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Existence Results for Sobolev Type Fractional Differential Equation with Nonlocal Integral Boundary Conditions

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Abstract: In this paper, a Sobolev type fractional differential equation with nonlocal integral boundary condition is investigated. The theory of resolvent operators, fractional calculus and fixed point techniques are used to study the existence results to the given equation. In the end, an example is provided to illustrate the applications of the abstract results.

Keywords: *fractional differential equations; fixed point theorems; resolvent operator; nonlocal boundary conditions.*

Mathematics Subject Classification (2010): 34A08, 34B10, 34G20.

1 Introduction

In a few decades, fractional differential equations have received much attention of researchers mainly due to their extensive interesting applications in physics, mechanics and engineering such as electrochemistry, control theory, signal and image processing, porous media, electromagnetism etc.(see [23], [24], [29]). The fact, that fractional derivative (integral) is an operator which includes integer order derivatives (integrals) as special case and describes the hereditary properties and memory effects of various materials, is the reason why fractional differential equations are more precise in the modeling of many phenomena. Many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic models [20] and nonlinear oscillations of earthquakes [21] can be described

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by the fractional differential equations. For a good introduction and applications to fractional differential equations we refer the reader to [25], [30] and [33]. Recently, boundary value problems for nonlinear fractional differential equations have been investigated by many researchers, see [1]- [5], [26]- [28], [34] and [36].

The Sobolev type fractional differential equations can be considered as an abstract formulation of partial differential equations which occurs in various applications such as the flow of fluid through fissured rocks [6], thermodynamics [14], and shear in second order fluids [22], [35]. There are many papers dealing with the investigation on the existence of solutions for Sobolev type differential equations in Banach spaces see [7]- [11].

In [18] Hernàndez et al. talked about an error in some papers regarding the problem of existence of a solution for abstract fractional differential equation and proposed a different approach to treat a general class of abstract fractional differential equation based on the theory of resolvent operators. But the results in [18] were not relevant for the problems with nonlocal conditions. Then in [19] Hernàndez et al. studied the theory of abstract fractional differential equations with nonlocal conditions and proved the existence results using resolvent operators. In [10], [11] Balachandran et al. studied the existence of mild solution for fractional integro-differential equation with nonlocal conditions and abstract fractional integro-differential equation of Sobolev type respectively by using the theory of resolvent operator. In [12] Belmekki et al. established the sufficient conditions for existence and uniqueness results for semilinear fractional differential equations with finite delay via resolvent operators. In [13] Belmekki et al. extended the results given in [12] to cover the case of infinite delay. Recently in [16] Chadha et al. discussed the existence results of history valued neutral fractional differential equation with the help of the theory of resolvent operators. For more details on resolvent operators see [15], [17], [31].

Up to now, to the best of our knowledge, there is a little gap in the literature on the Sobolev type fractional differential equation of order $1 < \beta \leqslant 2$ with nonlocal integral boundary condition using resolvent operators. Motivated by the above papers, to fill this gap, in this paper we consider the following Sobolev type fractional differential equation with nonlocal integral boundary conditions

$$\begin{cases} {}^{C}\mathbf{D}^{\beta}[Bx(t)] = Ax(t) + \mathcal{F}(t, x(t)), \quad 1 < \beta \leqslant 2, \quad t \in (0,1), \\ x(0) = 0, \quad x(\varepsilon) = c\int_{\eta}^{1} x(s)ds, \qquad 0 < \varepsilon < \eta < 1, \end{cases} \tag{1}$$

where ${}^{C}\mathbf{D}^{\beta}$ is the Caputo fractional derivative of order β. A is a closed linear unbounded operator, B is linear operator. $\mathcal{F} : [0,1] \times X \to X$ is continuous function. c is a positive real constant. The nonlocal integral boundary condition $x(\varepsilon) = c\int_{\eta}^{1} x(s)ds$ shows that the value of the unknown function at a nonlocal point $\varepsilon \in (0,1)$ with $0 < \varepsilon < \eta < 1$ is proportional to the integration over a sub-strip $(\eta, 1)$ of an unknown function.

