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Controllability of Neutral Functional Differential Equations Driven by Fractional Brownian Motion with Infinite Delay

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Abstract: This paper is concerned with the controllability of neutral stochastic functional differential equations with infinite delay driven by fractional Brownian motion in a real separable Hilbert space. The controllability results are obtained using stochastic analysis and a fixed-point strategy.

Keywords: *Controllability; neutral functional differential equations; fractional powers of closed operators; infinite delay; fractional Brownian motion.*

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1 Introduction

For the practical applications in the areas such as biology, medicine, physics, finance, electrical engineering, telecommunication networks, and so on, the theory of stochastic evolution equations has attracted research's great interest. For more details, one can see Da Prato and Zabczyk [5], and Ren and Sun [14] and the references therein. In many areas of science, there has been an increasing interest in the investigation of the systems incorporating memory or aftereffect, i.e., there is the effect of delay on state equations. Therefore, there is a real need to discuss stochastic evolution systems with delay. In many mathematical models the claims often display long-range memories, possibly due to extreme weather, natural disasters, in some cases, many stochastic dynamical systems depend not only on present and past states, but also contain the derivatives with delays. Neutral functional differential equations are often used to describe such systems.

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Moreover, control theory is an area of application-oriented mathematics which deals with basic principles underlying the analysis and design of control systems. Roughly speaking, controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Controllability plays a crucial role in a lot of control problems, such as the case of stabilization of unstable systems by feedback or optimal control [8, 9]. Conceived by Kalman, the controllability concept has been studied extensively in the fields of finite-dimensional systems, infinite-dimensional systems, hybrid systems, and behavioral systems. If a system cannot be controlled completely then different types of controllability can be defined such as approximate, null, local null and local approximate null controllability. For more details the reader may refer to [16–19] and the references therein. In this paper, we study the controllability of neutral functional stochastic differential equations of the form

$$\begin{cases} d[x(t)-g(t,x_t)] = [Ax(t)+f(t,x_t)+Bu(t)]dt+\sigma(t)dB^H(t),\ t\in[0,T], \\ x(t)=\varphi(t)\in L^2(\Omega,\mathcal{B}_h),\ for\ a.e.\ t\in(-\infty,0]. \end{cases} \tag{1}$$

Here, A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, in a Hilbert space X; B^H is a fractional Brownian motion with $H>\frac{1}{2}$ on a real and separable Hilbert space Y; and the control function $u(\cdot)$ takes values in $L^2([0,T],U)$, the Hilbert space of admissible control functions for a separable Hilbert space U; and B is a bounded linear operator from U into X.

The history $x_t:(-\infty,0]\to X$, $x_t(\theta)=x(t+\theta)$, belongs to an abstract phase space $\mathcal{B}_h$ defined axiomatically, and $f,g:[0,T]\times\mathcal{B}_h\to X$ are appropriated functions, and $\sigma:[0,T]\to\mathcal{L}_2^0(Y,X)$, are appropriate functions, where $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see section 2 below).

A general theory for the infinite-dimensional stochastic differential equations driven by a fractional Brownian motion (fBm) has begun to receive attention by various researchers, see e.g. [15]. For example, Dung studied the existence and uniqueness of impulsive stochastic Volterra integro-differential equation driven by fBm in [6]. Using the Riemann-Stieltjes integral, Boufoussi et al. [1] proved the existence and uniqueness of a mild solution to a related problem and studied the dependence of the solution on the initial condition in infinite dimensional space. Very recently, Caraballo and Diop [3], Caraballo et al. [4], and Boufoussi and Hajji [2] have discussed the existence, uniqueness and exponential asymptotic behavior of mild solutions by using the Wiener integral.

To the best of the author's knowledge, an investigation concerning the controllability for neutral stochastic differential equations with infinite delay of the form (1) driven by a fractional Brownian motion has not yet been conducted. Thus, we will make the first attempt to study such problem in this paper. Our results are motivated by those in [10, 11] where the controllability of mild solutions to neutral stochastic functional integro-differential equations driven by fractional Brownian motion with finite delays is studied.

