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On the Global Asymptotic Stability of a Class of Nonlinear Switched Systems

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Abstract: In this paper, a class of nonlinear switched systems with separable nonlinearities is studied. With the aid of multiple Lyapunov functions method, conditions on switching law are derived under which the zero solutions of the considered systems are globally asymptotically stable. Some examples are presented to illustrate the obtained results.

Keywords: *hybrid systems; switching law; global asymptotic stability; multiple Lyapunov functions.*

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1 Introduction

Switched systems represent a subclass of hybrid systems and have strong engineering background in various applications. A significant number of real systems can be modeled as switched systems such as mechanical systems, chemical processes, vehicle control, traffic control, automotive industry, etc. [3, 11, 18, 23, 24].

A switched system has two components: a family of subsystems and a switching signal. Subsystems in the family are described by a set of indexed equations. The switching signal selects an active subsystem at every instant of time, i.e., the subsystem from the family that is currently being followed [18]. Switching signals are usually classified as time-dependent or state-dependent. Note that qualitative behaviour of a switched system depends not only on the behaviour of individual subsystems in the family, but also on the switching signal [24].

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In the past decades, different methods of analysis for switched systems were developed, and many significant results were obtained (see, for instance, [2–7, 9–11, 18, 24]). In particular, with the aid of the Lyapunov function approach, various conditions of asymptotic stability were derived. Stability is one of the fundamental concepts, and it plays the most important role in control systems design.

For the stability problem, the first question is whether a switched system is stable when there are no restrictions on switching signal (stability analysis under arbitrary switching). On the other hand, many switched systems may fail to preserve stability under arbitrary switching, but may be stable under restricted switching signals. In the second case, it is required to find corresponding restrictions.

Many constructive approaches were developed for the stability analysis of switched systems, for example, the method of differential inequalities (scalar, vector or matrix) [4, 12, 20], the dwell time approach [6, 10, 11], the method of common or multiple Lyapunov functions [6, 7, 9–11, 14, 18, 24], etc. These methods are powerful and effective tool for the finding switching signals providing the required properties.

Stability analysis is complicated if the considered system is essentially nonlinear or/and contains some uncertainties [1, 2, 7, 8]. Along with the asymptotic stability, the problems of ultimate boundedness and finite-time stability are considered in many papers [3–5, 20].

In addition to the solving the problem of stability, it is important to estimate the attraction domain of a given equilibrium position [16]. It should be noted that the size of the region of attraction depends, generally, on switchig law [4]. Of a particular interest is the situation where the equilibrium position is globally asymptotically stable.

In this paper, the problem of global asymptotic stability for a class of nonlinear switched systems with separable nonlinearities is studied. It is assumed that every subsystem from the considered family admits globally asymptotically stable zero solution. We will look for conditions on switching law which guarantee the preservation of global asymptotic stability for the corresponding switched system. We will employ multiple Lyapunov functions in our analysis. As an additional result, estimates of the convergence rate of solutions to the origin will be obtained.

2 Statement of the Problem

Consider the system with separable nonlinearities

$$\dot{\mathbf{x}} = \mathbf{P}_\sigma \mathbf{f}(\mathbf{x}). \tag{1}$$

Here $\mathbf{x} = (x_1, \ldots, x_n)^T$; $\mathbf{f}(\mathbf{x}) = (f_1(x_1), \ldots, f_n(x_n))^T$, scalar functions $f_i(x_i)$ are defined and continuous for $x_i \in (-\infty, +\infty)$ and satisfy the conditions $x_i f_i(x_i) > 0$ for $x_i \neq 0$, $i = 1, \ldots, n$; $\sigma = \sigma(t)$ is a piecewise constant function defining the switching law, $\sigma(t)$: $[0, +\infty) \to Q = \{1, \ldots, N\}$; $\mathbf{P}_s = \{p_{ij}^{(s)}\}_{i,j=1}^n$ are constant matrices, $s = 1, \ldots, N$.

Thus, at each time instant one of the subsystems

$$\dot{\mathbf{x}} = \mathbf{P}_s \mathbf{f}(\mathbf{x}), \quad s = 1, \ldots, N, \tag{2}$$

is active. Subsystems of the form (2) belong to well-known class of the Persidskii type systems [21]. They are widely used for modeling of various practical systems and processes, see [2, 3, 13, 15, 17].

Let θ_i, $i = 1, 2, \ldots$, be the switching times, $0 < \theta_1 < \theta_2 < \ldots$, and $\theta_0 = 0$. Assume that the function $\sigma(t)$ is right-continuous. Without loss of generality, we suppose that the interval $(0, +\infty)$ contains the infinite number of switching instants. Hereinafter, we consider non Zeno sequences [18], i.e., sequences that switch at most a finite number of times in any finite time interval.

From the properties of functions $f_1(x_1), \dots, f_n(x_n)$ it follows that system (1) has the zero solution. We will look for conditions providing global asymptotic stability of the solution.

In [7], the problem of the existence of a common Lyapunov function for family (2) was studied. Several approaches to the construction of such function were proposed. It is known [18, 24] that the existence of a common Lyapunov function guarantees the asymptotic stability of the zero solution of (1) for any switching law.

In the present contribution, we will assume that we failed to construct a common Lyapunov function for subsystems (2). In this case, to prove stability of a switched system, one should restrict the class of admissible switching signals [10, 11, 18, 24]. The general approach for finding such restrictions is based on the use of multiple Lyapunov functions [10, 11].

In what follows we will impose some additional conditions on the right-hand sides of subsystems (2).