2 Preliminaries

In this segment, we have some basic notations, definitions, theorems and lemmas of fractional calculus and resolvent operators which will be used in the further sections. Let $(X, \|.\|)$ be a Banach space and $\mathcal{C} = C([0,1], X)$ be the Banach space of all continuous functions from $[0,1]$ to X equipped with the norm $\|x\| = \sup\limits_{t\in[0,1]} \|x(t)\|_X$. X_H denotes the domain of $H := B^{-1}A$ endowed with the graph norm $\|x\|_H = \|x\| + \|Hx\|$. Let $L^p(J, X)$ be the Banach space of all Bochner measurable functions $x : J \to X$ such that $\|x(t)\|_X^p$

is integrable equipped with the norm

$$\|x\|_{L^p(J,X)} = \left(\int_J \|x(s)\|_X^p ds \right)^{1/p}.$$

Definition 2.1 [33] The fractional integral of order β for a function $\mathcal{F} \in L^1(\mathbb{R}^+)$ is defined by

$$I_{0+}^{\beta}\mathcal{F}(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\mathcal{F}(s)ds, \quad t > 0, \quad \beta > 0.$$

Definition 2.2 [24] The Caputo fractional derivative of order β for a function $\mathcal{F} \in C^{m-1}(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ is defined by

$${}^c\mathbf{D}_{0+}^{\beta}\mathcal{F}(t) = \frac{1}{\Gamma(m-\beta)} \int_0^t (t-s)^{m-\beta-1}\mathcal{F}^m(s)ds,$$

where $m-1 < \beta < m,\ m = [\beta]+1$ and $[\beta]$ denotes the integral part of the real number β.

Lemma 2.1 [30] *Let $q > 0$, then*

$$D^{-\beta}D^{\beta}\mathcal{F}(t) = \mathcal{F}(t) + C_1 t^{\beta-1} + C_2 t^{\beta-2} + \ldots + C_n t^{\beta-1},$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $n = [\beta] + 1$.

To prove the existence results we admit the following hypotheses:

(H1) The linear unbounded operator $A : D(A) \subset X \to X$ and linear bijective operator $B : D(B) \subset D(A) \subset X \to X$ are closed linear operators.

(H2) $B^{-1} : X \to D(B)$ is a continuous operator.

(H3) The function $\mathcal{F} : [0,1] \times X \to X$ is a continuous function such that

$$\|\mathcal{F}(t,x) - \mathcal{F}(t,y)\| \leqslant L\|x-y\|, \tag{2}$$

for all $x, y \in X$, $t \in [0,1]$ and L is a positive constant.

Lemma 2.2 *For any functions $\mathcal{F} \in C([0,1] \times X, X)$, the solution of Sobolev type fractional boundary value problem*

$$\begin{cases} {}^C\boldsymbol{D}^{\beta}[Bx(t)] = Ax(t) + \mathcal{F}(t, x(t)), & 1 < \beta \leqslant 2, \quad t \in (0,1), \\ x(0) = 0, \quad x(\varepsilon) = c\int_\eta^1 x(s)ds, & 0 < \varepsilon < \eta < 1, \end{cases} \tag{3}$$

is given by

$$x(t) = C_1 t + \frac{1}{\Gamma\beta} \int_0^t (t-\tau)^{\beta-1}(B^{-1}Ax(\tau) + B^{-1}\mathcal{F}(\tau, x(\tau)))d\tau, \tag{4}$$

where

$$\begin{aligned} C_1 = \frac{1}{\Lambda}\Bigg\{ & \frac{c}{\Gamma\beta}\int_\eta^1 \left[\int_0^s (s-\tau)^{\beta-1}(B^{-1}Ax(\tau) + B^{-1}\mathcal{F}(\tau,x(\tau)))d\tau\right] ds \\ & - \frac{1}{\Gamma\beta}\int_0^\varepsilon (\varepsilon-\tau)^{\beta-1}(B^{-1}Ax(\tau) + B^{-1}\mathcal{F}(\tau,x(\tau)))d\tau \Bigg\} \end{aligned} \tag{5}$$

with $\Lambda = \varepsilon - \frac{c}{2}(1-\eta^2) \neq 0$.