The outline of this paper is as follows: In Section 2 we introduce some notations, concepts, and basic results about fractional Brownian motion, the Wiener integral defined in general Hilbert spaces, phase spaces and properties of analytic semigroups and the fractional powers associated to its generator. In Section 3, we derive the controllability of neutral stochastic differential systems driven by a fractional Brownian motion.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A standard fractional Brownian motion (fBm) $\{\beta^H(t), t \in \mathbb{R}\}$ with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process with continuous sample paths such that

$$R_H(t,s) = \mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \qquad s,t \in \mathbb{R}. \tag{2.1}$$

Remark 2.1 In the case $H > \frac{1}{2}$, it follows from [12]that the second partial derivative of the covariance function

$$\frac{\partial R_H}{\partial t \partial s} = \alpha_H |t-s|^{2H-2},$$

where $\alpha_H = H(2H-2)$, is integrable, and we can write

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |u-v|^{2H-2} du dv. \tag{2.2}$$

The following result is fundamental to prove our result, it can be proved by similar arguments as those used to prove Lemma 2 in [4].

Lemma 2.1 *If $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^T \|\psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$, we have*

$$\mathbb{E}\|\int_0^t \psi(s) dB^H(s)\|^2 \le 2Ht^{2H-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

Next, we introduce some notations and basic facts about the theory of semigroups and fractional power operators. Let $A : D(A) \to X$ be the infinitesimal generator of an analytic semigroup, $(S(t))_{t\ge 0}$, of bounded linear operators on X. The theory of strongly continuous is thoroughly discussed in [13] and [7]. It is well-known that there exist $M \ge 1$ and $\lambda \in \mathbb{R}$ such that $\|S(t)\| \le Me^{\lambda t}$ for every $t \ge 0$. If $(S(t))_{t\ge 0}$ is a uniformly bounded, analytic semigroup such that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A, then it is possible to define the fractional power $(-A)^\alpha$ for $0 < \alpha \le 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X, and the expression

$$\|h\|_\alpha = \|(-A)^\alpha h\|$$

defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\|.\|_\alpha$, then the following properties hold (see [13], p. 74).

Lemma 2.2 *Suppose that A, X_α, and $(-A)^\alpha$ are as described above.*

(i) For $0 < \alpha \le 1$, X_α is a Banach space.

(ii) If $0 < \beta \le \alpha$, then the injection $X_\alpha \hookrightarrow X_\beta$ is continuous.

(iii) For every $0 < \alpha \le 1$, there exists $M_\alpha > 0$ such that

$$\|(-A)^\alpha S(t)\| \le M_\alpha t^{-\alpha} e^{-\lambda t}, \quad t > 0, \ \ \lambda > 0.$$

3 Controllability Result

In this section we derive controllability conditions for a class of neutral stochastic functional differential equations with infinite delays driven by a fractional Brownian motion in a real separable Hilbert space. Before starting, we introduce the concepts of a mild solution of the problem (1) and the meaning of controllability of neutral stochastic functional differential equation.

Definition 3.1 *An X-valued process $\{x(t) : t \in (-\infty, T]\}$ is a mild solution of (1) if*

1. *$x(t)$ is continuous on $[0, T]$ almost surely and for each $s \in [0, t)$ the function $AS(t-s)g(s, x_s)$ is integrable,*
2. *for arbitrary $t \in [0, T]$, we have*

$$\begin{aligned} x(t) &= S(t)(\varphi(0) - g(0, \varphi)) + g(t, x_t) \\ &+ \int_0^t AS(t-s)g(s, x_s)ds + \int_0^t S(t-s)f(s, x_s)ds \\ &+ \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)\sigma(s)dB^H(s), \qquad \mathbb{P}-a.s. \end{aligned} \tag{3.1}$$

3. *$x(t) = \varphi(t)$ on $(-\infty, 0]$ satisfying $\|\varphi\|^2_{\mathcal{B}_h} < \infty$.*

Definition 3.2 The neutral stochastic functional differential equation (1) is said to be controllable on the interval $(-\infty, T]$ if for every initial stochastic process φ defined on $(-\infty, 0]$, there exists a stochastic control $u \in L^2([0, T], U)$ such that the mild solution $x(\cdot)$ of (1) satisfies $x(T) = x_1$, where x_1 and T are the preassigned terminal state and time, respectively.

