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Dwell Time Stability Analysis for Nonlinear Switched Difference Systems

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Abstract: This paper addresses the stability problem for a set of switched nonlinear difference equations with parametric uncertainties. For the corresponding family of subsystems, a regularization procedure is suggested, and a multiple Lyapunov function is constructed. With the aid of the Lyapunov function, classes of switching signals are determined for which the asymptotic stability of a stationary solution of a given set of equations may be guaranteed. An application of the proposed approach to the stability analysis of multiconnected switched difference systems by nonlinear approximation is presented. An example is given to illustrate our results.

Keywords: *difference systems; switching law; stability; comparison equations; dwell-time; multiple Lyapunov functions; complex systems.*

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1 Introduction

A general outline of the theory of set equations is presented in the monograph [18], where it is shown that classical results of qualitative theory of equations under an appropriate adaptation can be applied to the analysis of equations in metric spaces. The most effective methods are the method of integral inequalities [19], the Lyapunov direct method [22, 28] and the comparison method based on the use of scalar [11, 12], vector [25] and matrix-valued Lyapunov functions [22].

Difference equations are of great interest due to their wide applications in the modeling of real world processes in which states of systems are measured not continuously but at some fixed instants of time [1, 3, 16, 20]. Sets of difference equations with switching are

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a new subject for research designed to describe more accurately situations where the phenomena under study possess variable structure. This paper focuses on the development of methods for analysis of such systems.

The stability problem of a stationary solution for a set of nonlinear switched difference equations with parametric uncertainties is studied. First, for the corresponding family of subsystems, a regularization procedure and an approach for finding partial Lyapunov functions are proposed. Next, with the aid of these partial functions, a multiple Lyapunov function [10] is constructed for the original set of switched equations. On the basis of a development of dwell-time approach [2, 10, 29], restrictions on the switching law are determined under which the asymptotic stability of the stationary solution can be guaranteed.

Furthermore, it is shown that the proposed approaches can be applied to the stability analysis of multiconnected switched difference systems describing interaction of essentially nonlinear homogeneous subsystems, and, for these systems, sufficient conditions of the asymptotic stability by nonlinear approximation can be obtained.

2 Preliminaries

Further we shall need the following notions and results, see [18] and the references cited therein.

Let $K_C(\mathbb{R}^q)$ denote a family of all nonempty compact and convex subsets in the Euclidean space $\mathbb{R}^q$; $K(\mathbb{R}^q)$ contain all nonempty compact subsets in $\mathbb{R}^q$, and $C(\mathbb{R}^q)$ be a subset of all nonempty closed subsets in $\mathbb{R}^q$. The distance between nonempty closed subsets A and B of the space $\mathbb{R}^q$ is specified by the formula

$$D[A, B] = \max\{d_H(A, B),\ d_H(B, A)\},$$

where $d_H(B, A) = \sup\{d(\mathbf{b}, A):\ \mathbf{b} \in B\}$ is a Hausdorff separation of the sets A and B, and $d(\mathbf{b}, A) = \inf\{\|\mathbf{b} - \mathbf{a}\|:\ \mathbf{a} \in A\}$ is a distance from the point $\mathbf{b}$ to the set A, $\|\cdot\|$ is the Euclidean norm of a vector.

The following operations can be defined on $K_C(\mathbb{R}^q)$:

$$A + B = \{\mathbf{a} + \mathbf{b}:\ \mathbf{a} \in A,\, \mathbf{b} \in B\}, \qquad \lambda A = \{\lambda\mathbf{a}:\ \mathbf{a} \in A\},$$

where $A, B \in K_C(\mathbb{R}^q)$, and λ is an arbitrary nonnegative number.

The pair $(C(\mathbb{R}^q), D)$ is a complete separable metric space, where $K(\mathbb{R}^q)$ and $K_C(\mathbb{R}^q)$ are closed subsets.

The set $W \in K_C(\mathbb{R}^q)$ is called the Hukuhara difference for the sets $A, B \in K_C(\mathbb{R}^q)$, if $A = B + W$.

Let F be a mapping of the domain Q of the space $\mathbb{R}^q$ into the metric space $(K_C(\mathbb{R}^q), D)$, i.e., $F:\ Q \to K_C(\mathbb{R}^q)$, which is equivalent to the inclusion $F(\mathbf{t}) \in K_C(\mathbb{R}^q)$ for all $\mathbf{t} \in Q$. Such mappings are called the multivalued mappings of Q into $\mathbb{R}^q$.

Let $\mathbb{R}^q_+$ be the nonnegative cone of $\mathbb{R}^q$; $\mathbb{N}$ denote a set of positive integers, $\mathbb{N}_+ = \mathbb{N} \cup \{0\}$, and we designate by $\mathbb{N}_{n_0}$ the set

$$\mathbb{N}_{n_0} = \{n_0, n_0 + 1, \ldots, n_0 + k, \ldots\},$$

where $k \in \mathbb{N}$ and $n_0 \in \mathbb{N}_+$.

Next, let us introduce the concept of homogeneity, see [27, 30], for the following analysis.

Definition 2.1 A function $f(\mathbf{x}) : \mathbb{R}^q \to \mathbb{R}$ is called homogeneous of the order ν with respect to the dilation $(m_1, \dots, m_q)$, where $\nu, m_1, \dots, m_q$ are positive rationals with the odd denominators, if

$$f(\lambda^{m_1} x_1, \dots, \lambda^{m_q} x_q) = \lambda^{\nu} f(\mathbf{x}) \tag{1}$$

for all $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^q$. In the case when $\nu, m_1, \dots, m_q$ are positive real numbers, and equality (1) holds for $\lambda \geq 0$ and $\mathbf{x} \in \mathbb{R}^q$, the function $f(\mathbf{x})$ is called positive homogeneous of the order ν with respect to the dilation $(m_1, \dots, m_q)$.

Definition 2.2 A vector field $\mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_q(\mathbf{x}))^T : \mathbb{R}^q \to \mathbb{R}^q$ is called positive homogeneous of the order μ with respect to the dilation $(m_1, \dots, m_q)$, where $m_i > 0$ and $\mu + m_i > 0$, $i = 1, \dots, q$, if $f_i(\lambda^{m_1} x_1, \dots, \lambda^{m_q} x_q) = \lambda^{\mu + m_i} f_i(x_1, \dots, x_q)$, $i = 1, \dots, q$, for all $\lambda \geq 0$ and $\mathbf{x} \in \mathbb{R}^q$.