Assumption 2.1 For every $s \in \{1, \dots, N\}$, there exist positive constants $\lambda_1^{(s)}, \dots, \lambda_n^{(s)}$ such that the matrix $\mathbf{P}_s^T \Lambda_s + \Lambda_s \mathbf{P}_s$ is negative definite. Here $\Lambda_s = \operatorname{diag}\{\lambda_1^{(s)}, \dots, \lambda_n^{(s)}\}$.

Remark 2.1 Conditions of the existence of required values of $\lambda_1^{(s)}, \dots, \lambda_n^{(s)}$ were investigated in [5, 7, 9, 14].

Remark 2.2 If Assumption 2.1 is fulfilled, then for every $s \in \{1, \dots, N\}$ the zero solution of the s-th subsystem from (2) is asymptotically stable for any admissible functions $f_1(x_1), \dots, f_n(x_n)$, and for this subsystem the function

$$V_s(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i^{(s)} \int_0^{x_i} f_i(\tau)\, d\tau \tag{3}$$

satisfies the requirements of the Lyapunov asymptotic stability theorem. If it is possible to choose values of $\lambda_1^{(s)}, \dots, \lambda_n^{(s)}$ the same for all $s = 1, \dots, N$, then a common Lyapunov function can be constructed for subsystems (2). However, conditions of the existence of such common Lyapunov function are more conservative than those of the existence of a partial Lyapunov function of the form (3) for every subsystem.

Assumption 2.2 Let functions $f_j(x_j)$ be of the form $f_j(x_j) = \beta_j x_j^{\mu_j}$, $j = 1, \dots, n$, where β_j be positive constants, and μ_j be positive rationals with odd numerators and denominators.

Remark 2.3 Without loss of generality, we will assume that $\beta_j = 1$, $j = 1, \dots, n$, and $\mu_1 \le \dots \le \mu_n$.

Thus, under Assumption 2.2, we consider the family of subsystems

$$\dot{x}_i = \sum_{j=1}^{n} p_{ij}^{(s)} x_j^{\mu_j}, \quad i = 1, \dots, n, \quad s = 1, \dots, N, \tag{4}$$

and the corresponding switched system

$$\dot{x}_i = \sum_{j=1}^{n} p_{ij}^{(\sigma)} x_j^{\mu_j}, \quad i = 1, \dots, n. \tag{5}$$

Remark 2.4 System (5) can be treated as a system of the first, in the broad sense [25], approximation for a nonlinear hybrid system.

If Assumption 2.1 is fulfilled, then for subsystems from family (4) there exist Lyapunov functions of the form

$$V_s(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i^{(s)} \frac{x_i^{\mu_i+1}}{\mu_i + 1}, \qquad s = 1, \dots, N, \tag{6}$$

and the zero solutions of these subsystems are globally asymptotically stable.

Our goal is to find classes of switching signals for which we can guarantee the global asymptotic stability of the zero solution of system (5).

Remark 2.5 The case where $\mu_1 = \dots = \mu_n$ was investigated in [4, 6, 11, 18]. Therefore, in the present paper we will assume that $\mu_1 < \mu_n$.

3 Preliminary Results

Let

$$c = \max_{s,j=1,\dots,N} \max_{i=1,\dots,n} (\lambda_i^{(s)} / \lambda_i^{(j)}).$$

Then $c \geq 1$, and

$$V_s(\mathbf{x}) \leq c V_j(\mathbf{x}), \quad s, j = 1, \dots, N, \tag{7}$$

for $\mathbf{x} \in \mathbb{R}^n$.

Remark 3.1 If $c = 1$, then $V_1(\mathbf{x}) \equiv \dots \equiv V_N(\mathbf{x})$, i.e., for subsystems (4) a common Lyapunov function is constructed. In this case the zero solution of (5) is globally asymptotically stable for any admissible switching law. Therefore, in what follows we assume that $c > 1$.

Denote $T_i = \theta_i - \theta_{i-1}$, $i = 1, 2, \dots$. Construct auxiliary sequences. Let $\psi_1(b, m) = \chi_1(m) = \varphi_1(b, m) = 0$,

$$\psi_k(b, m) = \sum_{i=1}^{k-1} T_{m+i} b^{k-i}, \qquad \chi_k(m) = \frac{1}{k} \sum_{i=1}^{k-1} T_{m+i}, \qquad \varphi_k(b, m) = \sum_{i=1}^{k-1} T_{m+i} b^{-i}$$

for $k = 2, 3, \dots$. Here $b = \text{const} > 0$; $m = 1, 2, \dots$.

Consider the conditions

$$\psi_k(b, m) \to +\infty \quad \text{as} \quad k \to \infty, \tag{8}$$

$$\chi_k(m) \to +\infty \quad \text{as} \quad k \to \infty, \tag{9}$$

$$\varphi_k(b, m) \to +\infty \quad \text{as} \quad k \to \infty. \tag{10}$$

It is worth mentioning that condition (8) needs to be checked only for $0 < b < 1$, and condition (10) only for $b > 1$.

Lemma 3.1 *If any of conditions* (8)–(10) *is fulfilled for* $m = 1$, *then it is fulfilled for all* $m = 1, 2, \ldots$.

To prove the lemma, it is sufficient to note that the equalities

$$\psi_{m+k-1}(b, 1) = \psi_k(b, m) + b^k \sum_{j=2}^{m} T_j \, b^{m-j},$$

$$\chi_{m+k-1}(1) = \frac{k}{m+k-1}\chi_k(m) + \frac{1}{m+k-1}\sum_{j=2}^{m} T_j,$$

$$\varphi_{m+k-1}(b, 1) = b^{1-m}\varphi_k(b, m) + \sum_{j=2}^{m} T_j \, b^{1-j}$$

hold for $m = 1, 2, \ldots$ and $k = 2, 3, \ldots$.