Proof. Using Lemma 2.1, the solution x of (3) can be written as

$$x(t) = C_1 t + C_2 + \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1} B^{-1}Ax(\tau)d\tau + \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1} B^{-1}\mathcal{F}(\tau, x(\tau))d\tau,$$

for some constants C_1, $C_2 \in \mathbb{R}$.
On applying boundary conditions, we get $C_2 = 0$ and

$$\begin{aligned} C_1 &= \frac{1}{\Lambda}\Bigg\{\frac{c}{\Gamma\beta}\int_\eta^1 \Bigg[\int_0^s (s-\tau)^{\beta-1}(B^{-1}Ax(\tau) + B^{-1}\mathcal{F}(\tau, x(\tau)))d\tau\Bigg] ds \\ &\quad -\frac{1}{\Gamma\beta}\int_0^\varepsilon (\varepsilon-\tau)^{\beta-1}(B^{-1}Ax(\tau) + B^{-1}\mathcal{F}(\tau, x(\tau))d\tau\Bigg\}. \end{aligned}$$

Equation (4) can also be written as

$$x(t) = k(t) + \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1} B^{-1}Ax(\tau)d\tau, \tag{6}$$

where $k(t) = C_1 t + \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1}B^{-1}\mathcal{F}(\tau, x(\tau))d\tau$.
Let $B^{-1}A = H$. To demonstrate existence results, let us assume that integral equation (6) has an associated resolvent operator $\{\mathcal{S}(t),\, t \geqslant 0\}$ on X.

Definition 2.3 [31] A one parameter family of bounded linear operators $\{\mathcal{S}(t),\, t \geqslant 0\}$ on X is called a resolvent operator for (6) if the following conditions are satisfied.

1. $\mathcal{S}(t)$ is strongly continuous on $\mathbb{R}_+$ and $\mathcal{S}(0) = I$,
2. $\mathcal{S}(t)D(H) \subset D(H)$ and $H\mathcal{S}(t)x = \mathcal{S}(t)Hx\ \forall\, x \in D(H)$ and $t \geqslant 0$,
3. for every $x \in D(H)$ and $t \geqslant 0$,

$$\mathcal{S}(t)x = x + \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1} H\mathcal{S}(\tau)x d\tau. \tag{7}$$

Definition 2.4 [31] A resolvent operator $\{\mathcal{S}(t),\, t \geqslant 0\}$ for (6) is called differentiable if $\mathcal{S}(.)x \in W^{1,1}_{loc}(\mathbb{R}^+, X)$ ($W^{1,1}_{loc}(\mathbb{R}^+, X)$ is the space of all functions having distributional derivatives)for all $x \in D(H)$ and there exists $\phi_H \in L^1_{loc}(\mathbb{R}^+)$ such that $\|\mathcal{S}'(t)x\| \leqslant \phi_H(t)\|x\|_{X_H}\ \forall x \in D(H)$.

Definition 2.5 [31] A resolvent operator $\{\mathcal{S}(t),\, t \geqslant 0\}$ for (6) is called analytic if the operator $S(t) : (0, \infty) \to L(X)$ ($L(X)$ denotes the space of all bounded linear operators from X to X) admits an analytic extension to a sector $\Sigma_{0,\theta} = \{\lambda \in \mathbb{C} : |arg(\lambda)| < \theta_0\}$ for some $0 < \theta_0 \leqslant \pi/2$.