Our main result in this paper is based on the following fixed point theorem.

Theorem 3.1 *(Karasnoselskii's fixed point theorem) Let V be a bounded closed and convex subset of a Banach space X and let Π_1, Π_2 be two operators of V into X satisfying:*

1. *$\Pi_1(x) + \Pi_2(x) \in V$ whenever $x \in V$,*
2. *Π_1 is a contraction mapping, and*
3. *Π_2 is completely continuous.*

Then, there exists a $z \in V$ such that $z = \Pi_1(z) + \Pi_2(z)$.

In order to establish the controllability of (1), we impose the following conditions on the data of the problem:

($\mathcal{H}$.1) The analytic semigroup, $(S(t))_{t\geq 0}$, generated by A is compact for $t > 0$, and there exist constants M, $M_{1-\beta}$ such that

$$\|S(t)\|^2 \leq M \quad \text{and} \quad \|(-A)^{1-\beta}S(t)\| \leq \frac{M_{1-\beta}}{t^{1-\beta}}, \text{ for all } t \in [0, T]$$

(see Lemma 2.2).

($\mathcal{H}$.2) The map $f : [0,T] \times \mathcal{B}_h \to X$ satisfies the following conditions:

(i) The function $t \longmapsto f(t,x)$ is measurable for each $x \in \mathcal{B}_h$, the function $x \longmapsto f(t,x)$ is continuous for almost all $t \in [0,T]$,

(ii) for every positive integer k there exists $p_k \in L^1([0,T], \mathbb{R}^+)$, such that

$$\|f(t,x)\|^2 \le p_k(t), \text{ for all } \|x\|^2_{\mathcal{B}_h} \le k \text{ almost surely and for a.e. } t \in [0,T],$$

and

$$\liminf_{k \longrightarrow +\infty} \frac{1}{k} \int_0^T p_k(\tau) d\tau = \gamma < \infty.$$

($\mathcal{H}$.3) The function $g : [0,T] \times \mathcal{B}_h \longrightarrow X$ is continuous and there exist constants $0 < \beta < 1$, $M_g > 0$ and $\nu > 0$ such that the function g is X_β-valued and satisfies

i) $\|(-A)^\beta g(t,x) - (-A)^\beta g(t,y)\|^2 \le M_g \|x-y\|^2_{\mathcal{B}_h}$, almost surely, for a.e. $t \in [0,T]$, and for all $x, y \in \mathcal{B}_h$ with

$$\nu = 4M_g l^2 (\|(-A)^{-\beta}\|^2 + \frac{(M_{1-\beta} T^\beta)^2}{2\beta - 1}) < 1.$$

ii) $c_1 = \|(-A)^{-\beta}\|$ and $\overline{M}_g = \sup_{t \in [0,T]} \|(-A)^{-\beta} g(t,0)\|^2$.

($\mathcal{H}$.4) The function $\sigma : [0,\infty) \to \mathcal{L}_2^0(Y,X)$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty, \quad \forall T > 0.$$

($\mathcal{H}$.5) The linear operator W from U into X defined by

$$Wu = \int_0^T S(T-s) B u(s) ds$$

has an inverse operator W^{-1} that takes values in $L^2([0,T],U) \setminus kerW$, where

$$kerW = \{x \in L^2([0,T],U) : \ Wx = 0\}$$

(see [8]), and there exists finite positive constants M_b, M_w such that $\|B\|^2 \le M_b$ and $\|W^{-1}\|^2 \le M_w$.

The main result of this chapter is the following.

