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# Extending the Property of a System to Admit a Family of Oscillations to Coupled Systems

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**Abstract:** Coupled systems, each one admitting a family of nondegenerate periodic solutions, are considered. The period of oscillations in the family is supposed to depend on a unique parameter. Conditions imposed on weak couplings such that the coupled system admits a family of periodic solutions, which is similar to that of subsystems, are found. Differential equations of general form, as well as reversible mechanical systems are investigated. The existence of resonant orbits in the $N$-planet problem with one planet in a quasi-circular orbit is proved.

**Keywords:** *coupled system; differential equation; periodic solution; family; nondegenerate; reversible mechanical system; $N$-planet; resonant orbits.*

**Mathematics Subject Classification (2010):** 34A34, 37C27, 37C80, 70F10.

## 1 Introduction

Investigation of a dynamic model usually implies the consideration of substantial factors. The influence of other (minor) factors is regarded in the frame of the perturbation theory. This influence can either slightly change quantitatively dynamical characteristics of the system, or bring about a new quality. The latter case is usually associated with a bifurcation.

System perturbations result from weak influence of other systems. Taking this into account we consider a new model, which is closed one. The non-regarded influence is modelled by the couplings between the systems to constitute coupled systems. Since the intensity of non-regarded factors is weak, the couplings are expected to be small.

In [1] the closed model containing coupled subsystems (MCCS) is introduced. This model posesses dynamical properties (i.e. run-outs, enregy transfer, etc.) that cannot

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be explained in the frame of perturbation theory. Investigation of this model assumes the two main problem: 1) to find conditions on couplings such that the MCCS inherits dynamical properties of its subsystems; 2) to find qualitatively new effects resulting from couplings between subsystems.

In this paper the first problem is considered for coupled systems. Each system is supposed to admit a family of nondegenerate periodic solutions, where the period depends on a unique parameter. The goal is to find conditons on weak couplings such that the coupled system admits a family of periodic solutions, which is similar to that of subsystems. It is shown that these conditions are always satisfied in the case of symmetric periodic motions of reversible mechanical system. The $N$-planet problem with one planet in a quasi-circular orbit is considered. The existence of resonant orbits in this problem is proved.

The concept of MCCS arose from the classical perturbation theory. The MCCS describes the dynamics in various problems of classical and celestial mechanics, radioengineering, population dynamics, mechatronics and robotics, biology, medicine, etc. [1, 2]. MCCS can consist of subsystems of diverse nature, the subsystems being described by various type equations. Coupled oscillators (see, for example, [3, 4]) became the classical model that illustrates the complexity of behaviour in coupled systems.

In [1] the formal description of the MCCS is given. Since 2003 systematic investigations concerning the above problems for the MCCS have been carrying out, more than a dozen of papers have been published.

The autonomous models containing families of periodic solutions in subsystems were considered in [2]. In particular, for an MCCS consisting of $m$ subsystems the bifurcation scenario is given. This scenario assumes the bifurcation of the $2m$-family of periodic solutions such that the $m$-family of periodic solution arises in the MCCS.

Later MCCS with identical subsystems were considered in [5], where the existence of a family of periodic motions such that the period depends on a unique parameter is proven. This paper pushes further the investigations of [2, 5] to extend the results of [5] to MCCS containing different subsystems. It is shown that coupled reversible mechanical systems inherit completely the dynamic property of the subsystems. Thus the problem of extending the dynamic property of subsystems to the MCCS is completely solved.

MCCS belongs to the class of complex systems. Among the characteristic features of the model there are hierarchical and multi-level structure, multi-mode operation, nonlinearity, high order. MCCS is qualified also as a large-scale system. MCCS represents a network. It can be either autonomous or non-autonomous.

Weakly coupled MCCS is a system with a small parameter. Investigations of such systems can apply Yu. A. Mitropolsky's results (cf. [6]), in particular, the single-frequency approach to study nonlinear oscillations in multi-degree of freedom systems [7].

