \max\{x(0), x(t)\}$. From inequalities (11), (15), and (16), it follows that

$$W \geq D_2(x^2 + y^2 + z^2 + w^2), \tag{17}$$

where $D_2 = \dfrac{D_1}{2} e^{-\frac{\eta_1 + \eta_2}{\eta}}$. Also, it is easy to see that there is a positive constant U_2 such that

$$W \leq U_2(x^2 + y^2 + z^2 + w^2), \tag{18}$$

for all x, y, z and w, and all $t \geq 0$.

Next we show that $\dot{W}$ is negative definite function. The derivative of the function V, along any solution $(x(t), y(t), z(t), w(t))$ of system (10), with respect to t is after simplifying

$$2\dot{V}_{(10)} = -2\epsilon c(t) f(x) y^2 + V_4 + V_5 + V_6 + V_7 + 2(\beta y + z + \alpha w)e(t) + 2\frac{\partial V}{\partial t},$$

where

$$\begin{aligned}
V_4 =& -2\left(\frac{d_1 h_0}{c_0 m} c(t) f(x) - d(t) h'(x)\right) y^2 - 2\alpha d(t)\big(h_0 - h'(x)\big) yz,\\
V_5 =& -2\big(b(t) q(x) - \alpha c(t) f(x) - \beta a(t) p(x)\big) z^2,\\
V_6 =& -2\big(\alpha a(t) p(x) - 1\big) w^2,\\
V_7 =& -a(t)\theta_1\Big(z^2 + 2\alpha zw\Big) - b(t)\theta_2\Big(\alpha z^2 + 2\alpha zw + \beta y^2 + 2yz\Big)\\
&+ c(t)\theta_3\Big(y^2 + 2\alpha yz\Big).
\end{aligned}$$

By conditions (i), (ii), (H1), (H2) and inequality (12), (13) we obtain the following

$$\begin{aligned}
V_4 &\leq -2\left[d(t) h_0 - d(t) h'(x)\right] y^2 - 2\alpha d(t)\left[h_0 - h'(x)\right] yz\\
&\leq -2d(t)\left[h_0 - h'(x)\right] y^2 - 2\alpha d(t)\left[h_0 - h'(x)\right] yz\\
&\leq -2d(t)\left[h_0 - h'(x)\right]\left[\Big(y + \frac{\alpha}{2}z\Big)^2 - \Big(\frac{\alpha}{2}z\Big)^2\right]\\
&\leq \frac{\alpha^2}{2} d(t)\left[h_0 - h'(x)\right] z^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
V_4 + V_5 &\leq -2\left[b(t) q(x) - \alpha c(t) f(x) - \beta a(t) p(x) - \frac{\alpha^2}{4} d(t)\left[h_0 - h'(x)\right]\right] z^2\\
&\leq -2\left[b_0 q_0 - (\frac{1}{a_0 m} + \epsilon) c_1 M - (\frac{d_1 h_0}{c_0 m} + \epsilon) a_1 M - \frac{\alpha^2}{4}\left(a_0 m \delta_0\right)\right] z^2\\
&\leq -2\left[b_0 q_0 - \frac{M}{a_0 m} c_1 - \frac{d_1 h_0 a_1 M}{c_0 m} - \frac{\delta_0}{a_0 m} - \epsilon M\left(a_1 + c_1\right)\right] z^2\\
&\leq -2\left[b_0 q_0 - \delta_1 - \epsilon M\left(a_1 + c_1\right)\right] z^2 \leq 0,
\end{aligned}$$

and

$$V_6 \leq -2\left[\alpha a_0 m - 1\right] w^2 = -2\epsilon w^2 \leq 0.$$

Hence, there exists a positive constant D_3 such that

$$-2\epsilon c(t) f(x) y^2 + V_4 + V_5 + V_6 \leq -2D_3\left(y^2 + z^2 + w^2\right).$$