Theorem 3.2 *Suppose that* ($\mathcal{H}$.1) – ($\mathcal{H}$.5) *hold. Then, the system (1) is controllable on* $(-\infty, T]$ *provided that*

$$6l^2(1 + 6MM_bM_wT^2)\{8(c_1^2 + \frac{(M_{1-\beta}T^\beta)^2}{2\beta - 1})M_g + 8MT\gamma\} < 1. \tag{3.2}$$

Proof. Transform the problem(1) into a fixed-point problem. To do this, using the hypothesis ($\mathcal{H}$.5) for an arbitrary function $x(\cdot)$, define the control by

$$
\begin{aligned}
u(t) \;\; &= \;\; W^{-1}\{x_1 - S(T)(\varphi(0) - g(0,x_0)) - g(T,x_T)) \\
&- \;\; \int_0^T AS(T-s)g(s,x_s)ds - \int_0^T S(T-s)f(s,x_s)ds \\
&- \;\; \int_0^T S(T-s)\sigma(s)dB^H(s)\}(t).
\end{aligned}
$$

To formulate the controllability problem in the form suitable for application of the fixed point theorem, put the control $u(.)$ into the stochastic control system (3.1) and obtain a non linear operator Π on $\mathcal{B}_T$ given by

$$
\Pi(x)(t) = \begin{cases}
\varphi(t), \quad \text{if } \; t \in (-\infty, 0], \\
S(t)(\varphi(0) - g(0,\varphi)) + g(t,x_t) \\
+ \int_0^t AS(t-s)g(s,x_s)ds + \int_0^t S(t-s)f(s,x_s)ds \\
+ \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)\sigma(s)dB^H(s), \quad \text{if } \; t \in [0,T].
\end{cases}
$$

Then it is clear that to prove the existence of mild solutions to equation (1) is equivalent to find a fixed point for the operator Π. Clearly, $\Pi x(T) = x_1$, which means that the control u steers the system from the initial state φ to x_1 in time T, provided we can obtain a fixed point of the operator Π which implies that the system in controllable.

Let $y : (-\infty, T] \longrightarrow X$ be the function defined by

$$
y(t) = \begin{cases} \varphi(t), & \text{if } \; t \in (-\infty, 0], \\ S(t)\varphi(0), & \text{if } \; t \in [0,T], \end{cases}
$$

then, $y_0 = \varphi$. For each function $z \in \mathcal{B}_T$, set

$$
x(t) = z(t) + y(t).
$$

It is obvious that x satisfies the stochastic control system (3.1) if and only if z satisfies $z_0 = 0$ and

$$
\begin{aligned}
z(t) = \;\; & g(t, z_t + y_t) - S(t)g(0,\varphi) + \int_0^t AS(t-s)g(s, z_s + y_s)ds \\
& + \int_0^t S(t-s)f(s, z_s + y_s)ds + \int_0^t S(t-s)Bu_{z+y}(s)ds \\
& + \int_0^t S(t-s)\sigma(s)dB^H(s),
\end{aligned} \tag{3.3}
$$

where

$$
\begin{aligned}
u_{z+y}(t) \;\; &= W^{-1}\{x_1 - S(T)(\varphi(0) - g(0, z_0 + y_0)) - g(T, z_T + y_T)) \\
&- \int_0^T AS(T-s)g(s, z_s + y_s)ds - \int_0^T S(T-s)f(s, z_s + y_s)ds \\
&- \int_0^T S(T-s)\sigma(s)dB^H(s)\}(t).
\end{aligned}
$$

Set

$$
\mathcal{B}_T^0 = \{z \in \mathcal{B}_T : z_0 = 0\};
$$

for any $z \in B_T^0$, we have

$$\|z\|_{\mathcal{B}_T^0} = \|z_0\|_{\mathcal{B}_h} + \sup_{t\in[0,T]} (\mathbb{E}\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t\in[0,T]} (\mathbb{E}\|z(t)\|^2)^{\frac{1}{2}}.$$

