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Existence Result for Nonlinear Degenerated Parabolic Systems

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Abstract: An existence result of a solution for a class of nonlinear parabolic systems is established. The source term is less regular (bounded Radon measure) and no coercivity is made in the non-divergentiel lower order term $div(c(x,t)|u(x,t)|^{\gamma-2}u(x,t))$. The main contribution of our work is to prove the existence of a renormalized solution without the coercivity condition on the nonlinearities, so we used the Gagliardo-Nirenberg theorem to prove it.

Keywords: *Dirichlet problem; parabolic systems; Gagliardo-Nirenberg inequality; renormalized solutions.*

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1 Introduction

Given a bounded-connected open set Ω of $\mathbb{R}^N$ $(N \geq 2)$, with Lipschitz boundary $\partial\Omega$, $Q_T = \Omega \times (0,T)$ is the generic cylinder of an arbitrary finite hight, $T < \infty$. We prove the existence of a renormalized solution for the nonlinear parabolic systems

$$\begin{cases} \frac{\partial b_i(x,u_i)}{\partial t} - \operatorname{div}(a(x,t,u_i,\nabla u_i) - \phi_i(x,t,u_i) - F_i) = f_i(x,u_1,u_2) \quad in \quad Q_T, \\ u_i(x,t) = 0 \quad on \quad \partial\Omega \times (0,T), \\ b_i(x,u_i(x,0)) = b_i(x,u_{0,i}(x)) \quad in \quad \Omega, \end{cases} \tag{1}$$

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where $i = 1, 2$. Here the vector field $a : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function such that $A(u_i) = -div\Big(a(x,t,u_i,\nabla u_i)\Big)$ is a Leray-Lions operator defined on $L^p(0,T;W_0^{1,p}(\nu))$, $\phi_i(x,t,u_i)$ is a Carathéodory function (see assumptions (13)–(14)), and $b_i : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that for every $x \in \Omega$, $b_i(x,.)$ is a strictly increasing C^1-function, the data $u_{0,i}$ is in $L^1(\Omega)$ such that $b_i(., u_{0,i})$ in $L^1(\Omega)$. The data $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (see assumptions H4) and $F_i \in (L^{p'}(\nu))^N$. When problem (1) is investigated, there is a difficulty due to the fact that the data $b_1(x, u_0^1(x))$ and $b_2(x, u_0^2(x))$ only belong to L^1 and the functions $\Big(a(x,t,u_i,\nabla u_i)\Big)$, $\phi_i(x,t,u_i)$ and $f_i(x,u_1,u_2)$ do not belong to $(L^1_{loc}(Q_T))^N$ in general, so that proving existence of weak solution seems to be an arduous task, and we cannot use the Stocks formula in the a priori estimates of the nonlinearity $\phi_i(x,t,u_i)$. In order to overcome this difficulty, we work with the framework of renormalized solutions (see Definition 3.1). The notion of renormalized solutions was introduced by R.-J. DiPerna and P.-L. Lions [8] for the study of the Boltzmann equation. It was adapted to the study of some nonlinear elliptic or parabolic problems in fluid mechanics, see [6]. In the case where $b(x,u) = u$, the existence of renormalized solutions for (1) has been established by R.-Di Nardo [5]. In the case where $\phi(x;t;u) = 0$ and $f \in L^1(Q_T)$, the existence of renormalized solutions has been established by H. Redwane [13] in the classical Sobolev space, the existence results are already proved by the authors in the case where $f_i(x,u_1,u_2)$ is replaced by $f - div(g)$, where $f \in L^1(Q_T)$and $g \in (L^{p'}(Q_T))^N$. For the elliptic version of (1) we refer to [10].

One of the models of applications of these operators is the system of Boussinesq:

$$\frac{\partial u}{\partial t} + (u.\nabla))u - 2div(\mu(\theta)\varepsilon(u)) + \nabla p = F(\theta) \quad \text{in} \quad Q_T,$$

$$\frac{\partial b(\theta)}{\partial t} + u.\nabla b(\theta) - \triangle\theta = 2\mu(\theta)|\varepsilon(u)|^2 \quad \text{in} \quad Q_T,$$

$$u(t=0) = u_0,\ b(\theta)(t=0) = b(\theta_0) \quad \text{on} \quad \Omega,$$

$$u = 0 \quad \theta = 0 \quad \text{on} \quad \partial\Omega \times (0,T).$$

The first equation is the motion conservation equation, the unknowns are the fields of displacement $u : Q_T \to \mathbb{R}^N$ and temperature $\theta : Q_T \to \mathbb{R}$. The field $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)$ is the strain rate tensor.

It is our purpose, in this paper to generalize the result of [2, 5, 13] and we prove the existence of a renormalized solution of system (1).

The plan of the paper is as follows: In Section 2 we give basic assumptions. In Section 3 we give the definition of a renormalized solution of (1), and we establish (Theorem 3.1) the existence of such a solution.

2 Preliminaries and Auxiliary Results

We recall here some standard notations, properties and results which will be used throughout the paper.