The present paper is dedicated to the 100-year anniversary of Yu.A. Mitropolsky.

## 2 The Nondegenerate for a Periodic Solution Case

Consider the smooth equation

$$\dot{x} = X(x), \quad x \in R. \tag{1}$$

Denote by $x(x_1^0, \ldots, x_n^0, t)$ the general solution of (1). The necessary and sufficient conditions of the existence of a $T$-periodic solution are given by

$$f \equiv x(x_1^0, \ldots, x_n^0, T) - x^0 = 0, \tag{2}$$

where $x^0 = (x_1^0, \dots, x_n^0)$ is the initial point at $t = 0$.

Let equation (2) have a solution $x^0 = x^*,\ T = 2\pi$. Calculate the rank $Ra$ of the functional matrix for the function $f$ at the point $(x^*, T)$. Since (1) is autonomous, equation (2) possesses a monoparametric (denote the parameter by $\gamma$) family of solutions

$$x^0 = x^*(\gamma), \quad T = 2\pi. \tag{3}$$

Thus we obtain $Ra \leq n - 1$.

**Definition 2.1** The case of $Ra = n-1$ is referred to as nondegenerate for a periodic solution. The very solution is referred to as nondegenerate.

We use later on the following notion.

**Definition 2.2** The isolated periodic solution of an autonomous differential equation is called the cycle.

The following alternative holds [2].

**Theorem 2.1** *In the nondegenerate for a periodic solution case the following alternative takes place: the solution is either a cycle or belongs to a family of periodic solutions with the period depending on a unique parameter. If this alternative realizes for equation* (1) *then the nondegenerate for a periodic solution case takes place.*

In this paper the case of family in the alternative is analyzed. According to the law [8, 9] the period on the family depends on a unique parameter and $T = T(h)$. For ordinary points of the family $dT \neq 0$ and for critical points $dT = 0$ [10]. The nondegenerate for a periodic solution case excludes the critical point from consideration. In the case of family the periodic solution is associated with a double zero characteristic exponent (CE) in the Jordan cell [11]. Since $Ra = n - 1$, the remaining CE are nonzero.

Note that there always exists (cf. [11]) a particular solution of the form

$$x_s(t) = e^{\lambda_k t}\varphi_s(t), \quad \varphi_s(t+T) = \varphi_s(t), \quad s = 1, \dots, n,$$

for a $T$-periodic linear system of the $n$-th degree. Here $\lambda_k$ is CE; the total number of CE (regarding their multiplicity) being equal to $n$.

## 3 Extending the Dynamic Property

Consider $m$ smooth coupled systems

$$\dot{x}^s = X(x^s) + \varepsilon \tilde{X}^s(\varepsilon, x^1, \dots, x^m), \quad s = 1, \dots, m, \quad x^s \in R^{m_s}. \tag{4}$$

Here $\varepsilon$ is a nonnegative numerical parameter such that (4) breaks up into $m$ independent systems at $\varepsilon = 0$. Suppose that the $s$-th system admits a family of periodic solutions

$$x^s = \varphi^s(h_s, t + \gamma_s), \quad s = 1, \dots, m, \tag{5}$$

which contains two parameters $h_s$ and $\gamma_s$. Here the period $T_s = T_s(h_s)$ of (5) depends on $h_s$ and $\gamma_s$ and represents the shift of the initial point along the trajectory. The $s$-th system at a fixed $h_s = h_s^*$ admits a periodic solution that depends on $\gamma_s$ and given $h_s = h_s^*,\ s = 1, \dots, m$, the generating system (i.e. (4) at $\varepsilon = 0$) has an $m$-family of conditionally periodic solutions with $m$ frequencies. If $T_s(h_s^*) = T^*,\ s = 1,\dots,m$, this family is the family of $T^*$-periodic solutions with the parameter $\gamma = (\gamma_1,\dots,\gamma_m)$. In view of this the existence conditions of periodic motions for coupled systems are formulated in terms of $\gamma = (\gamma_1,\dots,\gamma_m)$ that corresponds to a chosen $h^* = (h_1^*,...,h_m^*)$, rather than in terms of $h_s$, as it is the case in [2] where arbitrary systems are considered.