Then, $(\mathcal{B}_T^0, \|.\|_{\mathcal{B}_T^0})$ is a Banach space. Define the operator $\widehat{\Pi} : \mathcal{B}_T^0 \longrightarrow \mathcal{B}_T^0$ by

$$(\widehat{\Pi}z)(t) = \begin{cases} 0, \text{ if } t \in (-\infty, 0], \\ g(t, z_t + y_t) - S(t)g(0,\varphi)) + \int_0^t AS(t-s)g(s, z_s + y_s)ds \\ + \int_0^t S(t-s)f(s, z_s + y_s)ds + \int_0^t S(t-s)Bu_{z+y}(s)ds \\ + \int_0^t S(t-s)\sigma(s)dB^H(s), \qquad \text{if } t \in [0,T]. \end{cases} \tag{3.4}$$

Set

$$\mathcal{B}_k = \{z \in \mathcal{B}_T^0 : \|z\|^2_{\mathcal{B}_T^0} \le k\}, \qquad \text{for some } k \ge 0,$$

then $\mathcal{B}_k \subseteq \mathcal{B}_T^0$ is a bounded closed convex set, and for $z \in \mathcal{B}_k$, we have

$$\begin{aligned} \|z_t + y_t\|^2_{\mathcal{B}_h} &\le 2(\|z_t\|^2_{\mathcal{B}_h} + \|y_t\|^2_{\mathcal{B}_h}) \\ &\le 4(l^2 \sup_{0\le s\le t} \mathbb{E}\|z(s)\|^2 + \|z_0\|^2_{\mathcal{B}_h} \\ &+ l^2 \sup_{0\le s\le t} \mathbb{E}\|y(s)\|^2 + \|y_0\|^2_{\mathcal{B}_h}) \\ &\le 4l^2(k + M\mathbb{E}\|\varphi(0)\|^2) + 4\|y\|^2_{\mathcal{B}_h} := q'. \end{aligned}$$

From our assumptions, using the fact that $(\sum_{i=1}^n a_i)^2 \le n\sum_{i=1}^n a_i^2$ for any positive real numbers a_i, $i = 1, 2, ..., n$, we have

$$\begin{aligned} \mathbb{E}\|u_{z+y}\|^2 \le \quad & 6M_w\{\|x_1\|^2 + M\mathbb{E}\|\varphi(0)\|^2 + 2Mc_1^2[M_g\|y\|^2_{\mathcal{B}_h} + \overline{M}_g] \\ & + 2(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})[M_g q' + \overline{M}_g] + MT\int_0^T p_{q'}(s)ds \\ & + 2MT^{2H-1}\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds\} := \mathcal{G}. \end{aligned} \tag{3.5}$$

It is clear that the operator Π has a fixed point if and only if $\widehat{\Pi}$ has one, so it turns to prove that $\widehat{\Pi}$ has a fixed point. To this end, we decompose $\widehat{\Pi}$ as $\widehat{\Pi} = \Pi_1 + \Pi_2$, where Π_1 and Π_2 are defined on $\mathcal{B}_T^0$, respectively by

$$(\Pi_1 z)(t) = \begin{cases} 0, \text{ if } t \in (-\infty, 0], \\ g(t, z_t + y_t) - S(t)g(0,\varphi)) + \int_0^t AS(t-s)g(s, z_s + y_s)ds \\ + \int_0^t S(t-s)\sigma(s)dB^H(s), \quad \text{if } t \in [0,T], \end{cases} \tag{3.6}$$

and

$$(\Pi_2 z)(t) = \begin{cases} 0, \text{ if } t \in (-\infty, 0], \\ \int_0^t S(t-s)f(s, z_s + y_s)ds \\ + \int_0^t S(t-s)Bu_{z+y}(s)ds, \text{ if } t \in [0,T]. \end{cases} \tag{3.7}$$

In order to apply the Karasnoselskii fixed point theorem for the operator $\widehat{\Pi}$, we prove the following assertions:

1. $\Pi_1(x) + \Pi_2(x) \in \mathcal{B}_k$ whenever $x \in \mathcal{B}_k$,
2. Π_1 is a contraction;
3. Π_2 is continuous and compact map.