From (14), and the Cauchy Schwartz inequality, we get

$$\begin{aligned}
V_7 \leq& \; a(t)|\theta_1|\Big(z^2 + \alpha(z^2 + w^2)\Big) + b(t)|\theta_2|\Big(\alpha z^2 + \alpha(z^2 + w^2) + \beta y^2 + (y^2 + z^2)\Big)\\
&+ c(t)|\theta_3|\Big(y^2 + \alpha(y^2 + z^2)\Big)\\
\leq& \; \lambda_1(|\theta_1| + |\theta_2| + |\theta_3|)\left(y^2 + z^2 + w^2 + H(x)\right)\\
\leq& \; 2\frac{\lambda_1}{D_0}\Big(|\theta_1| + |\theta_2| + |\theta_3|\Big) V,
\end{aligned}$$

where $\lambda_1 = \max\big\{a_1(1+\alpha), b_1(1+2\alpha+\beta), c_1(1+\alpha)\big\}$. We get also

$$\begin{aligned} 2\frac{\partial V}{\partial t} &= d'(t)\left[2\beta H(x) - \alpha h_0 y^2 + 2h(x)y + 2\alpha h(x)z\right] \\ &\quad + c'(t)\left[f(x)y^2 + 2\alpha f(x)yz\right] + b'(t)\left[\alpha q(x)z^2 + \beta q(x)y^2\right] \\ &\quad + a'(t)\left[p(x)z^2 + 2\beta p(x)yz\right]. \end{aligned}$$

Using condition (H1) and Lemma 2.1, we obtain

$$h^2(x) \le h_0 H(x),$$

consequently,

$$\begin{aligned} 2\left|\frac{\partial V}{\partial t}\right| &\le |d'(t)|\left[2\beta H(x) + \alpha h_0 y^2 + \left(h^2(x) + y^2\right) + \alpha\left(h^2(x) + z^2\right)\right] \\ &\quad + |c'(t)|\left[y^2 + \alpha\left(y^2 + z^2\right)\right] + |b'(t)|\left[\alpha z^2 + \beta y^2\right] \\ &\quad + |a'(t)|\left[z^2 + 2\beta\left(y^2 + z^2\right)\right] \\ &\le \lambda_2\left[|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|\right]\left(y^2 + z^2 + w^2 + H(x)\right) \\ &\le 2\frac{\lambda_2}{D_0}\left[|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|\right] V, \end{aligned}$$

such that $\lambda_2 = \max\big\{2\beta + \alpha h_0 + h_0, \alpha h_0 + 1, \alpha + 1\big\}$. By taking $\frac{1}{\eta} = \frac{1}{D_0}\max\big\{\lambda_1, \lambda_2\big\}$, we obtain

$$\begin{aligned} \dot{V}_{(10)} \le &- D_3(y^2+z^2+w^2) + \frac{1}{\eta}\Big(|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)| + |\theta_1| + |\theta_2| + |\theta_3|\Big)V \\ &+ \big(\beta y + z + \alpha w\big)e(t). \end{aligned} \tag{19}$$

From (iv), (H3), (16), (17), (19) and the Cauchy Schwartz inequality, we get

$$\begin{aligned} \dot{W}_{(10)} &= \left(\dot{V}_{(10)} - \frac{1}{\eta}\gamma(t)V\right) e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} \\ &\le \Big(-D_3\left(y^2 + z^2 + w^2\right) + \big(\beta y + z + \alpha w\big)e(t)\Big) e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} \qquad (20) \\ &\le \left(\beta|y| + |z| + \alpha|w|\right)|e(t)| \\ &\le D_4\left(|y| + |z| + |w|\right)|e(t)| \\ &\le D_4\left(3 + y^2 + z^2 + w^2\right)|e(t)| \\ &\le D_4\left(3 + \frac{1}{D_2}W\right)|e(t)| \\ &\le 3D_4|e(t)| + \frac{D_4}{D_2}W|e(t)|, \qquad (21) \end{aligned}$$

where $D_4 = \max\{\alpha, \beta, 1\}$. Integrating (21) from 0 to t, and using the condition (H4) and the Gronwall inequality, we obtain

$$\begin{aligned} W(t,x,y,z,w) &\le W\big(0,x(0),y(0),z(0),w(0)\big)+3D_4\eta_3 \\ &\quad +\frac{D_4}{D_2}\int_0^t W\big(s,x(s),y(s),z(s),w(s)\big)|e(s)|ds \\ &\le \Big(W\big(0,x(0),y(0),z(0),w(0)\big)+3D_4\eta_3\Big)e^{\frac{D_4}{D_2}\int_0^t |e(s)|ds} \\ &\le \Big(W\big(0,x(0),y(0),z(0),w(0)\big)+3D_4\eta_3\Big)e^{\frac{D_4}{D_2}\eta_3} = K_1<\infty. \quad (22) \end{aligned}$$