Let Ω be a bounded open set of $\mathbb{R}^N$ and $Q_T = \Omega \times (0,T)$, T is a positive real number. Let $\nu(x)$ be a nonnegative function on Ω such that $\nu(x) \in L^r(\Omega)$, $r \geq 1$, $\nu(x)^{-1} \in L^t(\Omega)$,

$p \geq 1 + 1/t$. We denote by $L^p(\Omega, \nu)$, or simply $L^p(\nu)$ if there is no confusion, $p \geq 1$, the space of measurable functions u on Ω such that

$$\|u\|_{L^p(\nu)} = \Big(\int_\Omega |u|^p \nu(x) dx \Big)^{\frac{1}{p}} < +\infty, \tag{2}$$

and by $W^{1,p}(\nu)$ the completion of the space $C^1(\overline{\Omega})$ with respect to the norm

$$\|u\|_{W^{1,p}(\nu)} = \|u\|_{L^p(\nu)} + \|\nabla u\|_{L^p(\nu)}. \tag{3}$$

Moreover, we denote by $W_0^{1,p}(\nu)$ the closure of $C_0^1(\overline{\Omega})$ in $W^{1,p}(\nu)$, provided with the induced topology defined by the induced norm, and by $W^{-1,p'}(\nu^{1-p'})$, $p' = \frac{p}{p-1}$, its dual space. $W^{1,p}(\nu)$ and $W_0^{1,p}(\nu)$ are reflexive Banach spaces if $1 < p < \infty$, (see [11]).

Denote $V = W_0^{1,p}(\nu)$, $H = L^2(\nu)$ and $V^* = W_0^{-1,p'}(\nu^{1-p'})$, with $p \geq 2$. The dual space of $X := L^p(0, T; W_0^{1,p}(\nu))$ denoted X^* is identified with $L^{p'}(0, T; V^*)$. Define $W_p^1(0, T, V, H) = \{v \in X : v' \in X^*\}$. Endowed with the norm

$$\|u\|_{W_p^1} = \|u\|_X + \|u'\|_{X^*},$$

$W_p^1(0, T, V, H)$ is a Banach space. Here u' stands for the generalized time derivative of u, that is,

$$\int_0^T u'(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt \text{ for all } \varphi \in C_0^\infty(0, T).$$

Lemma 2.1 *[14]*

1. *The evolution triple $V \hookrightarrow H \hookrightarrow V^*$ is verified.*
2. *The imbedding $W_p^1(0, T, V, H) \hookrightarrow C(0, T, H)$ is continuous.*
3. *The imbedding $W_p^1(0, T, V, H) \hookrightarrow L^p(Q_T, \nu)$ is compact.*

Lemma 2.2 *[1] Let $\{v_n\}$ be a bounded sequence in $L^p(0, T; V)$ such that*

$$\frac{\partial v_n}{\partial t} = \alpha_n + \beta_n \text{ in } \mathcal{D}'(Q_T)$$

with $\{\alpha_n\}$ and $\{\beta_n\}$ being two bounded sequences respectively in X^* and in $L^1(Q_T)$. Then $v_n \to v$ in $L^p_{loc}(Q_T, \nu)$. Furthermore, $v_n \to v$ strongly in $L^1(Q_T)$.

From now on, we assume that the following assumptions hold true

$$\nu(x)^{-1} \in L^t(\Omega),\ t \geq \frac{N}{p},\ 1 + \frac{1}{t} < p < N(1 + \frac{1}{t}), \tag{4}$$

$$\nu(x) \in L^r(\Omega),\ r > \frac{Nt}{pt - N}. \tag{5}$$

An important tool that we will use here, is the following weighted version of the Sobolev inequality (see Theorem 3.1 and Corollary 3.5 in [11]).

Proposition 2.1 *[11] Assume that (4) and (5) hold true. Let $\tilde{p}$ denote the number associated with p defined by*

$$\frac{1}{\tilde{p}} = r'\Big(\frac{1}{p}(1+\frac{1}{t}) - \frac{1}{N}\Big).$$

Then the imbedding of $W_0^{1,p}(\nu)$ into $L^{\tilde{p}}(\nu)$ is continuous. moreover, there exists a constant $C_0 > 0$ depending on N, p, ν, t, such that

$$\|u\|_{L^{\tilde{p}}(\nu)} \leq C_0 \||\nabla u|\|_{L^p(\nu)}, \forall u \in W_0^{1,p}(\nu). \tag{6}$$

Using this proposition, we can prove the following interpolation result.

Proposition 2.2 *Assume that (4) and (5) hold true. Let v be a function in $W_0^{1,p}(\nu) \cap L^s(\Omega)$ with $2 \leq p < N$ and $s > r'$. Then there exists a positive constant C, depending on N, p, ν, t and q, such that*

$$\|v\|_{L^\sigma(\nu)} \leq C \|\nabla v\|^{1-\theta}_{L^p(\nu)} \|v\|^{\theta}_{L^s(\Omega)}$$

for every θ and σ satisfying

$$0 \leq \theta \leq 1,\ 1 \leq \sigma \leq +\infty,\ \frac{1}{\sigma} = \theta + r'(1-\theta)\Big((1+\frac{1}{t})\frac{1}{p} - \frac{1}{N}\Big),\ r > \frac{Nt}{pt-N}.$$

Proof. For every $1 \leq \sigma \leq \tilde{p}$, we can write $\dfrac{1}{\sigma} = \theta + \dfrac{1-\theta}{\tilde{p}}$ for some $0 \leq \theta \leq 1$. then by the Hölder inequality and (6), one has

$$\|v\|_{L^\sigma(\nu)} \quad \leq C_0 \||\nabla v|\|^{1-\theta}_{L^p(\nu)} \|v\|^{\theta}_{L^1(\nu)} \leq C_0 \||\nabla v|\|^{1-\theta}_{L^p(\nu)} \|\nu\|^{\theta}_{L^{s'}(\Omega)} \|v\|^{\theta}_{L^s(\Omega)},$$

which gives the desired result.