Let the system of differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) \tag{2}$$

be given, where $\mathbf{x}(t) \in \mathbb{R}^q$ is the state vector, and components of the vector $\mathbf{F}(\mathbf{x})$ are continuous for all $\mathbf{x} \in \mathbb{R}^q$.

Definition 2.3 System (2) is called positive homogeneous if its vector field $\mathbf{F}(\mathbf{x})$ is positive homogeneous.

Moreover, we will use the following lemmas, see [6] and [14] respectively.

Lemma 2.1 *If a sequence $\{v_n\}$ satisfies the condition $0 \leq v_{n+1} \leq v_n - \alpha v_n^{1+\xi}$, $n \in \mathbb{N}_+$, with $\alpha > 0$, $\xi > 0$, $v_0 \geq 0$, and $\alpha(1+\xi) v_0^{\xi} \leq 1$, then*

$$v_n \leq v_0 \left(1 + \alpha \xi v_0^{\xi} n\right)^{-\frac{1}{\xi}} \quad for \quad n \in \mathbb{N}_+.$$

Lemma 2.2 *For any positive numbers x, y and ζ the estimate*

$$(x+y)^{\zeta} \geq 2^{\omega} \left(x^{\zeta} + y^{\zeta}\right)$$

holds, where $\omega = \min\{\zeta - 1; 0\}$.

3 Statement of the Problem

Consider a set of switched difference equations

$$X_{n+1} = F^{(\sigma)}(n, X_n, \alpha) \tag{3}$$

with initial conditions $X_{n_0} = X_0$, where $X_n \in K_C(\mathbb{R}^q)$ for all $n \geq n_0$; the function $\sigma = \sigma(n)$, with $\sigma(n) \in \{1, \dots, S\}$, defines the switching law; $\alpha \in \Im \subset \mathbb{R}^d$ is the uncertainty parameter; the mappings $F^{(s)} : \mathbb{N}_+ \times K_C(\mathbb{R}^q) \times \Im \to K_C(\mathbb{R}^q)$ are continuous with respect to X_n for every $n \in \mathbb{N}_+$ and $\alpha \in \Im$.

Thus, we assume that the system under consideration depends on an uncertain parameter. Moreover, while operating, the system switches between several operation modes, and, for every $n \geq n_0$, one of the subsystems

$$X_{n+1} = F^{(s)}(n, X_n, \alpha), \quad s = 1, \dots, S, \tag{4}$$

is active.

Let $X_n(n_0, X_0)$ be the solution of (3) satisfying the condition $X_{n_0} = X_0$.

For the set of equations (3) we introduce the following assumptions:

H_1. For equations (3) there exists a set of stationary solutions $\Theta_0 \in K_C(\mathbb{R}^q)$, i.e., $F^{(s)}(n, \Theta_0, \alpha) = \Theta_0$ for all $n \in \mathbb{N}_+$, $\alpha \in \Im$, $s = 1, \dots, S$.

H_2. For any $X_0 \in K_C(\mathbb{R}^q)$ and $Y_0 \in K_C(\mathbb{R}^q)$ there exists the Hukuhara difference $W_0 \in K_C(\mathbb{R}^q)$.

Definition 3.1 The stationary solution Θ_0 of the set of equations (3) is

(i) stable, if for any $n_0 \in \mathbb{N}_+$ and $\varepsilon > 0$ there exists a $\delta = \delta(n_0, \varepsilon) > 0$ such that the inequality $D[W_0, \Theta_0] < \delta$ implies the estimate $D[X_n, \Theta_0] < \varepsilon$ for all $n \geq n_0$, where $W_0 = X_0 - Y_0$, $X_0 \in K_C(\mathbb{R}^q)$, $Y_0 \in K_C(\mathbb{R}^q)$, and $X_n = X_n(n_0, X_0 - Y_0) = X_n(n_0, W_0)$ is the solution of (3);

(ii) attractive, if for any $n_0 \in \mathbb{N}_+$ there exists $\tilde{\delta}(n_0) > 0$, and for any $\xi > 0$ there exists $\tau(n_0, W_0, \xi) \in \mathbb{N}_+$ such that the inequality $D[W_0, \Theta_0] < \tilde{\delta}(n_0)$ implies the estimate $D[X_n, \Theta_0] < \xi$ for any $n \geq n_0 + \tau(n_0, W_0, \xi)$;

(iii) asymptotically stable, if it is both stable and attractive.

We will look for stability conditions for a stationary solution Θ_0 of the set of switched systems of difference equations (3).

It should be noted that the general stability theory of classical difference equations is well-developed, see [1, 3, 8, 15–17, 20] and references cited therein, whereas the stability theory of a set of difference equations is in a primitive state.

In particular, in [9] and [18] an extension of some results obtained for a set of continuous systems with Hukuhara derivative was proposed for a set of difference equations. Unsolved problem is that of constructing an appropriate Lyapunov function satisfying special properties providing the stability of a stationary solution.

In [4], an approach to the stability analysis for sets of difference equations of the form (3) has been developed in the case of absence of switching. In the present paper, we will extend this approach to the set of switched difference equations.

4 Construction of Matrix Lyapunov Functions and Comparison Equations

Let the symbol $\overline{\mathrm{co}}$ mean the closure of convex shell of the corresponding set.

Together with subsystems (4) we will consider the following families of sets of difference equations

$$X_{n+1} = F_M^{(s)}(n, X_n), \quad s = 1, \dots, S, \tag{5}$$

where $F_M^{(s)}(n, X_n) = \overline{\mathrm{co}} \bigcup\limits_{\alpha \in \Im} F^{(s)}(n, X_n, \alpha)$;

$$X_{n+1} = F_m^{(s)}(n, X_n), \quad s = 1, \dots, S, \tag{6}$$

where $F_m^{(s)}(n, X_n) = \overline{\mathrm{co}} \bigcap\limits_{\alpha \in \Im} F^{(s)}(n, X_n, \alpha)$;

$$X_{n+1} = F_\beta^{(s)}(n, X_n), \quad s = 1, \dots, S, \tag{7}$$

where $F_\beta^{(s)}(n, X_n) = F_M^{(s)}(n, X_n)\beta + F_m^{(s)}(n, X_n)(1 - \beta)$, $\beta \in [0, 1]$.