In view of inequalities (17) and (22), we get

$$(x^2+y^2+z^2+w^2)\le \frac{1}{D_2}W\le K_2, \tag{23}$$

where $K_2=\dfrac{K_1}{D_2}$. Clearly (23) implies that

$$|x(t)|\le\sqrt{K_2},\ |y(t)|\le\sqrt{K_2},\ |z(t)|\le\sqrt{K_2},\ |w(t)|\le\sqrt{K_2}\quad \text{for all}\quad t\ge 0.$$

Hence,

$$|x(t)|\le\sqrt{K_2},\ |x'(t)|\le\sqrt{K_2},\ |x''(t)|\le\sqrt{K_2},\ |x'''(t)|\le\sqrt{K_2}\quad \text{for all}\quad t\ge 0. \tag{24}$$

Square integrable solutions.
Now, we proof the square integrability of solutions and their derivatives. We define $F_t=F(t,x(t),y(t),z(t),w(t))$ as

$$F_t=W+\rho\int_0^t \big(y^2(s)+z^2(s)+w^2(s)\big)\,ds,$$

where $\rho>0$. It is easy to see that F_t is positive definite, since $W=W(t,x,y,z,w)$ is already positive definite. Using the following estimate

$$e^{-\frac{\eta_1+\eta_2}{\eta}}\le e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds}\le 1,$$

by (20) we have the following

$$\begin{aligned} \dot F_{t(10)} &\le -D_3\Big(y^2(t)+z^2(t)+w^2(t)\Big)e^{-\frac{\eta_1+\eta_2}{\eta}} \qquad (25) \\ &\quad +D_4\Big(|y(t)|+|z(t)|+|w(t)|\Big)|e(t)| \\ &\quad +\rho\Big(y^2(t)+z^2(t)+w^2(t)\Big). \end{aligned}$$

By choosing $\rho = D_3 e^{-\dfrac{\eta_1+\eta_2}{\eta}}$ we obtain

$$\begin{aligned}
\dot{F}_{t(10)} &\leq D_4\Big(3 + y^2(t) + z^2(t) + w^2(t)\Big)|e(t)| \\
&\leq D_4\Big(3 + \frac{1}{D_2}W\Big)|e(t)| \\
&\leq 3D_4|e(t)| + \frac{D_4}{D_2}F_t|e(t)|. \qquad (26)
\end{aligned}$$

Integrating the last inequality (26) from 0 to t, and using again the Gronwall inequality and the condition (H4), we get

$$\begin{aligned}
F_t &\leq F_0 + 3D_4\eta_3 + \frac{D_4}{D_2}\int_0^t F_s|e(s)|ds \\
&\leq \Big(F_0 + 3D_4\eta_3\Big)e^{\dfrac{D_4}{D_2}\int_0^t |e(s)|ds} \\
&\leq \Big(F_0 + 3D_4\eta_3\Big)e^{\dfrac{D_4}{D_2}\eta_3} = K_3 < \infty. \qquad (27)
\end{aligned}$$

Therefore,

$$\int_0^\infty y^2(s)ds < K_3 \quad , \quad \int_0^\infty z^2(s) < K_3 \text{ and } \int_0^\infty w^2(s)ds < K_3,$$

which implies that

$$\int_0^\infty x'^2(s)ds \leq K_3 \quad , \quad \int_0^\infty x''^2(s)ds \leq K_3 \quad , \quad \int_0^\infty x'''^2(s)ds \leq K_3. \qquad (28)$$