Let

$$x(\varepsilon, x^0, t) = (x^1(\varepsilon, x^0, t), \dots, x^m(\varepsilon, x^0, t)) \tag{6}$$

be the solution of the Cauchy problem of (4) with the initial point $x^0$ at $t = 0$. Take the derivative of (6) with respect to $\varepsilon$ at $\varepsilon = 0$ when (6) coinsides with the solution given by (5) at $h_s = h_s^*,\ s = 1,\dots,m$. This derivative satisfies the following linear nonhomogenious system with periodic coefficients

$$\frac{d}{dt}\left(\frac{\partial x^s}{\partial \varepsilon}\right) = P^s(h_s^*, t+\gamma_s)\left(\frac{\partial x^s}{\partial \varepsilon}\right) + \tilde{X}^s(0, \varphi^1(h_1^*, t+\gamma_1), \dots, \varphi^m(h_m^*, t+\gamma_m)), \quad s = 1,\dots,m, \tag{7}$$

where

$$P^s(h_s^*, t+\gamma_s) = \|p_{kj}^s(h_s^*, t+\gamma_s)\|_{k,j=1}^{m_s}, \quad p_{kj}^s(h_s^*, t+\gamma_s) = \left(\frac{\partial X_k^s}{\partial x_j^s}\right)_{x^s=\varphi^s(h_s^*, t+\gamma_s)}, \quad s = 1,\dots,m.$$

The homogenious part of (7) splits up into $m$ independent systems of $m_s$-th degree, each one having a unique $T^*$-periodic solution. The appropriate conjugate system splits up into $m$ subsystems as well. Denote the $T^*$-periodic solutions of those subsystems by $\psi^s(h_s^*, t+\gamma_s)$ $s = 1,\dots,m$. Consequently, the necessary condition of existence of a $T^*$-periodic solution for (4) can be written as

$$g_{h^*}(\gamma_1,\dots,\gamma_m) = 0, \tag{8}$$

where the components of $g$ are defined by

$$g_{h^*}^s(\gamma_1,\dots,\gamma_m) = \int_0^{T^*} \sum_{k=1}^{n} \tilde{X}_k^s(0, \varphi^1(h_1^*, t+\gamma_1), \dots, \varphi^m(h_m^*, t+\gamma_m))\psi_k^s(h_s^*, t+\gamma_m)dt, \quad s = 1,\dots,m.$$

Equation (8) determines the class of couplings that admit the existence of periodic solutions in coupled systems. It will be shown later that (8) turns out to be sufficient under some conditions.

Note that the equation $g = 0$ was used earlier [2] to find $h_s^*$ of the generating family with the parameter $\gamma$.

Let us formulate the theorem that establishes sufficient conditions of the existence of periodic solutions in coupled systems.

**Theorem 3.1** *Let equation* (8) *have a solution denoted by* $\gamma^*$*, i.e.* $g_{h^*}(\gamma^*) = 0$*. Let the rank of the functional matrix of the mapping* $\gamma \to g_{h^*}(\gamma)$ *at* $\gamma^*$ *be equal to* $m-1$*. Then* (4) *has a periodic solution.*

***Proof.*** The solution $x(\varepsilon, x^0, t+\gamma)$, being $T^*$-periodic, satisfies equation

$$F \equiv x(\varepsilon, x^0, T^*) - x^0 = 0, \quad F = (F^1, \dots, F^m). \tag{9}$$

Since (4) is autonomous, the solution $x^0$ of (9) depends on a unique parameter $\delta$ and $\gamma = \gamma(\delta)$.