As an immediate consequence of the previous result, we get

Corollary 2.1 *Let $v \in L^p((0,T), W_0^{1,p}(\nu)) \cap L^\infty((0,T), L^s(\Omega))$, with $2 \leq p < N$ and $s > r'$. Then $v \in L^\sigma(\nu)$ with $\sigma = \frac{p\tilde{p}+\tilde{p}-p}{\tilde{p}}$. Moreover,*

$$\int_{Q_T} \nu(x)|v|^\sigma dxdt \leq C \parallel v \parallel^{\frac{\tilde{p}-p}{\tilde{p}}}_{L^\infty(0,T,L^s(\Omega))} \int_{Q_T} \nu(x)|\nabla v|^p dxdt.$$

Proof. By virtue of Proposition 2.2, we can write

$$\int_\Omega \nu(x)|v|^\sigma dx \leq C \||\nabla v|\|^{(1-\theta)\sigma}_{L^p(\nu)} \parallel v \parallel^{\theta\sigma}_{L^s(\Omega)}.$$

Integrating between 0 and T, we get

$$\int_0^T \int_\Omega \nu(x)|v|^\sigma dxdt \leq C \int_0^T \||\nabla v|\|^{(1-\theta)\sigma}_{L^p(\nu)} \|v\|^{\theta\sigma}_{L^s(\Omega)} dt. \tag{7}$$

Since $v \in L^p((0,T), W_0^{1,p}(\nu)) \cap L^\infty((0,T), L^s(\Omega))$, we have

$$\int_0^T \int_\Omega \nu(x)|v|^\sigma dxdt \leq C \|v\|^{\theta\sigma}_{L^\infty(0,T,L^s(\Omega))} \int_0^T \||\nabla v(t)|\|^{(1-\theta)\sigma}_{L^p(\nu)} dt.$$

Now we choose θ such that $(1-\theta)\sigma = p$ and $\theta\sigma = \frac{\tilde{p}-p}{\tilde{p}}$. This choice yields

$$\theta = \frac{\tilde{p}-p}{p\tilde{p}+\tilde{p}-p}, \quad \sigma = \frac{p\tilde{p}+\tilde{p}-p}{\tilde{p}}.$$

Then, (7) becomes

$$\int_0^T \int_\Omega \nu(x)|v|^\sigma dxdt \leq C \parallel v \parallel^{\frac{\tilde{p}-p}{\tilde{p}}}_{L^\infty(0,T,L^s(\Omega))} \int_0^T \||\nabla v(t)|\|^p_{L^p(\nu)} dt.$$

3 Assumptions on Data

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, T be a positive real number, and $Q_T = \Omega \times (0,T)$.

3.1 Assumptions

Throughout this paper, we assume that the following assumptions hold true:

Assumptions (H1)

$$b_i : \Omega \times \mathbb{R} \to \mathbb{R} \quad \text{is a Carathéodory function such that for every } x \in \Omega, \tag{8}$$

$b_i(x,.)$ is a strictly increasing $\mathcal{C}^1(\mathbb{R})$-function with $b_i(x,0) = 0$, for any $k > 0$, there exist a constant $\lambda_i > 0$ and functions $A^i_k \in L^\infty(\Omega)$ and $B^i_k \in L^p(\Omega)$ such that: for almost every x in Ω

$$\lambda_i \leq \frac{\partial b_i(x,s)}{\partial s} \leq A^i_k(x) \quad \text{and} \quad \Big|\nabla_x\Big(\frac{\partial b_i(x,s)}{\partial s}\Big)\Big| \leq B^i_k(x) \quad \forall\ |s| \leq k. \tag{9}$$

Assumptions (H2) Let $a:\ Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that, for any $k > 0$, there exist ν_k and a function $h_k \in L^{p'}(\nu)$ with

$$|a(x,t,s,\xi)| \leq \nu(x)\Big(h_k(x,t) + |\xi|^{p-1}\Big) \quad \forall\ |s| \leq k, \tag{10}$$

$$a(x,t,s,\xi)\xi \geq \alpha\nu(x)|\xi|^p \ \text{ with } \alpha > 0, \tag{11}$$

$$(a(x,t,s,\xi) - a(x,t,s,\eta))(\xi - \eta) > 0 \ \text{ with } \xi \neq \eta. \tag{12}$$

Assumptions (H3) Let $\phi_i :\ Q_T \times \mathbb{R} \to \mathbb{R}^N$ be a Carathéodory function such that

$$|\phi_i(x,t,s)| \leq c_i(x,t)|s|^\gamma \nu(x), \tag{13}$$

$$c_i(x,t) \in L^\tau(\nu) \quad \text{with} \quad \tau = \frac{p(3\tilde{p}-p)}{(p-1)(\tilde{p}-p)}, \ \gamma = \frac{2(p-1)(p\tilde{p}+\tilde{p}-p)}{p(3\tilde{p}-p)} \tag{14}$$

for almost every $(x,t) \in Q_T$, for every $s \in \mathbb{R}$ and every $\xi,\ \eta \in \mathbb{R}^N$.