Next, multiply (9) by $x(t)$ and integrate by parts from 0 to t, we obtain

$$\int_0^t d(s)x(s)h(x(s))ds = I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + L_0, \qquad (29)$$

where

$$\begin{aligned}
I_1(t) &= x'(t)x''(t) - x(t)x'''(t) - \int_0^t x''^2(s)ds, \\
I_2(t) &= -a(t)p(x(t))x(t)x''(t) + \int_0^t a'(s)p(x(s))x(s)x''(s)ds \\
&\quad + \int_0^t a(s)p(x(s))x'(s)x''(s)ds, \\
I_3(t) &= -b(t)q(x(t))x(t)x'(t) + \int_0^t b'(s)q(x(s))x(s)x'(s)ds + \int_0^t b(s)q(x(s))x'^2(s)ds, \\
I_4(t) &= -\frac{1}{2}c(t)f(x(t))x^2(t) + \frac{1}{2}\int_0^t c'(s)f(x(s))x^2(s)ds + \frac{1}{2}\int_0^t c(s)f'(x(s))x'(s)x^2(s)ds, \\
I_5(t) &= \int_0^t e(s)x(s)ds,
\end{aligned}$$

and

$$\begin{aligned} L_0 &= x(0)x'''(0) - x'(0)x''(0) + a(0)p(x(0))x(0)x''(0) \\ &\quad + b(0)q(x(0))x(0)x'(0) + \frac{1}{2}c(0)f(x(0))x^2(0). \end{aligned}$$

From (24), (28) and the conditions (i), (ii), (iv), (H3) and (H4), we have

$$\begin{aligned} I_1(t) &\le 2K_2 + \int_0^t x''^2(s)ds, \\ I_2(t) &\le a_1MK_2 + MK_2\int_0^t |a'(s)|ds + a_1M\int_0^t x'(s)x''(s)ds, \\ &\le \frac{3}{2}a_1MK_2 + MK_2\int_0^t |a'(s)|ds, \\ I_3(t) &\le b_1q_1K_2 + q_1K_2\int_0^t |b'(s)|ds + b_1q_1\int_0^t x'^2(s)ds, \\ I_4(t) &\le \frac{1}{2}c_1MK_2 + \frac{1}{2}MK_2\int_0^t |c'(s)|ds, + \frac{1}{2}c_1K_2^{\frac{3}{2}}\int_0^t |f'(s)|ds, \\ I_5(t) &\le \sqrt{K_2}\int_0^t |e(s)|ds. \end{aligned}$$

It follows that

$$\lim_{t\to+\infty} I_1(t) \le 2K_2 + K_3 = L_1, \quad \lim_{t\to+\infty} I_2(t) \le \frac{3}{2}a_1MK_2 + MK_2\eta_1 = L_2,$$
$$\lim_{t\to+\infty} I_3(t) \le b_1q_1K_2 + q_1K_2\eta_1 + b_1q_1K_3 = L_3,$$
$$\lim_{t\to+\infty} I_4(t) \le \frac{1}{2}c_1MK_2 + \frac{1}{2}MK_2\eta_1 + \frac{1}{2}c_1K_2^{\frac{3}{2}}\eta_2 = L_4, \text{ and } \lim_{t\to+\infty} I_5(t) \le \sqrt{K_2}\eta_3 = L_5.$$

Thus,

$$\lim_{t\to+\infty} \big(I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t)\big) \le \sum_{i=1}^{5} L_i < \infty. \tag{30}$$

Consequently, (29), (30) and condition iii) give

$$\int_0^\infty x^2(s)ds \le \frac{1}{d_0\delta}\int_0^\infty d(s)x(s)h(x(s))ds \le \frac{1}{d_0\delta}\sum_{i=0}^{5} L_i < \infty,$$

which completes the proof of the theorem.