In what follows it is assumed that $F_m^{(s)}$, $F_M^{(s)}$ and $F_\beta^{(s)} \in K_c(\mathbb{R}^q)$.

For every $s \in \{1, \dots, S\}$, we introduce an auxiliary matrix function, see [4],

$$\mathbf{U}^{(s)}(n, \beta, X_n) = [U_{ij}^{(s)}(n, \beta, X_n)], \quad i, j = 1, 2, \tag{8}$$

where the element $U_{11}^{(s)}(n, X_n)$ is associated with the s-th set from the family (5), $U_{22}^{(s)}(n, X_n)$ is associated with the s-th set from the family (6), $U_{12}^{(s)}(n, \beta, X_n) = U_{21}^{(s)}(n, \beta, X_n)$ is associated with the s-th set from the family (7).

In terms of function (8) we construct a scalar function [22]

$$V_s(n, X_n, \beta, \theta_s) = \theta_s^T \mathbf{U}^{(s)}(n, \beta, X_n)\theta_s, \quad \theta_s \in \mathbb{R}_+^2, \tag{9}$$

and assume that $V_s : \mathbb{N}_+ \times K_C(\mathbb{R}^q) \times [0, 1] \times \mathbb{R}_+^2 \to \mathbb{R}_+$.

Function (9) is a partial Lyapunov function for the s-th subsystem from (4) if, together with the first difference

$$\Delta V_s(n, X_n, \beta, \theta_s) = V_s(n + 1, X_{n+1}, \beta, \theta_s) - V_s(n, X_n, \beta, \theta_s),$$

it solves the problem of stability of the stationary solution Θ_0 for the s-th subsystem.

Let the following assumptions be fulfilled.

H_3. For every $s \in \{1, \ldots, S\}$, there exists $\tilde{\theta}_s \in \mathbb{R}_+^2$ such that for the function $V_s(n, X_n, \beta, \tilde{\theta}_s)$ and for its first difference along trajectories of the s-th set of equations from (4) the estimates

$$a_s(D[X_n, \Theta_0]) \le V_s(n, X_n, \beta, \tilde{\theta}_s) \le b_s(D[X_n, \Theta_0]), \tag{10}$$

$$\Delta V_s \le w^{(s)}(n, V_s) \tag{11}$$

are valid for $n \in \mathbb{N}_+$, $X_n \in S(\rho)$, $\beta \in [0, 1]$. Here $\rho = \text{const} > 0$; $S(\rho) = \{X \in K_c(\mathbb{R}^q) : D[X, \Theta_0] < \rho\}$; $a(\cdot)$ and $b(\cdot)$ are class $\mathcal{K}$ (in the sense of Hahn) functions [28]; functions $w^{(s)}(n, r)$ are continuous with respect to $r \in [0, \tilde{\rho}]$ for every value of $n \in \mathbb{N}_+$, and $w^{(s)}(n, r)/r \to 0$ as $r \to 0$; $\tilde{\rho} = \text{const} > 0$.

H_4. The zero solutions of the equations

$$u_{n+1} = u_n + w^{(s)}(n, u_n), \qquad s = 1, \ldots, S, \tag{12}$$

are asymptotically stable.

Equations (12) are comparison ones for subsystems from the family (4). It is known, see [4], that under assumptions H_3 and H_4 the stationary solution Θ_0 of each subsystem is asymptotically stable.

To obtain stability conditions for the set of switched systems of difference equations (3), we will use multiple Lyapunov functions and the dwell-time approach.

5 Dwell Time Stability Analysis

Let us impose additional restrictions on the Lyapunov functions (9) and on the comparison equations (12).

H_5. There exist positive numbers c_{sl} such that

$$V_s(n, X_n, \beta, \tilde{\theta}_s) \le c_{sl} V_l(n, X_n, \beta, \tilde{\theta}_l) \tag{13}$$

for $n \in \mathbb{N}_+$, $X_n \in S(\rho)$, $\beta \in [0, 1]$; $s, l = 1, \ldots, S$.

H_6. Let equations (12) be of the form

$$u_{n+1} = u_n - \alpha^{(s)} u_n^{1+\xi^{(s)}}, \qquad s = 1, \dots, S, \tag{14}$$

where $\alpha^{(s)}$ and $\xi^{(s)}$ are positive constants.

Remark 5.1 Equations (14) can be considered as equations of the nonlinear approximation for (12).

Remark 5.2 The case where $\xi^{(s)} = 0$, $s = 1, \dots, S$, is well-investigated, see, for instance, [10, 13, 21]. Therefore, in this section we assume that $\xi^{(s)} > 0$, $s = 1, \dots, S$, i.e., the switched comparison equations (14) are essentially nonlinear.

Remark 5.3 Using Lemma 2.1 and taking into account Assumptions H_3, H_4 and H_6, one can obtain estimates for solutions of subsystems (4).

Without loss of generality, we assume that the interval $(0, +\infty)$ contains an infinite number of switching instants. Let τ_i, $i \in \mathbb{N}$, be the switching times, $0 < \tau_1 < \tau_2 < \dots$, and $\tau_0 = 0$.

Denote, for brevity, $\hat{\xi}_i = \xi^{(\sigma(\tau_i))}$, $\hat{\alpha}_i = \alpha^{(\sigma(\tau_i))}$, $i \in \mathbb{N}_+$; $\hat{c}_i = c_{\sigma(\tau_i)\sigma(\tau_{i-1})}$, $i \in \mathbb{N}$.

For every $m \in \mathbb{N}$ and $L_m \in \mathbb{R}_+$, define a sequence $\chi_n(L_m, m)$ by the formulae

$$\chi_0(L_m, m) = L_m,$$

$$\chi_n(L_m, m) = \hat{c}_{m+n-1}^{-\hat{\xi}_{m+n-1}} \left(\chi_{n-1}(L_m, m)\right)^{\hat{\xi}_{m+n-1}/\hat{\xi}_{m+n-2}} + \hat{\alpha}_{m+n-1}\hat{\xi}_{m+n-1} T_{m+n}$$

for $n \in \mathbb{N}$, where $T_i = \tau_i - \tau_{i-1}$, $i \in \mathbb{N}$.