Hence, the problem is to find the root $x^0$ of (9), which depends on $\varepsilon$ such that it satisfies the system

$$F_0^s \equiv x^s(0, x^{s0}, T^*) - x^{s0} = 0, \quad s = 1, \dots, m, \quad x^0 = (x^{10}, \dots, x^{m0}) \tag{10}$$

at $\varepsilon = 0$. System (10) splits up into $m$ subsistems. The $s$-th subsystem has a family of solutions $x^{s0} = x^{s0}(\gamma_s^*(\delta))$ with the parameter $\gamma_s^*(\delta)$.

At a given $\varepsilon$ (9) represents a system of $m$-th degree in $m$ variables, while its solution $x^0 = x^*(\varepsilon, \gamma(\delta))$ depends on $\varepsilon$.

Rearrange (9) as

$$F_0^s(x^{s0}, T^*) + \varepsilon G^s(\varepsilon, x^0, T^*) = 0, \quad s = 1, \dots, m, \tag{11}$$

where $F_0^s(x^{s*}(0, \gamma^*), T^*) = 0$. For (11) take the increments

$$y_k^s = x_k^{s0} - x_k^{s*}(0, \gamma_s^*), \quad k = 1, \dots, m_s, \quad s = 1, \dots, m. \tag{12}$$

By assumption, the rank $Ra^s$ of the functional matrix of $F_0^s$ at $x^{s0} = x^{s*}(0, \gamma_s^*)$ is equal to $m_s - 1$. Consequently, $m_s - 1$ increments $y_k^s$ of the $s$-th subsystem can be expressed as functions of the remaining increment, which we denote, to be specific, by $y_s^s$. Substitute the above functions into the equation for $y_s^s$, then the $s$-th subsystem yields a unique equation in $y_s^s$ instead of $m_s$ equations. By repeating this procedure for all $m$ subsystems we obtain a system of $m$ equations

$$\Phi^s(z, T^*) + \varepsilon \Psi^s(\varepsilon, z, T^*) = 0, \quad s = 1, \dots, m, \quad z = (y_1^1, \dots, y_m^m).$$

In this system functions $\Phi^s$ do not contain linear terms. According to (12) components of $z$ are of the $\varepsilon$-th order. Equations (12) mean the transition along the trajectory of the generating solution from the point $x^*(0, \gamma^*)$ to the initial point $x^*(0, \gamma)$ for a periodic solution of the system at $\varepsilon \neq 0$. This implies that $\gamma - \gamma^* \sim \varepsilon$, so we obtain the system

$$\Psi^s(0, y_1^1(\gamma), \dots, y_m^m(\gamma), T^*) = 0, \quad s = 1, \dots, m, \tag{13}$$

which coinsides with system of amplitude equations (8).

Let the rank of (13) at the point $\gamma = \gamma^*$ be equal to $m - 1$. Then system (9) has a solution, which depends on the parameter $\delta$. This means that there exists a $T^*$-periodic solution in (4).

Theorem 3.1 can be valid for an isolated point $h^*$. Such a point can be found by deriving an appropriate amplitude equation and by finding its simple roots [2].

Suppose that $h_s = h_s(\chi)$, $s = 1, \dots, m$ in (5). Then the generating system has an $(m+1)$-family of periodic solutions, and the vector $h^*(\chi)$ can be regarded as a parameter in Theorem 3.1.

So the following theorem holds.

**Theorem 3.2** *Let equation* (8) *have a solution* $g_{h^*}(\gamma^*) = 0$. *Let the rank of the functional matrix of the mapping* $\gamma \to g_{h^*}(\gamma)$ *at* $\gamma^*$ *be equal to* $m - 1$. *Then* (4) *has a 2-family of periodic solutions with the period depending on a unique parameter.*

Theorem 3.2 solves the problem of extending the property of having a family of periodic solutions with the period depending on a unique parameter to coupled systems. This result was announced in [12].