Lemma 3.2 *Let* $0 < b < 1$. *If condition* (8) *is fulfilled, then condition* (9) *is also fulfilled. In addition, if condition* (8) *is fulfilled uniformly with respect to* $m = 1, 2, \ldots$, *then condition* (9) *is also fulfilled uniformly with respect to* $m = 1, 2, \ldots$.

Proof. The equality $\psi_{k+1}(b, m) = b(\psi_k(b, m) + T_{m+k})$ holds for $k, m = 1, 2, \ldots$. Hence,

$$T_{m+k} = b^{-1}\psi_{k+1}(b, m) - \psi_k(b, m) = b^{-1}\left(\psi_{k+1}(b, m) - \psi_k(b, m)\right) + (b^{-1} - 1)\psi_k(b, m).$$

We obtain

$$\chi_k(m) = \frac{1}{bk}\sum_{i=1}^{k-1}\left(\psi_{i+1}(b, m) - \psi_i(b, m)\right) + \frac{1-b}{bk}\sum_{i=1}^{k-1}\psi_i(b, m)$$

$$= \frac{\psi_k(b, m)}{bk} + \frac{1-b}{bk}\sum_{i=1}^{k-1}\psi_i(b, m) \geq \frac{1-b}{b}\left(\frac{1}{k}\sum_{i=1}^{k}\psi_i(b, m)\right).$$

Let condition (8) be fulfilled. Then, for any $M > 0$, one can choose $N > 0$ such that $\psi_k(b, m) > M$ for $k \geq N$. Hence, $\chi_k(m) \geq (1-b)M/(2b)$ for $k \geq 2N$, and condition (9) is fulfilled.

If condition (8) is fulfilled uniformly with respect to $m = 1, 2, \ldots$, then the value of N can be chosen independent of m. Therefore, condition (9) is also fulfilled uniformly with respect to $m = 1, 2, \ldots$. This completes the proof.

Assume that the inequalities

$$\dot{V}_s \leq -\beta V_s^{1+\rho}(\mathbf{x}), \quad s = 1, \ldots, N, \tag{11}$$

hold in a domain $G \subset \mathbb{R}^n$. Here $\beta > 0$, $\rho > -1$, and $\dot{V}_s$ is the derivative of the function $V_s(\mathbf{x})$ with respect to the s-th subsystem from (4), $s = 1, \ldots, N$. Denote $b = c^{-\rho}$.

Let a switching law $\sigma(t)$ be given. Construct the multiple Lyapunov function $V_{\sigma(t)}(\mathbf{x})$ corresponding to the switching law. Choose $t_0 \geq 0$ and $\mathbf{x}_0 \in G$, and consider a solution $\mathbf{x}(t)$ of system (5) starting at $t = t_0$ from the point $\mathbf{x}_0$. Find a positive integer m such that $t_0 \in [\theta_{m-1}, \theta_m)$.

Assume that a number $\tilde{t}$ satisfies the conditions $\tilde{t} > t_0$ and $\mathbf{x}(t) \in G$ for $t \in [t_0, \tilde{t}]$. Integrating differential inequalities (11) and taking into account formulae (7), we arrive at the following estimates:

(i) if $\rho > 0$, then

$$\begin{aligned} &V_{\sigma(\theta_{m-1})}^{-\rho}(\mathbf{x}(\tilde{t})) \geq V_{\sigma(\theta_{m-1})}^{-\rho}(\mathbf{x}_0) + \beta\rho\left(\tilde{t} - t_0\right) \quad \text{for } \tilde{t} \in [t_0, \theta_m), \\ &V_{\sigma(\theta_{m+k-1})}^{-\rho}(\mathbf{x}(\tilde{t})) \geq b^k V_{\sigma(\theta_{m-1})}^{-\rho}(\mathbf{x}_0) + \beta\rho\Big(\left(\tilde{t} - \theta_{m+k-1}\right) \\ &+\psi_k(b, m) + b^k\left(\theta_m - t_0\right)\Big) \quad \text{for } \tilde{t} \in [\theta_{m+k-1}, \theta_{m+k}),\ k \geq 1; \end{aligned} \tag{12}$$

(ii) if $\rho = 0$, then

$$\begin{aligned} &V_{\sigma(\theta_{m-1})}(\mathbf{x}(\tilde{t})) \leq V_{\sigma(\theta_{m-1})}(\mathbf{x}_0)\mathrm{e}^{-\beta\left(\tilde{t} - t_0\right)} \quad \text{for } \tilde{t} \in [t_0, \theta_m), \\ &V_{\sigma(\theta_{m+k-1})}(\mathbf{x}(\tilde{t})) \leq V_{\sigma(\theta_{m-1})}(\mathbf{x}_0)\mathrm{e}^{k \ln c - \beta\left(\tilde{t} - t_0\right)} \quad \text{for } \tilde{t} \in [\theta_{m+k-1}, \theta_{m+k}),\ k \geq 1; \end{aligned} \tag{13}$$

(iii) if $-1 < \rho < 0$ and $\mathbf{0} \notin G$, then

$$\begin{aligned} &V_{\sigma(\theta_{m-1})}^{-\rho}(\mathbf{x}(\tilde{t})) \leq V_{\sigma(\theta_{m-1})}^{-\rho}(\mathbf{x}_0) + \beta\rho\left(\tilde{t} - t_0\right) \quad \text{for } \tilde{t} \in [t_0, \theta_m), \\ &V_{\sigma(\theta_{m+k-1})}^{-\rho}(\mathbf{x}(\tilde{t})) \leq b^k\Big(V_{\sigma(\theta_{m-1})}^{-\rho}(\mathbf{x}_0) + \beta\rho\big(b^{-k}\left(\tilde{t} - \theta_{m+k-1}\right) \\ &+\varphi_k(b, m) + \left(\theta_m - t_0\right)\big)\Big) \quad \text{for } \tilde{t} \in [\theta_{m+k-1}, \theta_{m+k}),\ k \geq 1. \end{aligned} \tag{14}$$