Assumptions (H4) We suppose for that for i=1,2 $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f_1(x,0,s) = f_2(x,s,0) = 0$ a.e $x \in \Omega$, $\forall s \in \mathbb{R}$. And for almost every $x \in \Omega$, for every $s_1, s_2 \in \mathbb{R}$

$$signe(s_i) f_i(x,s_1,s_2) \geq 0. \tag{15}$$

The growth assumptions on f_i are as follows: for each $k > 0$ there exist $\sigma_k > 0$ and a function F_k in $L^1(\Omega)$ such that

$$|f_1(x, s_1, s_2)| \leq F_k + \sigma_k |b_2(x, s_2)|. \quad \text{a.e} \quad x \in \Omega, \forall |s_1| \leq k, \quad \forall s_2 \in \mathbb{R}. \tag{16}$$

for each $k > 0$ there exist $\mu_k > 0$ and a function G_k in $L^1(\Omega)$ such that

$$|f_2(x, s_1, s_2)| \leq G_k(x) + \mu_k |b_1(x, s_1)|. \quad \text{a.e} \quad x \in \Omega, \forall |s_2| \leq k, \quad \forall s_1 \in \mathbb{R}. \tag{17}$$

$u_{0,i}$ is a measurable function such that $b_i(x, u_{0,i}) \in L^1(\Omega)$ for $i = 1, 2$.

4 Main Results

In this section, we study the existence of renormalized solutions to systems (1).

Definition 4.1 A couple of measurable functions (u_1, u_2) defined on Q_T is called a *renormalized* solution of (1) if for i=1,2. the function u_i satisfies

$$b_i(x, u_i) \in L^\infty(0, T; L^1(\Omega)), \tag{18}$$

$$T_k(u_i) \in L^p(0, T; W_0^{1,p}(\nu)) \text{ for any } k > 0, \tag{19}$$

$$\lim_{m \to +\infty} \frac{1}{m} \int_{\{(x,t) \in Q_T:\ |u_i(x,t)| \leq m\}} a(x, t, u_i, \nabla u_i) \nabla u_i \, dx \, dt = 0, \tag{20}$$

and if for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support

$$\begin{aligned} &\frac{\partial B_{i,S}(x, u_i)}{\partial t} - div\Big(a(x, t, u_i, \nabla u_i) S'(u_i)\Big) + S''(u_i) a(x, t, u_i, \nabla u_i) \nabla u_i \\ &\qquad + div\Big(\phi_i(x, t, u_i) S'(u_i)\Big) - S''(u_i) \phi_i(x, t, u_i) \nabla u_i \\ &\qquad = f_i(x, u_1, u_2) S'(u_i) - div(S'(u_i) F_i) + S''(u_i) F_i \nabla u_i \ in\ D'(Q_T), \end{aligned} \tag{21}$$

and

$$B_{i,S}(x, u_i)(t = 0) = B_{i,S}(x, u_{i,0}) \quad in \quad \Omega, \tag{22}$$

where $B_{i,S}(x, z) = \displaystyle\int_0^z \frac{\partial b_i(x, s)}{\partial s} S'(s) ds.$

Equation (21) is formally obtained through pointwise multiplication of equation (1) by $S'(u)$. However meanwhile $a(x, t, u_i, \nabla u_i)$ and $\phi_i(x, t, u_i)$ do not in general make sense in (1). Recall that for a renormalized solution, due to (19), each term in (21) has a meaning in $L^1(Q_T) + L^{p'}(0, T; W^{-1,p'}(\nu^{1-p'}))$ (see e.g. [6]).We have

$$\frac{\partial B_{i,S}(x, u_i)}{\partial t} \text{ belongs to } L^{p'}(0, T; W^{-1,p'}(\nu^{1-p'})) + L^1(Q_T). \tag{23}$$

$$B_{i,S}(x, u_i) \text{ belongs to } L^p(0, T; W_0^{1,p}(\nu)). \tag{24}$$

Then (23) and (24) imply that $B_{i,S}(x, u_i)$ belongs to $C^0([0, T]; L^1(\Omega))$ (for the proof of this trace result see [12],) so that the initial condition (22) makes sense.

Theorem 4.1 *Let $b(x,u_0) \in L^1(\Omega)$, assume that (H1)-(H4) hold true, then there exists at least a renormalized solution (u_1,u_2) of problem* (1) *in the sense of Definition* (4.1).

Proof. **Step 1.** Let us introduce the following regularization of the data: for i=1,2. For each $n > 0$

$$b_{i,n}(x,r) = b(x,T_n(r)) + \frac{r}{n} \quad \forall\ r \in \mathbb{R}, \tag{25}$$

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\ s \in \mathbb{R},\ \forall\ \xi \in \mathbb{R}^N, \tag{26}$$

$$\phi_{i,n}(x,t,r) = \phi_i(x,t,T_n(r)) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\ r \in \mathbb{R}. \tag{27}$$

Let $f_{1,n}(x,s_1,s_2) = f_1(x,T_n(s_1),T_n(s_2))$ a.e $x \in \Omega$, $\forall s_1,s_2 \in \mathbb{R}$.

$$\text{and} \quad f_{2,n}(x,s_1,s_2) = f_2(x,T_n(s_1),T_n(s_2)) \quad \text{a.e } x \in \Omega, \forall s_1,s_2 \in \mathbb{R}. \tag{28}$$

Let $u_{i,0n} \in \mathcal{C}_0^\infty(\Omega)$ such that

$$b_{i,n}(x,u_{i,0n}) \to b_i(x,u_{i,0}) \text{ strongly in } L^1(\Omega). \tag{29}$$