Theorem 5.1 *Let Assumptions* H_1–H_6 *be fulfilled. If there exists a positive constant* L *such that*

$$\chi_n(L, 1) \to +\infty \quad as \quad n \to +\infty, \tag{15}$$

then the stationary solution Θ_0 *of the set of equations* (3) *is asymptotically stable.*

Proof. Using partial Lyapunov functions $V_1(n, X_n, \beta, \tilde{\theta}_1), \dots, V_S(n, X_n, \beta, \tilde{\theta}_S)$, construct a multiple Lyapunov function $V_{\sigma(n)}(n, X_n, \beta, \tilde{\theta}_{\sigma(n)})$ corresponding to the switching law $\sigma(n)$.

Choose a number ε such that $0 < \varepsilon < \rho$, and

$$\alpha^{(s)} \left(1 + \xi^{(s)}\right) V_s^{\xi^{(s)}}(n, X_n, \beta, \tilde{\theta}_s) \le 1, \qquad s = 1, \dots, S,$$

for $n \in \mathbb{N}_+$, $X_n \in S(\varepsilon)$, $\beta \in [0, 1]$.

Consider the solution X_n of (3) satisfying the condition $X_{n_0} = W_0$, where $n_0 \in \mathbb{N}_+$, $W_0 \in S(\varepsilon)$. Find a positive integer m such that $n_0 \in [\tau_{m-1}, \tau_m)$. Let $X_n \in S(\varepsilon)$ for $n = n_0, \dots, \tilde{n}$.

If $n_0 < \tilde{n} \le \tau_m$, then applying Lemma 2.1 to the $\sigma(\tau_{m-1})$-th inequality from (11), we obtain that

$$V_{\sigma(\tau_{m-1})}^{-\hat{\xi}_{m-1}} \left(\tilde{n}, X_{\tilde{n}}, \beta, \tilde{\theta}_{\sigma(\tau_{m-1})}\right) \ge V_{\sigma(\tau_{m-1})}^{-\hat{\xi}_{m-1}} \left(n_0, W_0, \beta, \tilde{\theta}_{\sigma(\tau_{m-1})}\right) + \hat{\alpha}_{m-1}\hat{\xi}_{m-1}(\tilde{n} - n_0)$$
$$\ge V_{\sigma(\tau_{m-1})}^{-\hat{\xi}_{m-1}} \left(n_0, W_0, \beta, \tilde{\theta}_{\sigma(\tau_{m-1})}\right). \tag{16}$$

In the case of $\tilde{n} > \tau_m$, there exists a positive integer p satisfying the condition $\tau_{m+p-1} < \tilde{n} \le \tau_{m+p}$. It should be noted that $p \to +\infty$ as $\tilde{n} \to +\infty$. Applying successively Lemma 2.1 to the corresponding inequalities from (11) and taking into account estimates (13), we obtain

$$V_{\sigma(\tau_{m+p-1})}^{-\hat{\xi}_{m+p-1}}\left(\tilde{n}, X_{\tilde{n}}, \beta, \tilde{\theta}_{\sigma(\tau_{m+p-1})}\right) \ge V_{\sigma(\tau_{m+p-1})}^{-\hat{\xi}_{m+p-1}}\left(\tau_{m+p-1}, X_{\tau_{m+p-1}}, \beta, \tilde{\theta}_{\sigma(\tau_{m+p-1})}\right)$$
$$+\hat{\alpha}_{m+p-1}\hat{\xi}_{m+p-1}(\tilde{n} - \tau_{m+p-1})$$
$$\ge \hat{c}_{m+p-1}^{-\hat{\xi}_{m+p-1}}\left(V_{\sigma(\tau_{m+p-2})}^{-\hat{\xi}_{m+p-2}}\left(\tau_{m+p-1}, X_{\tau_{m+p-1}}, \beta, \tilde{\theta}_{\sigma(\tau_{m+p-2})}\right)\right)^{\hat{\xi}_{m+p-1}/\hat{\xi}_{m+p-2}}$$
$$\ge \ldots \ge \hat{c}_{m+p-1}^{-\hat{\xi}_{m+p-1}}\left(\chi_{p-1}\left(V_{\sigma(\tau_{m-1})}^{-\hat{\xi}_{m-1}}\left(n_0, W_0, \beta, \tilde{\theta}_{\sigma(\tau_{m-1})}\right), m\right)\right)^{\hat{\xi}_{m+p-1}/\hat{\xi}_{m+p-2}}. \tag{17}$$

From (10), (16) and (17), it follows that

$$D[X_{\tilde{n}}, \Theta_0] \le \max_{s=1,\ldots,S} a_s^{(-1)}\left(b_s\left(D[W_0, \Theta_0]\right)\right) \tag{18}$$

for $\tilde{n} = n_0, \ldots, \tau_m$, and

$$D[X_{\tilde{n}}, \Theta_0] \le \max_{s,k,j=1,\ldots,S} a_s^{(-1)}\left(c_{sk}\left(\chi_{p-1}\left(b_j^{-\xi^{(j)}}\left(D[W_0, \Theta_0]\right), m\right)\right)^{-1/\xi^{(k)}}\right) \tag{19}$$

for $\tilde{n} = \tau_{m+p-1} + 1, \ldots, \tau_{m+p}$ and $p \ge 1$. Here $a_s^{(-1)}(\cdot)$ is inverse of the function $a_s(\cdot)$, $s = 1, \ldots, S$.

Let there exist a positive constant L such that condition (15) is fulfilled. It is easy to check that if $L_m = \chi_{m-1}(L, 1)$, then $\chi_n(L_m, m) = \chi_{n+m-1}(L, 1)$. Hence, $\chi_n(L_m, m) \to +\infty$ as $n \to +\infty$.