4 Conditions of the Global Asymptotic Stability

Let Assumption 2.1 be fulfilled. Consider the partial Lyapunov functions (6) constructed for subsystems (4). It is easy to show that, for any positive numbers $\bar{H}$ and $\hat{H}$, one can find constants $\bar{\beta} > 0$ and $\hat{\beta} > 0$ such that

$$\dot{V}_s \leq -\bar{\beta} V_s^{1+\rho_n}(\mathbf{x}), \quad s = 1, \ldots, N, \tag{15}$$

for $\|\mathbf{x}\| < \bar{H}$, and

$$\dot{V}_s \leq -\hat{\beta} V_s^{1+\rho_1}(\mathbf{x}), \quad s = 1, \ldots, N, \tag{16}$$

for $\|\mathbf{x}\| > \hat{H}$. Here $\rho_n = (\mu_n - 1)/(\mu_n + 1)$, $\rho_1 = (\mu_1 - 1)/(\mu_1 + 1)$, and $\|\cdot\|$ is the Euclidean norm of a vector.

Denote $\bar{b} = c^{-\rho_n}$, $\hat{b} = c^{-\rho_1}$.

Theorem 4.1 *Let $1 \leq \mu_1 < \mu_n$. If*

$$\psi_k(\bar{b}, m) \to +\infty \quad as \quad k \to \infty \tag{17}$$

uniformly with respect to $m = 1, 2, \ldots$, then the zero solution of system (5) *is globally asymptotically stable.*

Proof. Choose a positive number ε, and find $\bar{\beta} > 0$ such that estimates (15) hold in the domain $G_1 = \{\mathbf{x} \in \mathbb{R}^n : \ \|\mathbf{x}\| < \varepsilon\}$.

The inequalities

$$\bar{a}_1\|\mathbf{x}\|^{\mu_n+1} \leq V_s(\mathbf{x}) \leq \bar{a}_2\|\mathbf{x}\|^{\mu_1+1}, \quad s = 1, \ldots, N, \tag{18}$$

are valid for $\mathbf{x} \in G_1$. Here $\bar{a}_1$ and $\bar{a}_2$ are positive constants.

Using estimates (12) with $G = G_1$, $\beta = \bar{\beta}$, $\rho = \rho_n$, $b = \bar{b}$ and taking into account inequalities (18), it is easy to prove (see [6]) that under the assumptions of Theorem 4.1 one can choose $\delta > 0$ such that if $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta$, then for a solution $\mathbf{x}(t)$ of (5) starting at $t = t_0$ from the point $\mathbf{x}_0$ the condition $\|\mathbf{x}(t)\| < \varepsilon$ hold for $t \geq t_0$, and $\|\mathbf{x}(t)\| \to 0$ as $t - t_0 \to +\infty$ uniformly with respect to $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta$. Hence, the zero solution of system (5) is uniformly asymptotically stable.

Let us show that the attraction domain of the zero solution coincides with the space $\mathbb{R}^n$.

Choose an arbitrary number $\varepsilon > 0$, and find the corresponding value of $\delta > 0$ according to the definition of uniform asymptotic stability. Let $\hat{H} \in (0, \delta)$. Then there exists $\hat{\beta} > 0$ such that estimates (16) are fulfilled in the domain $G_2 = \{\mathbf{x} \in \mathbb{R}^n : \ \|\mathbf{x}\| > \hat{H}\}$.

The inequalities

$$\hat{a}_1\|\mathbf{x}\|^{\mu_1+1} \leq V_s(\mathbf{x}) \leq \hat{a}_2\|\mathbf{x}\|^{\mu_n+1}, \quad s = 1, \ldots, N, \tag{19}$$

hold for $\mathbf{x} \in G_2$, where $\hat{a}_1$ and $\hat{a}_2$ are positive constants.

Consider a solution $\mathbf{x}(t)$ of system (5) starting at $t = t_0 \geq 0$ from a point $\mathbf{x}_0 \in G_2$. There exists a positive integer m such that $t_0 \in [\theta_{m-1}, \theta_m)$.

First, assume that $\mu_1 > 1$. Then $\hat{b} > \bar{b}$, and $\psi_k(\hat{b}, m) > \psi_k(\bar{b}, m)$ for all $k, m = 1, 2, \ldots$. Therefore, $\psi_k(\hat{b}, m) \to +\infty$ as $k \to \infty$ uniformly with respect to $m = 1, 2, \ldots$.

Using estimates (12) with $G = G_2$, $\beta = \hat{\beta}$, $\rho = \rho_1$, $b = \hat{b}$ and taking into account inequalities (19), one can find $\hat{T} \geq 0$ such that $\|\mathbf{x}(t_0 + \hat{T})\| < \delta$. Hence, $\|\mathbf{x}(t)\| \to 0$ as $t \to +\infty$.

Next, consider the case where $\mu_1 = 1$. Applying Lemma 3.2, we obtain that $\chi_k(m) \to +\infty$ as $k \to \infty$ uniformly with respect to $m = 1, 2, \ldots$. Note that $t - t_0 = (t - \theta_{m+k-1}) + k\chi_k(m) + (\theta_m - t_0))$ for $t \in [\theta_{m+k-1}, \theta_{m+k})$, $k \geq 1$.

Using estimates (13) with $G = G_2$, $\beta = \hat{\beta}$ and taking into account inequalities (19), it is easy to show the existence of a number $\hat{T} \geq 0$ such that $\|\mathbf{x}(t_0 + \hat{T})\| < \delta$. Hence, $\|\mathbf{x}(t)\| \to 0$ as $t \to +\infty$. This completes the proof.