Definition 2.6 A function $x \in \mathcal{C}$ is called a mild solution of the integral equation (6) if $\int_0^t (t-\tau)^{\beta-1}x(\tau)d\tau \in D(H)$ for all $t \in [0, 1]$, $k(t) \in \mathcal{C}$ and

$$x(t) = k(t) + \frac{H}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1}x(\tau)d\tau. \tag{8}$$

Lemma 2.3 [31] *If $\mathcal{S}(t)$ is the resolvent operator for (6).*

(i) *If x is a solution of (6) on $[0,1]$, then the function $t \to \int_0^t \mathcal{S}(t-s)k(s)ds$ is continuously differential on $[0,1]$ and*

$$x(t) = \frac{d}{dt}\int_0^t \mathcal{S}(t-s)k(s)ds, \forall t \in [0,1]. \tag{9}$$

(ii) *If $\mathcal{S}(t)$ is analytic and $k \in C^{\alpha}([0,1], X)$ for some $\alpha \in (0,1)$, then the function defined by*

$$x(t) = \mathcal{S}(t)(k(t)-k(0)) + \int_0^t \mathcal{S}'(t-s)[k(s)-k(t)]ds + \mathcal{S}(t)k(0), \forall t \in [0,1], \tag{10}$$

is a mild solution of (6).

(iii) *If $\mathcal{S}(t)$ is differentiable and $k \in C([0,1], X_H)$, then the function $x : [0,1] \to X$ given by*

$$x(t) = k(t) + \int_0^t \mathcal{S}'(t-s)k(s)ds, \forall t \in [0,1], \tag{11}$$

is a mild solution of (6).

3 Existence of Mild Solution

In this segment, we discuss the existence of mild solution for boundary value problem (1). Throughout this paper, we assume that the resolvent operator $\{\mathcal{S}(t), t \geqslant 0\}$ is a differential operator and function $\mathcal{F}$ is continuous in X_H.

By the help of Lemma (2.3)(iii), we introduce the mild solution of (6) given by

$$\begin{aligned} x(t) &= C_1 t + \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1} B^{-1}\mathcal{F}(\tau, x(\tau))d\tau \\ &\quad + \int_0^t \mathcal{S}'(t-s)\Big(C_1 s + \frac{1}{\Gamma\beta}\int_0^s (s-\tau)^{\beta-1}B^{-1}\mathcal{F}(\tau, x(\tau))d\tau\Big)ds. \end{aligned} \tag{12}$$

For simplification, let $N = \max_{t\in[0,1]} \mathcal{F}(t,0)$, $R = \|B^{-1}\|$, $P = \|B^{-1}A\|$.

Theorem 3.1 *Let $(H1)-(H4)$ hold with*

$$\delta = (1 + \|\phi_H\|_{L^1})\frac{(LR+P)}{|\Lambda|}\left[\frac{c(1-\eta^{\beta+1})}{\Gamma(\beta+2)} - \frac{\varepsilon^{\beta}}{\Gamma(\beta+1)}\right] < 1. \tag{13}$$

Then there exists a mild solution of (1) on $[0,1]$.

Proof. Let $\mathcal{B}_r = \{x \in \mathcal{C} : \|x\| \leqslant r\}$ such that

$$r \geqslant (1+\|\phi_H\|_{L^1})\left[\frac{(Pr + R(Lr+N))}{|\Lambda|}\left\{\frac{c(1-\eta^{1+\beta})}{\Gamma(\beta+2)} - \frac{\varepsilon^{\beta}}{\Gamma(\beta+1)}\right\} + \frac{R(Lr+N)}{\Gamma(\beta+1)}\right]. \tag{14}$$

Introduce the map $\Phi : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned} \Phi x(t) &= C_1 t + \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1} B^{-1}\mathcal{F}(\tau, x(\tau))d\tau \\ &\quad + \int_0^t \mathcal{S}'(t-s)\Big(C_1 s + \frac{1}{\Gamma\beta}\int_0^s (s-\tau)^{\beta-1}B^{-1}\mathcal{F}(\tau, x(\tau))d\tau\Big)ds. \end{aligned} \tag{15}$$