In view of (25), for i=1,2 $b_{i,n}$ is a Carathéodory function and satisfies (9), there exists $\lambda_i > 0$ such that:

$$\lambda_i + \frac{1}{n} \le \frac{\partial b_{i,n}(x,s)}{\partial s} \text{and} \quad |b_{i,n}(x,s)| \le max_{|s|\le n}|b_i(x,s)| \quad a.e.\ x \in \Omega,\ \forall s \in \mathbb{R}.$$

Let us now consider the regularized problem

$$\begin{cases} \frac{\partial b_{i,n}(x,u_{i,n})}{\partial t} - div(a_n(x,t,u_{i,n},\nabla u_{i,n}) - \phi_{i,n}(x,t,u_{i,n}) - F_i) = f_{i,n}(x,u_1,u_2) \ \ in \ \ Q_T, \\ u_{i,n}(x,t) = 0 \quad on \quad \partial\Omega \times (0,T), \\ b_{i,n}(x,u_{i,n})(t=0) = b_{i,n}(x,u_{i,0n}) \quad in \quad \Omega. \end{cases} \tag{30}$$

In view of (16)-(17), there exist $F_{1,n} \in L^1(\Omega)$ and $F_{2,n} \in L^1(\Omega)$ and $\sigma_n > 0$,$\mu_n > 0$ such that :

$$|f_{1,n}(x,s_1,s_2)| \le F_{1,n}(x) + \sigma_n \max_{|s|\le n} |b_i(x,s)|. \quad \text{a.e} \quad x \in \Omega, \quad \forall s_1,s_2 \in \mathbb{R}.$$

$$|f_{2,n}(x,s_1,s_2)| \le F_{2,n}(x) + \mu_n \max_{|s|\le n} |b_i(x,s)|. \quad \text{a.e} \quad x \in \Omega, \quad \forall s_1,s_2 \in \mathbb{R}.$$

As a consequence, proving the existence of weak solution $u_{i,n} \in L^p(0,T;W_0^{1,p}(\nu))$ of (30) is an easy task (see e.g. [**?**, 9]).

Step 2: The estimates derived in this step rely on standard techniques for problems of type (30). So we just sketch the proof of them (the reader is referred to [4]) for the elliptic version. Let $\tau_1 \in (0,T)$ and t be fixed in $(0,\tau_1)$. For i=1,2, using $T_k(u_{i,n})\chi_{(0,t)}$ as a test function in (30), we integrate between $(0,\tau_1)$, and by the condition (13) we have

$$\int_\Omega B_{i,k}^n(x,u_{i,n}(t))dx + \int_{Q_t} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_k(u_{i,n})\,dx\,ds \tag{31}$$

$$\le \int_{Q_t} c(x,t)|u_{i,n}|^\gamma \nu(x)|\nabla T_k(u_{i,n})|\,dx\,ds + \int_{Q_t} f_{i,n}(x,u_1^n,u_2^n)T_k(u_{i,n})\,dx\,ds$$

$$+\int_\Omega B_k^{i,n}(x,u_{i,0}^n)dx+\int_{Q_t} F_i\nabla T_k(u_i^n)dxds,$$

where $B_{i,k}^n(x,r)=\int_0^r T_k(s)\frac{\partial b_{i,n}(x,s)}{\partial s}ds$. Due to definition of $B_{i,k}^n$ we have:

$$0\le \int_\Omega B_{i,k}^n(x,u_{i,0n})dx\le k\int_\Omega |b_{i,n}(x,u_{i,0n})|dx=k||b_i(x,u_{i,0n})||_{L^1(\Omega)}\quad \forall k>0. \tag{32}$$

Using (31), (11) and (28) we obtain:

$$\int_\Omega B_{i,k}^n(x,u_{i,n}(t))dx+\alpha\int_{Q_t}\nu(x)|\nabla T_k(u_{i,n})|^p\,dx\,ds\le\int_{Q_t} c(x,t)|u_{i,n}|^\gamma\nu(x)|\nabla T_k(u_{i,\epsilon})|\,ds\,dx$$

$$+k(\|b_i(x,u_{i,0n})\|_{L^1(\Omega)}+\|f_{i,n}\|_{L^1(Q_T)})+\int_{Q_t}F_i\nabla T_k(u_{i,n})dxds. \tag{33}$$

Let $M_i=\Big(sup_n||f_{i,n}||_{L^1(Q_T)}+||b_i(x,u_{i,0n})||_{L^1(\Omega)}\Big)$. Noting that

$$B_{i,k}^n(x,s)=\int_0^s T_k(\sigma)\frac{\partial b_{i,n}(x,\sigma)}{\partial\sigma}d\sigma\ge\frac{\lambda_i+\frac1n}{2}|T_k(s)|^2>\frac{\lambda_i}{2}|T_k(s)|^2$$

we deduce from (31) and (32) that

$$\frac{\lambda_i}{2}\int_\Omega |T_k(u_{i,n})|^2\,dx+\alpha\int_{Q_t}\nu(x)|\nabla T_k(u_{i,n})|^p\,dx\,ds \tag{34}$$

$$\le M_ik+\int_{Q_t}c_i(x,t)|u_{i,n}|^\gamma\nu(x)|\nabla T_k(u_{i,n})|\,dx\,ds+\int_{Q_t}F_i\nabla T_k(u_{i,n})\,dx\,ds.$$