The proof of Theorem 3.2 repeats that of Theorem 3.1 with obvious modifications.

## 4 Coupled Reversible Mechanical Systems

### 4.1 Symmetric periodic motions

At first consider a separate system

$$\dot{u} = U(u, v), \quad \dot{v} = V(u, v), \tag{14}$$

$$U(u, -v) = -U(u, v), \quad V(u, -v) = V(u, v); u \in R^l, v \in R^n, l \geq n. \tag{15}$$

A series of models in classical and celestial mechanics are described by these equations [13]. Usually $u$ is the vector of generalized coordinates (quasicoordinates) and $v$ is the vector of generalized velocities (quasivelocities). System (14), (15) is the particular case of the reversible dynamical system [14]. It is called the reversible mechanical system.

In what follows the set $M = \{u, v : v = 0\}$, which is called the fixed set, is used.

System (14), (15) always possesses a pair of symmetric with respect to $M$ motions (see Figure 1, a). The solution of (14),(15) that crosses $M$ is called the symmetric motion (Figure 1, b). A symmetric motion can be periodic (symmetric periodic motion, SPM). An SPM crosses $M$ at least twice (Figure 1, c).

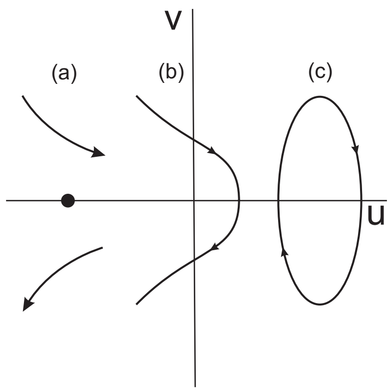


**Figure 1**: Motion types in a reversible mechanical system. a: a pair of symmetric with respect to $M$ motions; b: symmetric motion; c: symmetric periodic motion.

If system (14), (15) is $2\pi$-periodic in some or all components of $v$, the appropriate SPM can be either oscillation or rotation. These components take multiple of $\pi$ values on the fixed set [15].

Denote the symmetric motion by $v(u_1^0, \ldots, u_l^0, t)$, where $u^0$ is the initial point in $M$. Then the necessary and sufficient conditions of existence of a $T$-periodic SPM are given by [15]

$$v_s(u_1^0, \ldots, u_l^0, T/2) = 0, \quad s = 1, \ldots, n. \tag{16}$$

Let system (16) admit a solution

$$u_1^0 = u_1^*, \ldots, u_l^0 = u_l^*, \quad T = T^* = 2\pi. \tag{17}$$

Set up the matrix

$$A = \left\| \frac{\partial v_s(u_1^0, ..., u_l^0, T/2)}{\partial u_j^0} \right\|,$$

where the partial derivatives are taken at the values (17).

**Definition 4.1** The case of rank $A = n$ is called nondegenerate for a symmetric periodic motion; the very SPM is called nondegenerate.

Note that if $l > n$, a nondegenerate SPM will be degenerate in the sense of Definition 2.1.

A nondegenerate SPM is extended in the phase space over a family of $(l - n + 1)$-th degree. The condition rank $A = n$ means that the SPM with the initial point (17) is submerged in the family of SPM that depends on arbitrary $l - n$ initial values of the vector $u^0$ and on the period $T$ (cf. [16]). The law stating that the period depends on a unique parameter is valid over the family of SPM [8, 9]. If system (14) contains a numerical parameter $\mu$ and an SPM is nondegenerate at $\mu = 0$ then the property of nondegeneracy is extended for the appropriate SPM over the range $\mu \neq 0$. Given the existence of the SPM family in the system at $\mu = 0$, conditions of the extension of the SPM family over the range $\mu \neq 0$ are found [16]. The appropriate property of the SPM family is called stability with respect to parametric perturbations of the system.