Decompose the map Φ into Φ_1 and Φ_2 on $\mathcal{B}_r$ for $t \in [0,1]$ such that

$$\begin{aligned}
\Phi_1 x(t) &= \frac{t}{\Lambda}\bigg\{\frac{c}{\Gamma\beta}\int_\eta^1 \Big(\int_0^s (s-\tau)^{\beta-1}(B^{-1}Ax(\tau) + B^{-1}\mathcal{F}(\tau, x(\tau)))d\tau\Big)ds \\
&\quad -\frac{1}{\Gamma\beta}\int_0^\varepsilon (\varepsilon-\tau)^{\beta-1}(B^{-1}Ax(\tau) + B^{-1}\mathcal{F}(\tau, x(\tau)))d\tau\bigg\} \\
&\quad + \int_0^t \mathcal{S}'(t-s)\bigg[\frac{s}{\Lambda}\bigg\{\frac{c}{\Gamma\beta}\int_\eta^1 \Big(\int_0^v (v-\tau)^{\beta-1}(B^{-1}Ax(\tau)+B^{-1}\mathcal{F}(\tau, x(\tau)))d\tau\Big)dv \\
&\quad -\frac{1}{\Gamma\beta}\int_0^\varepsilon (\varepsilon-\tau)^{\beta-1}(B^{-1}Ax(\tau) + B^{-1}\mathcal{F}(\tau, x(\tau)))d\tau\bigg\}\bigg]ds. \\
\Phi_2 x(t) &= \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1}B^{-1}\mathcal{F}(\tau, x(\tau))d\tau \\
&\quad + \int_0^t \mathcal{S}'(t-s)\bigg(\frac{1}{\Gamma\beta}\int_0^s (s-\tau)^{\beta-1}B^{-1}\mathcal{F}(\tau, x(\tau))d\tau\bigg)ds.
\end{aligned}$$

Step 1. We show that $\Phi_1 x + \Phi_2 y \in \mathcal{B}_r$ for every $x,\, y \in \mathcal{B}_r$, we have

$$\begin{aligned}
\|\Phi_1 x + \Phi_2 y\| &\leqslant \sup_{t\in[0,1]}\bigg\{\frac{t}{|\Lambda|}\bigg\{\frac{c}{\Gamma\beta}\int_\eta^1 \Big(\int_0^s (s-\tau)^{\beta-1}(\|B^{-1}A\|\|x(\tau)\| \\
&\quad +\|B^{-1}\|\|\mathcal{F}(\tau, x(\tau)) - \mathcal{F}(\tau, 0) + \mathcal{F}(\tau, 0)\|)d\tau\Big)ds \\
&\quad -\frac{1}{\Gamma\beta}\int_0^\varepsilon (\varepsilon-\tau)^{\beta-1}(\|B^{-1}A\|\|x(\tau)\| \\
&\quad +\|B^{-1}\|\|\mathcal{F}(\tau, x(\tau)) - \mathcal{F}(\tau, 0) + \mathcal{F}(\tau, 0)\|)d\tau\bigg\} \\
&\quad + \int_0^t \mathcal{S}'(t-s)\bigg[\frac{s}{|\Lambda|}\bigg\{\frac{c}{\Gamma\beta}\int_\eta^1 \Big(\int_0^v (v-\tau)^{\beta-1}(\|B^{-1}A\|\|x(\tau)\| \\
&\quad +\|B^{-1}\|\mathcal{F}(\tau, x(\tau)) - \mathcal{F}(\tau, 0) + \mathcal{F}(\tau, 0)\|)d\tau\Big)dv \\
&\quad -\frac{1}{\Gamma\beta}\int_0^\varepsilon (\varepsilon-\tau)^{\beta-1}(\|B^{-1}A\|\|x(\tau)\| \\
&\quad +\|B^{-1}\|\|\mathcal{F}(\tau, x(\tau)) - \mathcal{F}(\tau, 0) + \mathcal{F}(\tau, 0)\|)d\tau\bigg\}\bigg]ds \\
&\quad +\frac{\|B^{-1}\|}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1}\|\mathcal{F}(\tau, y(\tau)) - \mathcal{F}(\tau, 0) + \mathcal{F}(\tau, 0)\|d\tau \\
&\quad + \int_0^t \|\mathcal{S}'(t-s)\|\bigg(\frac{\|B^{-1}\|}{\Gamma\beta}\int_0^s (s-\tau)^{\beta-1} \\
&\quad \|\mathcal{F}(\tau, y(\tau)) - \mathcal{F}(\tau, 0) + \mathcal{F}(\tau, 0)\|d\tau\bigg)ds\bigg\} \\
&\leqslant (1+\|\phi_H\|_{L^1})\bigg[\frac{(Pr + R(Lr+N))}{|\Lambda|}\bigg\{\frac{c(1-\eta^{\beta+1})}{\Gamma(\beta+2)} - \frac{\varepsilon^\beta}{\Gamma(\beta+1)}\bigg\} \\
&+ \frac{R(Lr+N)}{\Gamma(\beta+1)}\bigg] \leqslant r.
\end{aligned}$$