By Gagliardo-Nirenberg and Young inequalities we have:

$$\int_{Q_t}c_i(x,t)|u_{i,n}|^\gamma\nu(x)|\nabla T_k(u_{i,n})|\,dx\,ds$$

$$\le C_i\frac{\gamma(\tilde p-p)}{2(p\tilde p+\tilde p-p)}||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\sup_{t\in(0,\tau_1)}\int_\Omega|T_k(u_{i,n})|^2\,dx$$

$$+C_i\frac{2p\tilde p+(2-\gamma)(\tilde p-p)}{2(p\tilde p+\tilde p-p)}||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)}$$

$$\Big(\int_{Q_{\tau_1}}\nu(x)|\nabla T_k(u_{i,n})|^p\,dx\,ds\Big)^{(\frac1p+\frac{\gamma\tilde p}{p\tilde p+\tilde p-p})\frac{2(p\tilde p+\tilde p-p)}{2p\tilde p+(2-\gamma)(\tilde p-p)}}. \tag{35}$$

Since $\gamma=\frac{2(p-1)(p\tilde p+\tilde p-p)}{p(3\tilde p-p)}$ and by using (34) and (35), we obtain

$$\frac{\lambda_i}{2}\int_\Omega|T_k(u_{i,n})|^2\,dx+\alpha\int_{Q_t}\nu(x)|\nabla T_k(u_{i,n})|^p\,dx\,ds\le M_ik+$$

$$C_i\frac{\gamma(\tilde p-p)}{2(p\tilde p+\tilde p-p)}||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\sup_{t\in(0,\tau_1)}\int_\Omega|T_k(u_{i,n})|^2\,dx+\frac{\alpha^{\frac{-p'}{p}}}{p'}\|F_i\|_{(L^{p'}(\nu))^N}$$

$$+C_i\frac{2p\tilde{p}+(2-\gamma)(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)}||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\int_{Q_{\tau_1}}\nu(x)|\nabla T_k(u_{i,n})|^p\,dx\,ds$$

$$+\frac{\alpha}{p}\int_{Q_t}\nu(x)|\nabla T_k(u_{i,n})|^p\,dx\,ds$$

which is equivalent to

$$\Big(\frac{\lambda_i}{2}-C_i\frac{\gamma(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)}||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\Big)\sup_{t\in(0,\tau_1)}\int_\Omega|T_k(u_{i,n})|^2\,dx$$

$$+\alpha\int_{Q_{\tau_1}}\nu(x)|\nabla T_k(u_{i,n})|^p\,dx\,ds$$

$$-\Big(C_i\frac{2p\tilde{p}+(2-\gamma)(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)}||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)}+\frac{\alpha}{p}\Big)\int_{Q_{\tau_1}}\nu(x)|\nabla T_k(u_{i,n})|^p\,dx\,ds\le M_ik.$$

If we choose τ_1 such that

$$\Big(\frac{\lambda_i}{2}-C_i\frac{\gamma(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)}||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\Big)>0, \tag{36}$$

$$\Big(\frac{\alpha}{p'}-C_i\frac{2p\tilde{p}+(2-\gamma)(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)}||c_i(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\Big)>0, \tag{37}$$

and then denote by C_i the minimum between the constants $\Big(\frac{\lambda_i(p\tilde{p}+\tilde{p}-p)}{\gamma(\tilde{p}-p)||c_i(x,t)||_{L^\tau(Q_{\tau_1})}}\Big)$ and $\Big(\frac{2\alpha(p\tilde{p}+\tilde{p}-p)}{p'[2p\tilde{p}+(2-\gamma)(\tilde{p}-p)]||c_i(x,t)||_{L^\tau(Q_{\tau_1})}}\Big)$, we obtain

$$\sup_{t\in(0,\tau_1)}\int_\Omega|T_k(u_{i,n})|^2\,dx+\int_{Q_{\tau_1}}\nu(x)|\nabla T_k(u_{i,n})|^p\,dx\,dt\le C_iM_ik. \tag{38}$$

Then, by (38) and Lemma 3.1([**?**, 2]), we conclude that $T_k(u_{i,n})$ is bounded in $L^p(0,T,W_0^{1,p}(\nu))$ independently of n and for any $k\ge 0$, so there exists a subsequence still denoted by $u_{i,n}$ such that

$$T_k(u_{i,n})\rightharpoonup H_{i,k}\quad\text{weakly in}\quad L^p(0,T,W_0^{1,p}(\nu)). \tag{39}$$

Lemma 4.1 *(see [2])*

$$u_{i,n}\to u_i\ \ a.e.\ Q_T,\ \ b_i(x,u_i)\in L^\infty(0,T;L^1(\Omega)), \tag{40}$$

where u_i is a measurable function defined on Q_T for i=1,2.

$$\lim_{m\to+\infty}\limsup_{n\to+\infty}\frac{1}{m}\int_{\{|u_{i,n}|\le m\}}a(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt=0. \tag{41}$$

Step 4: In this step we prove that the weak limit $X_{i,k}$ of $a(x,t,T_k(u_{i,n})\nabla T_k(u_{i,n}))$ can be identified with $a(x,t,T_k(u_i),\nabla T_k(u_i))$, for i=1,2. In order to prove this result we recall the following lemma.