A more general matrix

$$A_1 = \left\| \frac{\partial v_s(u_1^0, ..., u_l^0, T/2)}{\partial u_j^0} \quad \frac{\partial v_s(u_1^0, ..., u_l^0, T/2)}{\partial t} \right\|$$

can be used instead of $A$ [17]. If rank $A_1 = n$ (generalized nondegeneracy condition) then the implicit function theorem guarantees the existence of solution of system (16) in the neighborhood of the point (17).

The case of rank $A = n$ (which implies rank $A_1 = n$) is described above. If rank $A = n - 1$ and rank $A_1 = n$ then system (14) has a $l - n + 1$ family of SPM with the period depending on $l - n + 1$ initial values of $u^0$.

### 4.2 SPM families in coupled reversible mechanical systems

Consider the model of coupled reversible mechanical systems [17]. The intensity of coupling is characterized by the small numeric parameter $\varepsilon$ such that the model decouples into independent systems of the form (14) at $\varepsilon = 0$. If so, matrices $A$ and $A_1$ depend on $\varepsilon$. When $\varepsilon = 0$, they are block diagonal with the blocks $A^{(j)}$ $(A_1^{(j)}))$ determined by the $j$-th system. The condition of nondegeneracy of SPM in all systems provides the condition of nondegeneracy of SPM in the coupled model [17]. The nondegeneracy condition for the SPM in all but one systems and the generalized nondegeneracy condition for the SPM in the remaining system yield the generalized nondegeneracy condition for the SPM in the coupled model. So the property of the reversible mechanical system to have SPM can be extended to coupled system in the following way.

**Theorem 4.1** *If the nondegeneracy condition for the SPM is satisfied for all but one reversible mechanical systems, while either the nondegeneracy condition for the SPM or the generalized nondegeneracy condition for the SPM is satisfied for the remaining system, then there exists a family of SPM of degree $2 + \sum(l - n)$ in the coupled model of reversible mechanical systems. Here $l$ and $n$ are dimensions of the vectors $u$ and $v$ in the systems, respectively.*

**Remark 4.1** If $l = n$ for all systems then the 2-family of SPM of the system is extended to 2-family of SPM of the coupled model.

### 4.3 Coupled reversible mechanical systems with couplings of general form

Reversible mechanical systems, being coupled, may loose the property of reversibility. This is the case when the couplings are represented by arbitrary functions of $u$ and $v$ such that conditions (15) are not satisfied for coupled systems. As a result, the coupled model is described by differential equations of general form. The following particular case can be distinguished: $l = n$ for all systems, all periodic motions involved are nondegenerate. In this case Theorem 3.2 can be applied to establish the extension of the property from the separate system to coupled systems. Besides, the result holds for both symmetric and non-symmetric periodic motions.

The generalization of Theorem 3.2 turns out to be valid for nondegenerate SPM even if $l \neq n$: the property of having a family of SPM is extended to coupled systems. The accurate statement requires preliminary transformations of coupled systems similar to those represented in [18]. This statement is beyond the scope of the paper.

## 5 Resonant Orbits in the $N$-planet Problem

The motion of $N+1$ gravitating bodies with one body (the Sun) being vastly superior in mass to other bodies (the planets) is studied in the frame of the $N$-planet problem. If the interaction between planets is neglected, the $N$-planet problem results in $N$ independent two-body problems (the Sun and the planet). The interaction between the planets can be treated as perturbations.

### 5.1 Parade of planets

In the Solar system the parade of planets phenomenon, where all planets or some of them line up in a straight line, is observed. In the frame of the $N$-planet problem this phenomenon is associated with the existence of symmetric periodic orbits [19].

The $N$-planet problem belongs to the class of reversible mechanical systems [17]. Elliptic orbits in the two-body problem are symmetric with respect to the major axis, the radial velocity being zero on the axis. The crossing of the fixed set by the image point means for the $N$-planet problem that the planets line up in the straight line (parade of planets). Since the parade of planets is periodic, this effect is observed on periodic orbits. Such orbits result in resonances in the planet system.