Thus $\Phi_1 x + \Phi_2 y \in \mathcal{B}_r$.

Step 2. We show that Φ_1 is a contraction. For $x, y \in \mathcal{B}_r$ and $t \in [0,1]$, we have

$$\begin{aligned}
\|\Phi_1 x - \Phi_1 y\| &\leqslant \sup_{t\in[0,1]} \bigg\{ \frac{t}{|\Lambda|} \bigg\{ \frac{c}{\Gamma\beta} \int_\eta^1 (\int_0^s (s-\tau)^{\beta-1} (\|B^{-1}A\| \|x(\tau) - y(\tau)\| \\
&\quad + \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau)) - \mathcal{F}(\tau, y(\tau))\|) d\tau) ds \\
&\quad - \frac{1}{\Gamma\beta} \int_0^\varepsilon (\varepsilon - \tau)^{\beta-1} (\|B^{-1}A\| \|x(\tau) - y(\tau)\| \\
&\quad + \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau)) - \mathcal{F}(\tau, y(\tau))\|) d\tau \bigg\} \\
&\quad + \int_0^t \mathcal{S}'(t-s) \bigg[\frac{s}{|\Lambda|} \bigg\{ \frac{c}{\Gamma\beta} \int_\eta^1 (\int_0^v (v-\tau)^{\beta-1} (\|B^{-1}A\| \|x(\tau) - y(\tau)\| \\
&\quad + \|B^{-1}\| \mathcal{F}(\tau, x(\tau)) - \mathcal{F}(\tau, y(\tau))\|) d\tau) dv \\
&\quad - \frac{1}{\Gamma\beta} \int_0^\varepsilon (\varepsilon - \tau)^{\beta-1} (\|B^{-1}A\| \|x(\tau) - y(\tau)\| \\
&\quad + \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau)) - \mathcal{F}(\tau, y(\tau))\|) d\tau \bigg\} \bigg] ds \bigg\} \\
&\leqslant (1 + \|\phi_H\|_{L^1}) \frac{(P + RL)}{|\Lambda|} \bigg(\frac{c(1 - \eta^{\beta+1})}{\Gamma(\beta+2)} - \frac{\varepsilon^\beta}{\Gamma(\beta+1)} \bigg) \|x - y\| \\
&\leqslant \delta \|x - y\|.
\end{aligned}$$

By assumption, $\delta < 1$ and therefore Φ_1 is a contraction.
Step 3. Next, we prove that Φ_2 is continuous and compact. The continuity of map Φ_2 can be obtained from the continuity of $\mathcal{F}$. Also for $t \in [0,1]$

$$\begin{aligned}
\|\Phi_2\| &\leqslant \sup_{t\in[0,1]} \bigg(\frac{1}{\Gamma\beta} \int_0^t (t-\tau)^{\beta-1} \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau))\| d\tau \\
&\quad + \int_0^t \|S'(t-s)\| \bigg(\frac{1}{\Gamma\beta} \int_0^s (s-\tau)^{\beta-1} \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau))\| d\tau \bigg) ds \bigg) \\
&\leqslant (1 + \|\phi_H\|_{L^1}) \frac{R(Lr + N)}{\Gamma(\beta+1)}.
\end{aligned}$$