Find a number δ_1 such that $0 < \delta_1 < \varepsilon$, and $b_j^{-\xi^{(j)}}\left(D[W_0, \Theta_0]\right) \ge L_m$ for $W_0 \in S(\delta_1)$, $j = 1, \ldots, S$. Using estimate (19), one can choose a positive integer p_0 satisfying the following condition: if $W_0 \in S(\delta_1)$ and $p \ge p_0$, then $X_{\tilde{n}} \in S(\varepsilon)$.

From (17) it follows that

$$D[X_{\tilde{n}}, \Theta_0] \le \max_{s,j=1,\ldots,S} a_s^{(-1)}\left(\bar{c}^p b_j\left(D[W_0, \Theta_0]\right)\right) \tag{20}$$

for $\tilde{n} = \tau_{m+p-1} + 1, \ldots, \tau_{m+p}$ and $p \ge 1$. Here $\bar{c} = \max\limits_{s,k=1,\ldots,S} c_{sk}$. Taking into account (18) and (20), one can find a number δ_2, $0 < \delta_2 < \varepsilon$, such that if $W_0 \in S(\delta_2)$ and $p < p_0$, then $X_{\tilde{n}} \in S(\varepsilon)$.

Let $\delta = \min\{\delta_1; \delta_2\}$. We obtain that $D[W_0, \Theta_0] < \delta$ implies the estimate $D[X_n, \Theta_0] < \varepsilon$ for all $n \ge n_0$.

Moreover, from (19) it follows that $D[X_n, \Theta_0] \to 0$ as $n \to +\infty$. Thus, the stationary solution Θ_0 of the set of equations (3) is asymptotically stable. This completes the proof.

Corollary 5.1 *Let Assumptions* H_1–H_6 *be fulfilled. If there exists a positive constant* L *such that* $\chi_n(L, m) \to +\infty$ *as* $n \to +\infty$ *uniformly with respect to* $m \in \mathbb{N}$*, then the stationary solution* Θ_0 *of the set of equations* (3) *is uniformly asymptotically stable.*

Corollary 5.2 *Let Assumptions* H_1–H_6 *be fulfilled. If* $T_i \to +\infty$ *as* $i \to +\infty$*, then the stationary solution* Θ_0 *of the set of equations* (3) *is uniformly asymptotically stable.*

Next, let us show that the use of Lemma 2.2 permits us to replace condition (15) in Theorem 5.1 by a condition though more conservative but more convenient for applications.

Construct a sequence ψ_n by the formulae $\psi_1 = \hat{\alpha}_1\hat{\xi}_1 T_2$,

$$\psi_n = \hat{\alpha}_n\hat{\xi}_n T_{n+1} + \sum_{i=1}^{n-1} 2^{\omega_{n,n-1}+\ldots+\omega_{n,n-i}} \left(\hat{c}_n \ldots \hat{c}_{n-i+1}\right)^{-\hat{\xi}_n} \left(\hat{\alpha}_{n-i}\hat{\xi}_{n-i} T_{n-i+1}\right)^{\hat{\xi}_n/\hat{\xi}_{n-i}}$$

for $n = 2, 3, \ldots$, where $\omega_{n,j} = \min\{\hat{\xi}_n/\hat{\xi}_j - 1;\, 0\}$, $j = 1, \ldots, n-1$.

Corollary 5.3 *Let Assumptions* H_1–H_6 *be fulfilled. If*

$$\psi_n \to +\infty \quad as \quad n \to +\infty, \tag{21}$$

then the stationary solution Θ_0 *of the set of equations* (3) *is asymptotically stable.*

Proof. With the aid of Lemma 2.2, it is easy to check that $\chi_n(L, 1) \geq \psi_n$ for any $L > 0$ and for any $n \in \mathbb{N}$.

Really, $\chi_0(L, 1) = L > 0$,

$$\chi_1(L, 1) = \hat{c}_1^{-\hat{\xi}_1} \left(\chi_0(L, 1)\right)^{\hat{\xi}_1/\hat{\xi}_0} + \hat{\alpha}_1\hat{\xi}_1 T_2 = \hat{c}_1^{-\hat{\xi}_1} L^{\hat{\xi}_1/\hat{\xi}_0} + \psi_1 \geq \psi_1,$$

and, for $n > 1$, we obtain

$$\begin{aligned}\chi_n(L, 1) &= \hat{c}_n^{-\hat{\xi}_n} \left(\chi_{n-1}(L, 1)\right)^{\hat{\xi}_n/\hat{\xi}_{n-1}} + \hat{\alpha}_n\hat{\xi}_n T_{n+1} \\ &= \hat{c}_n^{-\hat{\xi}_n} \left(\hat{c}_{n-1}^{-\hat{\xi}_{n-1}} \left(\chi_{n-2}(L, 1)\right)^{\hat{\xi}_{n-1}/\hat{\xi}_{n-2}} + \hat{\alpha}_{n-1}\hat{\xi}_{n-1} T_n\right)^{\hat{\xi}_n/\hat{\xi}_{n-1}} + \hat{\alpha}_n\hat{\xi}_n T_{n+1} \\ &\geq 2^{\omega_{n,n-1}} \left(\hat{c}_n\hat{c}_{n-1}\right)^{-\hat{\xi}_n} \left(\chi_{n-2}(L, 1)\right)^{\hat{\xi}_n/\hat{\xi}_{n-2}} + 2^{\omega_{n,n-1}} \hat{c}_n^{-\hat{\xi}_n} \left(\hat{\alpha}_{n-1}\hat{\xi}_{n-1} T_n\right)^{\hat{\xi}_n/\hat{\xi}_{n-1}} \\ &\quad + \hat{\alpha}_n\hat{\xi}_n T_{n+1} \geq \ldots \geq 2^{\omega_{n,n-1}+\ldots+\omega_{n,1}} \left(\hat{c}_n \ldots \hat{c}_1\right)^{-\hat{\xi}_n} L^{\hat{\xi}_n/\hat{\xi}_0} + \psi_n \geq \psi_n.\end{aligned}$$

Hence, from (21) follows the fulfilment of condition (15). This completes the proof.

Remark 5.4 The results of the present section can be extended to the case where Assumtion H_5 is replaced by the following one:

H_5'. There exist positive numbers c_{sl} and ν_{sl} such that

$$V_s(n, X_n, \beta, \tilde{\theta}_s) \leq c_{sl} V_l^{\nu_{sl}}(n, X_n, \beta, \tilde{\theta}_l)$$

for $n \in \mathbb{N}_+$, $X_n \in S(\rho)$, $\beta \in [0, 1]$; $s, l = 1, \ldots, S$.