In the stationary frame of reference the parade of planets is observed on elliptic orbits (orbits of the second type), while in the rotating frame of reference the parade of planets takes place on circular orbits (orbits of the first type).

Orbits of the first type were studied in [19,20], the parade of planets on the orbit of the second type was analyzed in [19]. A simple proof of existence of the orbit of the second type is given in [17]. The parade of planets turns out to occur on the $(N+1)$-parametric family of orbits close to elliptic, the period depending on a unique parameter (namely, on the energy integral). When the generating system comprises a two-body system with a circular orbit and other two-body systems with elliptic orbits, the existence of orbits of the second type has remained an open problem. The interest to this case is due to the fact that the eccentricity of Venus's orbit is 0.007, i.e. the orbit is close to circular. The solution to this problem is given in the paper.

In terms of the classification of oscillation modes in the model containing coupled subsystems [1] the above problem falls under the category of modes with a critical point. According to Theorem 4.1 for reversible mechanical systems the problem of existence of oscillations in this mode has a solution.

Note that the question on the number of periodic orbits of planet systems in the rotating frame of reference was raised in [19, 21, 22].

### 5.2 Two-body problem. Generalized nondegeneracy condition

Periodic orbits of two-body problems play an important role of the generating orbits in the $N$-planet problem. Noting that the orbits in the two-body problem are planar, consider only the planar problem. Write the equations in polar coordinates

$$\ddot{\rho} - \rho\dot{\theta}^2 + \frac{k}{\rho^2} = 0, \quad \frac{d}{dt}(\rho^2\dot{\theta}) = 0.$$

Introduce the notation $c = \rho^2\dot{\theta}$. Then

$$\ddot{\rho} - \frac{c^2}{\rho^3} + \frac{k}{\rho^2} = 0, \quad \dot{c} = 0, \quad \dot{\theta} = \frac{c}{\rho^2}. \tag{18}$$

System (18) admits a solution with the fixed $c = c_*$. In this case the first equation represents a conservative one degree of freedom system. This systems admits a family of oscillations with respect to $\rho$, the period $T(h)$ depending monotonically on the energy integral $h$. The only exception is the critical point $\rho_* = c_*/k$, which corresponds to the circular orbit; here $dT = 0$ [10]. The last condition implies that $\operatorname{rank} A \leq 1$ for the circular orbits

$$\rho = \rho_*, \quad \dot{\rho} = 0, \quad c = c_*, \quad \theta = (c_*/\rho_*^2)t. \tag{19}$$

Let us prove that $\operatorname{rank} A = 1$.

Derive the equations in variations for the circular orbit:

$$\delta\ddot{\rho} + \frac{k}{\rho_*^3}\delta\rho = 0, \quad \delta\dot{c} = 0, \quad \delta\dot{\theta} = \frac{\delta c}{\rho_*^2} - \frac{2c_*}{\rho_*^3}\delta\rho. \tag{20}$$

It is obvious that the first equation can be integrated independently. This equation has periodic solution $\delta\rho = \cos(k/\rho_*^3)t$, the corresponding Jordan cell in matrix $A$ breaks up so that $\operatorname{rank} A \leq 1$.

The other symmetric solution in system (20) is characterized by the initial point $\delta\rho(0) = 0$, $\delta\dot{\rho}(0) = 0$, $\delta c(0) = 1$, $\delta\theta(0) = 0$, so that $\delta\rho(t) \equiv 0$. At the half-period instant $t = T/2$ we have

$$\delta\theta(T) = (\rho_*^2)^{-1}T/2 \neq 0,$$

consequently, the second Jordan cell remains intact and $\operatorname{rank} A = 1$.