For the sake of convenience, the proof will be given in several steps.
Step 1. We claim that there exists a positive number k, such that $\Pi_1(x) + \Pi_2(x) \in \mathcal{B}_k$ whenever $x \in \mathcal{B}_k$. If it is not true, then for each positive number k, there is a function $z^k(.) \in \mathcal{B}_k$, but $\Pi_1(z^k) + \Pi_2(z^k) \notin \mathcal{B}_k$, that is $\mathbb{E}\|\Pi_1(z^k)(t) + \Pi_2(z^k)(t)\|^2 > k$ for some $t \in [0, T]$. However, on the other hand, we have

$$\begin{aligned} k \quad &< \mathbb{E}\|\Pi_1(z^k)(t) + \Pi_2(z^k)(t)\|^2 \\ &\leq 6\{2Mc_1^2(M_g\|y\|_{\mathcal{B}_h}^2 + \overline{M}_g) + 2(c_1^2 q' + \overline{M}_g) + 2\tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1}[M_g q' + \overline{M}_g] \\ &+ MM_bT^2\mathcal{G} + MT\textstyle\int_0^T p_{q'}(s)ds + 2MT^{2H-1}\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds\} \\ &\leq 6(1+6MM_bM_wT^2)\{2Mc_1^2(M_g\|y\|_{\mathcal{B}_h}^2 + \overline{M}_g) + 2(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})[M_g q' + \overline{M}_g]) \\ &+ MT\textstyle\int_0^T p_{q'}(s)ds + 2MT^{2H-1}\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds\} \\ &+ 6MM_bM_wT^2(\|x_1\|^2 + M\mathbb{E}\|\varphi(0)\|^2) \\ &\leq \overline{K} + 6(1+6MM_bM_wT^2)\{2(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})M_g q' + 2MT\textstyle\int_0^T p_{q'}(s)ds\}, \end{aligned}$$

where

$$\begin{aligned} \overline{K} \quad &= 6(1+6MM_bM_wT^2)\{2Mc_1^2(M_g\|y\|_{\mathcal{B}_h}^2 + \overline{M}_g) + 2(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})\overline{M}_g) \\ &+ 2MT^{2H-1}\textstyle\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds\} + 6MM_bM_wT^2(\|x_1\|^2 + M\mathbb{E}\|\varphi(0)\|^2)\}. \end{aligned}$$

Noting that $\overline{K}$ is independent of k. Dividing both sides by k and taking the lower limit as $k \longrightarrow \infty$, we obtain

$$q' = 4l^2(k + M\mathbb{E}\|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h} \longrightarrow \infty \;\text{ as } k \longrightarrow \infty,$$

$$\liminf_{k\longrightarrow\infty} \frac{\int_0^T p_{q'}(s)ds}{k} = \liminf_{k\longrightarrow\infty} \frac{\int_0^T p_{q'}(s)ds}{q'}.\frac{q'}{k} = 4l^2\gamma.$$

Thus, we have

$$6l^2(1+6MM_bM_wT^2)\{8(c_1^2 + \frac{(M_{1-\beta}T^\beta)^2}{2\beta-1})M_g + 8MT\gamma\} \geq 1.$$

This contradicts (3.2). Hence for some positive k,

$$(\Pi_1 + \Pi_2)(\mathcal{B}_k) \subseteq \mathcal{B}_k.$$

Step 2. Π_1 is a contraction. Let $t \in [0,T]$ and $z^1, z^2 \in \mathcal{B}_T^0$