Remark 4.1 If $1 < \mu_1 < \mu_n$, then the value of $\hat{T}$ in the proof of Theorem 4.1 is independent of t_0 and $\mathbf{x}_0$. Therefore, under the assumptions of Theorem 4.1, for any given neighborhood of the origin, one can find an estimate of the transient time of all solutions into the neighborhood, and this estimate will be independent of initial conditions of solutions. In the case where $1 = \mu_1 < \mu_n$, the value of $\hat{T}$ is independent of t_0, but it depends on $\mathbf{x}_0$.

Corollary 4.1 *Let* $1 \leq \mu_1 < \mu_n$. *If* $T_i \to +\infty$ *as* $i \to \infty$, *then the zero solution of system* (5) *is globally asymptotically stable.*

Remark 4.2 In the case where $1 \leq \mu_1 < \mu_n$ and condition (17) is fulfilled nonuniformly with respect to $m = 1, 2, \ldots$, we can guarantee only local and nonuniform asymptotic stability of the zero solution of system (5).

Theorem 4.2 *Let $0 < \mu_1 < 1 < \mu_n$. If condition* (17) *is fulfilled uniformly with respect to $m = 1, 2, \ldots$, and*

$$\varphi_k(\hat{b}, 1) \to +\infty \quad as \quad k \to \infty, \tag{20}$$

then the zero solution of system (5) *is globally asymptotically stable.*

Proof. In a similar way as in the proof of Theorem 4.1, we obtain that the zero solution of system (5) is uniformly asymptotically stable.

For an arbitrary chosen $\varepsilon > 0$, find constant $\delta > 0$ according to the definition of uniform asymptotic stability. Let $\hat{H} \in (0, \delta)$. Then there exists a constant $\hat{\beta} > 0$ such that estimates (16) hold in the domain $G = \{\mathbf{x} \in \mathbb{R}^n : \ \|\mathbf{x}\| > \hat{H}\}$.

Consider a solution $\mathbf{x}(t)$ of system (5) starting at $t = t_0 \geq 0$ from a point $\mathbf{x}_0 \in G$. Find positive integer m such that $t_0 \in [\theta_{m-1}, \theta_m)$.

Assume that $\mathbf{x}(t) \in G$ for all $t \geq t_0$. Then, for any $\tilde{t} > t_0$, estimates (14) are valid with the following specialization of parameters: $\beta = \hat{\beta}$, $\rho = \rho_1$, $b = \hat{b}$.

According to Lemma 3.1, condition (20) implies that $\varphi_k(\hat{b}, m) \to +\infty$ as $k \to \infty$ for any $m = 1, 2, \ldots$. Hence, from (14) it follows that if $\tilde{t}$ is sufficiently large, then $V^{-\rho_1}_{\sigma(\theta_{m+k-1})}(\mathbf{x}(\tilde{t})) < 0$. Thus, we arrive at the contradiction.

Therefore, there exists $\hat{T} \geq 0$ such that $\|\mathbf{x}(t_0 + \hat{T})\| < \delta$, and $\|\mathbf{x}(t)\| \to 0$ as $t \to +\infty$. This completes the proof.

Remark 4.3 The value of $\hat{T}$ in the proof of Theorem 4.2 depends on $\mathbf{x}_0$, and if $\varphi_k(\hat{b}, m) \to +\infty$ as $k \to \infty$ nonuniformly with respect to $m = 1, 2, \ldots$, then it depends on t_0 as well. Thus, the proof of Theorem 4.2 permits us to obtain an estimate of transient time of all solutions into a given neighborhood of the origin. However, this estimate depends on initial conditions of solutions.

Remark 4.4 If $0 < \mu_1 < 1 < \mu_n$, then $0 < \bar{b} < 1$ and $\hat{b} > 1$. In this case the fulfillment of condition (17), generally, does not guarantee the fulfillment of condition (20). Really, let $T_j = \hat{b}^{j/2}$, $j = 1, 2, \ldots$. Then, for any $0 < \bar{b} < 1$, condition (17) is fulfilled uniformly with respect to $m = 1, 2, \ldots$, whereas condition (20) is not fulfilled. Thus, condition (20) of Theorem 4.2 is not excessive one, and it can not be dropped.

Remark 4.5 In the case where $\mu_1 = \ldots = \mu_n = 1$, one can find a constant $L > 0$ such that if $T_i \geq L$, $i = 1, 2, \ldots$, then the zero solution of the corresponding switched system is globally asymptotically stable [11, 18]. Theorems 4.1 and 4.2 do not permit us to obtain a similar result for $\mu_n > 1$. For instance, if $T_i = L = \text{const} > 0$, $i = 1, 2, \ldots$, then the conditions of Theorems 4.1 and 4.2 are not fulfilled for any value of L.

Theorem 4.3 *Let $0 < \mu_1 < \mu_n = 1$. If condition* (20) *is fulfilled, and condition* (9) *is fulfilled uniformly with respect to $m = 1, 2, \ldots$, then the zero solution of system* (5) *is globally asymptotically stable.*

The proof of Theorem 4.3 is similar to those of Theorems 4.1 and 4.2.

Theorem 4.4 *Let $0 < \mu_1 < \mu_n < 1$. Then the zero solution of system* (5) *is asymptotically stable for any admissible switching law. Furthermore, if condition* (20) *is fulfilled, and there exist a constant $\varphi^* > 0$ and a positive integer $\bar{k} > 0$ such that $\varphi_k(\bar{b}, m) \geq \varphi^*$ for $k \geq \bar{k}$, $m = 1, 2, \ldots$, then the zero solution of system* (5) *is globally asymptotically stable.*

Proof. Let an admissible switching law and a positive number ε be given. Find $\bar{\beta} > 0$ such that inequalities (15) hold in the domain $G = \{\mathbf{x} \in \mathbb{R}^n : \ 0 < \|\mathbf{x}\| < \varepsilon\}$.