i.e. Φ_2 is uniformly bounded $\mathcal{B}_r$. Now we show that the set $\{\Phi_2 x(t) : x \in \mathcal{B}_r\}$ is relatively compact in Y for all $t \in [0,1]$. Clearly the set $\{\Phi_2 x(0) : x \in \mathcal{B}_r\}$ is compact. Fix $t \in (0,1]$, let δ be a real number satisfying $0 < \delta < 1$. For $x \in \mathcal{B}_r$, define the operator Φ_2^δ by

$$\begin{aligned}
\Phi_2^\delta x(t) &= \frac{1}{\Gamma\beta} \int_0^{t-\delta} (t-\tau)^{\beta-1} B^{-1} \mathcal{F}(\tau, x(\tau)) d\tau \\
&\quad + \int_0^{t-\delta} S'(t-s) \bigg(\frac{1}{\Gamma\beta} \int_0^s (s-\tau)^{\beta-1} B^{-1} \mathcal{F}(\tau, x(\tau)) d\tau \bigg) ds.
\end{aligned}$$

By assumption $(H4)$, $\mathcal{F}$ is completely continuous, the set $\{\Phi_2^\delta x(t) : x \in \mathcal{B}_r\}$ is precompact in X, for every $\delta \in (0,1]$. Furthermore, for every $x \in \mathcal{B}_r$, we have

$$\begin{aligned}
\|\Phi_2 x(t) - \Phi_2^\delta x(t)\| &\leqslant \frac{1}{\Gamma\beta} \int_{t-\delta}^t (t-\tau)^{\beta-1} \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau))\| d\tau \\
&\quad + \int_{t-\delta}^t \mathcal{S}'(t-s) \bigg(\frac{1}{\Gamma\beta} \int_0^s (s-\tau)^{\beta-1} \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau))\| d\tau \bigg) ds.
\end{aligned}$$

It shows that the precompact sets $\{\Phi_2^{\delta}x(t) : x \in \mathcal{B}_r\}$ are arbitrary close to the set $\{\Phi_2 x(t) : x \in \mathcal{B}_r\}$. Hence the set $\{\Phi_2 x(t) : x \in \mathcal{B}_r\}$ is precompact in X.

Step 4. Now, we show that $\{\Phi_2 x(t) : x \in \mathcal{B}_r\}$ is equicontinuous. Clearly $\{\Phi_2 x(t) : x \in \mathcal{B}_r\}$ are equicontinuous at $t = 0$. For $t < t + h \leqslant 1$, $h > 0$, we have

$$\begin{aligned}
\|\Phi_2 x(t+h) - \Phi_2 x(t)\| \leqslant\; & \frac{1}{\Gamma\beta}\bigg\| \int_0^{t+h} (t+h-\tau)^{\beta-1} B^{-1}\mathcal{F}(\tau, x(\tau))d\tau \\
& - \int_0^t (t-\tau)^{\beta-1} B^{-1}\mathcal{F}(\tau, x(\tau))d\tau \bigg\| \\
& + \frac{1}{\Gamma\beta}\bigg\| \int_0^{t+h} \mathcal{S}'(t+h-s) \int_0^s (s-\tau)^{\beta-1} B^{-1}\mathcal{F}(\tau, x(\tau))d\tau ds \\
& - \int_0^t \mathcal{S}'(t-s) \int_0^s (s-\tau)^{\beta-1} B^{-1}\mathcal{F}(\tau, x(\tau))d\tau ds \bigg\| \\
\leqslant\; & \frac{1}{\Gamma\beta}\int_0^t \Big[(t+h-\tau)^{\beta-1} - (t-\tau)^{\beta-1}\Big] \|B^{-1}\|\|\mathcal{F}(\tau, x(\tau))\| d\tau \\
& + \frac{1}{\Gamma\beta}\int_t^{t+h} (t+h-\tau)^{\beta-1}\|B^{-1}\|\|\mathcal{F}(\tau, x(\tau))\| d\tau \\
& + \int_0^h \|\mathcal{S}'(t+h-s)\| \frac{1}{\Gamma\beta}\int_0^s (s-\tau)^{\beta-1}\|B^{-1}\|\|\mathcal{F}(\tau, x(\tau))\| d\tau ds \\
& + \int_0^t \|\mathcal{S}'(t-s)\| \frac{\|B^{-1}\|}{\Gamma\beta}\bigg\| \int_0^{s+h} (s+h-\tau)^{\beta-1}\mathcal{F}(\tau, x(\tau))d\tau \\
& - \int_0^s (s-\tau)^{\beta-1}\mathcal{F}(\tau, x(\tau))d\tau \bigg\| ds.
\end{aligned}$$