$$\begin{aligned}
\mathbb{E}\|(\Pi_1 z^1)(t) - (\Pi_1 z^2)(t)\|^2 \quad & \leq 2\mathbb{E}\|g(t, z_t^1 + y_t) - g(t, z_t^2 + y_t)\|^2 \\
& +2\mathbb{E}\| \textstyle\int_0^t AS(t-s)(g(s, z_s^1 + y_s) - g(s, z_s^2 + y_s))ds\|^2 \\
& \leq 2M_g\|(-A)^{-\beta}\|^2\|z_s^1 - z_s^2\|_{\mathcal{B}_h}^2 \\
& +2T \textstyle\int_0^t \frac{M_{1-\beta}^2}{(t-s)^{2(1-\beta)}} M_g\|z_s^1 - z_s^2\|_{\mathcal{B}_h}^2 ds \\
& \leq 2M_g \left\{ \|(-A)^{-\beta}\|^2 + \frac{(M_{1-\beta}T^\beta)^2}{(2\beta-1)} \right\} (2l^2 \sup_{0\leq s\leq T} \\
& \mathbb{E}\|z^1(s) - z^2(s)\|^2 + 2(\|z_0^1\|_{\mathcal{B}_h}^2 + \|z_0^2\|_{\mathcal{B}_h}^2) \\
& \leq \nu \sup_{0\leq s\leq T} \mathbb{E}\|z^1(s) - z^2(s)\|^2) \quad (\text{ since } \ z_0^1 = z_0^2 = 0)
\end{aligned}$$

Taking supremum over t,

$$\|(\Pi_1 z^1)(t) - (\Pi_1 z^2)(t)\|_{\mathcal{B}_T^0} \leq \nu\|z^1 - z^2\|_{\mathcal{B}_T^0},$$

where

$$\nu = 4M_g l^2(\|(-A)^{-\beta}\|^2 + \frac{(M_{1-\beta}T^\beta)^2}{2\beta - 1}) < 1.$$

Thus Π_1 is a contraction on $\mathcal{B}_T^0$.
Step 3. Π_2 is completely continuous $\mathcal{B}_T^0$.
Claim 1. Π_2 is continuous on $\mathcal{B}_T^0$.
Let z^n be a sequence such that $z^n \longrightarrow z$ in $\mathcal{B}_T^0$. Then, there exists a number $k > 0$ such that $\|z^n(t)\| \leq k$, for all n and a.e. $t \in [0,T]$, so $z^n \in \mathcal{B}_k$ and $z \in \mathcal{B}_k$. By hypothesis $(\mathcal{H}.2)$, we have $f(t, z_t^n + y_t) \longrightarrow f(t, z_t + y_t)$ for each $t \in [0,T]$. Since
$\|f(t, z_t^n + y_t) - f(t, z_t + y_t)\|^2 \leq 2p_{q'}(t)$. From $(\mathcal{H}.3)$, Hölder inequality and the dominated convergence theorem, we have

$$\begin{aligned}
\mathbb{E}\|\Pi_2 z^n(t) - (\Pi_2 z)(t)\|^2 \quad & \leq 2\mathbb{E}\| \textstyle\int_0^t S(t-s)B[u_{z^n+y} - u_{z+y}]ds\|^2 \\
& +2\mathbb{E}\| \textstyle\int_0^t S(t-s)[f(s, z_s^n + y_s) - f(s, z_s + y_s)]ds\|^2 \\
& \leq 6M_w M_b MT \textstyle\int_0^T \{E\|g(T, z_T^n + y_T) - g(T, z_T + y_T)\|^2 \\
& +T \textstyle\int_0^T \mathbb{E}\|AS(T-s)g(s, z_s^n + y_s) - AS(T-s)g(s, z_s + y_s)\|^2 ds \\
& +MT \textstyle\int_0^T \mathbb{E}\|f(s, z_s^n + y_s) - f(s, z_s + y_s)\|^2 ds\}(\eta)d\eta \\
& +2MT \textstyle\int_0^T \mathbb{E}\|f(s, z_s^n + y_s) - f(s, z_s + y_s)\|^2 ds \\
& \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Thus, Π_2 is continuous.