Using estimates (14) with the following specialization of parameters: $\beta = \bar{\beta}$, $\rho = \rho_n$, $b = \bar{b}$, it is easy to prove that, for any $t_0 \geq 0$, one can choose numbers $\delta > 0$ and $\tilde{T} > 0$ such that if $0 < \|\mathbf{x}_0\| < \delta$, then $\|\mathbf{x}(t)\| = 0$ for $t \geq t_0 + \tilde{T}$. Here $\mathbf{x}(t)$ is a solution of (5) starting at $t = t_0$ from the point $\mathbf{x}_0$. Hence, the zero solution of system (5) is asymptotically stable.

Next, assume that condition (20) is fulfilled, and there exist a constant $\varphi^* > 0$ and a positive integer $\bar{k} > 0$ such that $\varphi_k(\bar{b}, m) \geq \varphi^*$ for $k \geq \bar{k}$, $m = 1, 2, \ldots$. In this case, δ and $\tilde{T}$ can be chosen independent of t_0. Thus, the zero solution of (5) is uniformly asymptotically stable. The subsequent proof is similar to those of Theorems 4.1–4.3.

Corollary 4.2 *Let* $0 < \mu_1 < \mu_n < 1$. *If* $\varphi_k(\hat{b}, m) \to +\infty$ *as* $k \to \infty$ *uniformly with respect to* $m = 1, 2, \ldots$, *then the zero solution of system* (5) *is globally asymptotically stable.*

To prove the corollary, it is sufficient to note that if $0 < \mu_1 < \mu_n < 1$, then $\varphi_k(\bar{b}, m) \geq \varphi_k(\hat{b}, m)$ for $k, m = 1, 2, \ldots$.

Remark 4.6 Theorem 4.4 does not guarantee the existence of a constant $L > 0$ such that if $T_i \geq L$, $i = 1, 2, \ldots$, then the zero solution of system (5) is globally asymptotically stable. However, for an arbitrary given bounded subset of $\mathbb{R}^n$, an appropriate choice of L permits us to guarantee that the subset is contained in the attraction domain of the zero solution.

Corollary 4.3 *Let* $0 < \mu_1 < \mu_n < 1$. *For any* $\Delta > 0$, *one can find a constant* $L > 0$ *such that if* $T_i \geq L$, $i = 1, 2, \ldots$, *then the set* $\{\mathbf{x}_0 \in \mathbb{R}^n : \ \|\mathbf{x}_0\| < \Delta\}$ *is contained in the attraction domain of the zero solution of system* (5) *for all* $t_0 \geq 0$.

Example 4.1 Consider the switched indirect control system

$$\begin{aligned} \dot{y}_1 &= a_1^{(\sigma)} y_1 + b_1^{(\sigma)} \eta^3, \\ \dot{y}_2 &= a_2^{(\sigma)} y_2 + b_2^{(\sigma)} \eta^3, \\ \dot{\eta} &= d_1^{(\sigma)} y_1 + d_2^{(\sigma)} y_2 + b_3^{(\sigma)} \eta^3 \end{aligned} \tag{21}$$

and the corresponding family of subsystems

$$\begin{aligned} \dot{y}_1 &= a_1^{(s)} y_1 + b_1^{(s)} \eta^3, \\ \dot{y}_2 &= a_2^{(\sigma)} y_2 + b_2^{(s)} \eta^3, \qquad\qquad s = 1, 2. \\ \dot{\eta} &= d_1^{(s)} y_1 + d_2^{(s)} y_2 + b_3^{(s)} \eta^3, \end{aligned} \tag{22}$$

Thus, $\sigma(t) : [0, +\infty) \to Q = \{1, 2\}$. Let $a_1^{(1)} = -7$, $a_2^{(1)} = -3$, $b_1^{(1)} = 1$, $b_2^{(1)} = 2$, $b_3^{(1)} = -4$, $d_1^{(1)} = 4$, $d_2^{(1)} = 5$, $a_2^{(2)} = -6$, $a_2^{(2)} = -3$, $b_1^{(2)} = 6$, $b_2^{(2)} = 1$, $b_3^{(2)} = -5$, $d_1^{(2)} = 2$, $d_2^{(2)} = 7$.

System (21) is a special case of system (1). Here $n = 3$, $N = 2$, $x_1 = y_1$, $x_2 = y_2$, $x_3 = \eta$, $f_1(x_1) = x_1$, $f_2(x_2) = x_2$, $f_3(x_3) = x_3^3$, $\mu_1 = \mu_2 = 1$, $\mu_3 = 3$,

$$\mathbf{P}_1 = \begin{pmatrix} -7 & 0 & 1 \\ 0 & -3 & 2 \\ 4 & 5 & -4 \end{pmatrix}, \qquad \mathbf{P}_2 = \begin{pmatrix} -6 & 0 & 6 \\ 0 & -3 & 1 \\ 2 & 7 & -5 \end{pmatrix}.$$

Let

$$\Lambda_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then the matrices $\mathbf{P}_s^T\Lambda_s + \Lambda_s\mathbf{P}_s$, $s = 1, 2$, are negative definite. Hence, partial Lyapunov functions for subsystems (22) can be chosen in the form

$$V_1 = \frac{3y_1^2}{2} + y_2^2 + \frac{\eta^4}{4}, \quad V_2 = \frac{y_1^2}{2} + 3y_2^2 + \frac{\eta^4}{2}. \tag{23}$$

At the same time, there is no a positive definite diagonal matrix $\Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3\}$ for which matrices

$$\mathbf{P}_s^T\Lambda + \Lambda\mathbf{P}_s, \qquad s = 1, 2, \tag{24}$$

are negative definite.