6 Stability Analysis of Multiconnected Switched Systems

Consider the system

$$\mathbf{x}_i(n+1) = \mathbf{x}_i(n) + \mathbf{F}_i^{(\sigma)}(\mathbf{x}_i(n)) + \sum_{j=1}^{k} \mathbf{\Psi}_{ij}^{(\sigma)}(n, \mathbf{x}(n)), \quad i = 1, \ldots, k, \tag{22}$$

which describes the dynamics of a complex system composed of k interconnected systems [19, 22]. Here $\mathbf{x}_i(n) = (x_{i1}(n), \dots, x_{iq_i}(n))^T$, $\mathbf{x}(n) = (\mathbf{x}_1^T(n), \dots, \mathbf{x}_k^T(n))^T$; $n \in \mathbb{N}_+$; function $\sigma = \sigma(n)$, with $\sigma(n) \in \{1, \dots, S\}$, defines the switching law; vector fields $\mathbf{F}_i^{(s)}(\mathbf{x}_i)$ are continuous for $\mathbf{x}_i \in \mathbb{R}^{q_i}$ and positive homogeneous of the order $\mu_i^{(s)}$ with respect to the dilation $(m_{i1}, \dots, m_{iq_i})$, where $\mu_i^{(s)}, m_{i1}, \dots, m_{iq_i}$ are positive numbers; vector functions $\mathbf{\Psi}_{ij}^{(s)}(n, \mathbf{x}) = \left(\Psi_{ij1}^{(s)}(n, \mathbf{x}), \dots, \Psi_{ijq_i}^{(s)}(n, \mathbf{x})\right)^T$ are defined for $n \in \mathbb{N}_+$, $\|\mathbf{x}\| < H$, $0 < H \le +\infty$, and continuous with respect to $\mathbf{x}$ for every fixed n; $i, j = 1, \dots, k$; $s = 1, \dots, S$. We assume that the estimates

$$|\Psi_{ijg}^{(s)}(n, \mathbf{x})| \le c_{ijg}^{(s)} r_j^{\alpha_{ijg}^{(s)}}(\mathbf{x}_j)$$

hold for $n \in \mathbb{N}_+$, $\|\mathbf{x}\| < H$, where $r_j(\mathbf{x}_j) = \sum_{p=1}^{q_j} |x_{jp}|^{1/m_{jp}}$, $c_{ijg}^{(s)} \ge 0$, $\alpha_{ijg}^{(s)} > 0$, $g = 1, \dots, q_i$; $i, j = 1, \dots, k$; $s = 1, \dots, S$.

Thus, at each time instant, one of the subsystems

$$\mathbf{x}_i(n+1) = \mathbf{x}_i(n) + \mathbf{F}_i^{(s)}(\mathbf{x}_i(n)) + \sum_{j=1}^{k} \mathbf{\Psi}_{ij}^{(s)}(n, \mathbf{x}(n)), \quad i = 1, \dots, k, \quad s = 1, \dots, S, \tag{23}$$

is active.

From the properties of the right-hand sides of (22) it follows that the system admits the zero solution. We will look for conditions of asymptotic stability of the solution.

For every $i \in \{1, \dots, k\}$, consider the family of isolated difference subsystems

$$\mathbf{x}_i(n+1) = \mathbf{x}_i(n) + \mathbf{F}_i^{(s)}(\mathbf{x}_i(n)), \quad s = 1, \dots, S, \tag{24}$$

and the corresponding family of subsystems of differential equations

$$\dot{\mathbf{z}}_i(t) = \mathbf{F}_i^{(s)}(\mathbf{z}_i(t)), \quad s = 1, \dots, S. \tag{25}$$

Let us impose some additional conditions on the right-hand sides of (22).

H_7. There exist numbers $h_1, \dots, h_k$ such that $h_i \ge 2\max\{m_{i1}, \dots, m_{iq_i}\}$, $i = 1, \dots, k$, and, for every $s \in \{1, \dots, S\}$, the inequalities

$$\frac{\alpha_{ijg}^{(s)}}{h_j + \mu_j^{(s)}} \ge \frac{\mu_i^{(s)} + m_{ig}}{h_i + \mu_i^{(s)}} \quad \text{for} \quad c_{ijg}^{(s)} \ne 0, \quad g = 1, \dots, q_i, \quad i, j = 1, \dots, k, \tag{26}$$

hold.

Remark 6.1 Assumption H_7 means that the orders of the right-hand sides of the isolated subsystems (24) are, in a certain sense, less than or equal to the orders of functions characterizing interconnections between the subsystems.

H_8. For every $i \in \{1, \dots, k\}$, the zero solutions of all subsystems (25) are asymptotically stable.

Remark 6.2 It is known, see [7, 26], that the fulfilment of Assumption H_8 implies that the zero solutions of all difference subsystems (24) are asymptotically stable as well.

H_9. For every $i \in \{1, \ldots, k\}$, for the family of subsystems (25), Lyapunov functions $v_{i1}(\mathbf{z}_i), \ldots, v_{iS}(\mathbf{z}_i)$ are constructed so that $v_{is}(\mathbf{z}_i)$ is twice continuously differentiable for $\mathbf{z}_i \in \mathbb{R}^{q_i}$ positive definite and positive homogeneous of the order $\gamma_i \geq 2\max\{m_{i1}, \ldots, m_{iq_i}\}$ with respect to the dilation $(m_{i1}, \ldots, m_{iq_i})$ function, and the derivative of $v_{is}(\mathbf{z}_i)$ with respect to the s-th subsystem from the family (25) is negative definite, $s = 1, \ldots, S$.

Remark 6.3 In [27, 30], it was proved that the fulfilment of Assumption H_8 implies the existence of the required Lyapunov functions.