Claim 2. Π_2 maps $\mathcal{B}_k$ into equicontinuous family.
Let $z \in \mathcal{B}_k$ and τ_1, $\tau_2 \in [0, T]$, we have

$$\begin{aligned}\mathbb{E}\|(\Pi_2 z)(\tau_2) - (\Pi_2 z)(\tau_1)\|^2 \leq\ & 4\mathbb{E}\| \textstyle\int_0^{\tau_1} (S(\tau_2 - s) - S(\tau_1 - s))f(s, z_s + y_s)ds\|^2 \\ & +4\mathbb{E}\| \textstyle\int_0^{\tau_1} (S(\tau_2 - s) - S(\tau_1 - s))Bu(s)ds\|^2 \\ & +4\mathbb{E}\| \textstyle\int_{\tau_1}^{\tau_2} S(\tau_2 - s)f(s, z_s + y_s)ds\|^2 \\ & +4\mathbb{E}\| \textstyle\int_{\tau_1}^{\tau_2} S(\tau_2 - s))Bu(s)ds\|^2.\end{aligned}$$

From (3.5), Hölder inequality, it follows that

$$\begin{aligned}\mathbb{E}\|(\Pi_2 z)(\tau_2) - (\Pi_2 z)(\tau_1)\|^2 \leq\ & 4T\| \textstyle\int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|^2 p_{q'}(s)ds \\ & +4TM_b\mathcal{G} \textstyle\int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|^2 ds \\ & +4T \textstyle\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|^2 p_{q'}(s)ds \\ & +4TM_b\mathcal{G} \textstyle\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|^2 ds.\end{aligned}$$

The right-hand side is independent of $z \in \mathcal{B}_k$ and tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$, since the compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Thus, Π_2 maps $\mathcal{B}_k$ into an equicontinuous family of functions. The equicontinuities for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 < 0 < \tau_2$ are obvious.
Claim 3. $(\Pi_2 \mathcal{B}_k)(t)$ is precompact set in X.
Let $0 < t \leq T$ be fixed, $0 < \epsilon < t$, for $z \in \mathcal{B}_k$, we define

$$(\Pi_{2,\epsilon} z)(t) = S(\epsilon) \int_0^{t-\epsilon} S(t - s - \epsilon)f(s, z_s + y_s)ds + S(\epsilon) \int_0^{t-\epsilon} S(t - s - \epsilon)Bu(s)ds.$$

Using the estimation (3.5) as above and by the compactness of $S(t)$ $(t > 0)$, we obtain $V_\epsilon(t) = \{(\Pi_{2,\epsilon} z)(t) :\ z \in \mathcal{B}_k\}$ is relative compact in X for every ϵ, $0 < \epsilon < t$. Moreover, for every $z \in \mathcal{B}_k$, we have

$$\begin{aligned}\mathbb{E}\|\Pi_2 z)(t) - \Pi_{2,\epsilon} z)(t)\|^2 \leq\ & 2T \textstyle\int_{t-\epsilon}^{t} \|S(t - s)\|^2 \mathbb{E}\|f(s, z_s + y_s)\|^2 ds \\ & +2TM_b\mathcal{G} \textstyle\int_{t-\epsilon}^{t} \|S(t - s)\|^2 ds \\ & \leq 2TM \textstyle\int_{t-\epsilon}^{t} p_{q'}(s)ds + 2TM_b\mathcal{G}M\epsilon.\end{aligned}$$

Therefore,

$$\mathbb{E}\|\Pi_2 z)(t) - \Pi_{2,\epsilon} z)(t)\|^2 \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0^+,$$

and there are precompact sets arbitrarily close to the set $V(t) = \{(\Pi_2 z)(t) :\ z \in B_k\}$, hence the set $V(t)$ is also precompact in X.

Thus, by Arzela-Ascoli theorem Π_2 is a compact operator. These arguments enable us to conclude that Π_2 is completely continuous, and by the fixed point theorem of Karasnoselskii there exists a fixed point $z(.)$ for $\widehat{\Pi}$ on $\mathcal{B}_k$. If we define $x(t) = z(t) + y(t)$, $-\infty < t \leq T$, it is easy to see that $x(.)$ is a mild solution of (1) satisfying $x_0 = \varphi$, $x(T) = x_1$. Then the proof is complete.

4 Conclusion

Our paper contains some controllability results for neutral stochastic functional differential equations with infinite delay driven by fractional Brownian motion in a real separable Hilbert space. The result proves that the fixed-point theorem can effectively be used in control problems to obtain sufficient conditions. We can extend the controllability result for neutral impulsive stochastic systems with different types of delays in our subsequent papers.

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