Really, without loss of generality, we may assume that $\lambda_3 = 1$. Then for the negative definiteness of matrices (24), it is necessary and sufficient the fulfilment of the conditions

$$\frac{48}{\lambda_1} + 3\lambda_1 + \frac{175}{\lambda_2} + 28\lambda_2 < 172, \quad \frac{2}{\lambda_1} + 18\lambda_1 + \frac{49}{\lambda_2} + \lambda_2 < 34.$$

Adding corresponding sides of these inequalities, we arrive at

$$\frac{50}{\lambda_1} + 21\lambda_1 + \frac{224}{\lambda_2} + 29\lambda_2 < 206.$$

However,

$$\frac{50}{\lambda_1} + 21\lambda_1 \geq 10\sqrt{42}, \quad \frac{224}{\lambda_2} + 29\lambda_2 \geq 8\sqrt{406}$$

for all $\lambda_1 > 0$, $\lambda_2 > 0$.

Thus, we can not construct a common Lyapunov function for family (22) in the form

$$V = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 \frac{\eta^4}{2}.$$

For Lyapunov functions (23), the estimates $V_i \leq 3V_j$, $i, j = 1, 2$, holds for $y_1, y_2, \eta \in (-\infty, +\infty)$. Hence, in this case, $c = 3$, $\bar{b} = 1/\sqrt{3}$. Applying Theorem 4.1, we obtain that if

$$\sum_{i=1}^{k-1} 3^{(i-k)/2} T_{m+i} \to +\infty \quad \text{as} \quad k \to \infty$$

uniformly with respect to $m = 1, 2, \ldots$, then the zero solution of system (21) is globally asymptotically stable.

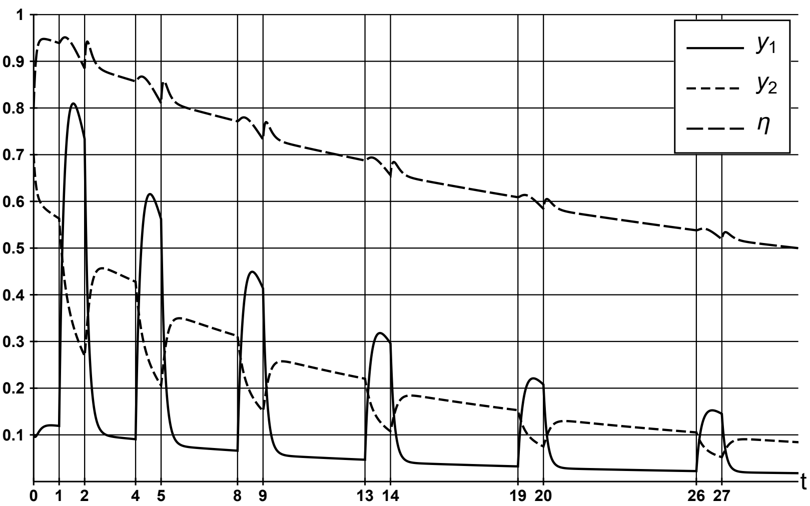

Figure 1: The state response of system (21).

The results of a computer simulation are presented in Figure 1. It is assumed that $T_i = i$ for $i = 1, 3, 5, \ldots$, and $T_i = 1$ for $i = 2, 4, 6, \ldots$. In this case,

$$\sum_{i=1}^{k-1} 3^{(i-k)/2} T_{m+i} > (T_{m+k-2} + T_{m+k-1})/3 \geq (k-1)/3 \to +\infty \quad \text{as} \quad k \to \infty$$

uniformly with respect to $m = 1, 2, \ldots$.

We consider the solution of (21) starting at $t = 0$ from the point $(y_1, y_2, \eta)^T = (0.1, 0.7, 0.8)^T$. In Fig. 1, the dependence of components of the solution on time is presented.

Finally in this section, consider the case where Assumption 2.2 is replaced by the following one.

Assumption 4.1 Functions $f_j(x_j)$ in system (1) can be represented in the form $f_j(x_j) = \beta_j x_j^{\mu_j} + h_j(x_j)$, where β_j are positive constants, μ_j are positive rationals with odd numerators and denominators, functions $h_j(x_j)$ are continuous for $x_j \in (-\infty, +\infty)$ and satisfy the condition $h_j(x_j)/x_j^{\mu_j} \to 0$ as $x_j \to 0$, $j = 1, \ldots, n$.

Remark 4.7 As well as for Assumption 2.2, we will suppose that $\beta_j = 1$, $j = 1, \ldots, n$, and $\mu_1 \leq \ldots \leq \mu_n$.

Theorem 4.5 *Let Assumptions 1.1 and 4.1 be fulfilled. Then under the conditions of any of Theorems 4.1–4.4 the zero solution of system* (1) *is asymptotically stable.*

Remark 4.8 Theorem 4.5 guarantees only local asymptotic stability. However, if the estimates $|h_j(x_j)| \leq \eta_j |x_j|^{\mu_j}$ hold for $x_j \in (-\infty, +\infty)$, where η_j are positive constants, $j = 1, \ldots, n$, then, for sufficiently small values of η_j, the fulfilment of conditions of any of Theorems 4.1–4.4 provides global asymptotic stability of the zero solution of system (1).

5 An Optimization of the Choice of Lyapunov Functions

Conditions of the global asymptotic stability obtained in the previous section depend on the value of constant c in inequalities (7). The smaller the value of c, the less conservative are restrictions on switching law determined in Theorems 4.1–4.4. Therefore, the problem of finding Lyapunov functions for which value of c is smallest is actual.