Remark 6.4 In view of homogeneous functions properties, see [30], the estimates

$$a_{1i}^{(s)} r_i^{\gamma_i}(\mathbf{z}_i) \leq v_{is}(\mathbf{z}_i) \leq a_{2i}^{(s)} r_i^{\gamma_i}(\mathbf{z}_i), \quad \left|\frac{\partial v_{is}(\mathbf{z}_i)}{\partial z_{ig}}\right| \leq a_{3ig}^{(s)} r_i^{\gamma_i - m_{ig}}(\mathbf{z}_i),$$

$$\left(\frac{\partial v_{is}(\mathbf{z}_i)}{\partial \mathbf{z}_i}\right)^T \mathbf{F}_i^{(s)}(\mathbf{z}_i) \leq -a_{4i}^{(s)} r_i^{\gamma_i + \mu_i^{(s)}}(\mathbf{z}_i)$$

hold for $\mathbf{z}_i \in \mathbb{R}^{q_i}$, where $a_{1i}^{(s)}$, $a_{2i}^{(s)}$, $a_{3ig}^{(s)}$, $a_{4i}^{(s)}$, $s = 1, \ldots, S$, are positive constants depending on chosen Lyapunov functions; $g = 1, \ldots, q_i$; $i = 1, \ldots, k$.

In what follows, we will assume, without loss of generality, that $\gamma_i = h_i$, $i = 1, \ldots, k$, where numbers $h_1, \ldots, h_k$ satisfy the conditions specified in Assumption H_7.

H_{10}. For every $s \in \{1, \ldots, S\}$, the inequality system

$$-a_{4i}^{(s)} \xi_i^{\gamma_i + \mu_i^{(s)}} + \sum_{g=1}^{q_i} a_{3ig}^{(s)} \xi_i^{\gamma_i - m_{ig}} \sum_{j=1}^{k} c_{ijg}^{(s)} \xi_j^{\alpha_{ijg}^{(s)}} < 0, \quad i = 1, \ldots, k, \tag{27}$$

admits a positive solution.

Remark 6.5 Assumption H_{10} is the Martynyuk-Obolenskii condition [23, 24] of asymptotic stability for the zero solutions of the corresponding Wazewskij systems

$$\dot{z}_i(t) = -a_{4i}^{(s)} z_i^{\gamma_i + \mu_i^{(s)}}(t) + \sum_{g=1}^{q_i} a_{3ig}^{(s)} z_i^{\gamma_i - m_{ig}}(t) \sum_{j=1}^{k} c_{ijg}^{(s)} z_j^{\alpha_{ijg}^{(s)}}(t), \quad i = 1, \ldots, k, \quad s = 1, \ldots, S.$$

From the results of [5] it follows that if Assumptions H_7–H_{10} are fulfilled, then, for every $s \in \{1, \ldots, S\}$, one can find positive numbers $\zeta_1^{(s)}, \ldots, \zeta_k^{(s)}$ for which the first difference of the function

$$V_s(\mathbf{z}) = \sum_{i=1}^{k} \zeta_i^{(s)} v_{is}(\mathbf{z}_i) \tag{28}$$

with respect to solutions of the corresponding subsystem from family (23) will be negative definite.

It is easy to show the existence of positive numbers $\beta^{(1)}, \ldots, \beta^{(S)}$, $\alpha^{(1)}, \ldots, \alpha^{(S)}$ and $\bar{H}$ such that $\bar{H} \in (0, H)$, and for the first difference of $V_s(\mathbf{z})$ with respect to solutions of the s-th subsystem from (23) the inequalities

$$\Delta V_s\big|_{(s)} \leq -\beta^{(s)} \sum_{i=1}^{k} r_i^{\gamma_i + \mu_i^{(s)}}(\mathbf{x}_i(n)) \leq -\alpha^{(s)} V_s^{1+\xi^{(s)}}(\mathbf{x}(n))$$

hold for $\|\mathbf{x}(n)\| < \bar{H}$. Here $\xi^{(s)} = \max\limits_{i=1,\dots,k} \mu_i^{(s)}/\gamma_i$, $s = 1, \dots, S$.

Thus, for subsystems (23) we obtain comparison equations of the form (14). Hence, for the subsequent stability analysis of (22) one can use the results of Section 5.

7 Example

Let system (22) be of the form

$$\begin{cases} x_1(n+1) &=& x_1(n) + x_2(n), \\ x_2(n+1) &=& x_2(n) - a_\sigma x_1^3(n) - b_\sigma |x_2(n)|^{1/2}\, x_2(n) + \psi_1^{(\sigma)}(x_3(n)), \\ x_3(n+1) &=& x_3(n) - d_\sigma x_3^{\lambda_\sigma}(n) + \psi_2^{(\sigma)}(x_2(n)). \end{cases} \tag{29}$$

Here $x_1(n), x_2(n), x_3(n)$ are scalar variables; $\sigma = \sigma(n) \in \{1, 2\}$; $a_1 = b_2 = 2$, $a_2 = b_1 = 1$, $d_1 = 8$, $d_2 = 4$, $\lambda_1 = 3$, $\lambda_2 = 5$; functions $\psi_1^{(s)}(z_3)$ and $\psi_2^{(s)}(z_2)$ are continuous for $|z_3| < H$ and $|z_2| < H$ respectively and satisfy the conditions

$$|\psi_1^{(s)}(z_3)| \le c_s |z_3|^{\alpha_s}, \qquad |\psi_2^{(s)}(z_2)| \le e_s |z_2|^{\beta_s}, \qquad s = 1, 2,$$

where $\alpha_1 = 12/5$, $\alpha_2 = 4$, $\beta_1 = 15/8$, $\beta_2 = 31/8$, and c_1, c_2, e_1, e_2 are positive parameters.