Which tends to zero as $h \to 0$, therefore the set $\{\Phi_2 x(t) : x \in \mathcal{B}_r\}$ is equicontinuous. Thus Φ_2 is relatively compact for $t \in [0, 1]$. By Arzela-Ascoli's theorem Φ_2 is compact. Hence by Krasnoselskii fixed point theorem [32] there exists a fixed point $x \in \mathcal{C}$ such that $\Phi x = x$ which is a mild solution of the boundary value problem (1).

4 Example

Let $X = L^2(0, \pi)$, $1 < \beta \leqslant 2$ and $t \in [0, 1]$. Consider the following partial differential equation with fractional derivative

$$\begin{cases} \frac{\partial^\beta}{\partial t^\beta}\left(w(t,x) - \frac{\partial^2}{\partial x^2}w(t,x)\right) = \frac{\partial^2}{\partial x^2}w(t,x) + \frac{w(t,x)}{1+w(t,x)}, \\ w(t,0) = w(t,\pi) = 0, \\ w(0,x) = 0,\ w(\varepsilon, x) = c\int_\eta^1 w(t,s)ds. \end{cases} \tag{16}$$

Define the operators $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ by

$$Aw = w'', \quad Bw = w - w'',$$

where

$$D(A) = D(B) = \{w \in X, w, w' \text{are absolutely continuous}, w'' \in X, w(0) = w(\pi) = 0\}.$$

Then A and B can be written as

$$\begin{aligned} Aw &= \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A), \\ Bw &= \sum_{n=1}^{\infty} (1+n^2)(w, w_n) w_n, \quad w \in D(B), \end{aligned}$$

where $w_n(x) = \sqrt{2/\pi} \sin nx$, $n = 1, 2, \ldots$, is the original set of vectors A. Moreover, we have

$$B^{-1}w = \sum_{n=1}^{\infty} \frac{1}{1+n^2}(w, w_n) w_n,$$

$$Hw = B^{-1}Aw = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2}(w, w_n) w_n.$$

The equation (16) can be reformulated as the following Sobolev type fractional differential equation with nonlocal integral boundary condition

$$\begin{cases} D^\beta (Bw(t)) = Aw(t) + \mathcal{F}(t, w(t)), \quad 1 < \beta \leqslant 2,\, t \in (0,1), \\ w(0) = 0,\, w(\varepsilon) = c \int_\eta^1 w(s) ds, \quad 0 < \varepsilon < \eta < 1. \end{cases} \tag{17}$$

Clearly all the assumptions $(H1) - (H4)$ are satisfied.

Theorem 4.1 *Suppose* $(H1) - (H4)$ *hold and* A *generates a differential resolvent operator* $\{\mathcal{S}(t)\}$ *with*

$$\delta = (1 + \|\phi_H\|_{L^1}) \frac{(LR+P)}{|\Lambda|} \left[\frac{c(1-\eta^{\beta+1})}{\Gamma(\beta+2)} - \frac{\varepsilon^\beta}{\Gamma(\beta+1)} \right] < 1.$$

Then the problem (17) has a solution.

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