Let Lyapunov functions $V_1(\mathbf{x})$, ..., $V_N(\mathbf{x})$ of the form (6) be constructed for subsystems (4). Then the estimates

$$V_s(\mathbf{x}) \le c_{sj} V_j(\mathbf{x}), \quad s, j = 1, \ldots, N,$$

hold for $\mathbf{x} \in \mathbb{R}^n$, where $c_{sj} = \max\limits_{i=1,\ldots,n} (\lambda_i^{(s)}/\lambda_i^{(j)})$. Hence, the value of constant c in inequalities (7) is defined by the formula $c = \max\limits_{s,j=1,\ldots,N} c_{sj}$.

It should be noted that, for arbitrary positive constants $b_1, \ldots, b_N$, functions $\widetilde{V}_s(\mathbf{x}) = b_s V_s(\mathbf{x})$, $s = 1, \ldots, N$, are also Lyapunov functions for the considered subsystems. For these functions estimates (7) take the form

$$\widetilde{V}_s(\mathbf{x}) \le \tilde{c}\, \widetilde{V}_j(\mathbf{x}), \quad s, j = 1, \ldots, N,$$

where $\tilde{c} = \max\limits_{s,j=1,\ldots,N} (c_{sj} b_s)/b_j$. As a result, we arrive at the optimization problem: it is required to choose positive constants $b_1, \ldots, b_N$ for which value of $\tilde{c}$ is minimal. This problem can be reduced to the following nonlinear programming problem [19]:

$$\text{Minimize}: \ \tilde{c},$$

$$\text{subject to}: \ \frac{c_{sj} b_s}{b_j} \le \tilde{c}, \quad s, j = 1, \ldots, N. \tag{25}$$

Conditions of the existence of positive constants $b_1, \ldots, b_N$ satisfying inequalities of the form (25) were investigated in [22]. According to the results of this paper, system (25) admits a positive solution if and only if, for any set of indices $i_1, \ldots, i_k$ ($i_m \in \{1, \ldots, N\}$, $i_m \ne i_l$ for $m \ne l$; $m, l = 1, \ldots, k$, $1 \le k \le N$), the condition $c_{i_1 i_2} c_{i_2 i_3} \ldots c_{i_k i_1} \le \tilde{c}^k$ is fulfilled. Hence, $\min \tilde{c} = \max(c_{i_1 i_2} c_{i_2 i_3} \ldots c_{i_k i_1})^{1/k}$, where the maximum is calculated on all pointed out sets of indices $i_1, \ldots, i_k$.

It is worth mentioning that in [22] a constructive procedure for finding required constants $b_1, \ldots, b_N$ was proposed.

Example 5.1 Let family (4) consist of three subsystems of the second order. Hence, $N = 3$ and $n = 2$. Assume that the following Lyapunov functions

$$V_1(\mathbf{x}) = \frac{x_1^{\mu_1+1}}{\mu_1+1} + \frac{x_2^{\mu_2+1}}{\mu_2+1}, \ \ V_2(\mathbf{x}) = \frac{x_1^{\mu_1+1}}{\mu_1+1} + 2\frac{x_2^{\mu_2+1}}{\mu_2+1}, \ \ V_3(\mathbf{x}) = \frac{x_1^{\mu_1+1}}{\mu_1+1} + 3\frac{x_2^{\mu_2+1}}{\mu_2+1} \tag{26}$$

are constructed for these subsystems.

The estimates

$$V_1(\mathbf{x}) \le V_2(\mathbf{x}), \quad V_1(\mathbf{x}) \le V_3(\mathbf{x}),$$

$$V_2(\mathbf{x}) \le 2V_1(\mathbf{x}), \quad V_2(\mathbf{x}) \le V_3(\mathbf{x}),$$

$$V_3(\mathbf{x}) \le 3V_1(\mathbf{x}), \quad V_3(\mathbf{x}) \le \frac{3}{2}V_2(\mathbf{x})$$

are valid for $\mathbf{x} \in \mathbb{R}^2$. Therefore, $c = 3$.

Applying the proposed approach, we obtain

$$\min \tilde{c} = \max \left\{ \sqrt{2};\ \sqrt{3};\ \sqrt{\frac{3}{2}};\ \sqrt[3]{3} \right\} = \sqrt{3}.$$

In this case inequalities (25) take the form

$$\frac{b_1}{b_2} \le \sqrt{3}, \quad \frac{b_1}{b_3} \le \sqrt{3}, \quad \frac{2b_2}{b_1} \le \sqrt{3}, \quad \frac{b_2}{b_3} \le \sqrt{3}, \quad \frac{3b_3}{b_1} \le \sqrt{3}, \quad \frac{(3/2)b_3}{b_2} \le \sqrt{3}.$$

Choose, for instance, $b_1 = \sqrt{3}$, $b_2 = b_3 = 1$. As a result, we find the Lyapunov functions

$$\widetilde{V}_1(\mathbf{x}) = \sqrt{3}V_1(\mathbf{x}), \quad \widetilde{V}_2(\mathbf{x}) = V_2(\mathbf{x}), \quad \widetilde{V}_3(\mathbf{x}) = V_3(\mathbf{x}).$$

With the aid of these functions, one can derive less conservative stability conditions than those which can be obtained with the use of functions (26).

6 Conclusion

In this paper, the problem of global asymptotic stability for a class of nonlinear switched systems with separable nonlinearities was investigated. Sufficient conditions on the switching law which garantee the required property for the given equilibrium position are obtained.

It is worth mentioning that the approaches proposed in the paper can be used as well for the analysis of hybrid models of population dynamics and neutral networks. This will be our future work.

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