Lemma 4.2 *For i=1,2, the subsequence of $u_{i,n}$ satisfies for any $k \geq 0$:*

$$\limsup_{n\to+\infty} \int_{Q_T} \int_0^t a(x,s,u_{i,n},\nabla T_k(u_{i,n}))\nabla T_k(u_{i,n})\,ds\,dx\,dt \leq \int_{Q_T} \int_0^t X_{i,k} \nabla T_k(u_i)\,dx\,ds\,dt, \tag{42}$$

$$\lim_{n\to+\infty} \int_{Q_T} \int_0^t \Big(a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n})) - a(x,t,T_k(u_{i,n}),\nabla T_k(u_i)) \Big) \Big(\nabla T_k(u_{i,n}) - \nabla T_k(u_i) \Big) = 0, \tag{43}$$

$$X_{i,k} = a(x,t,T_k(u_i),\nabla T_k(u_i)) \quad a.e. \ in \ Q_T, \tag{44}$$

and as n tends to $+\infty$

$$a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\nabla T_k(u_{i,n}) \rightharpoonup a(x,t,T_k(u_i),\nabla T_k(u_i))\nabla T_k(u_i) \tag{45}$$

weakly in $L^1(Q_T)$.

For i=1,2. We introduce a time regularization of the $T_k(u_i)$ for $k > 0$ in order to perform the monotonicity method.

Lemma 4.3 *(see H. Redwane [13]) Let $k \geq 0$ be fixed. Let S be an increasing $C^\infty(\mathbb{R})-$function such that $S(r) = r$ for $|r| \leq k$, and $suppS'$ is compact. Then*

$$\liminf_{\mu\to+\infty} \lim_{n\to+\infty} \int_0^T \int_0^t < \frac{\partial b_{i,n}(x,u_{i,n})}{\partial t}, S'(u_{i,n})(T_k(u_{i,n}) - (T_k(u_i))_\mu) > \geq 0,$$

where $< .,. >$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'}(\nu^{1-p'})$ and $L^\infty(\Omega) \cap W_0^{1,p}(\nu)$.

Let S_m be a sequence of increasing C^∞-function such that:

$$S_m(r) = r \text{ for } |r| \leq m, \ supp(S'_m) \subset [-2m,2m] \text{ and } \|S''_m\|_{L^\infty(\mathbb{R})} \leq \frac{3}{m} \text{ for any } m \geq 1.$$

For i=1,2. We use the sequence $(T_k(u_i))_\mu$ of approximation of $T_k(u_i)$, and plug the test function $S'_m(u_{i,n})(T_k(u_{i,n}) - (T_k(u_i))_\mu)$ for $m > 0$ and $\mu > 0$. For fixed $k \geq 0$, let $W^n_\mu = T_k(u_{i,n}) - (T_k(u_i))_\mu$. We obtain upon integration over $(0,t)$ and then over $(0,T)$:

$$\begin{aligned} &\int_0^T \int_0^t < \frac{\partial b_{i,n}(x,u_{i,n})}{\partial t}, S'_m(u_{i,n})W^n_\mu > \,ds\,dt \\ &+ \int_{Q_T} \int_0^t a_n(x,s,u_{i,n},\nabla u_{i,n}) S'_m(u_{i,n}) \nabla W^n_\mu \,ds\,dt\,dx \\ &+ \int_{Q_T} \int_0^t a_n(x,s,u_{i,n},\nabla u_{i,n}) S''_m(u_{i,n}) \nabla u_{i,n} \nabla W^n_\mu \,ds\,dt\,dx \\ &- \int_{Q_T} \int_0^t \phi_{i,n}(x,s,u_{i,n}) S'_m(u_{i,n}) \nabla W^n_\mu \,ds\,dt\,dx \\ &- \int_{Q_T} \int_0^t S''_m(u_{i,n}) \phi_{i,n}(x,s,u_{i,n}) \nabla u_{i,n} \nabla W^n_\mu \,ds\,dt\,dx = \int_{Q_T} \int_0^t f_{i,n} S'_m(u_{i,n}) W^n_\mu \,dx\,ds\,dt \end{aligned} \tag{46}$$

$$+\int_{Q_T}\int_0^t F_i S_m'(u_{i,n})\nabla W_\mu^n\,ds\,dt\,dx+\int_{Q_T}\int_0^t F_i S_m''(u_{i,n})\nabla u_{i,n}\nabla W_\mu^n\,ds\,dt\,dx.$$

We pass to the limit in (46) as $n\to+\infty,\ \mu\to+\infty$ and then $m\to+\infty$ for k being a fixed real number. We use Lemma (4.3) and proceed as in ([4, 13]), then it possible to conclude that

$$\liminf_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t <\frac{\partial b_{i,n}(x,u_{i,n})}{\partial t},W_\mu^n>\,ds\,dt\geq 0\qquad\text{for any } m\geq k, \tag{47}$$

$$\lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{n\to+\infty}\int_{Q_T}\int_0^t a_n(x,t,u_{i,n},\nabla u_{i,n})S_m''(u_{i,n})\nabla u_{i,n}\nabla W_\mu^n\,ds\,dt\,dx=0, \tag{48}$$

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_{Q_T}\int_0^t f_{i,n}S_m'(u_{i,n})W_\mu^n\quad ds\,dt\,dx=0, \tag{49}$$

$$\lim_{\mu\to+\infty}\int_{Q_T}\int_0^t F_iS_m'(u_{i,n})\nabla W_\mu^n\quad ds\,dt\,dx=0, \tag{50}$$

$$\lim_{\mu\to+\infty}\int_{Q_T}\int_0^t F_iS_m''(u_{i,n})\nabla u_{i,n}W_\mu^n\quad ds\,dt\,dx=0. \tag{51}$$

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_{Q_T}\int_0^t \phi_{i,n}(x,t,u_{i,n})S_m'(u_{i,n})\nabla W_\mu^n\,ds\,dt\,dx=0, \tag{52}$$

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_{Q_T}\int_0^t S_m''(u_n)\phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}\nabla W_\mu^n\,ds\,dt\,dx=0. \tag{53}$$

For the proof of(52) and (53) the reader is referred to ([2]),(44) and (45) hold true. Note that, taking the limit as n tends to $+\infty$ in (41) and using (45) show that u satisfies (20). Now we want to prove that u satisfies the equation (21).

Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that $suppS'\subset[-k,k]$ where k is a real positive number. Pointwise multiplication of the approximate equation (30) by $S'(u_n)$ leads to

$$\begin{aligned}\frac{\partial B_{i,S}^n(x,u_{i,n})}{\partial t}&-div\Big(a_n(x,t,u_{i,n},\nabla u_{i,n})S'(u_{i,n})\Big)+S''(u_{i,n})a(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\\&+div\Big(\phi_{i,n}(x,t,u_{i,n})S'(u_{i,n})\Big)-S''(u_{i,n})\phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}=f_{i,n}S'(u_{i,n})\\&-div(F_iS'(u_{i,n}))+S''(u_{i,n})F_i\nabla u_{i,n}\quad\text{in } D'(Q_T),\end{aligned} \tag{54}$$

where $B_{i,S}^n(x,r)=\displaystyle\int_0^r\frac{\partial b_{i,n}(x,s)}{\partial s}S'(s)\,ds$. In what follows we pass to the limit as n tends to $+\infty$ in each term of (54). Since the fact that $u_{i,n}$ converges to u_i a.e. in Q_T implies that $B_{i,S}^n(x,u_{i,n})$ converges to $B_{i,S}(x,u_i)$ a.e. in Q_T and $L^\infty(Q_T)$ is weak-$*$, we have that $\dfrac{\partial B_{i,S}^n(x,u_{i,n})}{\partial t}$ converges to $\dfrac{\partial B_{i,S}(x,u_i)}{\partial t}$ in $D'(Q_T)$. We observe that the term $a_n(x,t,u_{i,n},\nabla u_{i,n})S'(u_{i,n})$ can be identified with $a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))S'(u_{i,n})$ for $n\geq k$, so using the pointwise convergence of $u_{i,n}$ to u_i in Q_T and the weak convergence of $T_k(u_{i,n})$to $T_k(u_i)$in $L^p(0,T;W_0^{1,p}(\nu))$, we get
$a_n(x,t,u_{i,n},\nabla u_{i,n})S'(u_{i,n})\rightharpoonup a(x,t,T_k(u_{i,n}),\nabla T_k(u_i))S'(u_i)$ in $L^{p'}(\nu^{1-p'})$, and $S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\rightharpoonup S''(u_i)a(x,t,T_k(u_{i,n}),\nabla T_k(u_i))\nabla T_k(u_i)$

in $L^1(Q_T)$. Furthermore, $\phi_{i,n}(x,t,u_{i,n})S'(u_{i,n}) = \phi_{i,n}(x,t,T_k(u_{i,n}))S'(u_{i,n})$ a.e. in Q_T. By (27) we obtain $|\phi_{i,n}(x,t,T_k(u_{i,n}))S'(u_{i,n})| \leq \nu(x)|c_i(x,t)|k^\gamma$, it follows that $\phi_{i,n}(x,t,T_k(u_{i,n}))S'(u_{i,n}) \to \phi_{i,n}(x,t,T_k(u_i))S'(u_i)$ strongly in $L^{p'}(\nu^{1-p'})$.

In a similar way

$$S''(u_{i,n})\phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} = S''(T_k(u_{i,n}))\phi_{i,n}(x,t,T_k(u_{i,n}))\nabla T_k(u_{i,n}) \quad \text{a.e. in } Q_T.$$

Using the weak convergence of $T_k(u_{i,n})$ in $L^p(0,T;W_0^{1,p}(\nu))$ it is possible to prove that $S''(u_{i,n})\phi_n(x,t,u_{i,n})\nabla u_{i,n} \to S''(u_i)\phi_i(x,t,u_i)\nabla u_i$ in $L^1(Q_T)$, and $S''(u_{i,n})F_i\nabla u_{i,n}$ converges to $S''(u_i)F_i\nabla u_i$ in $L^1(Q_T)$. Since $|S'(u_{i,n})| \leq C$, it follows that $F_iS''(u_{i,n})$ converges to $F_iS''(u_i)$ strongly in $L^{p'}(\nu)$. Finally by (28) we deduce that $f_nS'(u_{i,n})$ converges to $f_iS'(u_i)$ in $L^1(Q_T)$. It remains to prove that $B_{i,S}(x,u_i)$ satisfies the initial condition $B_{i,S}(x,u_i)(t=0) = B_{i,S}(x,u_{i,0})$ in Ω. To this end, firstly note that $B^n_{i,S}(x,u_{i,n})$ is bounded in $L^p(0,T;W_0^{1,p}(\nu))$. Secondly, the above considerations of the behavior of the terms of this equation show that $\frac{\partial B^n_{i,S}(x,u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T) + L^{p'}(0,T;W^{-1,p'}(\nu^{1-p'}))$. As a consequence, $B^n_{i,S}(u_{i,n})(t=0) = B^n_{i,S}(x,u_{i,0n})$ converges to $B_{i,S}(x,u_i)(t=0)$ strongly in $L^1(\Omega)$ (for the proof of this trace result see [12]). On the other hand, the smoothness of S implies that $B_{i,S}(x,u_i)(t=0) = B_{i,S}(x,u_{i,0})$ in Ω. The proof of Theorem 3.1 is complete.

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