Remark 2.2 If $e(t) = 0$, similarly to the above proof, the inequality (3.10) becomes

$$\begin{aligned} \dot{W}_{(10)} &= \left(\dot{V}_{(10)} - \frac{1}{\eta}\gamma(t)V\right)e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} \\ &\le -D_3\left(y^2 + z^2 + w^2\right)e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} \\ &\le -\mu\left(y^2 + z^2 + w^2\right), \end{aligned}$$

where $\mu = D_3 e^{-\frac{\eta_1+\eta_2}{\eta}}$. It can also be observed that the only solution of system (10) for which $\dot{W}_{(10)}(t,x,y,z,w) = 0$ is the solution $x = y = z = w = 0$. The above discussion guarantees that the trivial solution of equation (9) is uniformly asymptotically stable, and the same conclusion as in the proof of Theorem 2.1 can be drawn for square integrability of solutions of equation (9).

3 Example

We consider the following fourth order non-autonomous differential equation

$$\begin{aligned} & x'''' + \left(e^{-t}\sin t + 2\right)\left(\left(\frac{x + 4e^{x} + 4e^{-x}}{4\left(e^{x} + e^{-x}\right)}\right)x''\right)' \\ & + \left(\frac{\cos t + 7t^2 + 7}{1 + t^2}\right)\left(\left(\frac{\sin x + 6e^{x} + 6e^{-x}}{e^{x} + e^{-x}}\right)x'\right)' \\ & + \left(e^{-2t}\sin^3 t + 2\right)\left(\frac{x\cos x + 5x^4 + 5}{5\left(1 + x^4\right)}\right)x' \\ & + \left(\frac{\cos^2 t + t^2 + 1}{10\left(1 + t^2\right)}\right)\left(\frac{x}{x^2 + 1}\right) = \frac{2\sin t}{t^2 + 1}, \end{aligned} \tag{31}$$

by taking
$p(x) = \dfrac{x + 4e^{x} + 4e^{-x}}{4\left(e^{x} + e^{-x}\right)}$, $q(x) = \dfrac{\sin x + 3e^{x} + 3e^{-x}}{e^{x} + e^{-x}}$, $f(x) = \dfrac{x\cos x + 5x^4 + 5}{5\left(1 + x^4\right)}$,
$h(x) = \dfrac{x}{x^2 + 1}$, $a(t) = e^{-t}\sin t + 2$, $b(t) = \dfrac{\cos t + 4t^2 + 4}{1 + t^2}$,
$c(t) = e^{-2t}\sin^3 t + 2$, $d(t) = \dfrac{\cos^2 t + t^2 + 1}{10\left(1 + t^2\right)}$ and $e(t) = \dfrac{2\sin t}{t^2 + 1}$. It follows easily that
$m = \dfrac{9}{10}$, $M = \dfrac{11}{10}$, $q_0 = \dfrac{5}{2}$, $q_1 = \dfrac{7}{2}$, $h_0 = \dfrac{11}{5}$, $\delta_0 = \dfrac{3}{2}$, $a_0 = 1$, $a_1 = 3$, $b_0 = 3$, $b_1 = 5$ $c_0 = 1$, $c_1 = 3$, $d_0 = \dfrac{1}{10}$, and $d_1 = \dfrac{1}{5}$. We find $h_0 - \dfrac{a_0 m \delta_0}{d_1} = -4$,
$55 \leq h'(x) \leq \dfrac{h_0}{2} = 1.1$ and $b_0 q_0 = \dfrac{15}{2} > \dfrac{6\,946\,7}{10000} = \dfrac{d_1 h_0 a_1 M}{c_0 m} + \dfrac{c_1 M + \delta_0}{c_0 m} = \delta_1$.