Accoding to (19)

$$\frac{\partial \dot{\rho}(T/2)}{\partial t} \equiv 0, \quad \frac{\partial \theta(T/2)}{\partial t} = \frac{c}{\rho_*^2} \neq 0$$

on the circular orbits at $t = T/2$, so that the condition $\operatorname{rank} A_1 = 2$ holds. This means that the generalized nondegeneracy condition is satisfied on circular orbits.

### 5.3 Family of resonant orbits in the $N$-planet problem

Let us write the equation of motion for the problem in the cylindric coordinates [23, p. 365]

$$\ddot{\rho}_s - \rho_s \dot{\theta}_s^2 = \frac{\partial \Omega_s}{\partial \rho_s}, \quad \frac{d}{dt}(\rho_s^2 \dot{\theta}_s) = \frac{\partial \Omega_s}{\partial \theta_s}, \quad \ddot{z}_s = \frac{\partial \Omega_s}{\partial z_s}, \quad s = 1, \ldots, N, \tag{21}$$

where

$$\Omega_s = \frac{f(m_0 + m_s)}{\sqrt{\rho_s^2 + z_s^2}} + \Omega_{s1},$$

$$\Omega_{s1} = f \sum_{j=1 (s \neq j)}^{N} m_j \left[ \frac{1}{\Delta_{sj}} - \frac{\rho_s \rho_j \cos(\theta_s - \theta_j) + z_s z_j}{(\rho_s^2 + z_s^2)^{3/2}} \right],$$

$$\Delta_{sj}^2 = \rho_s^2 + \rho_j^2 - 2\rho_s \rho_j \cos(\theta_s - \theta_j) + (z_s - z_j)^2,$$

$m_0$ is the Sun's mass, $m_s$ are the planets' masses, $m_0 \gg m_s$, $f$ is the gravitational constant.

System (21) is invariant with respect to the change of variables

$$\rho \to \rho, \ \theta \to \pm\theta, \ z \to z(-z), \ t \to -t$$

and belongs to the class of reversible mechanical systems. Consider the planar problem ($z \equiv 0$). At $\Omega_{s1} = 0$ ($s = 1, \ldots, N$) the system splits up into $N$ planar two-body problems, which represent the generating system. Suppose that one planet moves along a circular orbit, while other planets move along elliptic orbits. Then the reversible mechanical system for the $N$-planet problem admits an $N$-family of SPM. The nondegeneracy condition is satisfied for elliptic orbits of $N-1$ two-body problems, while the generalized nondegeneracy condition is satisfied for the circular orbit of the remaining two-body problem. Perturbations $\Omega_{s1}$ depend only on $x_s, \ y_s$, such that (21) remains reversible. Hence, Theorem 4.1 can be applied to establish the extension of the family in the generating system to the $N$-planet problem.

So we can conclude that in the $N$-planet problem there exist resonant orbits close to orbits in two-body problems such that the planets line up in a straight line (parade of planets). The parade of planets is observed on the $N$-family of orbits, where the energy integral $h$ is one of parameters and the period depends only on $h$.

Let us summarize the above reasoning by

**Theorem 5.1** *In the $N$-planet problem there exists an $N$-family of planar symmetric resonant periodic orbits close to orbits of the two-body problem, one of the orbits being circular, and the others being elliptic. The planets in such orbits line up (periodically in time) in a straight line (parade of planets).*

## 6 Conclusion

A great interest to coupled and network systems is being observed at present time. There is a vast variety of the considered models and problem statements. One of the problems to solve is to extend dynamic properties of a separate system to coupled systems. This problem is solved in this paper for the smooth autonomous model containing coupled subsystems. The separate subsystem is supposed to admit a family of periodic solutions with the period depending on a unique parameter. For coupled systems described by ordinary differential equations the problem of extending dynamic properties is solved by finding appropriate couplings. In the case of reversible mechanical systems the dynamic property is completely extended to coupled systems. The obtained results are applied to the $N$-planet problem with one planet in a circular orbit and the other planets in elliptic orbits. The existence of $N$-resonant orbits, on which the parade of planets is observed, is established.

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