Thus, switching in (29) occurs between the subsystems

$$\begin{cases} x_1(n+1) &=& x_1(n) + x_2(n), \\ x_2(n+1) &=& x_2(n) - 2x_1^3(n) - |x_2(n)|^{1/2}\, x_2(n) + \psi_1^{(1)}(x_3(n)), \\ x_3(n+1) &=& x_3(n) - 8x_3^3(n) + \psi_2^{(1)}(x_2(n)), \end{cases} \tag{30}$$

and

$$\begin{cases} x_1(n+1) &=& x_1(n) + x_2(n), \\ x_2(n+1) &=& x_2(n) - x_1^3(n) - 2|x_2(n)|^{1/2}\, x_2(n) + \psi_1^{(2)}(x_3(n)), \\ x_3(n+1) &=& x_3(n) - 4x_3^5(n) + \psi_2^{(2)}(x_2(n)). \end{cases} \tag{31}$$

System (29) can be treated as a complex system describing the interaction of two ($k = 2$) systems

$$\begin{cases} x_1(n+1) &=& x_1(n) + x_2(n), \\ x_2(n+1) &=& x_2(n) - a_\sigma x_1^3(n) - b_\sigma |x_2(n)|^{1/2}\, x_2(n), \end{cases}$$

and

$$x_3(n+1) = x_3(n) - d_\sigma x_3^{\lambda_\sigma}(n).$$

The differential systems

$$\begin{cases} \dot{z}_1 &=& z_2, \\ \dot{z}_2 &=& -a_s z_1^3 - b_s |z_2|^{1/2}\, z_2, \quad s = 1, 2, \end{cases} \tag{32}$$

are homogeneous ones of the order $1/2$ with respect to the dilation $(1/2, 1)$, and the differential equations

$$\dot{z}_3 = -d_s z_3^{\lambda_s}, \quad s = 1, 2, \tag{33}$$

are homogeneous ones of the orders 2 and 4 with respect to the dilation 1.

Construct inequalities (26) corresponding to complex system (29). We obtain

$$\max\left\{\frac{8}{5(h_2+2)};\ \frac{5}{3(h_2+4)}\right\} \le \frac{2}{2h_1+1} \le \min\left\{\frac{8}{5(h_2+2)};\ \frac{8}{3(h_2+4)}\right\}.$$

These inequalities admit positive solutions. For example, one can choose $h_1 = h_2 = 2$. Hence, Assumption H_7 is fulfilled.

Lyapunov functions for systems (32) and equations (33) can be constructed in the forms

$$v_{1s}(z_1, z_2) = \frac{a_s}{4} z_1^4 + \frac{1}{2} z_2^2 + \frac{1}{10}\,|z_1|\, z_1 z_2, \quad s = 1, 2,$$

and

$$v_{2s}(z_3) = \frac{1}{2} z_3^2, \quad s = 1, 2,$$

respectively. Thus, Assumptions H_8 and H_9 are fulfilled as well.

In the present case inequalities (27) take the form

$$-0.1\xi_1^{5/2} + c_1\xi_1\xi_2^{12/5} < 0, \quad -8\xi_2^4 + e_1\xi_2\xi_1^{15/8} < 0 \tag{34}$$

for $s = 1$, and

$$-0.06\xi_1^{5/2} + c_2\xi_1\xi_2^4 < 0, \quad -4\xi_2^6 + e_2\xi_2\xi_1^3 < 0 \tag{35}$$

for $s = 2$. System (34) admits a positive solution if and only if

$$c_1 e_1^{4/5} < 8^{4/5}/10 \approx 0.52, \tag{36}$$

whereas system (35) admits a positive solution for any positive values of c_2 and e_2.

Assume that inequality (36) is valid. Let, for instance, $c_1 = e_2 = 1/2$, $c_2 = e_1 = 2/3$. Thus, Assumption H_{10} is fulfilled.

It is easy to check that if

$$V_s(\mathbf{z}) = \frac{a_s}{4} z_1^4 + \frac{1}{2} z_2^2 + \frac{1}{10}\,|z_1|\, z_1 z_2 + \frac{1}{4} z_3^2, \quad s = 1, 2,$$

then there exists $\bar{H} > 0$ such that

$$\Delta V_1\big|_{(30)} \le -0.004\, V_1^2(\mathbf{x}(n)), \quad \Delta V_2\big|_{(31)} \le -0.32\, V_2^3(\mathbf{x}(n))$$

for $\|\mathbf{x}(n)\| < \bar{H}$. Here $\mathbf{z} = (z_1, z_2, z_3)^T$, $\mathbf{x}(n) = (x_1(n), x_2(n), x_3(n))^T$.

Moreover, the estimates $V_1(\mathbf{z}) \le 2V_2(\mathbf{z})$, $V_2(\mathbf{z}) \le V_1(\mathbf{z})$ hold for all $\mathbf{z} \in \mathbb{R}^3$.

Next, with the aid of the results of Section 5, it easy to derive sufficient conditions of asymptotic stability of the zero solution of system (29).

Assume, for definiteness, that subsystem (30) is active for $n = \tau_{2i}, \ldots, \tau_{2i+1} - 1$, whereas subsystem (31) is active for $n = \tau_{2i+1}, \ldots, \tau_{2i+2} - 1$; $i \in \mathbb{N}_+$.

Consider the sequence $\chi_0 = L = \text{const} > 0$,

$$\chi_{2i+1} = (\chi_{2i})^2 + 0.64\, T_{2i+2}, \quad \chi_{2i+2} = \frac{1}{2}\,(\chi_{2i+1})^{1/2} + 0.004\, T_{2i+3}, \quad i \in \mathbb{N}_+.$$

If there exists $L > 0$ such that

$$\chi_n \to +\infty \quad \text{as} \quad n \to +\infty, \tag{37}$$

then, by Theorem 5.1, the zero solution of system (29) is asymptotically stable.

For instance, condition (37) is fulfilled in the case when

$$T_1^2 + 0.64\,T_2 \geq 4\,p_1^2, \quad (p_i + 0.004\,T_{2i+1})^2 + 0.64\,T_{2i+2} \geq 4\,p_{i+1}^2, \quad i \in \mathbb{N},$$

where $\{p_i\}_{i=1}^{+\infty}$ is a sequence of positive numbers, such that $p_i \to +\infty$ as $i \to +\infty$.

8 Conclusion

In the present paper, for a set of switched difference equations, a regularization procedure with respect to the uncertainty parameter of the original system is developed. On the basis of the procedure, an approach to constructing Lyapunov functions and comparison systems for the corresponding family of subsystems is suggested. By means of the multiple Lyapunov function method, classes of switching law are determined for which the asymptotic stability of a stationary solution of the set of switched equations can be guaranteed. The developed approaches are applied to the stability analysis of a nonlinear multiconnected switched difference system.

An interesting problem for further research is that of estimating attraction domains of stationary solutions and finding restrictions on switching laws providing preassigned estimates.

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