We have

$$\begin{aligned} \int_{-\infty}^{+\infty} \left|p'(x)\right| dx &= \frac{1}{4}\int_{-\infty}^{+\infty}\left|\frac{1}{e^{x} + e^{-x}} + x\frac{e^{-x} - e^{x}}{\left(e^{x} + e^{-x}\right)^2}\right| dx \\ &\leq \frac{1}{4}\int_{-\infty}^{0}\left(\frac{1}{e^{x} + e^{-x}} - x\frac{e^{-x} - e^{x}}{\left(e^{x} + e^{-x}\right)^2}\right) dx \\ &\quad + \frac{1}{4}\int_{0}^{+\infty}\left(\frac{1}{e^{x} + e^{-x}} - x\frac{e^{-x} - e^{x}}{\left(e^{x} + e^{-x}\right)^2}\right) dx = \frac{\pi}{4}, \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{+\infty} \left|q'(x)\right| dx &= \int_{-\infty}^{+\infty}\left|\frac{\left(e^{x} + e^{-x}\right)\cos x - \left(e^{x} - e^{-x}\right)\sin x}{\left(e^{x} + e^{-x}\right)^2}\right| dx \\ &\leq \int_{-\infty}^{+\infty}\left(\frac{1}{e^{x} + e^{-x}} + \frac{x}{\left(e^{x} + e^{-x}\right)^2}\left(e^{x} - e^{-x}\right)\right) dx = \pi, \text{ and} \end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}\left|f'(x)\right|dx &= \frac{1}{5}\int_{-\infty}^{+\infty}\left|\frac{(\cos x - x\sin x)\left(x^4+1\right)-4x^4\cos x}{\left(x^4+1\right)^2}\right|dx\\ &= \frac{1}{5}\int_{-\infty}^{+\infty}\left|\frac{\cos x}{x^4+1}-4x^4\frac{\cos x}{\left(x^4+1\right)^2}-x\frac{\sin x}{x^4+1}\right|dx\\ &\le \frac{1}{5}\int_{-\infty}^{+\infty}\left(\frac{5}{x^4+1}+\frac{x^2}{x^4+1}\right)dx=\frac{6}{5}\sqrt{2}\pi.\end{aligned}$$

Consequently,

$$\int_{-\infty}^{+\infty}\left(\left|p'(s)\right|+\left|q'(s)\right|+\left|f'(s)\right|\right)ds<\infty.$$

A simple computation gives

$$\int_0^{+\infty}\left|e(t)\right|dt=\int_0^{+\infty}\left|\frac{2\sin t}{t^2+1}\right|dt\le\int_0^{+\infty}\frac{2}{t^2+1}dt=\pi,$$

$$\begin{aligned}\int_0^{+\infty}\left|a'(t)\right|dt &= \int_0^{+\infty}\left|(\cos t)e^{-t}-(\sin t)e^{-t}\right|dt\le\int_0^{+\infty}2e^{-t}dt=2,\\ \int_0^{+\infty}\left|b'(t)\right|dt &= \int_0^{+\infty}\left|-\frac{\sin t}{t^2+1}-2t\frac{\cos t}{\left(t^2+1\right)^2}\right|dt\le\int_0^{+\infty}\left(\frac{1}{t^2+1}+\frac{2\left|t\right|}{\left(t^2+1\right)^2}\right)dt\\ &\le \int_0^{+\infty}\left(\frac{1}{t^2+1}+\frac{t^2+1}{\left(t^2+1\right)^2}\right)dt=\int_0^{+\infty}\frac{2}{t^2+1}dt=\pi,\\ \int_0^{+\infty}\left|c'(t)\right|dt &= \int_0^{+\infty}\left|3\left(\cos t\sin^2 t\right)e^{-2t}-2\left(\sin^3 t\right)e^{-2t}\right|dt\le\int_0^{+\infty}5e^{-2t}dt=\frac{5}{2},\end{aligned}$$

and

$$\begin{aligned}\int_0^{+\infty}\left|d'(t)\right|dt &= \int_0^{+\infty}\left|-2(\cos t)\frac{\sin t}{t^2+1}-2t\frac{\cos^2 t}{\left(t^2+1\right)^2}\right|dt\\ &\le \int_0^{+\infty}\left(\frac{2}{t^2+1}+\frac{2\left|t\right|}{\left(t^2+1\right)^2}\right)dt\le\int_0^{+\infty}\frac{3}{t^2+1}dt=\frac{3\pi}{2}.\end{aligned}$$

Therefore,

$$\int_0^{+\infty}\left(\left|a'(t)\right|+\left|b'(t)\right|+\left|c'(t)\right|+\left|d'(t)\right|\right)dt<+\infty.$$

Thus all the assumptions of Theorem 2.1 hold, so solutions of (31) are bounded and square integrable.

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